The Behavioral Approach to Systems and Control:
Introduction and Recent Advances

Dissipative Systems

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Outline

- **Lyapunov functions**
- **Dissipative systems**
- **Examples**
- **Dissipative PDE's**
- **Global and local dissipation inequalities**

Questions

- What do we mean by a **dissipative system**?
- It involves the **storage function.** How is it constructed?
- How does this theory look like for **PDE's**? What does it mean e.g. in Maxwell's equations?
- Where does it enter is stability analysis? In robust control?
- How is it applied in thermodynamics? In circuit synthesis?

Lyapunov functions

Lyapunov functions

Consider the classical dynamical system, the *'flow'*,

$$\Sigma : \frac{d}{dt}x = f(x)$$

with $x \in \mathbb{X} = \mathbb{R}^{\mathtt{n}}$ the *state*, and $f : \mathbb{X} \to \mathbb{X}$ the *vectorfield*.

Denote the set of solutions $x : \mathbb{R} \to \mathbb{X}$ by $\mathscr{B}$, the *'behavior'*.

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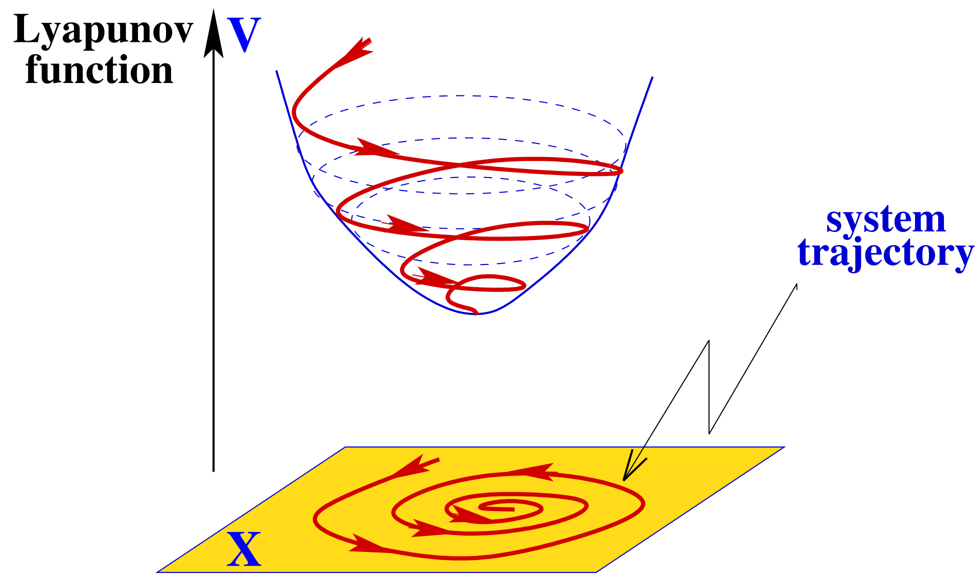
$$V : \mathbb{X} \to \mathbb{R}$$

is said to be a *Lyapunov function* for Σ if along $x \in \mathscr{B}$

$$\frac{d}{dt} V(x(\cdot)) \leq 0$$

Equivalently, if $\overset{\bullet}{V}^{\Sigma} := \nabla V \cdot f \leq 0.$

Typical Lyapunov theorem



$$V(x) > 0 \textbf{ and } \overset{\bullet}{V}^{\Sigma}(x) < 0 \textbf{ for } 0 \neq x \in \mathbb{X}$$

$$\Rightarrow$$

$\forall\ x \in \mathscr{B},$ **there holds** $x(t) \to 0$ **for** $t \to \infty$ **'global stability'**

Lyapunov

Lyapunov f'ns play a remarkably central role in the field.

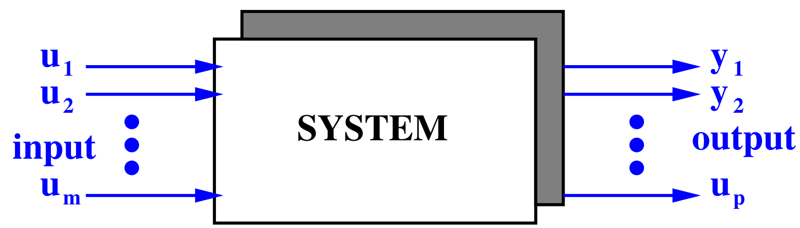
Aleksandr Mikhailovich Lyapunov (1857-1918)

Introduced Lyapunov's 'second method' in his thesis (1899).

The classical notion of a dissipative systems

Open systems

'Open' systems are a much more appropriate starting point for the study of dynamics. For example,



$\rightsquigarrow$ **the dynamical system**

$$\Sigma : \quad \frac{d}{dt} x = f(x,u), \quad y = h(x,u).$$

$u \in \mathbb{U} = \mathbb{R}^{\mathtt{m}}, y \in \mathbb{Y} = \mathbb{R}^{\mathtt{p}}, x \in \mathbb{X} = \mathbb{R}^{\mathtt{n}}$: **input, output, state.**

Behavior $\mathscr{B} =$ **all sol'ns** $(u,y,x) : \mathbb{R} \to \mathbb{U} \times \mathbb{Y} \times \mathbb{X}$.

Dissipative dynamical systems

Let $s : \mathbb{U} \times \mathbb{Y} \to \mathbb{R}$ **be a function, called the *supply rate*.**

Σ is said to be *dissipative* w.r.t. the supply rate s if $\exists$

$$V : \mathbb{X} \to \mathbb{R},$$

called the *storage function*, such that

$$\frac{d}{dt} V(x(\cdot)) \leq s(u(\cdot), y(\cdot))$$

$\forall \ (u(\cdot), y(\cdot), x(\cdot)) \in \mathscr{B}$.

Dissipation inequality

$$\frac{d}{dt} V(x(\cdot)) \leq s(u(\cdot), y(\cdot))$$

$\forall \quad (u(\cdot), y(\cdot), x(\cdot)) \in \mathscr{B}$.

This inequality is called the ***dissipation inequality.***

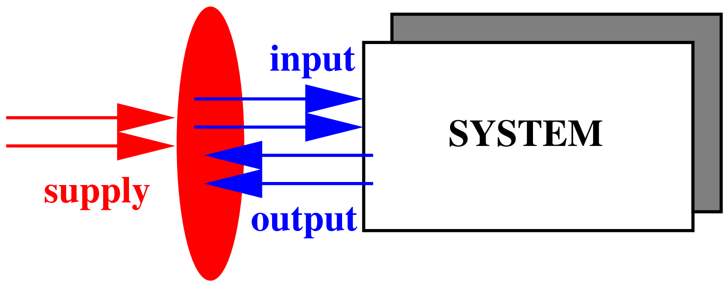
Equivalent to

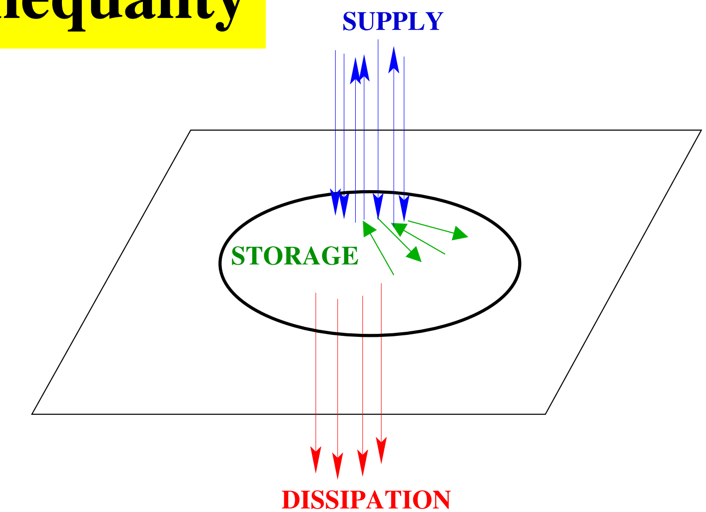
$$\overset{\bullet}{V}^{\Sigma}(x,u) := \nabla V(x) \cdot f(x,u) \leq s(u, h(x,u))$$

for all $(u,x) \in \mathbb{U} \times \mathbb{X}$**.**

If equality holds: ***'conservative' system.***

Dissipation inequality





$s(u, y)$ models something like the power delivered to the system when the input value is u and output value is y.

$V(x)$ then models the internally stored energy.

Dissipativity $:\Leftrightarrow$

rate of increase of internal energy $\leq$ power delivered.

Dissipation inequality

Special case: 'closed' system: $s = 0$ then

dissipativeness $\leftrightarrow$ V is a Lyapunov function.

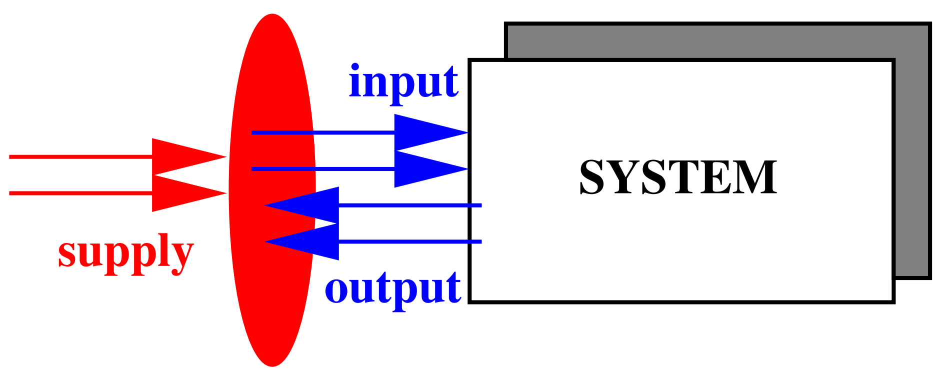
Dissipativity is the natural generalization to open systems of Lyapunov theory.

Stability for closed systems $\simeq$ Dissipativity for open systems.

The construction of storage functions

Basic question:

Given (a representation of) Σ, the dynamics,
and given s, the supply rate,
is the system dissipative w.r.t. s, i.e.
does there exist a storage function V such that
the dissipation inequality holds?



Monitor power in, known dynamics, what is the stored energy?

The construction of storage functions

The construction of storage f'ns is very well understood, particularly for finite dimensional linear systems and quadratic supply rates.

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Leads to the KYP-lemma, LMI's, ARIneq, ARE, semi-definite programming, spectral factorization, Lyapunov functions, $\mathscr{H}_\infty$ and robust control, positive and bounded real functions, electrical circuit synthesis, stochastic realization theory.

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The storage function V is in general far from unique. There are two 'canonical' storage functions:

the available storage and the required supply.

For conservative systems, V is unique. There are other cases.

Dissipative systems

Dissipative systems and storage functions play a remarkably central role in the field.

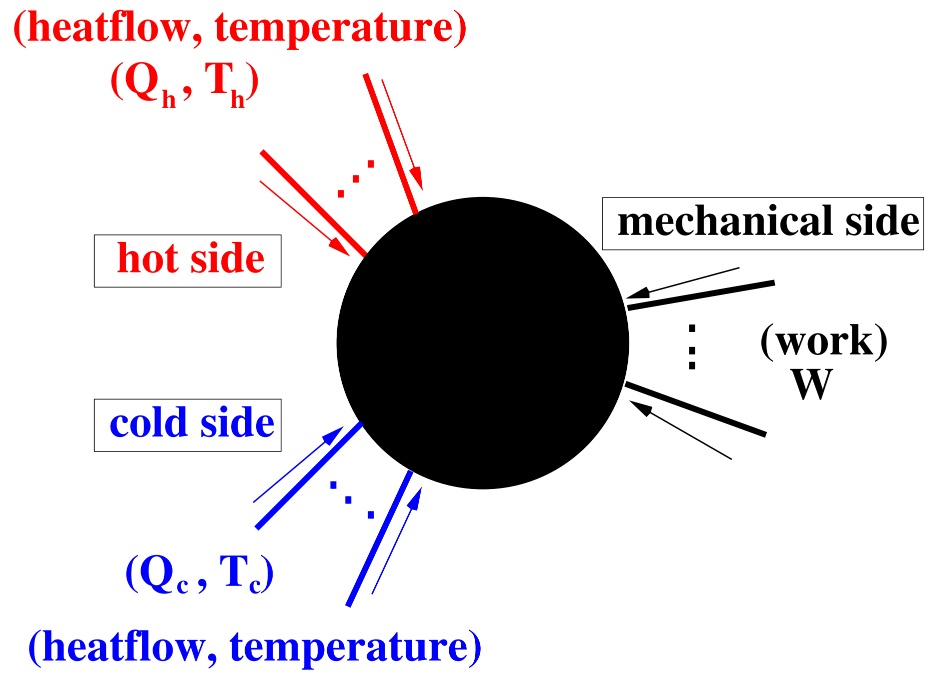
Shortcomings

Shortcomings

The classical framework falls short in very important situations, for example,

- **it assumes an (often fictitious) input/output partition, and a state representation to start with.**
- **it covers thermodynamics only in simple cases;**
- **it deals with electrical circuit syntesis in an awkward way;**
- **it does not apply to distributed systems;**
- **etc., etc.**

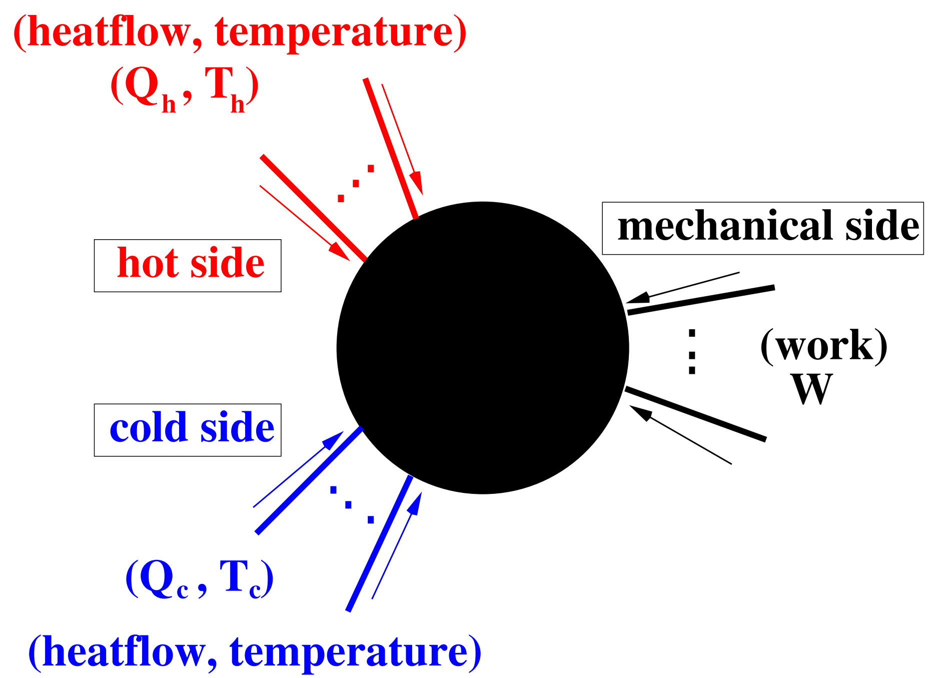
Thermodynamics



Not all histories W, Q_h, T_h, Q_c, T_c are possible. Must satisfy:

1. **The first law: conservation of energy**
2. **The second law:**
   heat and work are nevertheless not exchangeable

Thermodynamics

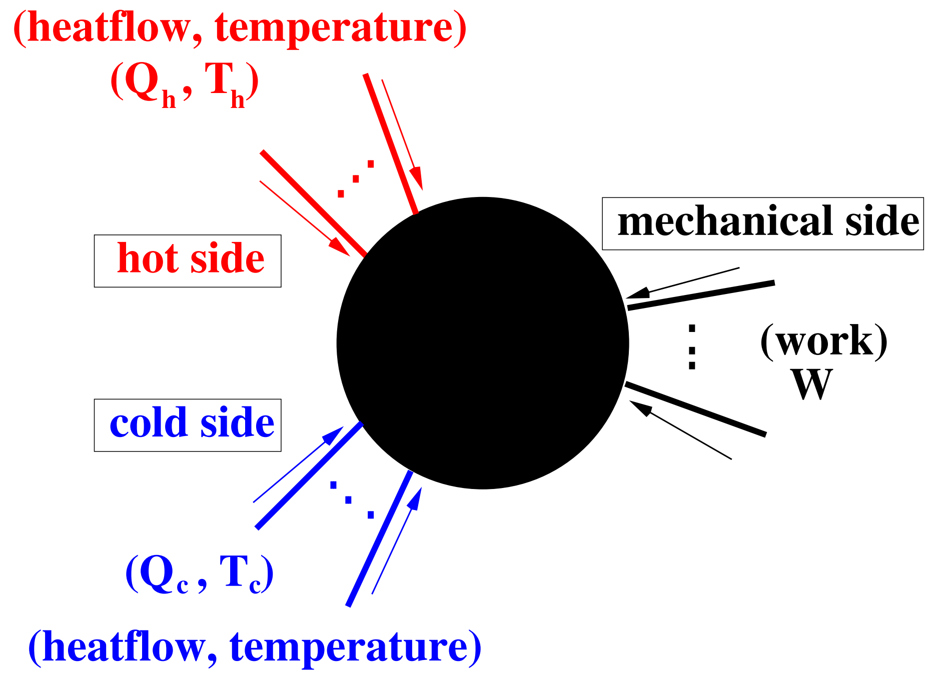


Thermodynamics is the only theory of a general nature of which I am convinced that it will never be overthrown.

Albert Einstein

The law that entropy always increases – the second law of thermodynamics – holds, I think, the supreme position among the laws of nature. **Arthur Eddington**

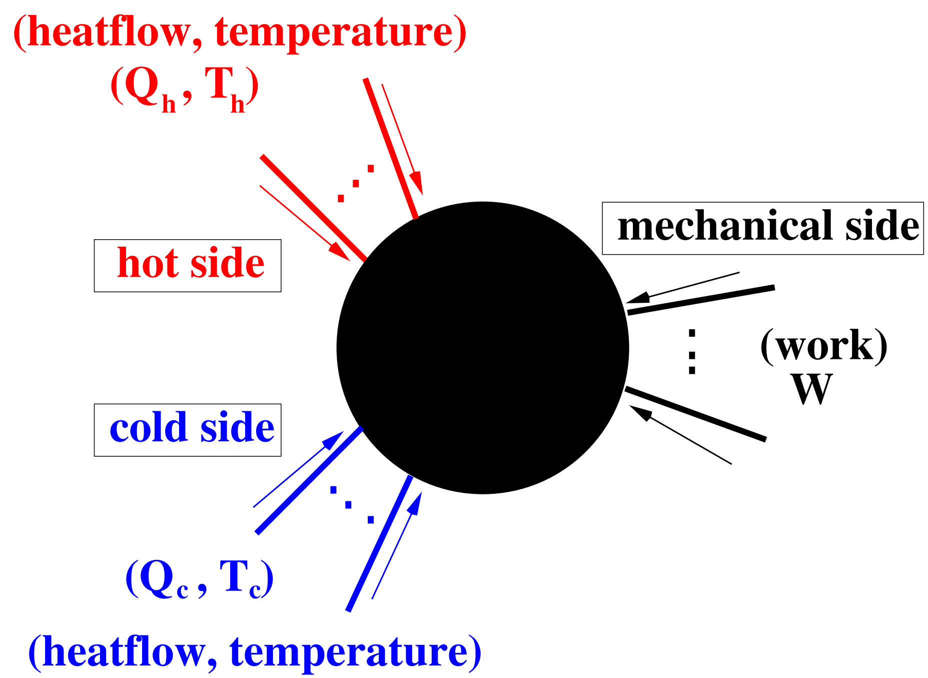
Thermodynamics



Paradigmatic example of open, dissipative, dynamical

- **Deals with histories.**
- **The first and second law express something about the interaction with an 'arbitrary' environment.**
- **The first law expresses conservativeness, in a the second law dissipativeness in a certain sense.**

Thermodynamics



Not all histories W, Q_h, T_h, Q_c, T_c are possible. Must satisfy:

1. **The first law: conservation of energy**
2. **The second law: heat and work are nevertheless not exchangeable**

How can we express these laws in an non-ambiguous way?

Inappropriateness of inputs, outputs; unavailability of states.

The realization problem

Given a set of building blocks,
and a way to interconnect these building blocks,
what behaviors can be obtained?

Example 1: State representation algorithms.

Building blocks: adders, amplifiers, forks, integrators (as in analog computers)

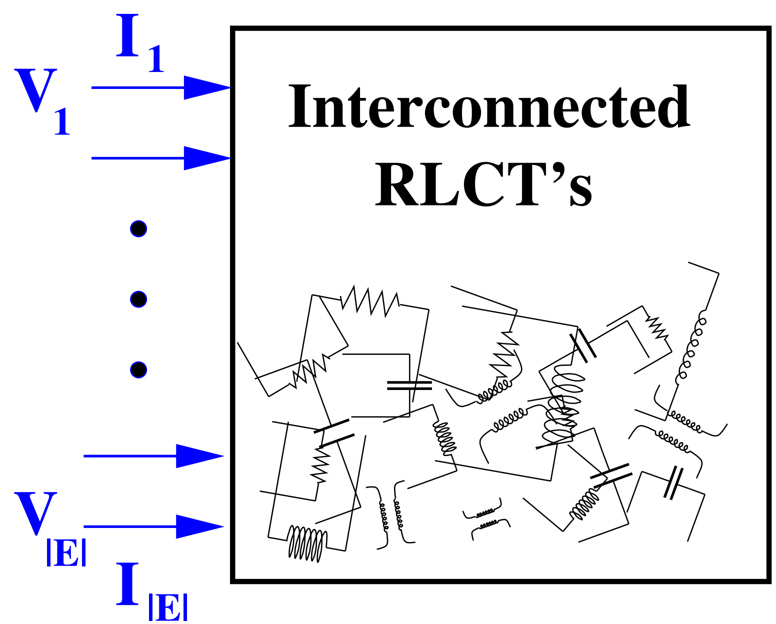
$$\rightsquigarrow \quad \textbf{LTIDS} \quad \dot{x} = Ax + Bu, \quad y = Cx + Du.$$

Example 2: Electrical circuit synthesis. Building blocks:

resistors, capacitors, inductors, connectors, transformers, gyrators.

Circuit synthesis

Realizability: ***Which external behaviors can be obtained by interconnecting a finite number of R's, C's, L's, & T's ? (or without T's, or with also G's?)***



Synthesis: ***If a behavior is realizable, give a wiring diagram (an architecture) that leads to the desired external behavior.***

Circuit synthesis

This problem is best dealt with, if we do not consider a state representation, nor an input/output partition.

In fact, the input/output partition is a result.

Hybridicity

There exists an I/O repr. for which the input and output var.

$$(u_1, u_2, \ldots, u_{|E|}), \quad (y_1, y_2, \cdots, y_{|E|})$$

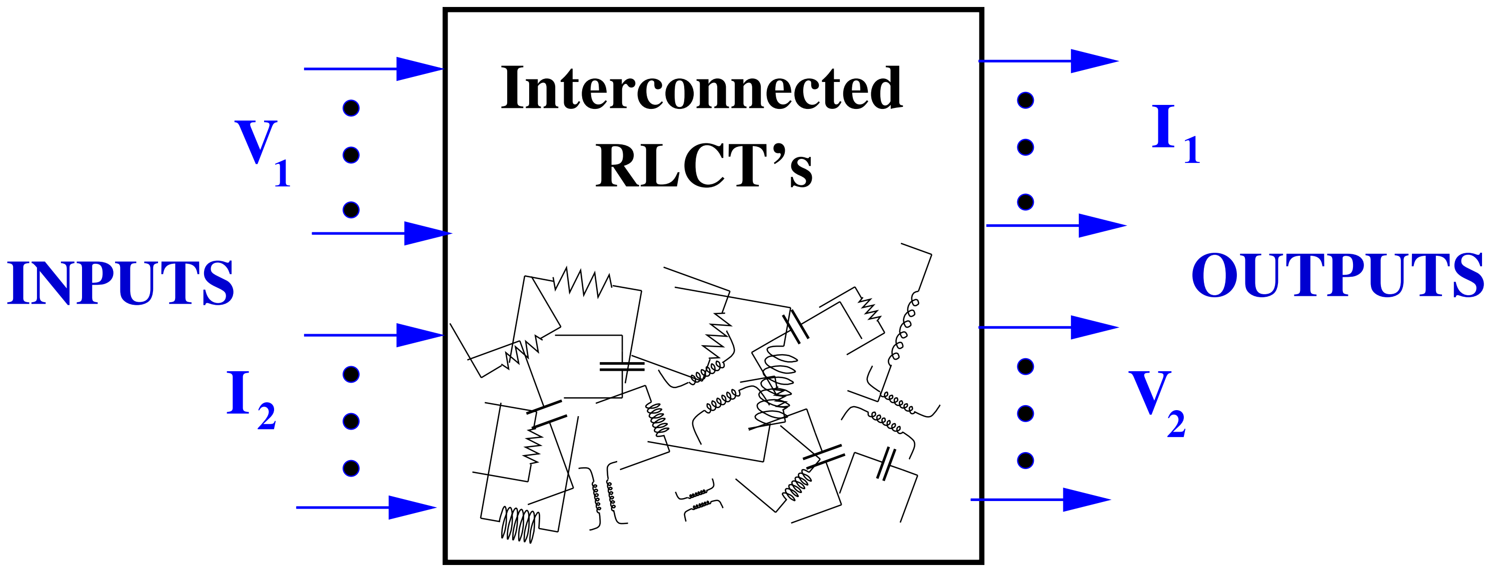
pair as follows:

$$\{u_k, y_k\} = \{V_k, I_k\}$$

In other words, each terminal is either current controlled or voltage controlled.

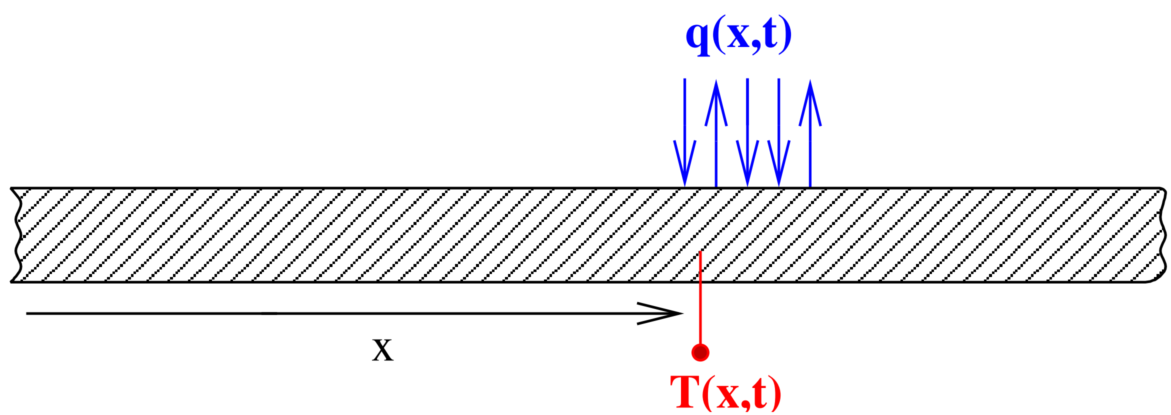
Circuit synthesis

Hybridicity



Distributed systems

First principles motivating example: heat diffusion



The PDE

$$\frac{\partial}{\partial t} T = \frac{\partial^2}{\partial x^2} T + q$$

describes the evolution of the temperature $T(x,t)$ ($x \in \mathbb{R}$ position, $t \in \mathbb{R}$ time) in a medium and the heat $q(x,T)$ supplied to / radiated away from it.

Distributed systems

$$\frac{\partial}{\partial t} T = \frac{\partial^2}{\partial x^2} T + q$$

For all sol'ns T, q **with** $T(x,t) = \text{constant} > 0$ **(and therefore** $q = 0$**) outside a compact set, there holds:**

First law:

$$\int_{\mathbb{R}^2} q(x,t)\, dx\, dt \quad = \quad 0,$$

Second law:

$$\int_{\mathbb{R}^2} \frac{q(x,t)}{T(x,t)}\, dx\, dt \quad \leq \quad 0.$$

Distributed systems

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First law:

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Second law:

$$\int_{\mathbb{R}^2} \frac{q(x,t)}{T(x,t)}\, dx\, dt \;\leq\; 0.$$

$\Rightarrow$

$$\mathbf{max}_{x,t}\{T(x,t) \mid q(x,t) \geq 0\} \geq \mathbf{min}_{x,t}\{T(x,t) \mid q(x,t) \leq 0\}.$$

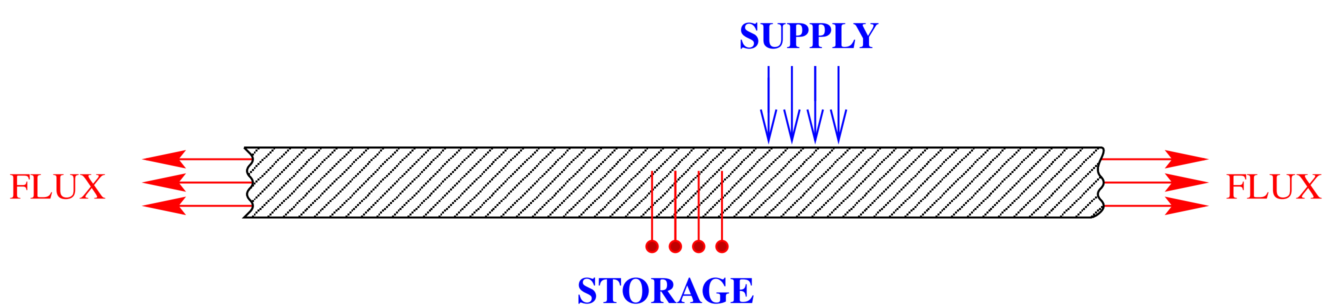
Cannot transport heat from a ‘cold source’ to a ‘hot sink’.

Distributed systems

$$\frac{\partial}{\partial t} T = \frac{\partial^2}{\partial x^2} T + q$$

$$\int_{\mathbb{R}^2} q(x,t)\, dx\, dt \;=\; 0, \qquad \int_{\mathbb{R}^2} \frac{q(x,t)}{T(x,t)}\, dx\, dt \;\leq\; 0.$$

Can these 'global' laws be expressed as 'local' laws?



rate of change in storage + spatial flux $\leq$ supply rate

Distributed systems

$$\frac{\partial}{\partial t} T = \frac{\partial^2}{\partial x^2} T + q$$

To be invented:

an 'extensive' quantity for the first law: internal energy

an 'extensive' quantity for the second law: entropy

Distributed systems

Define the following variables:

$$E = T \quad : \quad \textbf{the stored energy density,}$$

$$S = \ln(T) \quad : \quad \textbf{the entropy density,}$$

$$F_E = -\frac{\partial}{\partial x} T \quad : \quad \textbf{the energy flux,}$$

$$F_S = -\frac{1}{T}\frac{\partial}{\partial x} T \quad : \quad \textbf{the entropy flux,}$$

$$D_S = \left(\frac{1}{T}\frac{\partial}{\partial x} T\right)^2 \quad : \quad \textbf{the rate of entropy production.}$$

Distributed systems

$$\frac{\partial}{\partial t} T = \frac{\partial^2}{\partial x^2} T + q$$

$\Rightarrow$ ***Local versions*** **of the first and second law:**

rate of change in storage + spatial flux $\leq$ supply rate

Conservation of energy:

$$\frac{\partial}{\partial t} E + \frac{\partial}{\partial x} F_E = q,$$

Entropy production:

$$\frac{\partial}{\partial t} S + \frac{\partial}{\partial x} F_S = \frac{q}{T} + D_S \quad \Rightarrow \quad \frac{\partial}{\partial t} S + \frac{\partial}{\partial x} F_S \geq \frac{q}{T}.$$

Distributed systems

Our problem:

- **Extend notion of dissipative system to cover this case**
- **theory behind ad hoc constructions of E, F_E and S, F_S.**

Systems described by PDE's

PDE's: polynomial notation

Consider, for example, the PDE:

$$\begin{aligned} w_1(x_1,x_2) + \frac{\partial^2}{\partial x_2^2} w_1(x_1,x_2) + \frac{\partial}{\partial x_1} w_2(x_1,x_2) &= 0 \\ w_2(x_1,x_2) + \frac{\partial^3}{\partial x_2^3} w_1(x_1,x_2) + \frac{\partial^4}{\partial x_1^4} w_2(x_1,x_2) &= 0 \end{aligned}$$

PDE's: polynomial notation

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$$
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\end{aligned}
$$

$$\updownarrow$$

Notation:

$$\xi_1 \leftrightarrow \frac{\partial}{\partial x_1}, \quad \xi_2 \leftrightarrow \frac{\partial}{\partial x_2}, w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}, \quad R(\xi_1,\xi_2) = \begin{bmatrix} 1+\xi_2^2 & \xi_1 \\ \xi_2^3 & 1+\xi_1^4 \end{bmatrix}.$$

$$\boxed{R\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right) w = 0.}$$

Linear differential distributed systems

$\mathbb{T} = \mathbb{R}^{\mathtt{n}}$, **the set of independent variables,**

typically $\mathtt{n} = 4$**: time and space,**

$\mathbb{W} = \mathbb{R}^{\mathtt{w}}$, **the set of dependent variables,**

$\mathscr{B} =$ **the solutions of a linear constant coefficient PDE.**

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Let $R \in \mathbb{R}^{\bullet \times \mathtt{w}}[\xi_1, \cdots, \xi_{\mathtt{n}}]$, **and consider**

$$R\left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathtt{n}}}\right) w = 0. \quad (*)$$

Define the associated behavior

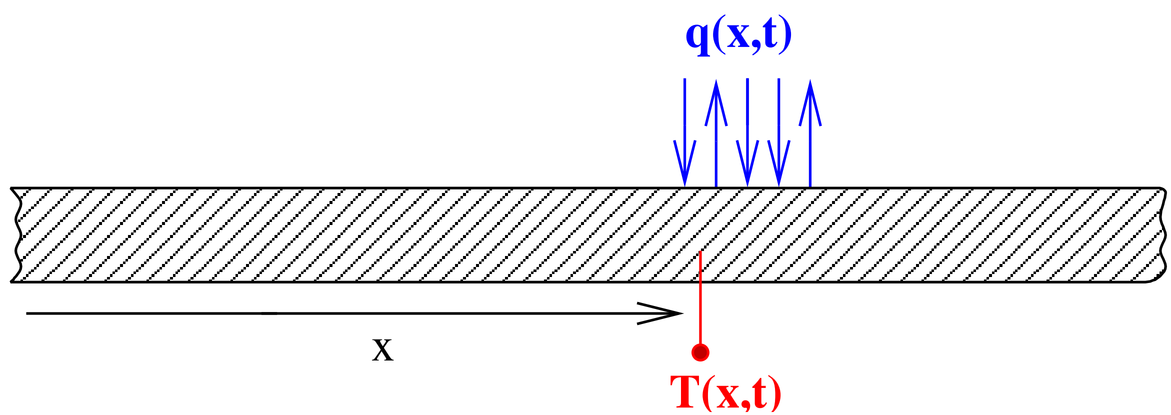
$$\mathscr{B} = \{w \in \mathscr{C}^{\infty}(\mathbb{R}^{\mathtt{n}}, \mathbb{R}^{\mathtt{w}}) \mid (*) \textbf{ holds } \}.$$

Notation for $\mathtt{n}$**-D linear differential systems:**

$$(\mathbb{R}^{\mathtt{n}}, \mathbb{R}^{\mathtt{w}}, \mathscr{B}) \in \mathscr{L}^{\mathtt{w}}_{\mathtt{n}}, \quad \textbf{or } \mathscr{B} \in \mathscr{L}^{\mathtt{w}}_{\mathtt{n}}.$$

Examples

Heat diffusion in a bar



$\rightsquigarrow$ **the PDE**

$$\boxed{\frac{\partial}{\partial t}T = \frac{\partial^2}{\partial x^2}T + q}$$

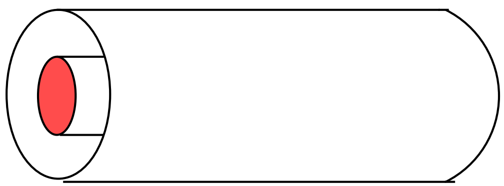
($x \in \mathbb{R}$, position, $t \in \mathbb{R}$, time), **(2-D system)**

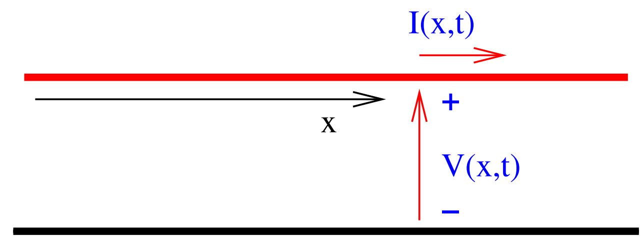
describes the evolution of the temperature $T(x,t)$

and the heat $q(x,T)$ supplied to / radiated away.

Examples

The voltage $V(x,t)$ and current $I(x,t)$ in a *coaxial cable*





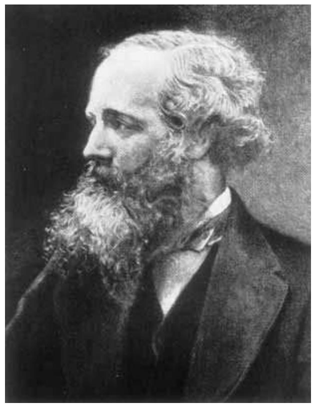
$$\begin{aligned} \frac{\partial}{\partial x} V &= RI - L\frac{\partial}{\partial t} I, \\ \frac{\partial}{\partial x} I &= GV - C\frac{\partial}{\partial t} V. \end{aligned}$$

R **the resistance,** L **the inductance,** C **the capacitance of the cable,** G **the conductance of the dielectric medium, all per unit length.**

(2-D system)

Examples

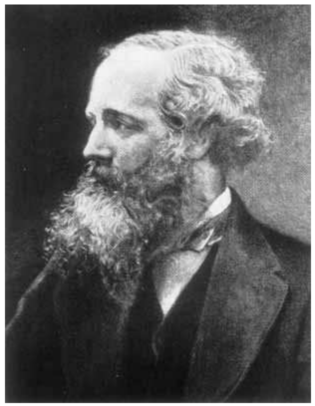
Maxwell's equations



$$\begin{aligned}
\nabla \cdot \vec{E} &= \frac{1}{\varepsilon_0}\rho \,, \\
\nabla \times \vec{E} &= -\frac{\partial}{\partial t}\vec{B}, \\
\nabla \cdot \vec{B} &= 0 \,, \\
c^2 \nabla \times \vec{B} &= \frac{1}{\varepsilon_0}\vec{j} + \frac{\partial}{\partial t}\vec{E}.
\end{aligned}$$

Examples

Maxwell's equations



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$\mathbb{T} = \mathbb{R} \times \mathbb{R}^3$ **(time and space)** $\rightsquigarrow \mathtt{n} = 4$ **(4-D system) ,**

$w = (\vec{E}, \vec{B}, \vec{j}, \rho)$ **(electric field, magnetic field, current, charge),**

$\mathbb{W} = \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}, \rightsquigarrow \mathtt{w} = 10$**,**

$\mathscr{B} =$ **set of solutions to these PDE's.** $\rightsquigarrow \mathscr{B} \in \mathscr{L}_4^{10}$**.**

Note: 10 variables, 8 equations! $\Rightarrow \exists$ free variables.

Elimination theorem

Theorem:

If the behavior of $(w_1, \ldots, w_{\mathtt{k}}, w_{\mathtt{k}+1}, \ldots, w_{\mathtt{w}})$ obeys a constant coefficient linear PDE, then so does the behavior of $(w_1, \ldots, w_{\mathtt{k}})$!

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Which PDE's describe $(\rho, \vec{E}, \vec{j})$ in ME's? Eliminate $\vec{B}$ $\rightsquigarrow$

$$\begin{aligned}
\nabla \cdot \vec{E} &= \frac{1}{\varepsilon_0}\rho\ , \\
\varepsilon_0 \frac{\partial}{\partial t} \nabla \cdot \vec{E} + \nabla \cdot \vec{j} &= 0, \\
\varepsilon_0 \frac{\partial^2}{\partial t^2}\vec{E} + \varepsilon_0 c^2 \nabla \times \nabla \times \vec{E} + \frac{\partial}{\partial t}\vec{j} &= 0.
\end{aligned}$$

Image representation

$$R\left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathtt{n}}}\right) w = 0$$

is called a kernel representation of the associated $\mathscr{B} \in \mathscr{L}_{\mathtt{n}}^{\mathtt{w}}$.

Image representation

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is called a kernel representation of the associated $\mathscr{B} \in \mathscr{L}^{\mathtt{w}}_{\mathtt{n}}$.
Another representation: image representation

$$w = M\left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathtt{n}}}\right) \ell$$

Elimination thm $\Rightarrow \quad \mathrm{im}\left(M\left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathtt{n}}}\right)\right) \in \mathscr{L}^{\mathtt{w}}_{\mathtt{n}}$!

Do all behaviors admit an image representation???

Image representation

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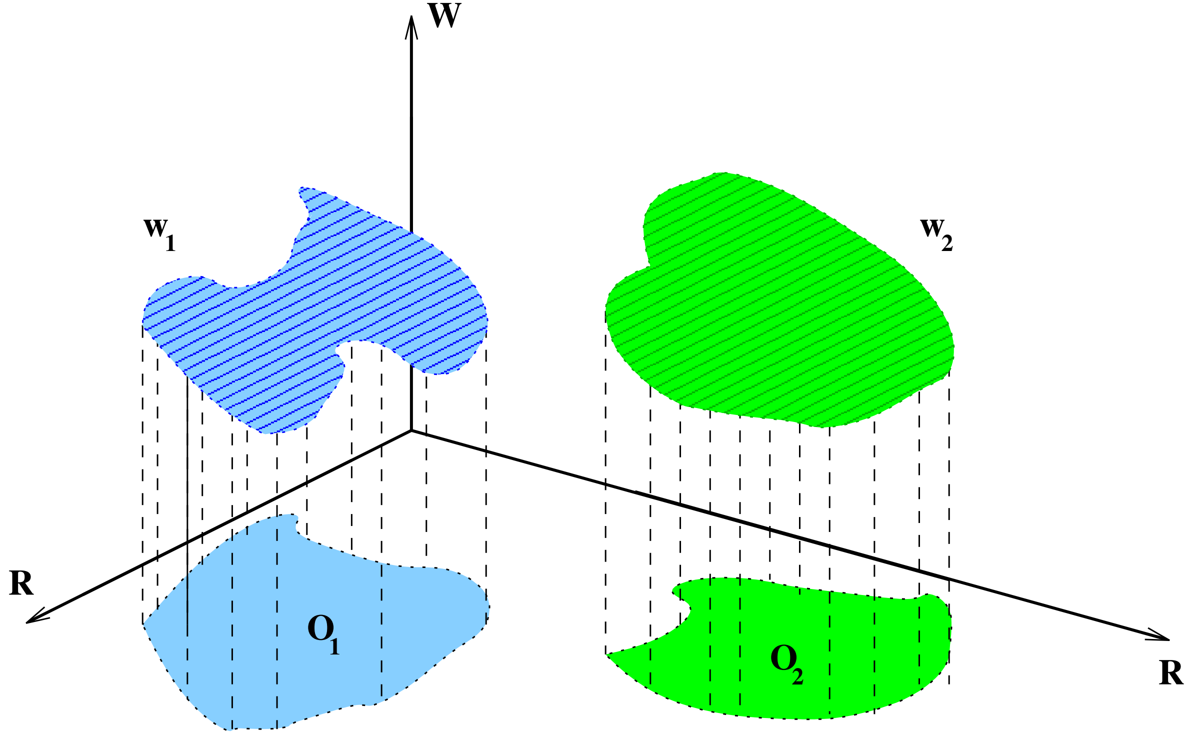
Do all behaviors admit an image representation???

$\mathscr{B} \in \mathscr{L}_{\mathtt{n}}^{\mathtt{w}}$ admits an image representation iff it is 'controllable'.

Controllability

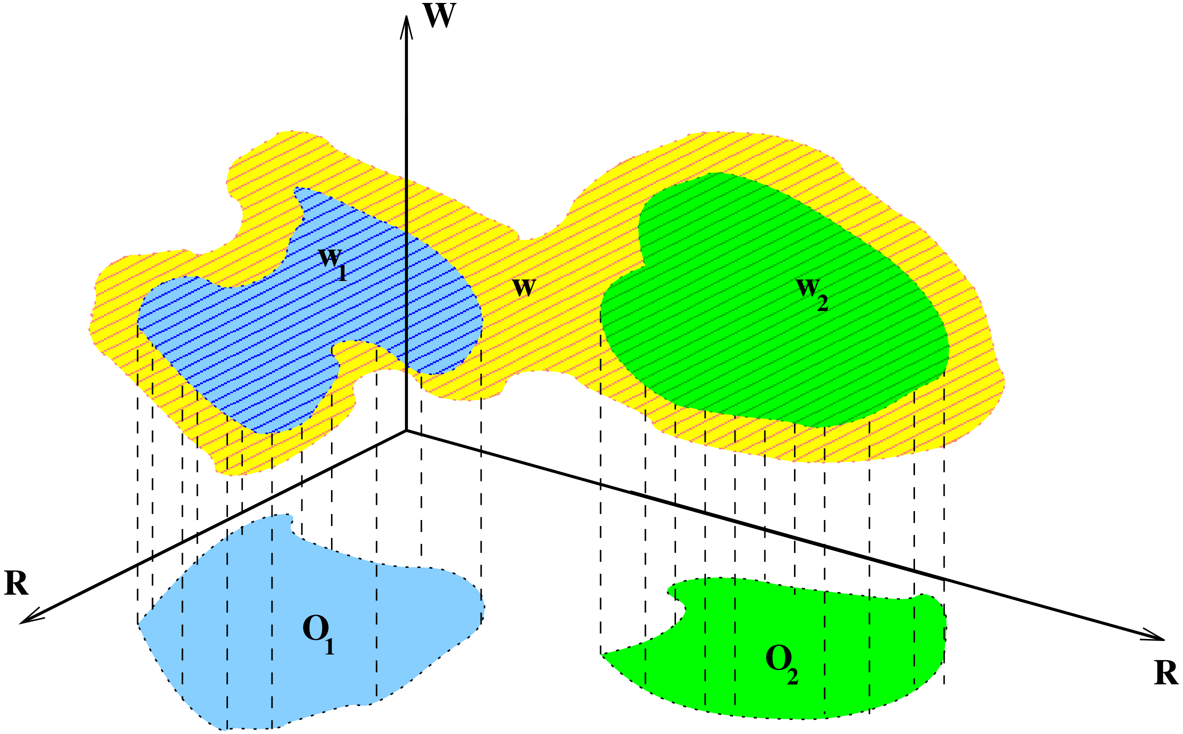
Def'n in pictures:



$$w_1, w_2 \in \mathscr{B}.$$

Controllability

Def'n in pictures:



w **'patches'** $w_1, w_2 \in \mathscr{B}$.

$\exists \;\; w \in \mathscr{B} \;\; \forall \;\; w_1, w_2 \in \mathscr{B}$: **Controllability :⇔ 'patchability'.**

Controllability

Theorem: The following are equivalent:

1. $\mathscr{B} \in \mathscr{L}_{\mathtt{n}}^{\mathtt{w}}$ is **controllable**
2. $\mathscr{B}$ **admits an image representation**
3. $\cdots$

Are Maxwell's equations controllable ?

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The following equations
in the ***scalar potential*** $\phi : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$ **and**
the ***vector potential*** $\vec{A} : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$
generate exactly the solutions to Maxwell's equations:

$$\begin{aligned}
\vec{E} &= -\frac{\partial}{\partial t}\vec{A} - \nabla\phi, \\
\vec{B} &= \nabla \times \vec{A}, \\
\vec{j} &= \varepsilon_0 \frac{\partial^2}{\partial t^2}\vec{A} - \varepsilon_0 c^2 \nabla^2 \vec{A} + \varepsilon_0 c^2 \nabla\left(\nabla \cdot \vec{A}\right) + \varepsilon_0 \frac{\partial}{\partial t}\nabla\phi, \\
\rho &= -\varepsilon_0 \frac{\partial}{\partial t}\nabla \cdot \vec{A} - \varepsilon_0 \nabla^2 \phi.
\end{aligned}$$

Proves controllability. Illustrates the interesting connection

controllability $\Leftrightarrow \exists$ potential!

Observability

Observability **of the image representation**

$$w = M\left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathtt{n}}}\right) \ell$$

is defined as: ℓ **can be deduced from** w**,**

i.e. $M\left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathtt{n}}}\right)$ **should be injective.**

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Not all controllable systems admit an observable im. repr'n.
For $\mathtt{n} = 1$**, they do. For** $\mathtt{n} > 1$**, exceptionally so.**

The latent variable ℓ **in an im. repr'n may be 'hidden'.**

Example: **Maxwell's equations do not allow a potential representation with an observable potential.**

Dissipative distributed systems

Notation

Multi-index notation:

$$x = (x_1, \ldots, x_{\mathtt{n}}), k = (k_1, \ldots, k_{\mathtt{n}}), \ell = (\ell_1, \ldots, \ell_{\mathtt{n}}),$$
$$\xi = (\xi_1, \cdots, \xi_{\mathtt{n}}), \zeta = (\zeta_1, \ldots, \zeta_{\mathtt{n}}), \eta = (\eta_1, \ldots, \eta_{\mathtt{n}}),$$

$$\frac{d}{dx} = \left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_{\mathtt{n}}}\right), \frac{d^k}{dx^k} = \left(\frac{\partial^{k_1}}{\partial x_1^{k_1}}, \ldots, \frac{\partial^{k_{\mathtt{n}}}}{\partial x_{\mathtt{n}}^{k_{\mathtt{n}}}}\right),$$
$$dx = dx_1 dx_2 \ldots dx_{\mathtt{n}},$$

$$R\left(\frac{d}{dx}\right) w = 0 \quad \textbf{for} \quad R\left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathtt{n}}}\right) w = 0,$$
$$w = M\left(\frac{d}{dx}\right) \ell \quad \textbf{for} \quad w = M\left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathtt{n}}}\right) \ell,$$

etc.

Notation

$$\nabla\cdot := \frac{\partial}{\partial x_1} + \cdots + \frac{\partial}{\partial x_{\mathtt{n}}}.$$

For simplicity of notation, and for concreteness, we often take $\mathtt{n} = 4$, independent variables, t, time, and x, y, z, space.

$$\nabla\cdot := \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z},$$ **'spatial flux'**

QDF's

The quadratic map acting on $w : \mathbb{R}^{\mathtt{n}} \to \mathbb{R}^{\mathtt{w}}$ and its derivatives, defined by

$$w \mapsto \Sigma_{k,\ell} \left(\frac{d^k}{dx^k} w \right)^\top \Phi_{k,\ell} \left(\frac{d^\ell}{dx^\ell} w \right)$$

is called ***quadratic differential form*** **(QDF) on** $\mathscr{C}^\infty(\mathbb{R}^{\mathtt{n}}, \mathbb{R}^{\mathtt{w}})$**.**
$\Phi_{k,\ell} \in \mathbb{R}^{\mathtt{w} \times w}$; **WLOG:** $\Phi_{k,\ell} = \Phi_{\ell,k}^\top$**.**

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$\Phi_{k,\ell} \in \mathbb{R}^{\mathtt{w} \times w}$; **WLOG:** $\Phi_{k,\ell} = \Phi_{\ell,k}^\top$**.**

Introduce the $2\mathtt{n}$-variable polynomial matrix Φ

$$\Phi(\zeta, \eta) = \sum_{k,\ell} \Phi_{k,\ell} \zeta^k \eta^\ell .$$

Denote the QDF as Q_Φ. QDF's are parametrized by $\mathbb{R}[\zeta, \eta]$.

Dissipative distributed systems

We henceforth consider only controllable linear differential systems and QDF's for supply rates.

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Definition: $\mathscr{B} \in \mathscr{L}_{\mathtt{n}}^{\mathtt{w}}$, **controllable, is said to be**

***dissipative* with respect to the supply rate** Q_Φ

(a QDF) if

$$\int_{\mathbb{R}^{\mathrm{n}}} Q_\Phi(w)\ dx \geq 0$$

for all $w \in \mathscr{B}$ **of compact support, i.e., for all** $w \in \mathscr{B} \cap \mathscr{D}$**.**

$\mathscr{D} := \mathscr{C}^\infty$ **and 'compact support'.**

Dissipative distributed systems

Assume $\mathtt{n} = 4$:
independent variables $x, y, z; t$: space and time.

Idea: $Q_\Phi(w)(x,y,z;t) \; dxdydz \; dt$:

'energy' supplied to the system
in the space-cube $[x, x+dx] \times [y, y+dy] \times [z, z+dz]$
during the time-interval $[t, t+dt]$.

Dissipativity $:\Leftrightarrow$

$$\boxed{\int_{\mathbb{R}} \left[\int_{\mathbb{R}^3} Q_\Phi(w)(x,y,z,\, t) \; dxdydz\right] \, dt \geq 0} \quad \forall\, w \in \mathscr{B} \cap \mathscr{D}.$$

A dissipative system absorbs net energy.

Example: EM fields

Maxwell's eq'ns define a dissipative (in fact, a conservative) system w.r.t. the QDF $-\vec{E} \cdot \vec{j}$

Indeed, if $\vec{E}, \vec{j}$ **are of compact support and satisfy**

$$\begin{aligned} \varepsilon_0 \frac{\partial}{\partial t} \nabla \cdot \vec{E} + \nabla \cdot \vec{j} &= 0, \\ \varepsilon_0 \frac{\partial^2}{\partial t^2} \vec{E} + \varepsilon_0 c^2 \nabla \times \nabla \times \vec{E} + \frac{\partial}{\partial t} \vec{j} &= 0, \end{aligned}$$

then

$$\int_{\mathbb{R}} \left[\int_{\mathbb{R}^3} \left(-\vec{E} \cdot \vec{j} \right) \; dxdydz \right] \, dt = 0 \, .$$

The storage and the flux

Local dissipation law

Dissipativity $:\Leftrightarrow$

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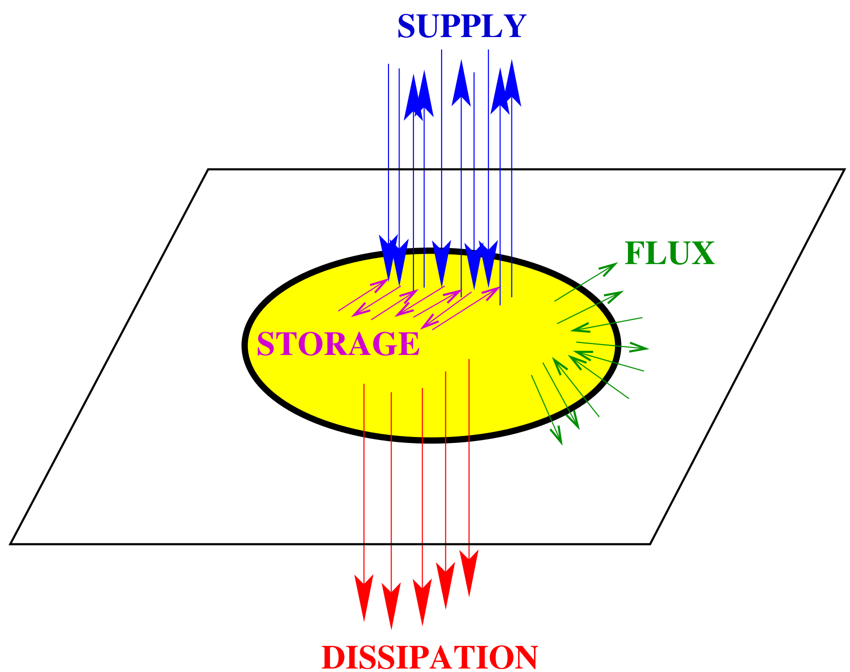
Can this be reinterpreted as:

As the system evolves, some of the energy supplied is locally stored, some locally dissipated, and some redistributed over space?

Local dissipation law

!! Invent storage and flux, locally defined in time and space, such that in every spatial domain there holds:

$$\frac{d}{dt}\ \textbf{Storage} + \textbf{Spatial flux} \leq \textbf{Supply.}$$



Supply = partly stored + partly radiated + partly dissipated.

Main result (stated for $\mathtt{n} = 4$)

Thm: $\mathtt{n} = 4 : x, y, z; t :$ **space/time;** $\mathscr{B} \in \mathscr{L}_4^{\mathtt{w}}$**, controllable.**

Then $\int_{\mathbb{R}} \left[\int_{\mathbb{R}^3} Q_\Phi(w) \ dxdydz\right] \ dt \geq 0$ **for all** $w \in \mathscr{B} \cap \mathscr{D}$

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$\exists$ **an im. repr.** $w = M\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \frac{\partial}{\partial t}\right)\ell$ **of** $\mathscr{B}$,

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and QDF's S**, the *storage*, and** F_x, F_y, F_z**, the *flux*,**

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$\exists$ **an im. repr.** $w = M\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \frac{\partial}{\partial t}\right)\ell$ **of** $\mathscr{B}$,
and QDF's S, **the *storage*, and** F_x, F_y, F_z, **the *flux*,**
such that the *local dissipation law*

$$\boxed{\frac{\partial}{\partial t}S(\ell) + \frac{\partial}{\partial x}F_x(\ell) + \frac{\partial}{\partial y}F_y(\ell) + \frac{\partial}{\partial z}F_z(\ell) \leq Q_\Phi(w)}$$

holds for all (w, ℓ) **that satisfy** $w = M\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \frac{\partial}{\partial t}\right)\ell$.

Hidden variables

The local law involves
possibly unobservable, - i.e., hidden!
latent variables (the ℓ's).

This gives physical notions as stored energy, entropy, etc., an enigmatic physical flavor.

Energy stored in EM fields

Maxwell's equations are dissipative (in fact, conservative) with respect to $-\vec{E}\cdot\vec{j},$ **the rate of energy supplied.**

Energy stored in EM fields

Maxwell's equations are dissipative (in fact, conservative) with respect to $-\vec{E}\cdot\vec{j}$, **the rate of energy supplied.**

Introduce the *stored energy density,* S, and

the *energy flux density* (the *Poynting vector*), $\vec{F}$,

$$S\left(\vec{E},\vec{B}\right) := \frac{\varepsilon_0}{2}\vec{E}\cdot\vec{E} + \frac{\varepsilon_0 c^2}{2}\vec{B}\cdot\vec{B},$$

$$\vec{F}\left(\vec{E},\vec{B}\right) := \varepsilon_0 c^2 \vec{E}\times\vec{B}.$$

Local conservation law for Maxwell's equations:

$$\boxed{\frac{\partial}{\partial t}S\left(\vec{E},\vec{B}\right) + \nabla\cdot\vec{F}\left(\vec{E},\vec{B}\right) = -\vec{E}\cdot\vec{j}.}$$

Involves $\vec{B}$, **unobservable from** $\vec{E}$ **and** $\vec{j}$.

The proof

Outline of the proof

Using controllability and image representations, we may assume, WLOG: $\mathscr{B} = \mathscr{C}^\infty(\mathbb{R}^{\mathtt{n}}, \mathbb{R}^{\mathtt{w}})$

To be shown

Global dissipation $:\Leftrightarrow$

$$\int_{\mathbb{R}^{\mathtt{n}}} Q_\Phi(w) \geq 0 \textbf{ for all } w \in \mathscr{D}$$

$$\Updownarrow$$

$$\exists\ \Psi : \quad \nabla \cdot Q_\Psi(w) \leq Q_\Phi(w) \textbf{ for all } w \in \mathscr{C}^\infty$$

$\Leftrightarrow$: **Local dissipation**

$$\int_{\mathbb{R}^{\mathtt{n}}} Q_\Phi(w) \geq 0 \textbf{ for all } w \in \mathscr{D}$$

$$\Updownarrow \quad \textbf{(Parseval)}$$

$$\Phi(-i\omega, i\omega) \geq 0 \textbf{ for all } \omega \in \mathbb{R}^{\mathtt{n}}$$

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$$\exists\ D: \quad \Phi(-\xi, \xi) = D^\top(-\xi) D(\xi)$$

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$\Updownarrow$ **(clearly)**

$$\exists\ \Psi : \quad \nabla \cdot Q_\Psi(w) \leq Q_\Phi(w) \textbf{ for all } w \in \mathscr{C}^\infty$$

Outline of the proof

Assuming factorizability, we indeed obtain:

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$\Leftrightarrow$: **Local dissipation**

However, ... this argument is valid only for $\mathtt{n} = 1$**...**

The factorization equation (FE)

The factorization equation

Consider

$$X^{\top}(-\xi)X(\xi) = Y(\xi) \quad \textbf{(FE)}$$

with $Y \in \mathbb{R}^{\bullet\times\bullet}[\xi]$ **given, and** X **the unknown.** ***Solvable??***

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$\cong$

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Under what conditions on Y **does there exist a solution** X**?**

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Under what conditions on Y **does there exist a solution** X**?**

Scalar case: write the real polynomial Y **as a sum of squares**

$$Y = x_1^2 + x_2^2 + \cdots + x_{\mathtt{k}}^2.$$

$$X^\top(\xi)X(\xi) = Y(\xi) \quad \textbf{(FE)}$$

Y is a given polynomial matrix; X is the unknown.

For $\mathtt{n} = 1$ and $Y \in \mathbb{R}[\xi]$, solvable (with $X \in \mathbb{R}^2[\xi]$) iff

$$Y(\alpha) \geq 0 \qquad \textbf{for all } \alpha \in \mathbb{R}.$$

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For $\mathtt{n} = 1$ and $Y \in \mathbb{R}^{\bullet\times\bullet}[\xi]$, it is well-known (but non-trivial) that (FE) is solvable (with $X \in \mathbb{R}^{\bullet\times\bullet}[\xi]$!) iff

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For $\mathtt{n}>1$ and under the symmetry and positivity condition

$$Y(\alpha)= Y^\top(\alpha)\geq 0 \qquad \textbf{for all } \alpha\in\mathbb{R}^{\mathtt{n}},$$

this equation can nevertheless in general not be solved over the polynomial matrices, for $X\in\mathbb{R}^{\bullet\times\bullet}[\xi]$,

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For $\mathtt{n} = 1$ and $Y \in \mathbb{R}^{\bullet\times\bullet}[\xi]$, it is well-known (but non-trivial) that (FE) is solvable (with $X \in \mathbb{R}^{\bullet\times\bullet}[\xi]$!) iff

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For $\mathtt{n} > 1$ and under the symmetry and positivity condition

$$Y(\alpha) = Y^\top(\alpha) \geq 0 \qquad \textbf{for all } \alpha \in \mathbb{R}^{\mathtt{n}},$$

this equation can nevertheless in general not be solved over the polynomial matrices, for $X \in \mathbb{R}^{\bullet\times\bullet}[\xi]$,
but it can be solved over the matrices of rational functions,
i.e., for $X \in \mathbb{R}^{\bullet\times\bullet}(\xi)$.

Hilbert's 17-th

This factorizability is a consequence of Hilbert's 17-th pbm!

!! Solve $p = p_1^2 + p_2^2 + \cdots + p_{\mathtt{k}}^2$, p **given**

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!! Solve $p = p_1^2 + p_2^2 + \cdots + p_{\mathtt{k}}^2$, p **given**

A polynomial $p \in \mathbb{R}[\xi_1, \cdots, \xi_{\mathtt{n}}]$, **with** $p(\alpha_1, \ldots, \alpha_{\mathtt{n}}) \geq 0$ **for all** $(\alpha_1, \ldots, \alpha_{\mathtt{n}}) \in \mathbb{R}^{\mathtt{n}}$ **can in general not be expressed as a SOS of polynomials, with the** p_i**'s** $\in \mathbb{R}[\xi_1, \cdots, \xi_{\mathtt{n}}]$.

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This factorizability is a consequence of Hilbert's 17-th pbm!

!! Solve $p = p_1^2 + p_2^2 + \cdots + p_{\mathtt{k}}^2$, p **given**

A polynomial $p \in \mathbb{R}[\xi_1, \cdots, \xi_{\mathtt{n}}]$, **with** $p(\alpha_1, \ldots, \alpha_{\mathtt{n}}) \geq 0$ **for all** $(\alpha_1, \ldots, \alpha_{\mathtt{n}}) \in \mathbb{R}^{\mathtt{n}}$ **can in general <u>not</u> be expressed as a SOS of polynomials, with the** p_i**'s** $\in \mathbb{R}[\xi_1, \cdots, \xi_{\mathtt{n}}]$**.**
But a rational function (and hence a polynomial) $p \in \mathbb{R}(\xi_1, \cdots, \xi_{\mathtt{n}})$, **with** $p(\alpha_1, \ldots, \alpha_{\mathtt{n}}) \geq 0$, **for all** $(\alpha_1, \ldots, \alpha_{\mathtt{n}}) \in \mathbb{R}^{\mathtt{n}}$, **can be expressed as a SOS of** ($\mathtt{k} = 2^{\mathtt{n}}$) **rational functions, with the** p_i**'s** $\in \mathbb{R}(\xi_1, \cdots, \xi_{\mathtt{n}})$**.**

Outline of the proof

$\Rightarrow$ **solvability of the factorization eq'n**

$$\Phi(-i\omega, i\omega) \geq 0 \text{ for all } \omega \in \mathbb{R}^{\mathtt{n}}$$

$\Updownarrow$ **(Factorization equation)**

$$\exists \; D : \quad \Phi(-\xi, \xi) = D^{\top}(-\xi) D(\xi)$$

over the rational functions, i.e., with D **a matrix with elements in** $\mathbb{R}(\xi_1, \cdots, \xi_{\mathtt{n}})$.

Outline of the proof

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over the rational functions, i.e., with D a matrix with elements in $\mathbb{R}(\xi_1, \cdots, \xi_{\mathrm{n}})$.

The need to introduce rational f'ns in this factorization and an image repr. of $\mathscr{B}$ (to reduce the pbm to $\mathscr{C}^{\infty}$) are the causes of the unavoidable presence of (possibly unobservable, i.e., 'hidden') latent variables in the local dissipation law.

Uniqueness

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1. **The non-uniqueness of the latent variable ℓ in various (non-observable) image representations of $\mathscr{B}$.**
2. **of D in the factorization equation**

$$\Phi(-\xi,\xi) = D^\top(-\xi)D(\xi)$$

3. **(in the case $\mathtt{n} > 1$) of the solution Ψ of**

$$(\zeta+\eta)^\top\Psi(\zeta,\eta) = \Phi(\zeta,\eta) - D^\top(\zeta)D(\eta)$$

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For conservative systems, $\Phi(-\xi,\xi) = 0$, whence $D = 0$, but, when $\mathtt{n} > 1$, the third source of non-uniqueness remains.

Uniqueness

The non-uniqueness is very real, even for EM fields.

Uniqueness

The non-uniqueness is very real, even for EM fields. Cfr.

The ambiguity of the field energy

... There are, in fact, an infinite number of different possibilities for u **[the internal energy]** ***and*** S **[the flux] ...** ***It is sometimes claimed that this problem can be resolved using the theory of gravitation ... as yet nobody has done such a delicate experiment ... So we will follow the rest of the world - besides, we believe that it*** **[our choice]** ***is probably perfectly right.***

The Feynman Lectures on Physics,
Volume II, page 27-6.

Summary

- **The theory of dissipative systems centers around the construction of the storage function**
- **global dissipation $\Leftrightarrow \exists$ local dissipation law**
- **Involves possibly hidden latent variables**
  (e.g. $\vec{B}$ in Maxwell's eq'ns)
- **The proof $\cong$ Hilbert's 17-th problem**
- **Neither controllability nor observability are good generic system theoretic assumptions for physical models**

End of lecture 6

The DDS work was done jointly with Harish Pillai from the IIT Bombay.

<u>Reference</u>: H.P. and J.C. Willems, Dissipative distributed systems, *SIAM Journal on Control and Optimization* volume 40, pages 1406–1430, 2002.

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