Lecture 4

INTERCONNECTION AND CONTROL

OUTLINE

Part I: Interconnection of dynamical systems

1. **Interconnection**
2. **Regular interconnection**
3. **Feedback interconnection**
4. **Control as interconnection**
5. **Implementability**

OUTLINE

Part II: Stabilization and pole placement

1. Problem formulation

2. Stabilizability and controllability

3. Solution of the stabilization and pole placement problem

4. Feedback implementability of the controllers

PART I

INTERCONNECTION OF DYNAMICAL SYSTEMS

FULL INTERCONNECTION

Two dynamical systems:

$$\Sigma_1 = (\mathbb{T}, \mathbb{W}, \mathfrak{B}_1),$$

$$\Sigma_2 = (\mathbb{T}, \mathbb{W}, \mathfrak{B}_2).$$

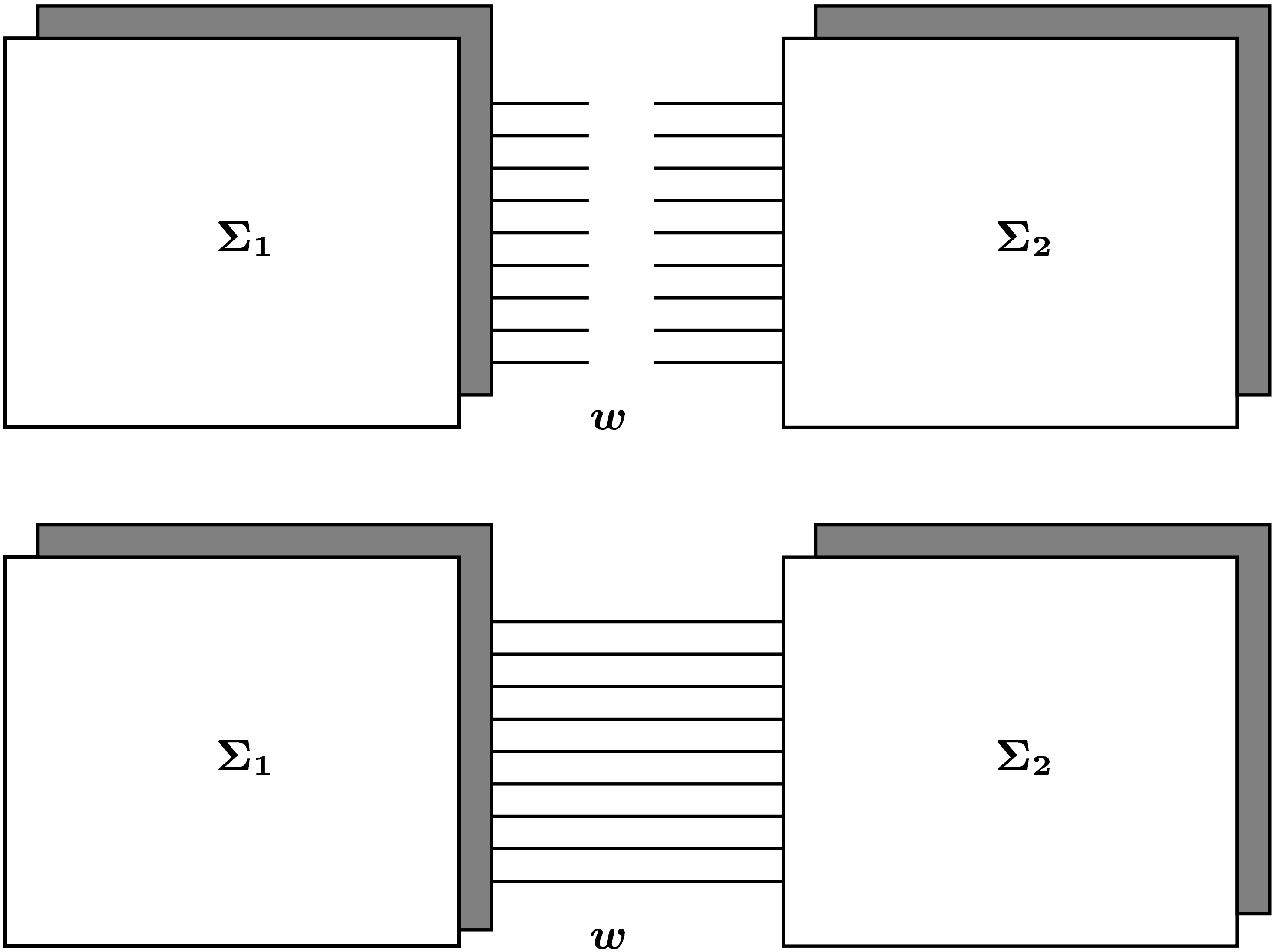
Common time axis $\mathbb{T}$, common signal space $\mathbb{W}$.

The full interconnection of Σ_1 and Σ_2 is defined by:

$$\Sigma_1 \wedge \Sigma_2 := (\mathbb{T}, \mathbb{W}, \mathfrak{B}_1 \cap \mathfrak{B}_2).$$

Note: the behavior of $\Sigma_1 \wedge \Sigma_2$ is the intersection of the behaviors of Σ_1 and Σ_2.

The full interconnection consists of those trajectories that are compatible with both the laws of Σ_1 ánd Σ_2.

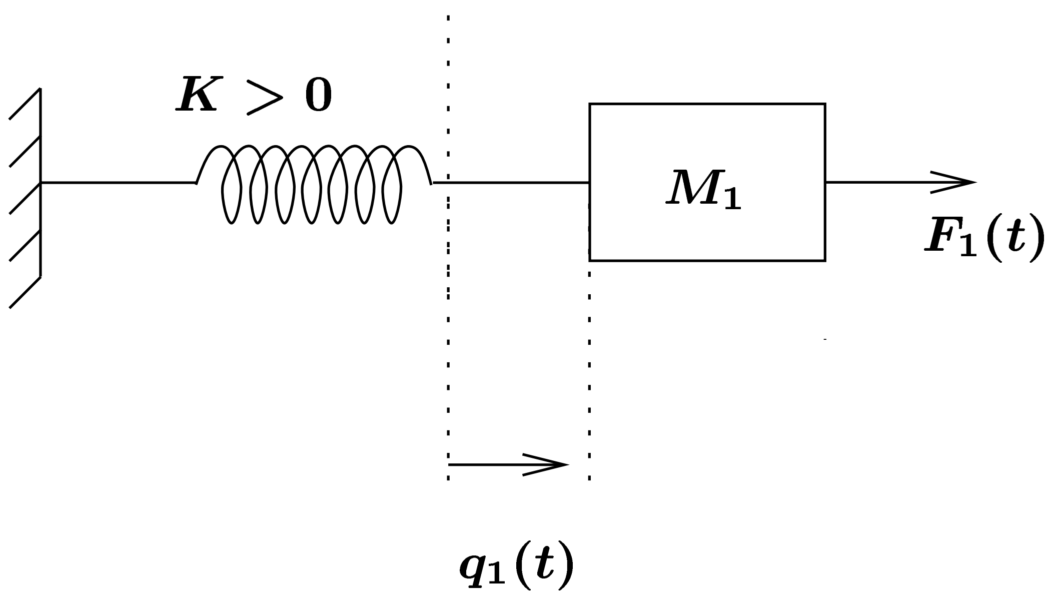


Σ_1
Σ_2
w
Σ_1
Σ_2
w

EXAMPLE

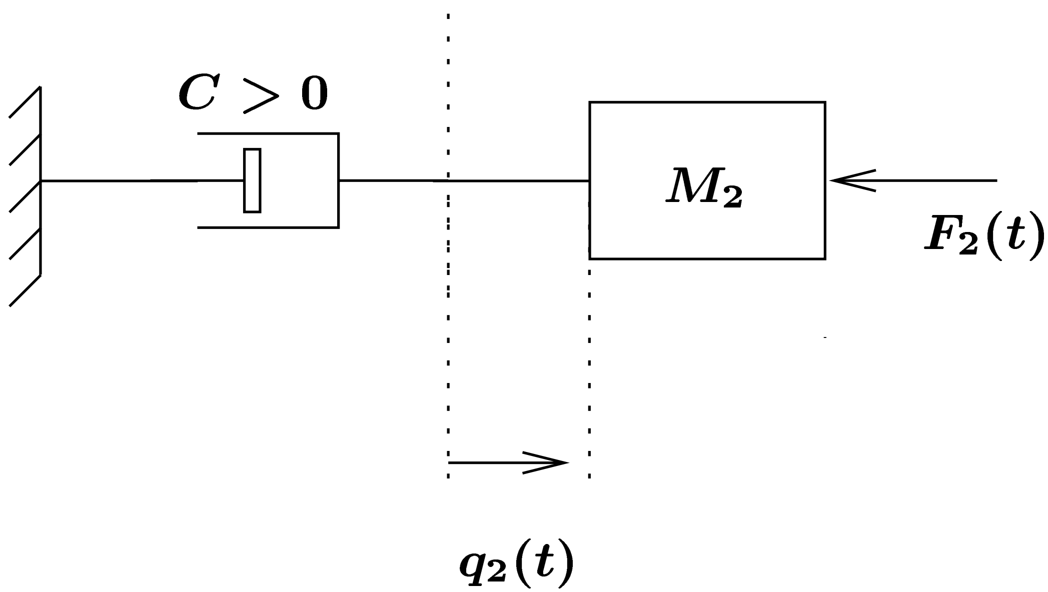
$\Sigma_1 = (\mathbb{R}, \mathbb{R}^2, \mathfrak{B}_1)$ **with**

$$\mathfrak{B}_1 = \{(q_1, F_1) \mid M_1 \frac{d^2 q_1}{dt^2} + K q_1 = F_1\}$$



$\Sigma_2 = (\mathbb{R}, \mathbb{R}^2, \mathfrak{B}_2)$ **with**

$$\mathfrak{B}_2 = \{(q_2, F_2) \mid M_2 \frac{d^2 q_2}{dt^2} + C \frac{dq_2}{dt} = -F_2\}$$

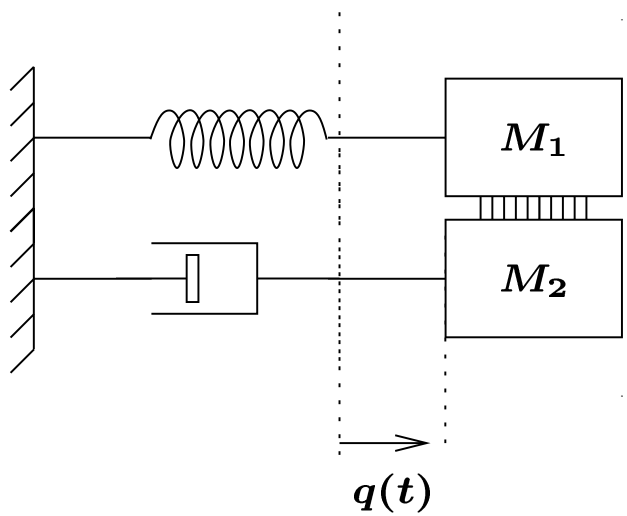


Full interconnection of Σ_1 and Σ_2 :

$\Sigma_1 \wedge \Sigma_2 = (\mathbb{R}, \mathbb{R}^2, \mathfrak{B}_1 \cap \mathfrak{B}_2)$,

$$\mathfrak{B}_2 \cap \mathfrak{B}_2 = \{(q, F) \mid \begin{bmatrix} M_1 \frac{d^2}{dt^2} + K & -1 \\ M_2 \frac{d^2}{dt^2} + C \frac{d}{dt} & 1 \end{bmatrix} \begin{bmatrix} q \\ F \end{bmatrix} = 0\}$$

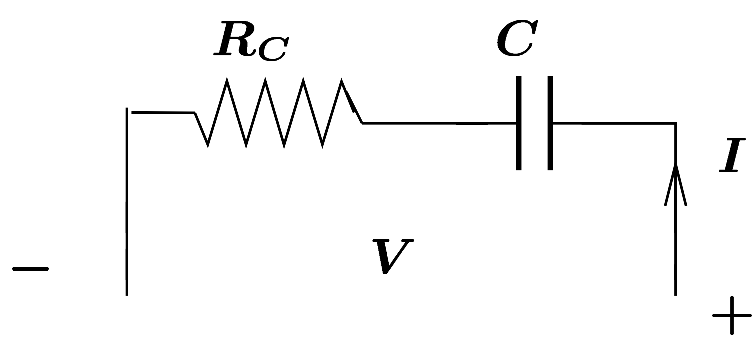
Note: $(q, F) \in \mathfrak{B}_1 \cap \mathfrak{B}_2 \Rightarrow (M_1 + M_2) \frac{d^2 q}{dt^2} + C \frac{dq}{dt} + Kq = 0.$



EXAMPLE

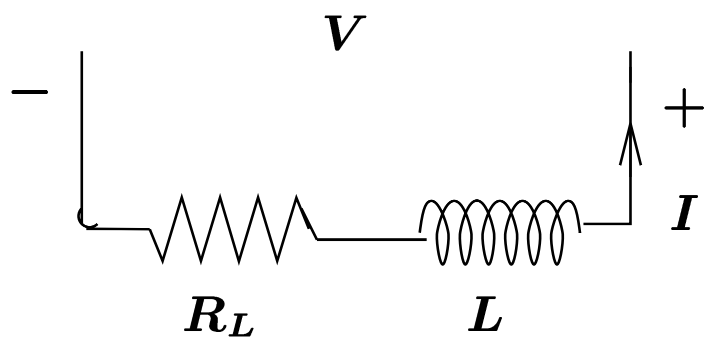
$\Sigma_1 = (\mathbb{R}, \mathbb{R}^2, \mathfrak{B}_1)$ **with**

$$\mathfrak{B}_1 = \{(V_1, I_1) \mid \frac{dV_1}{dt} = R_C \frac{dI_1}{dt} + \frac{1}{C} I_1\}$$



$\Sigma_2 = (\mathbb{R}, \mathbb{R}^2, \mathfrak{B}_2)$ with

$$\mathfrak{B}_2 = \{(V_2, I_2) \mid V_2 = -R_L I_2 - L\frac{dI_2}{dt}\}$$

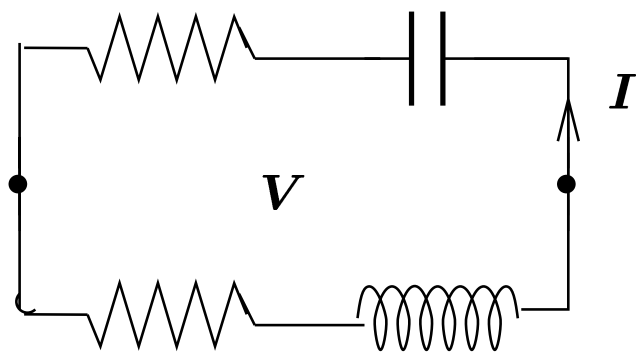


Full interconnection of Σ_1 and Σ_2 :

$\Sigma_1 \wedge \Sigma_2 = (\mathbb{R}, \mathbb{R}^2, \mathfrak{B}_1 \cap \mathfrak{B}_2)$,

$$\mathfrak{B}_1 \cap \mathfrak{B}_2 = \{(V, I) \mid \begin{bmatrix} \frac{d}{dt} & -R_C \frac{d}{dt} - \frac{1}{C} \\ 1 & L\frac{d}{dt} + R_L \end{bmatrix} \begin{bmatrix} V \\ I \end{bmatrix} = 0\}$$

Note: $(V, I) \in B_1 \cap \mathfrak{B}_2 \Rightarrow \frac{d^2 I}{dt^2} + \frac{R_L + R_C}{L} \frac{dI}{dt} + \frac{1}{LC} I = 0.$



INTERCONNECTION THROUGH SPECIFIC COMPONENTS

Often the interconnection of systems takes place only through certain components of the manifest variable:

$$\Sigma_1 = (\mathbb{T}, \mathbb{W}_1 \times \mathbb{W}_2, \mathfrak{B}_1),$$

$$\Sigma_2 = (\mathbb{T}, \mathbb{W}_2 \times \mathbb{W}_3, \mathfrak{B}_2).$$

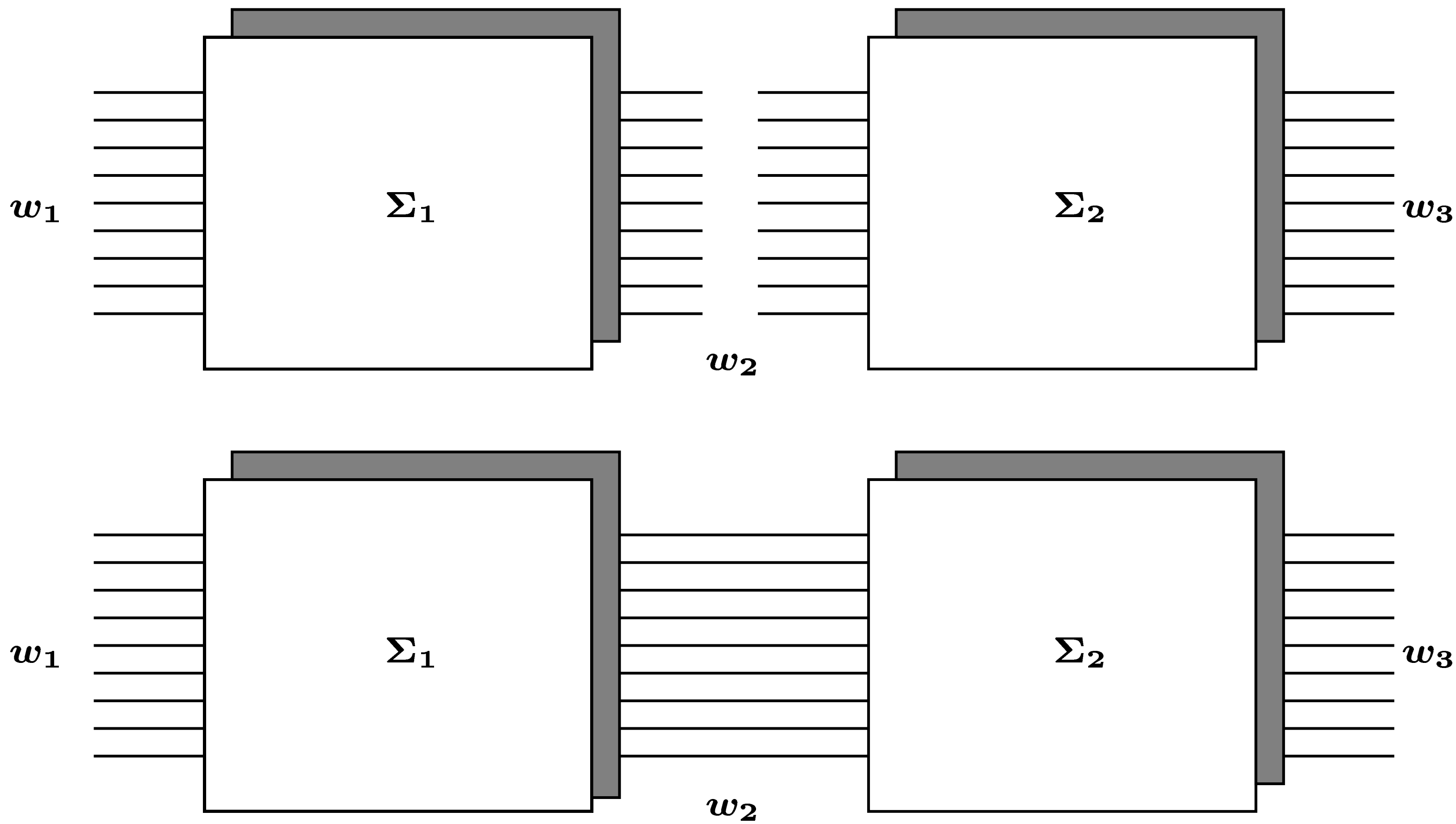
Common time axis $\mathbb{T}$, and common factor $\mathbb{W}_2$ in the signal space $\mathbb{W}$.

The interconnection of Σ_1 and Σ_2 through w_2 is defined by:

$$\Sigma_1 \wedge_{w_2} \Sigma_2 := (\mathbb{T}, \mathbb{W}_1 \times \mathbb{W}_2 \times \mathbb{W}_3, \mathfrak{B}),$$

where

$$\mathfrak{B} := \{(w_1, w_2, w_3) \mid (w_1, w_2) \in \mathfrak{B}_1 \text{ and } (w_2, w_3) \in \mathfrak{B}_2\}.$$



w_1
Σ_1
Σ_2
w_3
w_2
w_1
Σ_1
Σ_2
w_3
w_2

EXAMPLE: FEEDBACK

$\Sigma_1 = (\mathbb{R}, \mathbb{R}^{\mathtt{z}+\mathtt{d}} \times \mathbb{R}^{\mathtt{u}+\mathtt{y}}, \mathfrak{B}_1)$, with latent variable representation (latent variable x_1)

$$\frac{d}{dt}x_1 = Ax_1 + Bu + Ed,\ y = C_1x_1,\ z = C_2x_1.$$

We take $w_1 = (z, d)$ and $w_2 = (u, y)$.

$\Sigma_1 = (\mathbb{R}, \mathbb{R}^{\mathtt{u}+\mathtt{y}}, \mathfrak{B}_2)$, with latent variable representation (latent variable x_2)

$$\frac{d}{dt}x_2 = Kx_2 + Ly,\ u = Mx_2 + Ny.$$

We take $w_2 = (u, y)$, w_3 is absent.

The interconnection of Σ_1 and Σ_2 through (u, y) is equal to the usual feedback interconnection, with latent variable representation (latent variable (x_1, x_2)):

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} A + BNC_1 & BM \\ LC_1 & K \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} E \\ 0 \end{bmatrix} d,$$

$$\begin{bmatrix} z \\ u \\ y \end{bmatrix} = \begin{bmatrix} C_2 & 0 \\ NC_1 & 0 \\ C_1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Interconnection through specific components can be considered as full interconnection:

$$\boldsymbol{\Sigma_1} = (\mathbb{T}, \mathbb{W}_\mathbf{1} \times \mathbb{W}_\mathbf{2}, \mathfrak{B}_1) \longrightarrow \boldsymbol{\Sigma'_1} = (\mathbb{T}, \mathbb{W}_\mathbf{1} \times \mathbb{W}_\mathbf{2} \times \mathbb{W}_\mathbf{3}, \mathfrak{B}_1 \times \mathbb{W}_\mathbf{3}^{\mathbb{T}})$$

$$\boldsymbol{\Sigma_2} = (\mathbb{T}, \mathbb{W}_\mathbf{2} \times \mathbb{W}_\mathbf{3}, \mathfrak{B}_2) \longrightarrow \boldsymbol{\Sigma'_2} = (\mathbb{T}, \mathbb{W}_\mathbf{1} \times \mathbb{W}_\mathbf{2} \times \mathbb{W}_\mathbf{3}, \mathbb{W}_\mathbf{1}^{\mathbb{T}} \times \mathfrak{B}_2)$$

The interconnection of $\boldsymbol{\Sigma}_1$ and $\boldsymbol{\Sigma}_2$ through w_2 is equal to the full interconnection of $\boldsymbol{\Sigma}'_1$ and $\boldsymbol{\Sigma}'_2$:

$$\boldsymbol{\Sigma}_1 \wedge_{w_2} \boldsymbol{\Sigma}_2 = \boldsymbol{\Sigma}'_1 \wedge \boldsymbol{\Sigma}'_2$$

REGULAR INTERCONNECTION

Two linear time-invariant differential systems:

$$\Sigma_1 = (\mathbb{R}, \mathbb{R}^{\mathtt{w}}, \mathfrak{B}_1),$$

$$\Sigma_2 = (\mathbb{R}, \mathbb{R}^{\mathtt{w}}, \mathfrak{B}_2).$$

The full interconnection $\Sigma_1 \wedge \Sigma_2$ is called a regular full interconnection if the output cardinality of $\Sigma_1 \wedge \Sigma_2$ is equal to the sum of the output cardinalities of Σ_1 and Σ_2:

$$\mathrm{p}(\mathfrak{B}_1 \cap \mathfrak{B}_2) = \mathrm{p}(\mathfrak{B}_1) + \mathrm{p}(\mathfrak{B}_2).$$

In the case of interconnection through specific components:

$$\Sigma_1 = (\mathbb{R}, \mathbb{R}^{\mathtt{w}_1} \times \mathbb{R}^{\mathtt{w}_2}, \mathfrak{B}_1),$$

$$\Sigma_2 = (\mathbb{R}, \mathbb{R}^{\mathtt{w}_2} \times \mathbb{R}^{\mathtt{w}_3}, \mathfrak{B}_2).$$

The interconnection $\Sigma_1 \wedge_{w_2} \Sigma_2$ is called a regular interconnection if the output cardinality of $\Sigma_1 \wedge_{w_2} \Sigma_2$ is equal to the sum of the output cardinalities of Σ_1 and Σ_2:

$$\mathtt{p}(\mathfrak{B}) = \mathtt{p}(\mathfrak{B}_1) + \mathtt{p}(\mathfrak{B}_2),$$

with

$$\mathfrak{B} = \{(w_1, w_2, w_3) \mid (w_1, w_2) \in \mathfrak{B}_1 \text{ and } (w_2, w_3) \in \mathfrak{B}_2\}.$$

EXAMPLE: FEEDBACK

Recall the previous example on the feedback interconnection of finite dimensional linear systems.

$$\mathtt{p}(\boldsymbol{\Sigma_1}) = \mathtt{y} + \mathtt{z}.$$

$$\mathtt{p}(\boldsymbol{\Sigma_2}) = \mathtt{u},$$

$$\mathtt{p}(\boldsymbol{\Sigma_1} \wedge_{(u,y)} \boldsymbol{\Sigma_2}) = \mathtt{z} + \mathtt{u} + \mathtt{y}.$$

Conclusion: the usual feedback interconnection of finite dimensional linear systems is a regular interconnection.

FEEDBACK INTERCONNECTION

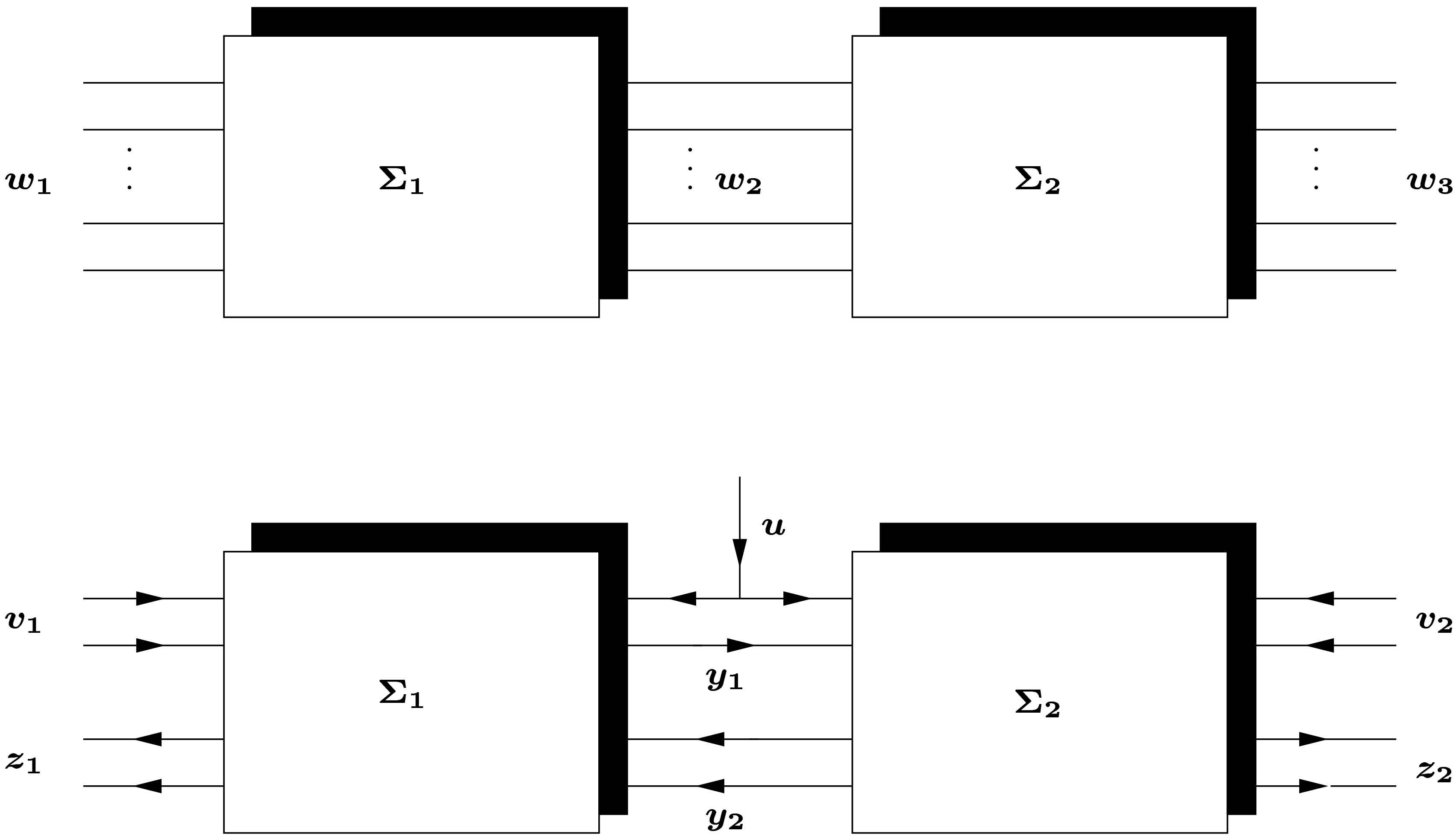
Two linear differential systems:

$$\Sigma_1 = (\mathbb{R}, \mathbb{R}^{w_1} \times \mathbb{R}^{w_2}, \mathfrak{B}_1) \in \mathfrak{L}^{w_1+w_2},$$

$$\Sigma_2 = (\mathbb{R}, \mathbb{R}^{w_2} \times \mathbb{R}^{w_3}, \mathfrak{B}_2) \in \mathfrak{L}^{w_2+w_3}.$$

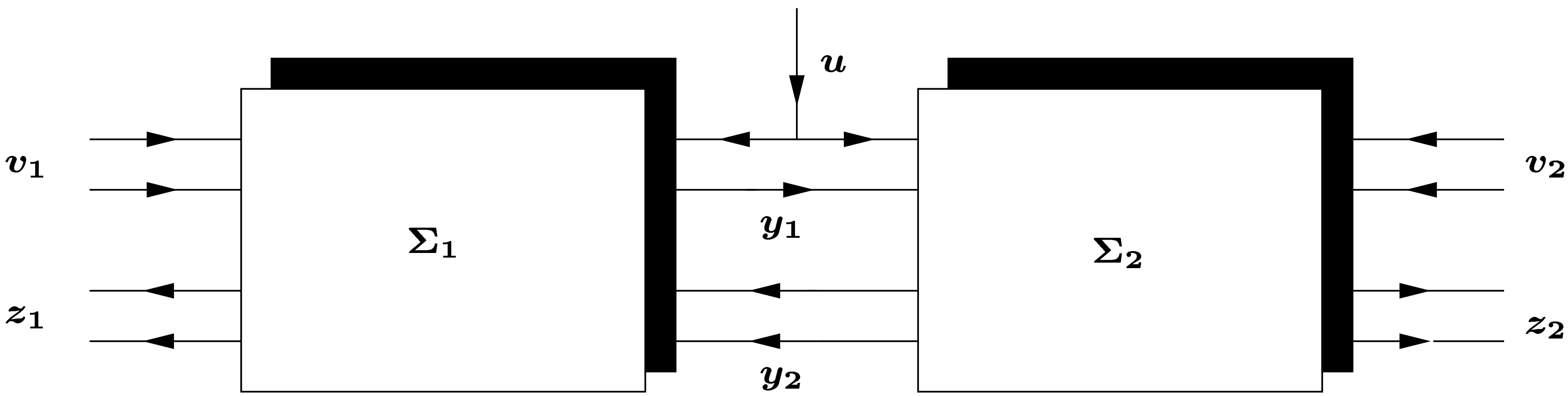
The interconnection of Σ_1 and Σ_2 through w_2 is called a feedback interconnection if (modulo permutation of components) there exist partitions $w_1 = (v_1, z_1)$, $w_2 = (u, y_1, y_2)$, and $w_3 = (v_2, z_2)$ such that

- **in Σ_1, (u, y_2, v_1) is input and (y_1, z_1) is output,**
- **in Σ_2, (u, y_1, v_2) is input and (y_2, z_2) is output,**
- **in $\Sigma \wedge_{w_2} \Sigma_2$, (u, v_1, v_2) is input and (y_1, y_2, z_1, z_2) is output.**



w_1
Σ_1
w_2
Σ_2
w_3
u
v_1
Σ_1
y_1
Σ_2
v_2
z_1
y_2
z_2

REGULAR ⇔ FEEDBACK



Obviously, every feedback interconnection is regular:

$$\mathrm{p}(\Sigma_1) = \mathrm{y}_1 + \mathrm{z}_1,\ \mathrm{p}(\Sigma_2) = \mathrm{y}_2 + \mathrm{z}_2,$$

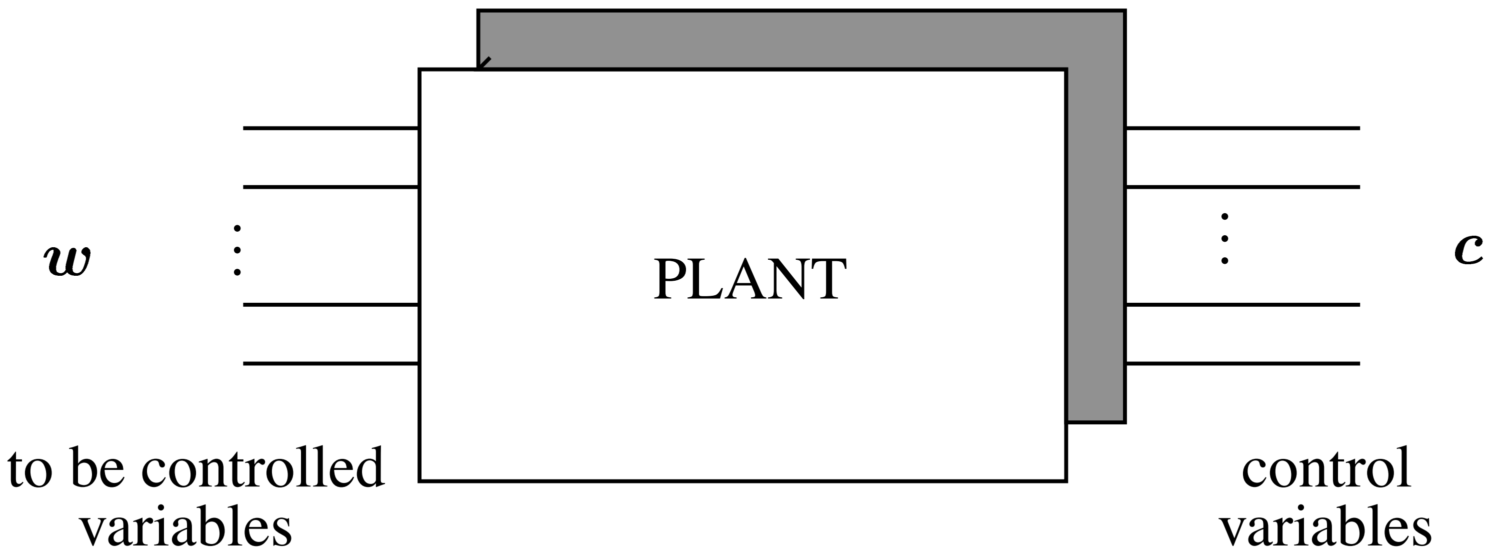
$$p(\Sigma_1 \wedge_{w_2} \Sigma_2) = \mathrm{y}_1 + \mathrm{y}_2 + \mathrm{z}_1 + \mathrm{z}_2.$$

Also the converse holds:

Theorem: Let $\Sigma_1 \in \mathfrak{L}^{\mathtt{w}_1+\mathtt{w}_2}$, $\Sigma_2 \in \mathfrak{L}^{\mathtt{w}_2+\mathtt{w}_3}$. The interconnection of Σ_1 and Σ_2 through w_2 is a regular interconnection if and only if it is a feedback interconnection.

CONTROL AS INTERCONNECTION

Plant to be controlled:



two kinds of variables:

- **variables to be controlled w (taking values in $\mathbb{W}$),**
- **control variables c (taking values in $\mathbb{W}_c$).**

The control variables are those variables through which we will interconnect the plant to a controller.

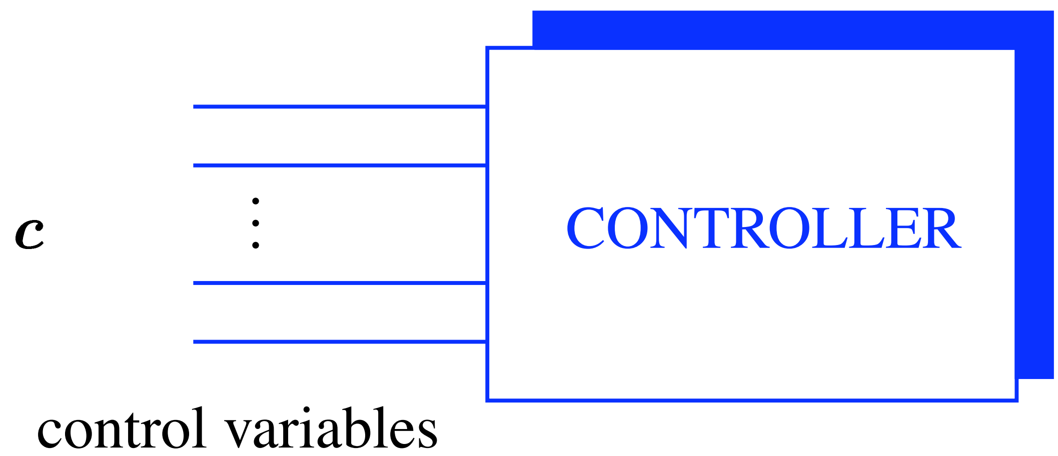
The plant is a dynamical system

$$\Sigma_{\mathrm{p}} = (\mathbb{T}, \mathbb{W} \times \mathbb{W}_c, \mathcal{P}_{\mathrm{full}}),$$

with full plant behavior

$$\mathcal{P}_{\mathrm{full}} := \{(w, c) \mid (w, c) \text{ satisfies the plant equations}\}.$$

Controllers:



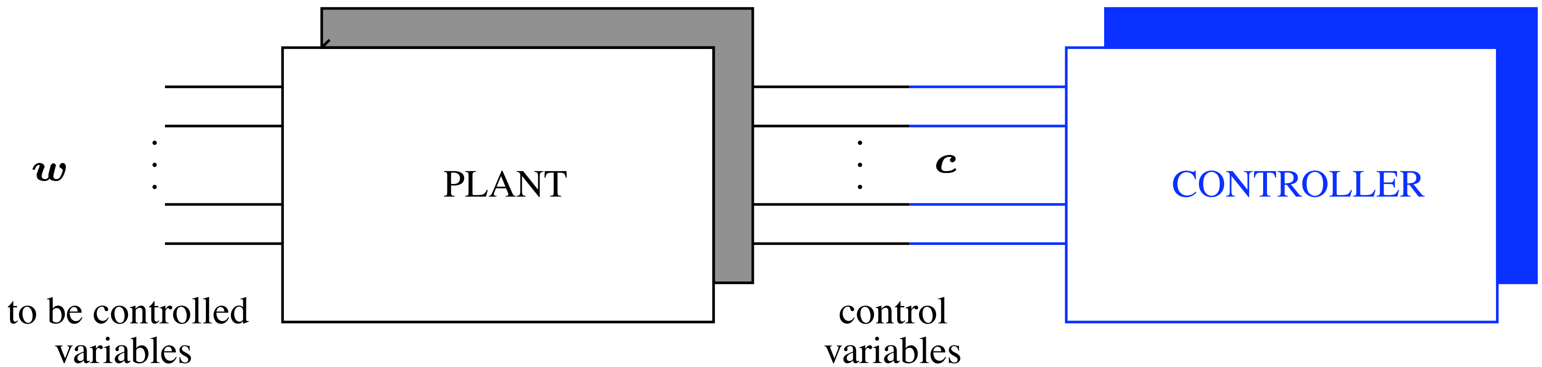
A controller for Σ_p is a dynamical system

$$\Sigma_c = (\mathbb{T}, \mathbb{W}_c, \mathcal{C}),$$

with controller behavior

$$\mathcal{C} = \{c \mid c \text{ satisfies the controller equations}\}.$$

Controlled plant:



The controlled plant is the interconnection of Σ_{p} and Σ_{c} through c:

$$\Sigma_{\mathrm{p}} \wedge_c \Sigma_{\mathrm{c}} = (\mathbb{T}, \mathbb{W} \times \mathbb{W}_c, \mathcal{K}_{\mathrm{full}}),$$

with **full controlled behavior**

$$\mathcal{K}_{\mathrm{full}} = \{(w, c) \mid (w, c) \in \mathcal{P}_{\mathrm{full}} \text{ and } c \in \mathcal{C}\}.$$

GENERAL CONTROL PROBLEM

We define the **manifest controlled behavior** by

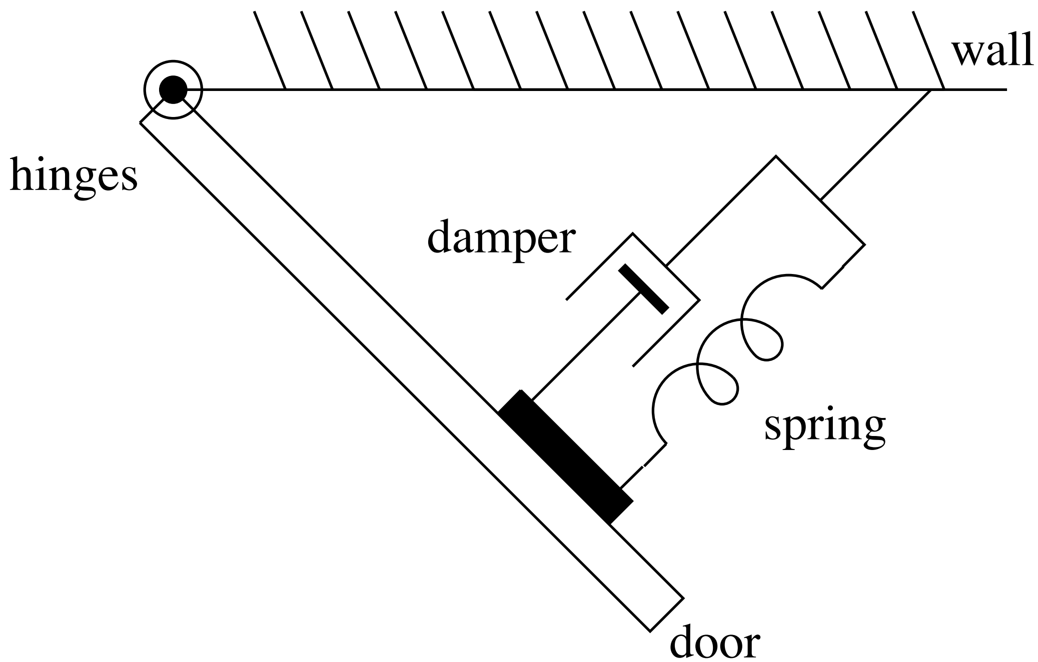
$$\mathcal{K} := \{w \mid \text{ there exists } c \text{ such that } (w, c) \in \mathcal{K}_{\mathrm{full}}\}.$$

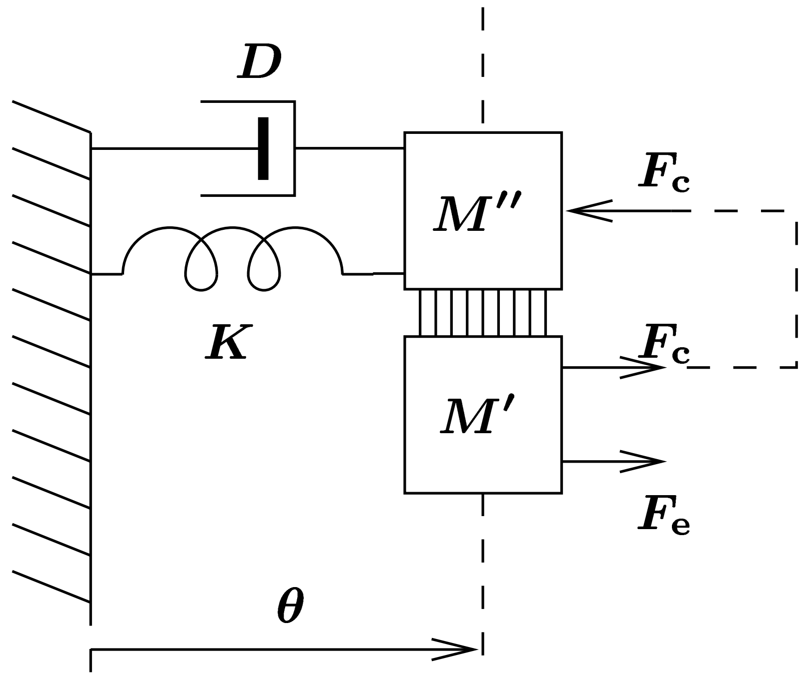
General control problem: given the plant Σ_{p}

- specify a family $\mathcal{A}$ of admissible controllers,
- describe a set of specifications on the controlled plant, i.e. desired properties of the manifest controlled behavior $\mathcal{K}$,
- find a controller $\Sigma_{\mathrm{c}} \in \mathcal{A}$ such that the manifest controlled behavior $\mathcal{K}$ satisfies these specifications.

EXAMPLE

Door closing mechanism:





Equation of motion of the door:

$$M' \frac{d^2\theta}{dt^2} = F_c + F_e.$$

No friction in the hinges, M' mass of the door, F_c force to be exerted by the door closing device, F_e exogenous force.

To be controlled variable: $w = (\theta, F_e)$.

Control variable: $c = (\theta, F_c)$

Plant:

$$\Sigma_{\mathrm{p}} = (\mathbb{R}, \mathbb{R}^2 \times \mathbb{R}^2, \mathcal{P}_{\mathrm{full}}),$$

with $\mathcal{P}_{\mathrm{full}}$ **all** $(w, c) = ((\theta, F_e), (\theta, F_c))$ **that satisfy the equation of motion of the door.**

Door closing mechanism modeled as mass-spring-damper combination:

$$M'' \frac{d^2\theta}{dt^2} + D\frac{d\theta}{dt} + K\theta = -F_c$$

M'' **mass of the door closing mechanism,** D **damping coefficient,** K **spring constant.**

Controller: $\Sigma_{\mathrm{c}} = (\mathbb{R}, \mathbb{R}^2, \mathcal{C})$, **with** $\mathcal{C}$ **all** $c = (\theta, F_c)$ **that satisfy the equation of motion of the door closing mechanism.**

Controlled plant: $\Sigma_p \wedge_c \Sigma_c = (\mathbb{R}, \mathbb{R}^2 \times \mathbb{R}^2, \mathcal{K}_{\text{full}})$ with full controlled behavior $\mathcal{K}_{\text{full}}$: all $((\theta, F_e), (\theta, F_c))$ that satisfy the equations of motion of the door ánd the door closing mechanism.

Manifest controlled behavior:

$$\mathcal{K} = \{(\theta, F_e) \mid (M' + M'')\frac{d^2\theta}{dt^2} + D\frac{d\theta}{dt} + K\theta = F_e\}$$

Specifications on the controlled system: the system $\mathcal{K}$ should have small overshoot, fast settling time, not-to-high steady state gain from F_e to θ.

Finding a suitable controller means finding suitable values for M', K and D.

IMPLEMENTABILITY

Let $\Sigma_{\mathrm{p}} = (\mathbb{R}, \mathbb{R}^{\mathtt{w}+\mathtt{c}}, \mathcal{P}_{\mathrm{full}})$ be a linear differential system, i.e., $\mathcal{P}_{\mathrm{full}} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}}$.

Let the controller $\Sigma_{\mathrm{c}} = (\mathbb{R}, \mathbb{R}^{\mathtt{c}}, \mathcal{C})$ be a linear differential system, i.e., $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$.

Let $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}$. If $\mathcal{K}$ is equal to the manifest controlled behavior obtained by interconnecting Σ_{p} and Σ_{c}, i.e.

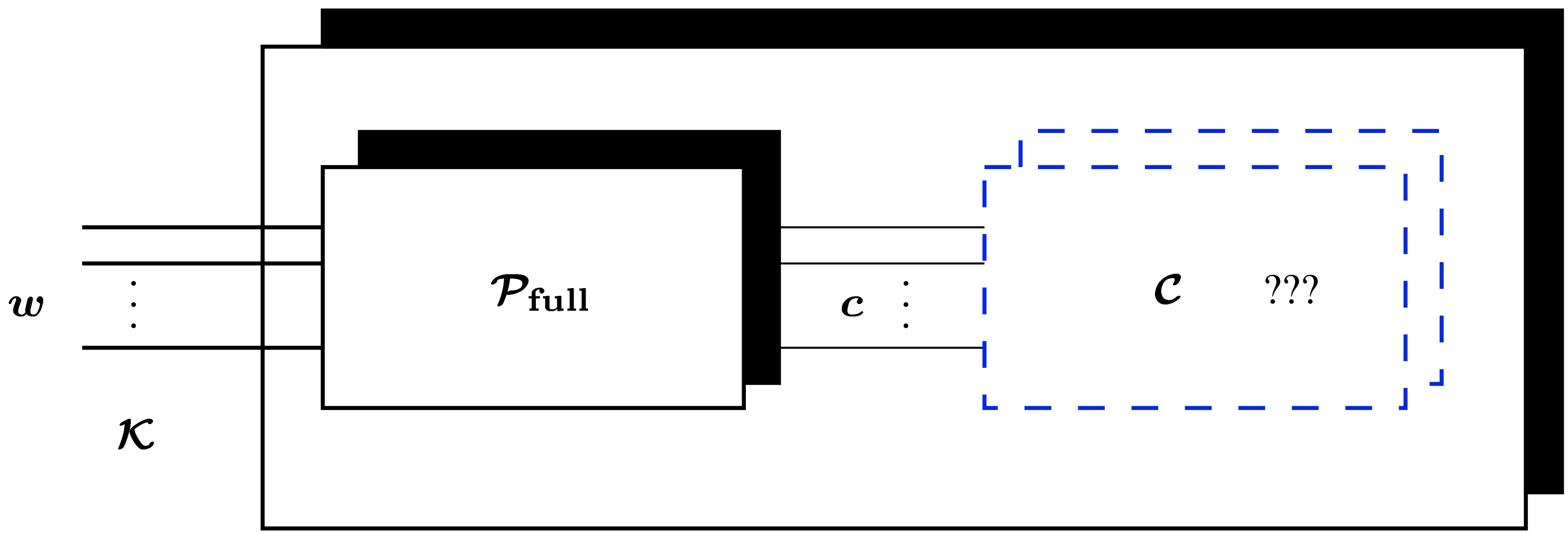
$$\mathcal{K} = \{w \mid \text{ there exists } c \in \mathcal{C} \text{ such that } (w, c) \in \mathcal{P}_{\mathrm{full}}\},$$

then we say: $\boxed{\mathcal{C} \text{ implements } \mathcal{K}}$.

Let $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}$. If there exists $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$ such that $\mathcal{C}$ implements $\mathcal{K}$, then we say: $\boxed{\mathcal{K} \text{ is implementable}}$.

Given $\mathcal{P}_{\text{full}} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}}$, we ask ourselves the question:

which $\mathcal{K}$'s in $\mathfrak{L}^{\mathtt{w}}$ are implementable?



MANIFEST PLANT BEHAVIOR AND HIDDEN BEHAVIOR

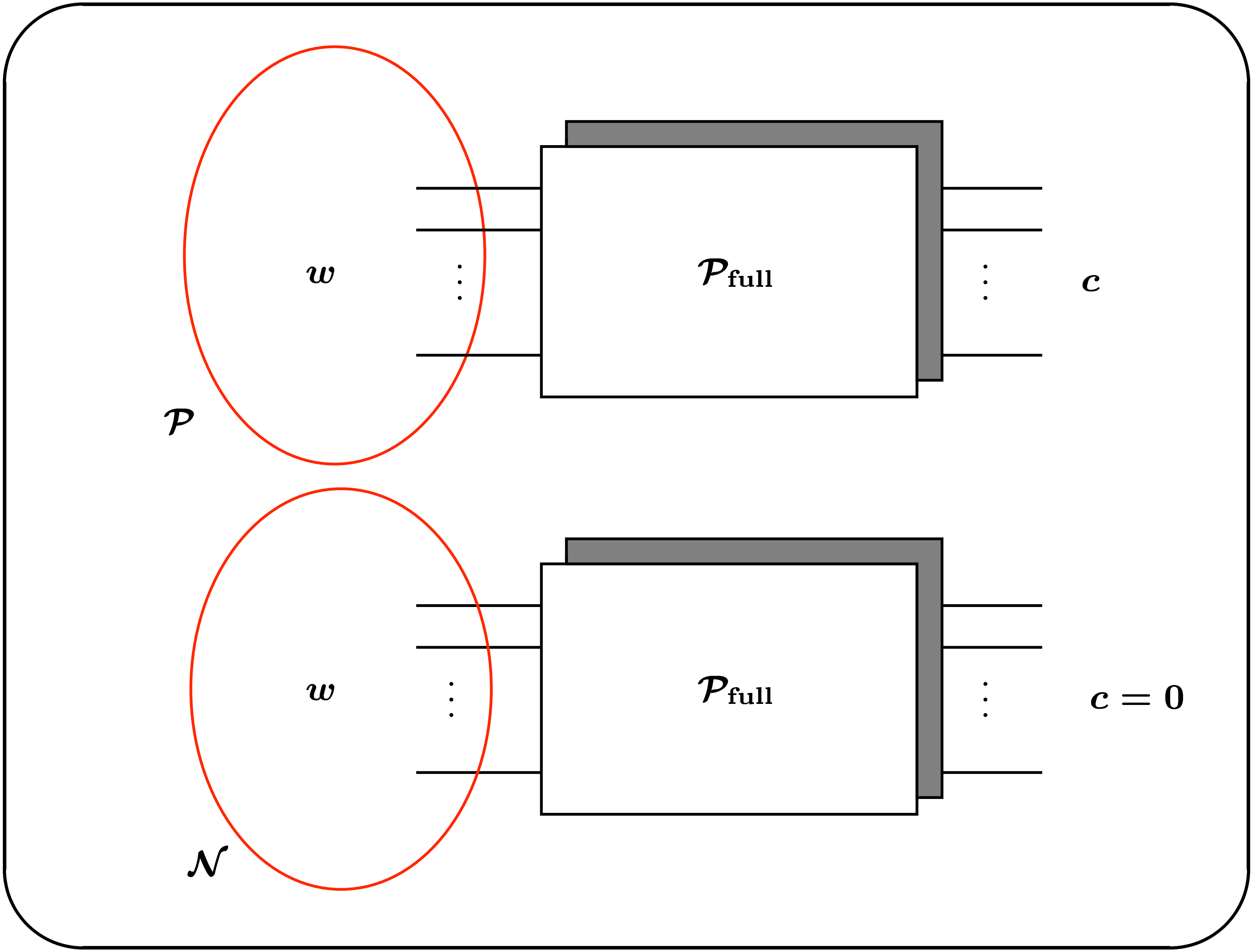
Key concepts in the characterization of the implementable $\mathcal{K}$'s are the plant behavior and hidden behavior associated with $\mathcal{P}_{\text{full}}$:

Given $\mathcal{P}_{\text{full}}$, the manifest plant behavior is defined as the system $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}}$ obtained by eliminating the control variable c:

$$\mathcal{P} := \{w \mid \text{there exists } c \text{ such that } (w, c) \in \mathcal{P}_{\text{full}}\}.$$

The hidden behavior is defined as the system $\mathcal{N} \in \mathfrak{L}^{\mathtt{w}}$ consisting of the to-be controlled variable trajectories w that are compatible with the control variable c set to zero:

$$\mathcal{N} := \{w \mid (w, 0) \in \mathcal{P}_{\text{full}}\}.$$


w
$\mathcal{P}_{\mathbf{full}}$
c
$\mathcal{P}$
w
$\mathcal{P}_{\mathbf{full}}$
c = 0
$\mathcal{N}$

CONDITIONS FOR IMPLEMENTABILITY

It is easily seen that for all $\mathcal{P}_{\text{full}}$ we have $\mathcal{N} \subset \mathcal{P}$.

The plant behavior and the hidden behavior determine whether a given $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}$ is implementable. In fact, the implementable $\mathcal{K}$'s are exactly those that are wedged in between $\mathcal{N}$ and $\mathcal{K}$:

Theorem: **Let $\mathcal{P}_{\text{full}} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}}$, $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}$. Then we have: $\mathcal{K}$ is implementable if and only if**

$$\mathcal{N} \subset \mathcal{K} \subset \mathcal{P}.$$

REGULAR IMPLEMENTABILITY

Let $\mathcal{P}_{\text{full}} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}}$.

Let $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$, $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}$. If $\mathcal{C}$ implements $\mathcal{K}$, and if the interconnection of $\mathcal{C}$ and $\mathcal{P}_{\text{full}}$ is regular, i.e.

$$\mathtt{p}(\mathcal{K}_{\text{full}}) = \mathtt{p}(\mathcal{P}_{\text{full}}) + \mathtt{p}(\mathcal{C}),$$

then we say: $\mathcal{C}$ regularly implements $\mathcal{K}$.

Let $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}$. If there exists $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$ such that $\mathcal{C}$ regularly implements $\mathcal{K}$, then we say: $\mathcal{K}$ is regularly implementable.

CONDITIONS FOR REGULAR IMPLEMENTABILITY

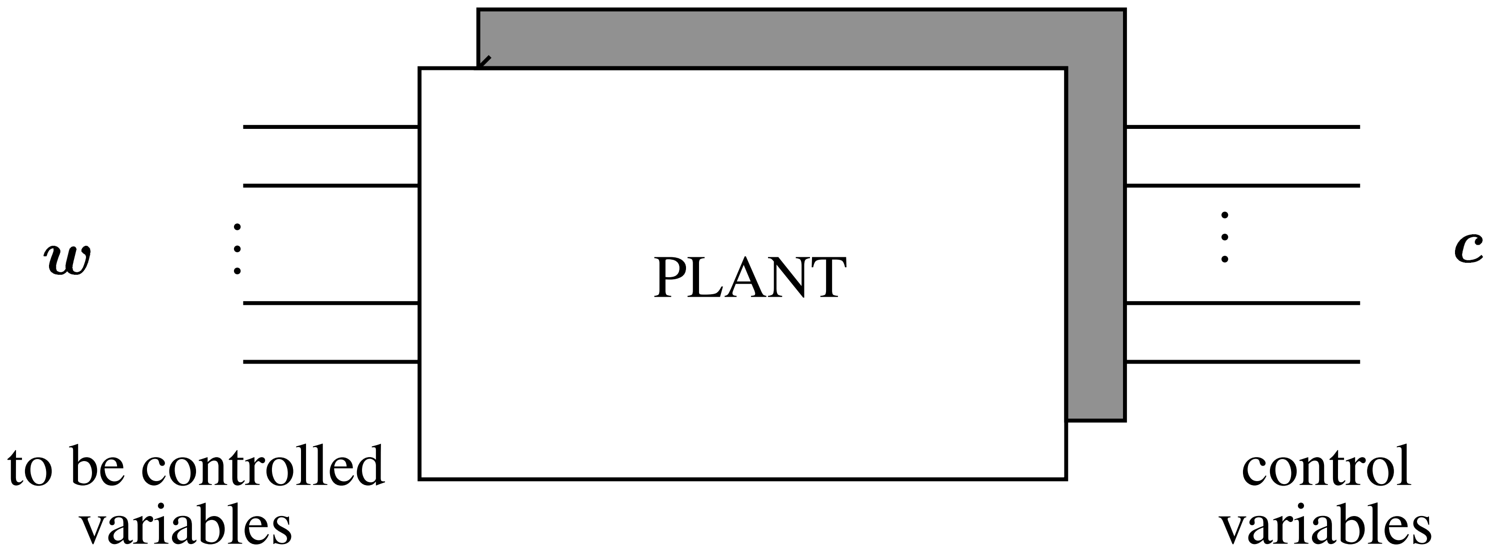
If the manifest plant behavior $\mathcal{P}$ associated with $\mathcal{P}_{\text{full}}$ is controllable, then every implementable $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}$ is regularly implementable:

Theorem: Let $\mathcal{P}_{\text{full}} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}}$ and $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}$. Assume that the manifest plant behavior $\mathcal{P}$ is controllable. Then $\mathcal{K}$ is regularly implementable if and only if $\mathcal{N} \subset \mathcal{K} \subset \mathcal{P}$.

PART II

POLE-PLACEMENT AND STABILIZATION

Plant to be controlled:



Our plant has two kinds of variables:

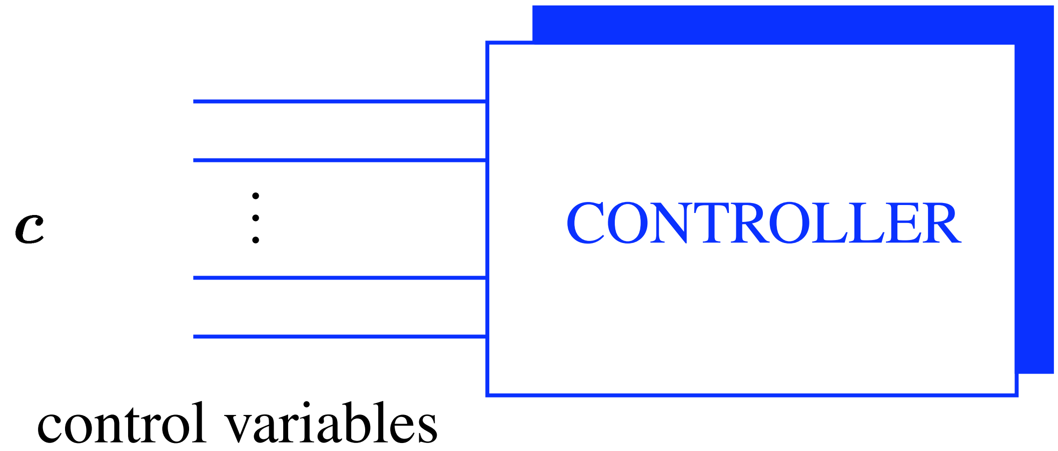
- **to be controlled variables w,**
- **control variables c.**

Full plant behavior:

$$\mathcal{P}_{\text{full}} := \{(w,c) \mid (w,c) \text{ satisfies the plant equations}\}$$

We assume that $\mathcal{P}_{\text{full}}$ is a linear differential system, i.e.,

$$\mathcal{P}_{\text{full}} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}}.$$



The control variables c are those variables in the full plant that we are allowed to put constraints on. In particular, we allow constraints of the form

$$C(\frac{d}{dt})c = 0,$$

with $C \in \mathbb{R}^{\bullet \times \mathtt{c}}[\xi]$.

In other words: a controller is a linear differential system $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$, with manifest variable c:

$$\mathcal{C} = \{c \mid c \text{ satisfies the controller equations}\}.$$

Given a full plant $\mathcal{P}_{\text{full}}$, and a controller $\mathcal{C}$, we have

- **the full controlled behavior given by**

$$\mathcal{K}_{\text{full}} = \{(w, c) \mid (w, c) \in \mathcal{P}_{\text{full}} \text{ and } c \in \mathcal{C}\},$$

- **the manifest controlled behavior given by**

$$\mathcal{K} = \{w \mid \text{ there exists } c \in \mathcal{C} \text{ such that } (w, c) \in \mathcal{P}_{\text{full}}\}$$

Note: $\mathcal{K}_{\text{full}} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}}$ **and** $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}$**.**

DESIGN SPECIFICATIONS

Design specifications are desired properties of the manifest controlled behavior $\mathcal{K}$.

In this lecture:

- **Stability of $\mathcal{K}$: the stabilization problem.**
- **Stability of $\mathcal{K}$ with arbitrary transient settling time and frequencies of oscillation: the pole placement problem.**

STABILITY

$\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ is called **stable** if for all $w \in \mathfrak{B}$ we have:

$$\lim_{t \to \infty} w(t) = 0.$$

Note: if $\mathfrak{B}$ is stable, then it is autonomous.

Stability in terms of representations:

Proposition: Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$. Let $R \in \mathbb{R}^{\mathtt{w} \times \mathtt{w}}[\xi]$ be such that $R(\frac{d}{dt})w = 0$ is a kernel representation of $\mathfrak{B}$. Then $\mathfrak{B}$ is stable if and only if R is Hurwitz, i.e., the polynomial $\det(R)$ has all its roots in $\mathbb{C}^{-} = \{\lambda \in \mathbb{C} \mid \mathrm{Re}(\lambda) < 0\}$.

THE STABILIZATION PROBLEM

Given $\mathcal{P}_{\text{full}} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}}$, the stabilization problem is to find a controller $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$ such that

- **the interconnection of $\mathcal{P}_{\text{full}}$ and $\mathcal{C}$ is regular, and**
- **the manifest controlled behavior $\mathcal{K}$ is stable.**

In other words:

Given $\mathcal{P}_{\text{full}} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}}$, find $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}$ such that $\mathcal{K}$ is regularly implementable and stable.

THE CHARACTERISTIC POLYNOMIAL OF A SYSTEM

Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ be autonomous. Then there exists $R \in \mathbb{R}^{\mathtt{w} \times \mathtt{w}}[\xi]$, $\det(R) \neq 0$, such that $R(\frac{d}{dt})w = 0$ is a kernel representation of $\mathfrak{B}$.

Obviously, for any non-zero $\alpha \in \mathbb{R}$, αR also yields a kernel representation of $\mathfrak{B}$.

Hence: we can choose R such that $\det(R)$ is a monic polynomial. This monic polynomial is denoted by $\chi_{\mathfrak{B}}$, and is called the

the characteristic polynomial of $\mathfrak{B}$.

$\chi_{\mathfrak{B}}$ only depends on $\mathfrak{B}$, and not on the polynomial matrix R we have used to define it: if R_1 and R_2 both represent $\mathfrak{B}$, then there exists a unimodular U such that $R_2 = UR_1$. Hence if $\det(R_1)$ and $\det(R_2)$ are monic, then $\det(R_1) = \det(R_2)$.

THE POLE PLACEMENT PROBLEM

Given $\mathcal{P}_{\text{full}} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}}$, the pole placement problem is to find, for every monic polynomial $r \in \mathbb{R}[\xi]$, a controller $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$ such that

- **the interconnection of $\mathcal{P}_{\text{full}}$ and $\mathcal{C}$ is regular, and**
- **the characteristic polynomial $\chi_{\mathcal{K}}$ of the controlled behavior $\mathcal{K}$ is equal to r.**

In other words:

Given $\mathcal{P}_{\text{full}} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}}$, for every monic polynomial $r \in \mathbb{R}[\xi]$ find $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}$ such that $\mathcal{K}$ is regularly implementable and $\chi_{\mathcal{K}} = r$.

STABILIZABILITY

Recall that $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ is called stabilizable if for all $w \in \mathfrak{B}$ there exists $w' \in \mathfrak{B}$ such that

- $w'(t) = w(t)$ for $t < 0$,
- $\lim_{t \to \infty} w'(t) = 0$.

Proposition: Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$, and let $R \in \mathbb{R}^{\bullet \times \mathtt{w}}[\xi]$ be such that $R(\frac{d}{dt})w = 0$ is a minimal kernel representation of $\mathfrak{B}$. Then $\mathfrak{B}$ is stabilizable if and only if there exists $R' \in \mathbb{R}^{\bullet \times \mathtt{w}}[\xi]$ such that

$$\begin{bmatrix} R \\ R' \end{bmatrix}$$

is Hurwitz.

Interpretation of stabilizability in terms of full interconnection

$\mathfrak{B}$ is represented by $R(\frac{d}{dt})w = 0$. Let $\mathfrak{B}'$ be the system represented by $R'(\frac{d}{dt})w = 0$. The interconnection $\mathfrak{B} \cap \mathfrak{B}'$ is then represented by

$$\begin{bmatrix} R(\frac{d}{dt}) \\ R'(\frac{d}{dt}) \end{bmatrix} w = 0.$$

If R, R' have full row rank then $\begin{bmatrix} R \\ R' \end{bmatrix}$ is nonsingular if and only if

$$\mathrm{rank}(R) + \mathrm{rank}(R') = \mathrm{rank} \begin{bmatrix} R \\ R' \end{bmatrix},$$

Equivalently:

$$\mathtt{p}(\mathfrak{B} \cap \mathfrak{B}') = \mathtt{p}(\mathfrak{B}) + \mathtt{p}(\mathfrak{B}').$$

Thus we get the following characterization of stabilizability in terms of stabilization by regular full interconnection:

Proposition: Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$. Then $\mathfrak{B}$ is stabilizable if and only if there exists $\mathfrak{B}' \in \mathfrak{L}^{\mathtt{w}}$ such that the full interconnection $\mathfrak{B} \cap \mathfrak{B}'$ is stable and regular.

Note: the entire manifest variable w is used as a control variable.

CONTROLLABILITY

Recall the definition of controllability: $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ **is controllable if for all** $w', w'' \in \mathfrak{B}$ **there exists** $w \in \mathfrak{B}$ **and** $T \geq 0$ **such that**

$$\begin{aligned} w|_{(-\infty,0)} &= w'|_{(-\infty,0)} \\ w|_{[T,\infty)} &= w''|_{[T,\infty)} \end{aligned}$$

Proposition: **Let** $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$**, and let** $R \in \mathbb{R}^{\bullet \times \mathtt{w}}[\xi]$ **be such that** $R(\frac{d}{dt})w = 0$ **is a minimal kernel representation of** $\mathfrak{B}$**. Then** $\mathfrak{B}$ **is controllable if and only if for every monic polynomial** $r \in \mathbb{R}[\xi]$ **there exists** $R' \in \mathbb{R}^{\bullet \times \mathtt{w}}[\xi]$ **such that**

$$\det\left(\begin{bmatrix} R \\ R' \end{bmatrix}\right) = r.$$

Interpretation of controllability in terms of full interconnection

This yields the following characterization of controllability in terms of pole placement by regular full interconnection:

Proposition: Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$. Then $\mathfrak{B}$ is controllable if and only if for each monic polynomial $r \in \mathbb{R}[\xi]$ there exists $\mathfrak{B}' \in \mathfrak{L}^{\mathtt{w}}$ such that the full interconnection $\mathfrak{B} \cap \mathfrak{B}'$ regular, autonomous, and $\chi_{\mathfrak{B}\cap\mathfrak{B}'} = r$.

Note again: the entire manifest variable w is used as a control variable.

SOLUTION OF THE STABILIZATION PROBLEM

Recall: given $\mathcal{P}_{\text{full}} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}}$, the stabilization problem is to find $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}$ such that $\mathcal{K}$ is regularly implementable and stable.

Recall the notions of manifest plant behavior:

$$\mathcal{P} := \{w \mid \text{there exists } c \text{ such that } (w, c) \in \mathcal{P}_{\text{full}}\},$$

and hidden behavior:

$$\mathcal{N} := \{w \mid \ (w, 0) \in \mathcal{P}_{\text{full}}\}.$$

Theorem: **Let $\mathcal{P}_{\text{full}} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}}$. There exists a regularly implementable, stable $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}$ if and only if**

- **$\mathcal{N}$ is stable,**
- **$\mathcal{P}$ is stabilizable.**

STABLE HIDDEN BEHAVIOR ⇔ DETECTABILITY

Note: $\mathcal{N}$ **is stable if and only if**

$$(w, 0) \in \mathcal{P}_{\text{full}} \Rightarrow \lim_{t\to\infty} w(t) = 0.$$

By linearity, this is equivalent with:

$$(w_1, c), (w_2, c) \in \mathcal{P}_{\text{full}} \Rightarrow \lim_{t\to\infty} (w_1(t) - w_2(t)) = 0.$$

Conclusion: $\mathcal{N}$ **is stable ⇔ in** $\mathcal{P}_{\text{full}}$**,** w **is detectable from** c**.**

Reformulation of the theorem:

Let $\mathcal{P}_{\text{full}} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}}$**. There exists a regularly implementable, stable** $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}$ **if and only if**

- **in** $\mathcal{P}_{\text{full}}$**,** w **is detectable from** c**,**
- **the system obtained by eliminating** c **from** $\mathcal{P}_{\text{full}}$ **is stabilizable.**

Proof of Theorem:

Minimal kernel representation of $\mathcal{P}_{\text{full}}$: $R_1(\frac{d}{dt})w + R_2(\frac{d}{dt})c = 0$.

By suitable unimodular premultiplication of $[R_1 \;\; R_2]$, $\mathcal{P}_{\text{full}}$ is represented by

$$\begin{bmatrix} R_{11}(\frac{d}{dt}) & R_{12}(\frac{d}{dt}) \\ R_{21}(\frac{d}{dt}) & 0 \end{bmatrix} \begin{bmatrix} w \\ c \end{bmatrix} = 0.$$

with R_{12} full row rank.

Plant behavior $\mathcal{P}$: eliminate $c \Rightarrow R_{21}(\frac{d}{dt})w = 0$,

Hidden behavior $\mathcal{N}$: set c equal to zero $\Rightarrow \begin{bmatrix} R_{11}(\frac{d}{dt}) \\ R_{21}(\frac{d}{dt}) \end{bmatrix} w = 0.$

(only if) $\mathcal{K}$ **regularly implementable** $\Rightarrow \mathcal{N} \subset \mathcal{K} \subset \mathcal{P}$.

$\mathcal{K}$ **stable** $\Rightarrow$ $\mathcal{N}$ **stable.**

There exists $\mathcal{C} \in \mathfrak{L}^{\mathrm{c}}$ **that implements** $\mathcal{K}$**. Minimal kernel representation** $C(\frac{d}{dt})c = 0$.

Minimal kernel representation of $\mathcal{K}_{\mathrm{full}}$**:**

$$\begin{bmatrix} R_{11}(\frac{d}{dt}) & R_{12}(\frac{d}{dt}) \\ R_{21}(\frac{d}{dt}) & 0 \\ 0 & C(\frac{d}{dt}) \end{bmatrix} \begin{bmatrix} w \\ c \end{bmatrix}.$$

Latent variable representation of $\mathcal{K}$**:**

$$\begin{bmatrix} R_{11}(\frac{d}{dt}) \\ R_{21}(\frac{d}{dt}) \\ 0 \end{bmatrix} w = -\begin{bmatrix} R_{12}(\frac{d}{dt}) \\ 0 \\ C(\frac{d}{dt}) \end{bmatrix} c$$

(latent variable c**).**

Note: $\mathcal{K} = \mathcal{P} \cap \mathcal{P}'$**, with** $\mathcal{P}' \in \mathfrak{L}^{\mathtt{w}}$ **represented by**

$$\begin{bmatrix} R_{11}(\frac{d}{dt}) \\ 0 \end{bmatrix} w = - \begin{bmatrix} R_{12}(\frac{d}{dt}) \\ C(\frac{d}{dt}) \end{bmatrix} c$$

(latent variable c**).**

Interconnection of $\mathcal{P}_{\text{full}}$ **and** $\mathcal{C}$ **regular** $\Rightarrow$ **Interconnection** $\mathcal{P} \cap \mathcal{P}'$ **regular.**

$\mathcal{P} \cap \mathcal{P}'$ **stable** $\Rightarrow$ $\mathcal{P}$ **stabilizable.**

(if) $\mathcal{N}$ **is represented by** $\begin{bmatrix} R_{11}(\frac{d}{dt}) \\ R_{21}(\frac{d}{dt}) \end{bmatrix} w = 0,$

$\mathcal{P}$ **is represented by** $R_{21}(\frac{d}{dt})w = 0.$

$\mathcal{N}$ **stable** $\Rightarrow \begin{bmatrix} R_{11} \\ R_{21} \end{bmatrix} = \begin{bmatrix} R'_{11} \\ R'_{21} \end{bmatrix} G$**, with** $\begin{bmatrix} R'_{11}(\lambda) \\ R'_{21}(\lambda) \end{bmatrix}$ **full column rank for all** $\lambda \in \mathbb{C}$**, and** G **Hurwitz.**

Hence: $\mathcal{P}_{\text{full}}$ **has a representation of the form**

$$\begin{bmatrix} G(\frac{d}{dt}) & R'_{12}(\frac{d}{dt}) \\ 0 & R'_{22}(\frac{d}{dt}) \end{bmatrix} \begin{bmatrix} w \\ c \end{bmatrix} = 0.$$

Note: $(w, c) \in \mathcal{P}_{\text{full}} \Rightarrow \boxed{G(\frac{d}{dt})w = -R'_{12}(\frac{d}{dt})c}$

(reconstruction of $G(\frac{d}{dt})w$ **using** c**).**

$\mathcal{P}$ stabilizable and $R_{21} = R'_{21}G$, G Hurwitz $\Rightarrow R'_{21}(\lambda)$ full row rank for $\lambda \in \mathbb{C}^+ \Rightarrow$ there exists $C_0 \in \mathbb{R}^{\bullet \times \mathtt{w}}[\xi]$ such that $\begin{bmatrix} R'_{21} \\ C_0 \end{bmatrix}$ is Hurwitz.

Hence $\begin{bmatrix} R_{21} \\ C_0 G \end{bmatrix}$ is Hurwitz.

Define now $\mathcal{K} := \mathcal{P} \cap \mathcal{P}'$, with $\mathcal{P}'$ repr. by $C_0(\frac{d}{dt})G(\frac{d}{dt})w = 0$

Then $\mathcal{K}$ is stable.

Since $G(\frac{d}{dt})w = -R'_{12}(\frac{d}{dt})c$ for $(w, c) \in \mathcal{P}_{\text{full}}$, $\mathcal{K}$ is regularly implemented by the controller $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$ represented by $C_0(\frac{d}{dt})R'_{12}(\frac{d}{dt})c = 0$.

SOLUTION OF THE POLE PLACEMENT PROBLEM

Recall: given $\mathcal{P}_{\text{full}} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}}$, the pole placement problem is to find, for every monic polynomial $r \in \mathbb{R}[\xi]$, a behavior $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}$ such that $\mathcal{K}$ is regularly implementable and $\chi_{\mathcal{K}} = r$.

Theorem: Let $\mathcal{P}_{\text{full}} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}}$. For every $r \in \mathbb{R}[\xi]$ there exists a regularly implementable $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}$ such that $\chi_{\mathcal{K}} = r$ if and only if

- $\mathcal{N} = 0$,
- **$\mathcal{P}$ is controllable.**

ZERO HIDDEN BEHAVIOR ⇔ OBSERVABILITY

Note: $\mathcal{N} = 0$ if and only if

$$(w, 0) \in \mathcal{P}_{\text{full}} \Rightarrow w = 0.$$

By linearity, this is equivalent with:

$$(w_1, c), (w_2, c) \in \mathcal{P}_{\text{full}} \Rightarrow w_1 = w_2.$$

Conclusion: $\mathcal{N} = 0 \Leftrightarrow$ in $\mathcal{P}_{\text{full}}$, w is observable from c.

Reformulation of the theorem:

Let $\mathcal{P}_{\text{full}} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}}$. For every $r \in \mathbb{R}[\xi]$ there exists a regularly implementable $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}$ such that $\chi_{\mathcal{K}} = r$ if and only if

- in $\mathcal{P}_{\text{full}}$, w is observable from c,
- the system obtained by eliminating c from $\mathcal{P}_{\text{full}}$ is controllable.

FROM GENERAL RESULT TO PARTICULAR REPRESENTATIONS

Statement of the main results do not use representations of $\mathcal{N}$ and $\mathcal{P}$. Hence: applicable to any particular representation of the full plant $\mathcal{P}_{\text{full}}$. Procedure:

- **for a given representation of $\mathcal{P}_{\text{full}}$, compute representations of its hidden behavior $\mathcal{N}$ and its manifest plant behavior $\mathcal{P}$.**
- **Next: express the representation-free conditions of the main result in terms of the parameters of these representations.**
- **Use the general construction of the controlled behavior $\mathcal{K}$ to set up algorithms in terms of the parameters of these representations.**

Example: applying this procedure to $\mathcal{P}_{\text{full}}$ represented by $\frac{d}{dt}x = Ax + Bu,\ \ y = Cx$, with $w = (x, u, y)$ and $c = (u, y)$ yields the well-known conditions on A, B and C.

FEEDBACK IMPLEMENTABILITY OF STABILIZING CONTROLLERS

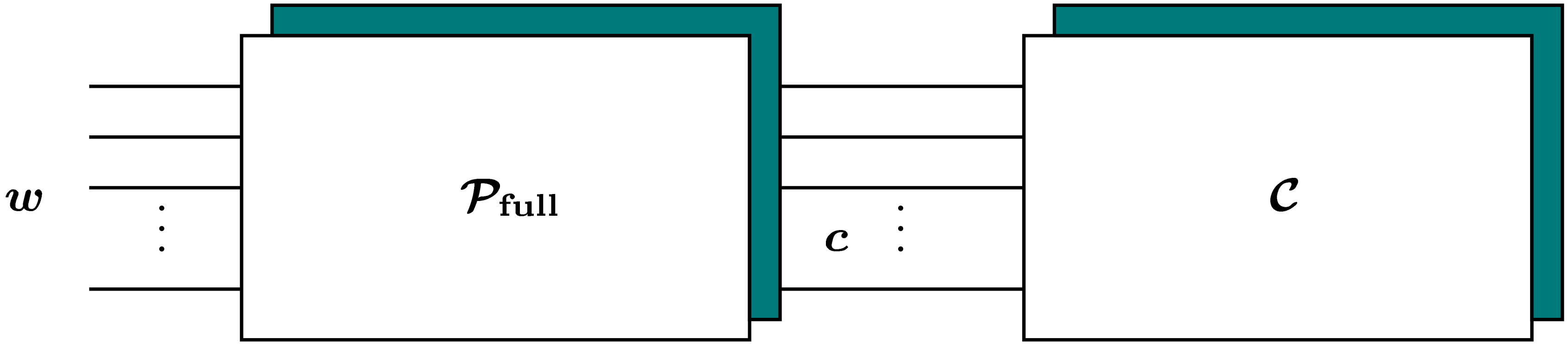
Recall: regular interconnection ⇔ feedback interconnection.

In the stabilization problem, the manifest controlled behavior $\mathcal{K}$ becomes autonomous, so in the full controlled behavior $\mathcal{K}_{\text{full}}$, w does not contain free components.

Hence:

<u>COROLLARY:</u> Let $\mathcal{P}_{\text{full}} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}}$. Let $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$ be a controller such that the interconnection of $\mathcal{P}_{\text{full}}$ and $\mathcal{C}$ is regular and such that $\mathcal{K}$ is stable. Then there exists (modulo reordering of components) a partition of the control variable, $c = (u_1, u_2, y)$, such that

- **in $\mathcal{P}_{\text{full}}$, (u_1, u_2) is input and (y, w) is output,**
- **in $\mathcal{C}$, (u_1, y) is input and u_2 is output,**
- **in $\mathcal{K}_{\text{full}}$, u_1 is input and (y, u_2, w) is output.**


w
$\mathcal{P}_{\text{full}}$
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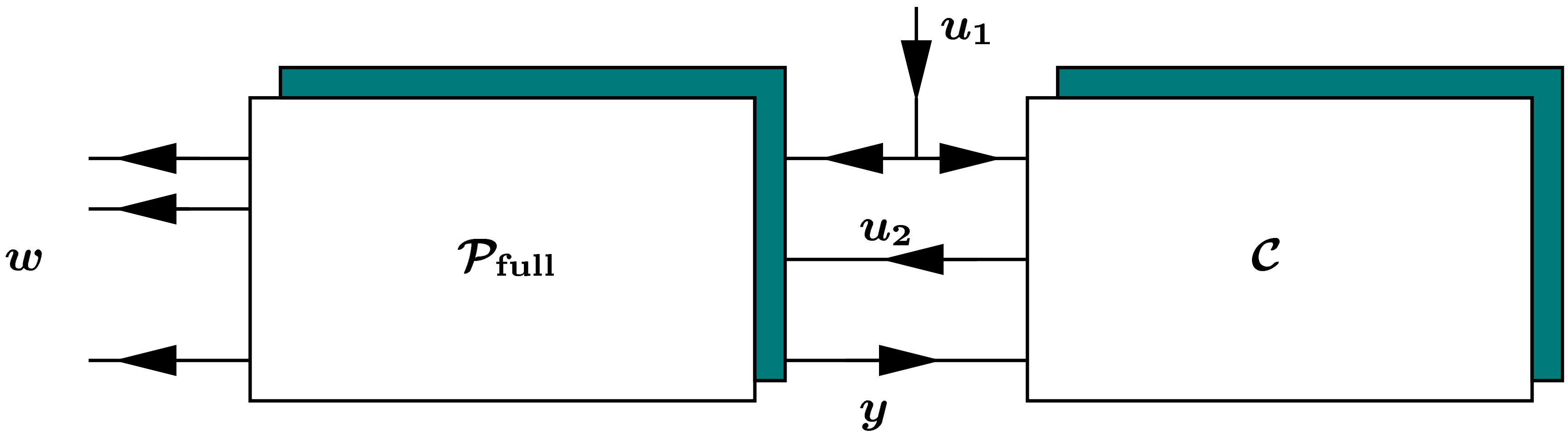
$\mathcal{C}$


u_1
w
$\mathcal{P}_{\text{full}}$
u_2
$\mathcal{C}$
y

RECAP

- **We have given definitions of interconnection of dynamical systems.**
- **The interconnection of two linear differential systems is called regular if the output cardinality of the interconnection is equal to the sum of the output cardinalities of the two systems.**
- **Feedback interconnection is a special kind of interconnection**
- **Feedback interconnection $\Leftrightarrow$ regular interconnection.**
- **We consider control problems as problems to achieve interconnections that satisfy the design specifications.**
- **Given a plant $\mathcal{P}_{\text{full}}$, a behavior $\mathcal{K}$ is implementable if there exists a controller $\mathcal{C}$ such that $\mathcal{K}$ is equal to the manifest behavior of the interconnection of $\mathcal{P}_{\text{full}}$ and $\mathcal{C}$.**

- **A behavior is implementable if and only if it is wedged in between the hidden behavior $\mathcal{N}$ and the manifest plant behavior $\mathcal{P}$.**
- **For a given plant $\mathcal{P}_{\mathrm{full}}$, the stabilization problem is to find a regularly implementable, stable behavior $\mathcal{K}$.**
- **For a given plant $\mathcal{P}_{\mathrm{full}}$, the pole placement problem is to find, for every monic polynomial r, a regularly implementable, autonomous behavior $\mathcal{K}$ whose characteristic polynomial equals r.**
- **The stabilization problem admits a solution if and only if, in the full plant $\mathcal{P}_{\mathrm{full}}$, the manifest variable w is detectable from the control variable c, and the manifest plant behavior $\mathcal{P}$ is stabilizable.**
- **The pole placement problem admits a solution if and only if in the full plant $\mathcal{P}_{\mathrm{full}}$, w is observable from c, and $\mathcal{P}$ is controllable.**