Interconnection of stochastic systems

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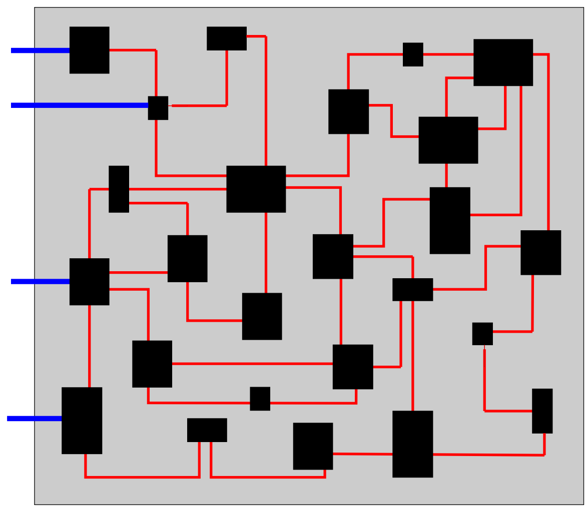
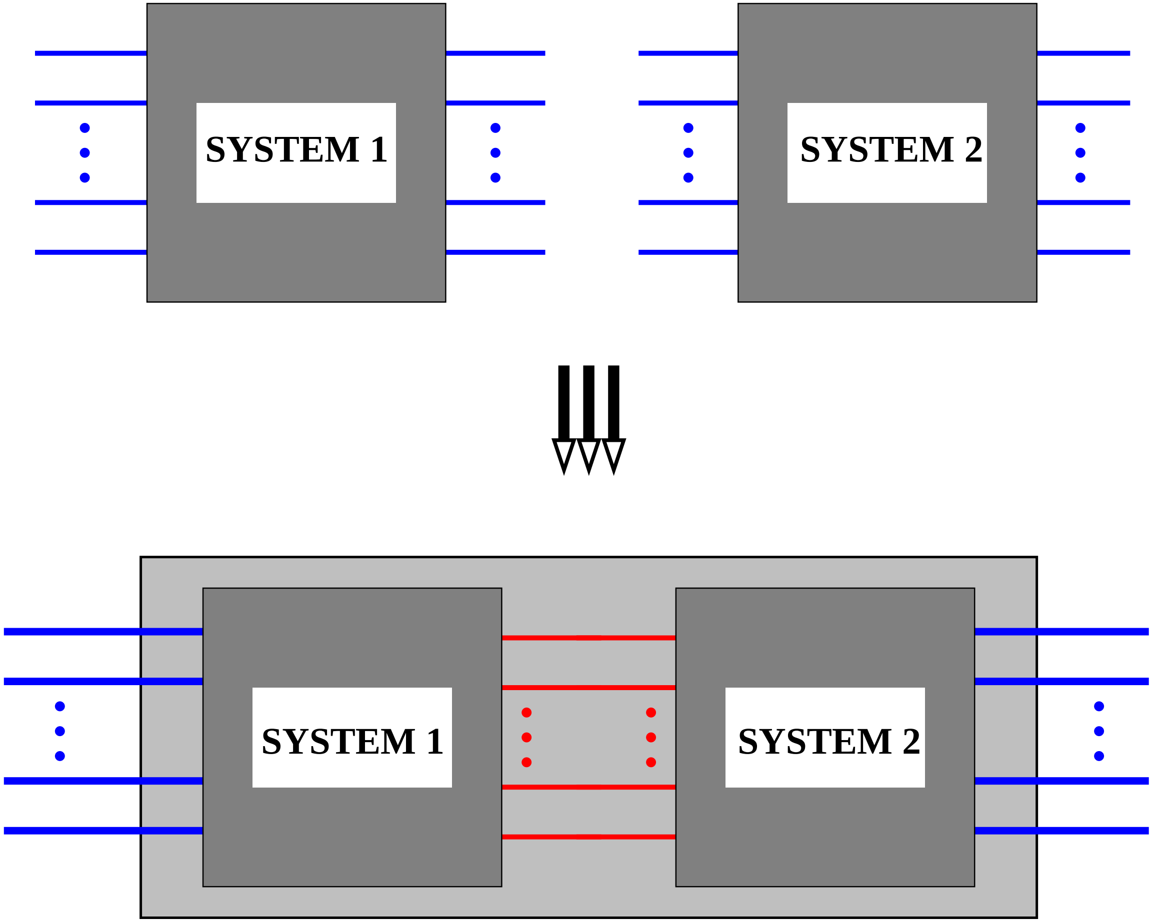
Message

Interconnection is a basic system operation.



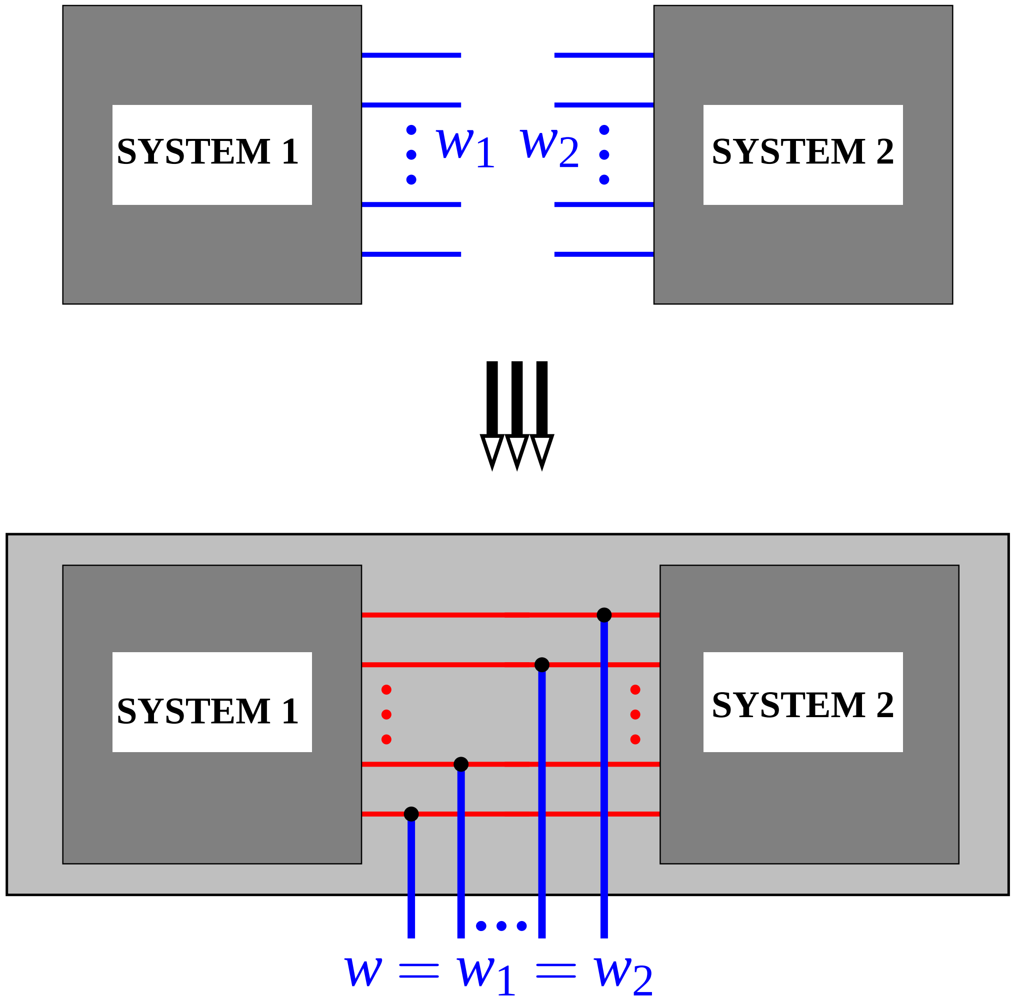
It is not dealt with (very well) in probability theory.

Interconnection

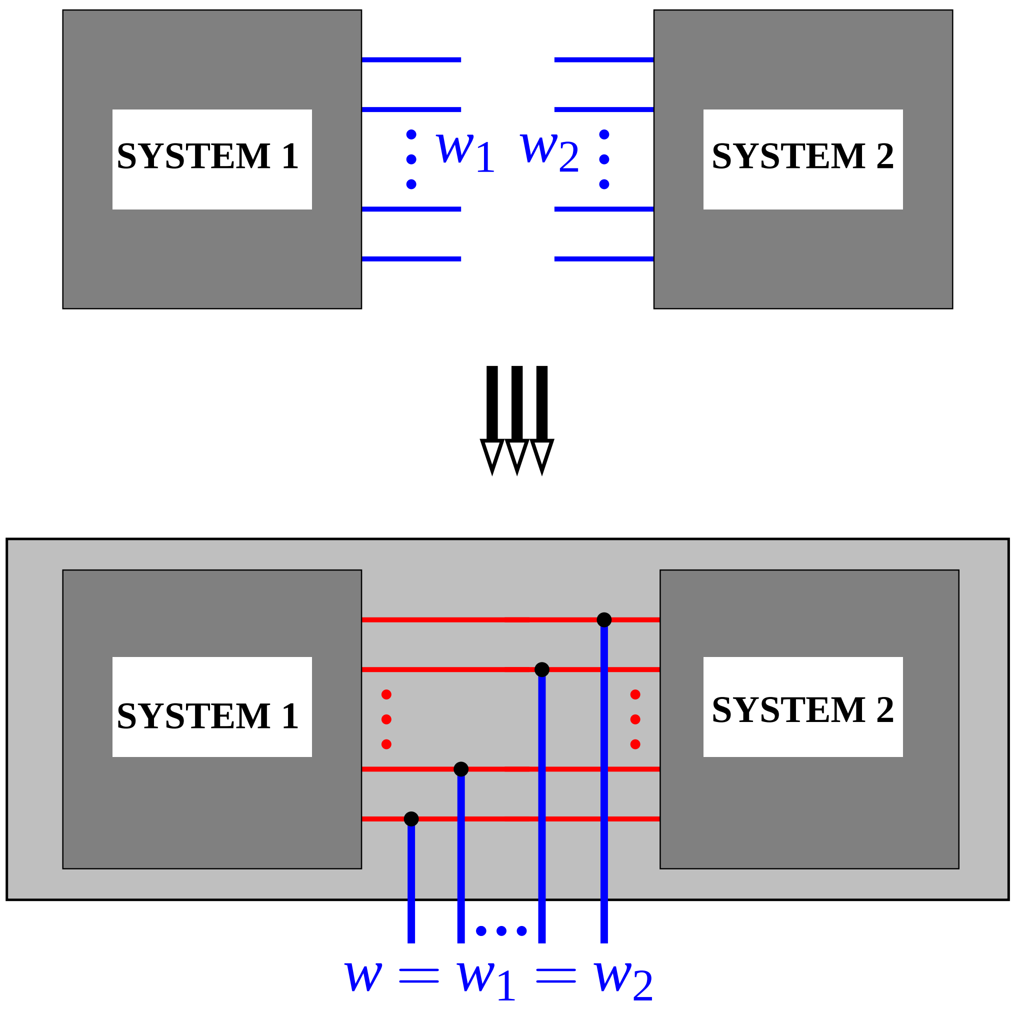


Interconnection = variable sharing

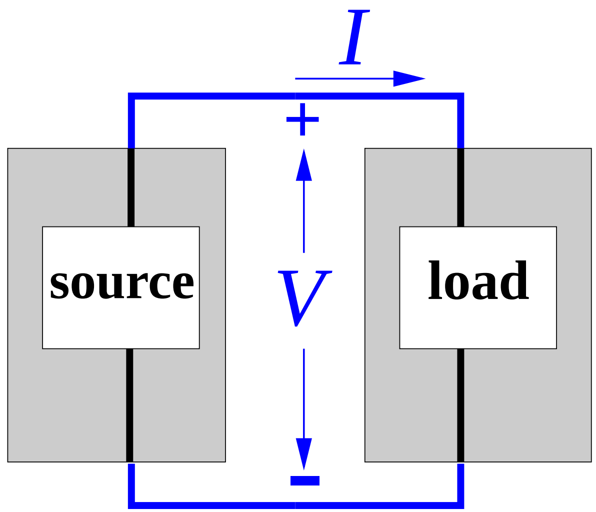
Interconnection



Interconnection



Example



$$w = \begin{bmatrix} V \\ I \end{bmatrix}$$

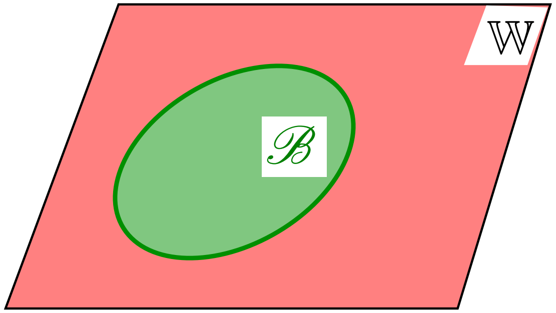
Interconnection of deterministic systems

Formalization

A deterministic system $\Sigma = (\mathbb{W}, \mathscr{B})$, **with**

$\mathbb{W}$ **the 'outcome space'**

$\mathscr{B} \subseteq \mathbb{W}$ **the 'behavior'**



Formalization

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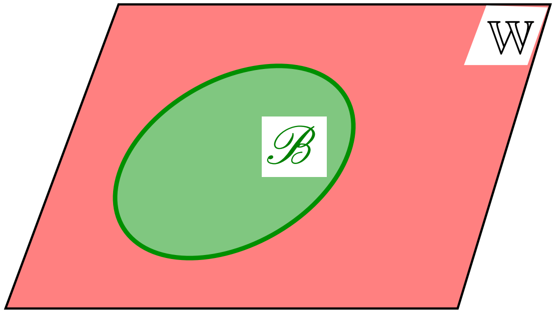
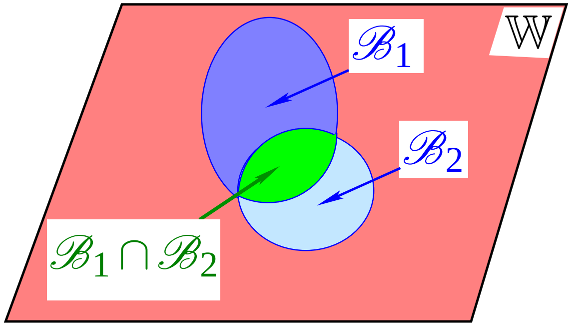


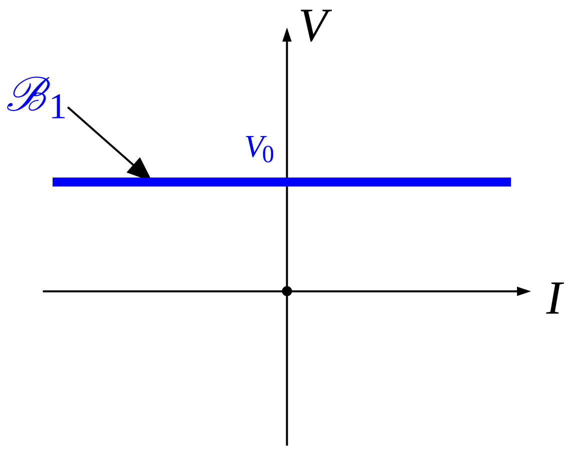
Interconnection of

$\Sigma_1 = (\mathbb{W}, \mathscr{B}_1)$ **and** $\Sigma_2 = (\mathbb{W}, \mathscr{B}_2)$

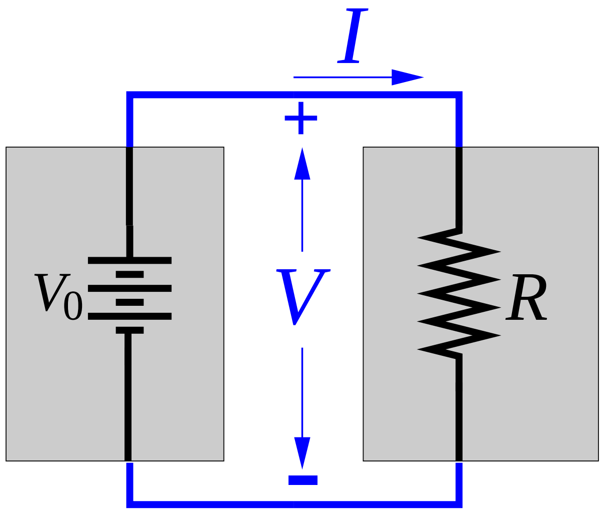
$\Sigma_1 \wedge \Sigma_2 := (\mathbb{W}, \mathscr{B}_1 \cap \mathscr{B}_2)$



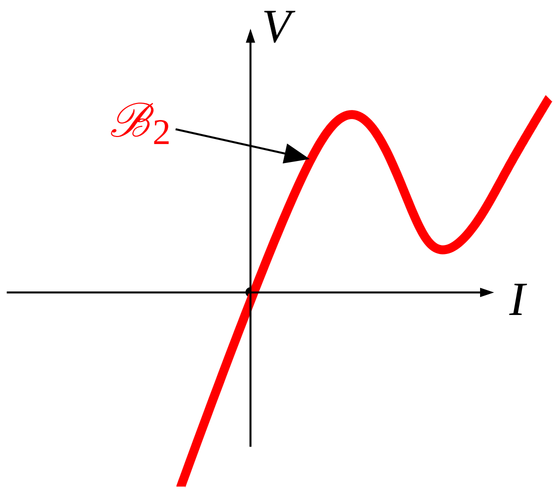
Example



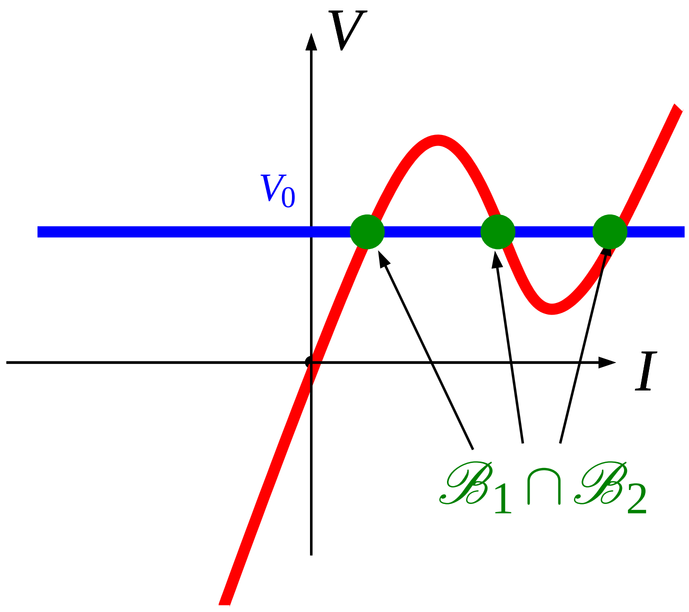
$\mathscr{B}_1$
V_0
V
I



I
+
V_0
V
R



$\mathscr{B}_2$
V
I



V
V_0
I
$\mathscr{B}_1 \cap \mathscr{B}_2$

Interconnection of stochastic systems

Stochastic systems

A stochastic system $\Sigma := (\mathbb{W}, \mathscr{E}, P)$, **with**

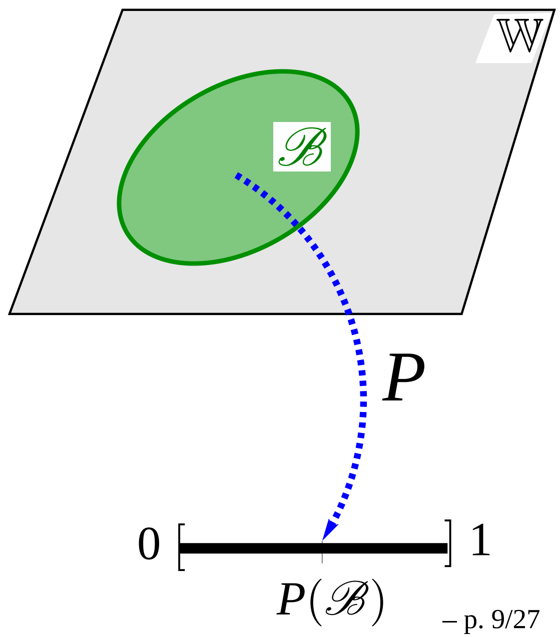
$\mathbb{W}$ **the 'outcome space'**

$\mathscr{E}$ **a σ-algebra of subsets of $\mathbb{W}$ the 'events'**

$P : \mathscr{E} \to [0,1]$ **the 'probability'**

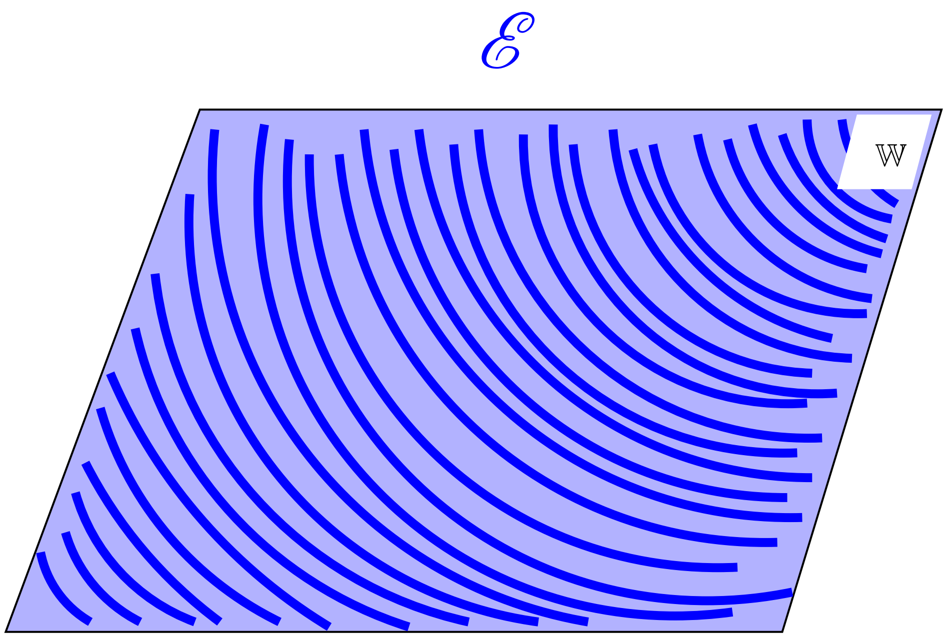
$\mathscr{E}$ **and** P **satisfy the Kolmogorov axioms.**

$P(\mathscr{B})$**: the probability that the behavior is** $\mathscr{B} \subseteq \mathbb{W}, \mathscr{B} \in \mathscr{E}$**.**



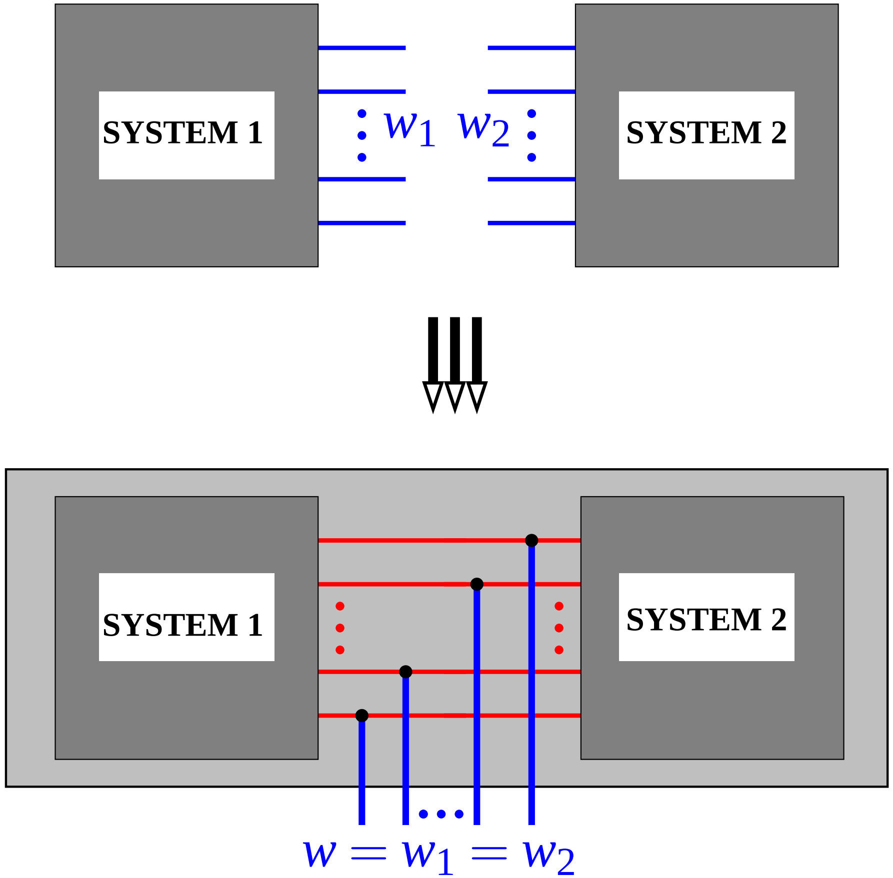
Example σ-algebras

- $\mathbb{W}$ **finite,** $\mathscr{E}$ **= all subsets of** $\mathbb{W}$**.**
- $\mathbb{W} = \mathbb{R}^n$, $\mathscr{E}$ **the 'Borel' sets** $\cong$ **all subsets of** $\mathbb{R}^n$**.**
- **measurable sets** $\cong$ **union of partitioning sets**



$\mathscr{E}$ **in terms of a partition of** $\mathbb{W}$**.**

Difficulty with stochastic interconnection



We have to avoid getting in a jam because of things as

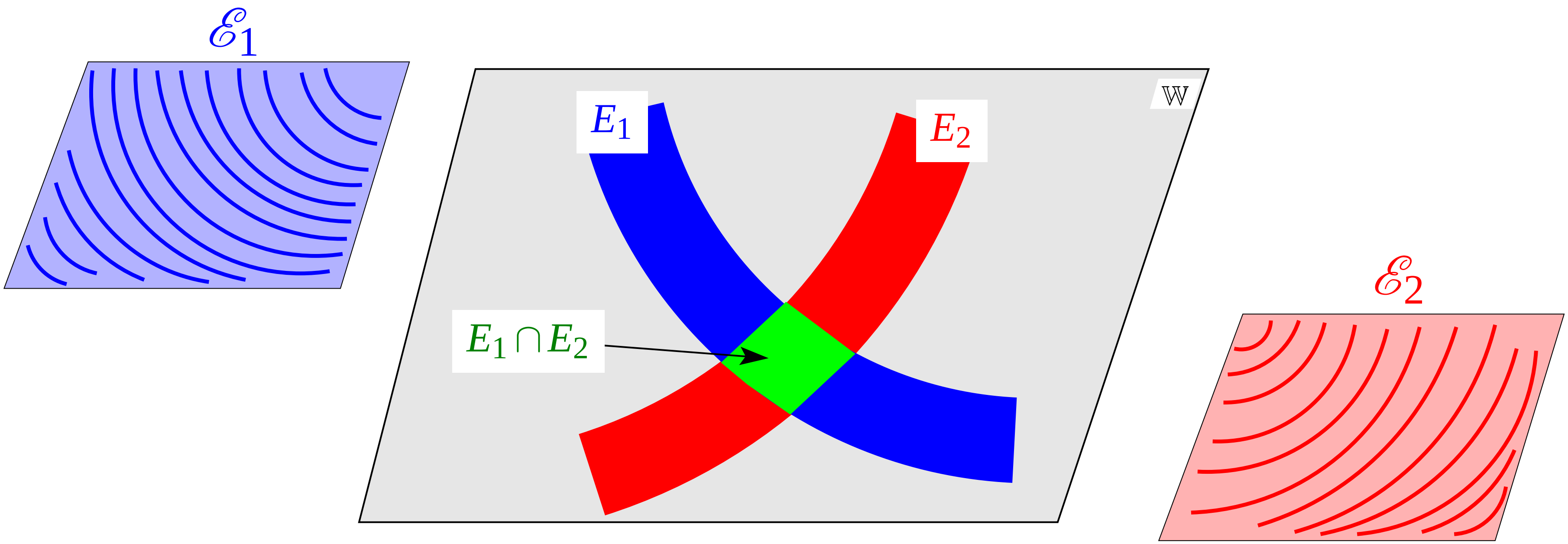
$P_1(E) \neq P_2(E)$ **for some** $E \subseteq \mathbb{W}$

Complementarity of σ-algebras

$\mathscr{E}_1$ **and** $\mathscr{E}_2$ **are complementary** ***σ-algebras***

$:\Leftrightarrow$ **for all nonempty** $E_1, E_1' \in \mathscr{E}_1, E_2, E_2' \in \mathscr{E}_2$

$$[\![E_1 \cap E_2 = E_1' \cap E_2']\!] \Rightarrow [\![E_1 = E_1' \text{ and } E_2 = E_2']\!].$$



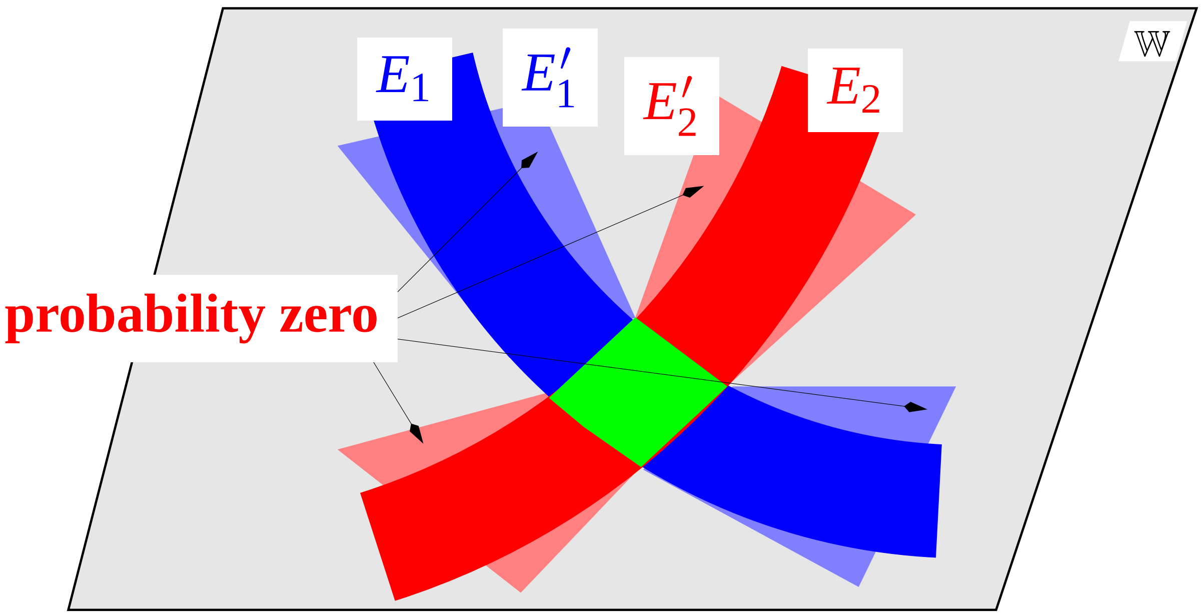
Intersection ⇒ intersectants.

Complementarity of stochastic systems

$(\mathbb{W}, \mathscr{E}_1, P_1)$ **and** $(\mathbb{W}, \mathscr{E}_2, P_2)$ **are complementary systems**

:⇔ for all $E_1, E_1' \in \mathscr{E}_1, E_2, E_2' \in \mathscr{E}_2$

$$[\![E_1 \cap E_2 = E_1' \cap E_2']\!] \Rightarrow [\![P_1(E_1)P_2(E_2) = P_1(E_1')P_2(E_2')]\!].$$



Intersection ⇒ product of probabilities of intersectants.

Interconnection

Let $(\mathbb{W}, \mathscr{E}_1, P_1)$ and $(\mathbb{W}, \mathscr{E}_2, P_2)$ be *independent* and *complementary* stochastic systems.

Their *interconnection* is defined as $(\mathbb{W}, \mathscr{E}, P)$ with

$\mathscr{E}$:= the σ-algebra generated by the *'rectangles'*

$$\{E_1 \cap E_2 \mid E_1 \in \mathscr{E}_1, E_2 \in \mathscr{E}_2\},$$

and P defined for rectangles by

$$P(E_1 \cap E_2) := P_1(E_1) P_2(E_2).$$

and extended to $\mathscr{E}$ via the Hahn-Kolmogorov thm.

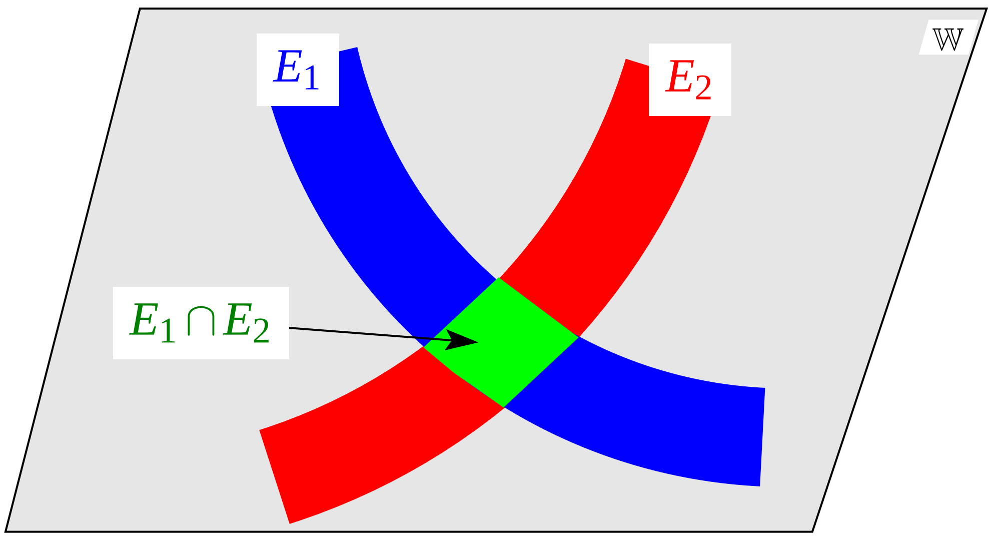
Interconnection of complementary systems

$$\Sigma_1 = (\mathbb{W}, \mathscr{E}_1, P_1), \quad \Sigma_2 = (\mathbb{W}, \mathscr{E}_2, P_2)$$

$$\Sigma_1 \wedge \Sigma_2 = (\mathbb{W}, \mathscr{E}, P)$$

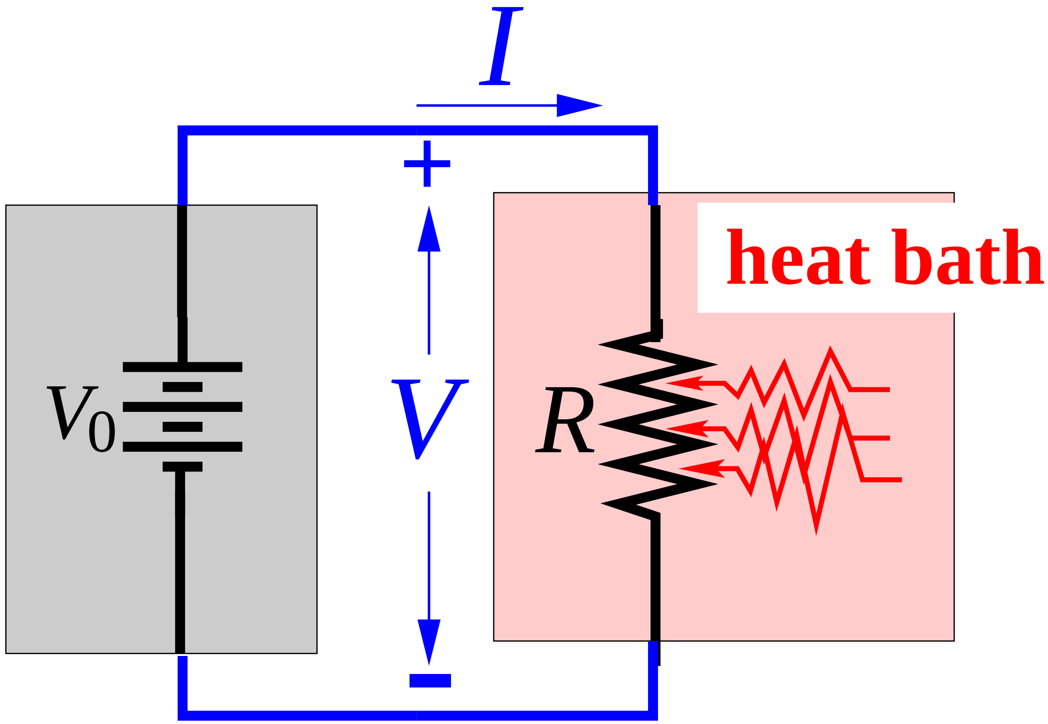
with $\mathscr{E}$ **= the** σ**-algebra generated by** $\mathscr{E}_1 \cup \mathscr{E}_2$

and P **generated by** $P(E_1 \cap E_2) = P_1(E_1)P_2(E_2)$.



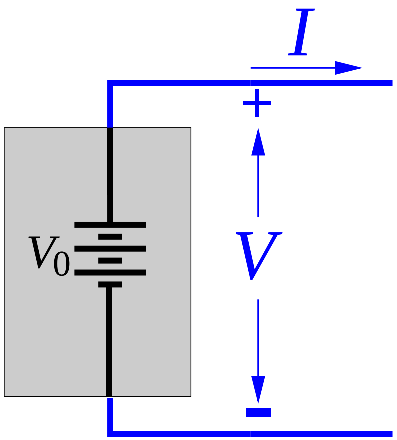
Example

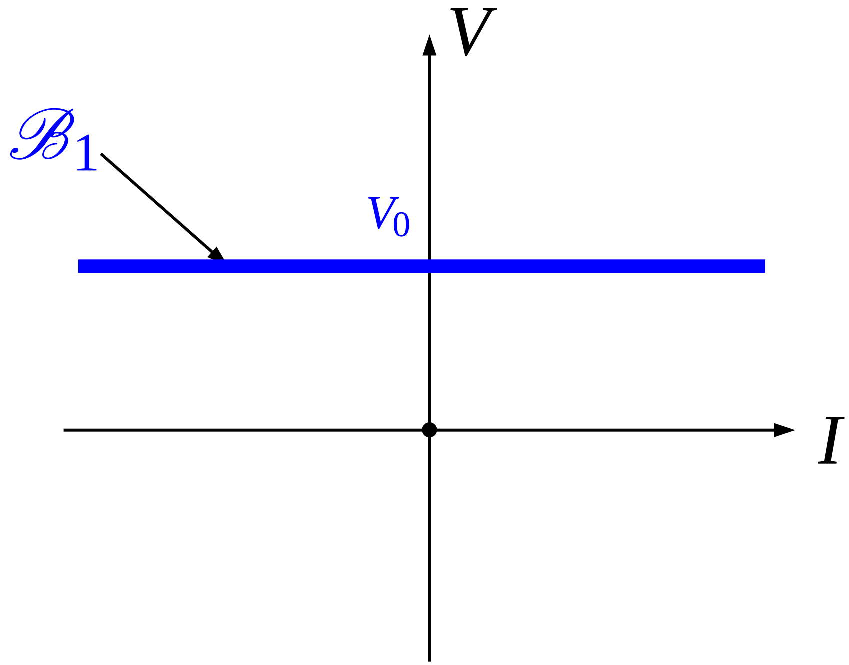
Noisy resistor terminated by a voltage source



Outcomes $\begin{bmatrix} V \\ I \end{bmatrix}$**, outcome space** $\mathbb{W} = \mathbb{R}^2$**;**
events: subsets of $\mathbb{R}^2$

The voltage source





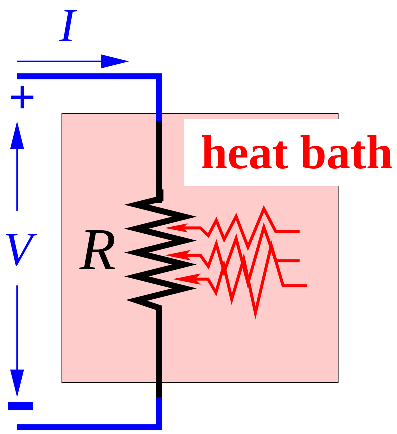
$\Sigma_1 = (\mathbb{R}^2, \mathscr{E}_1, P_1)$,

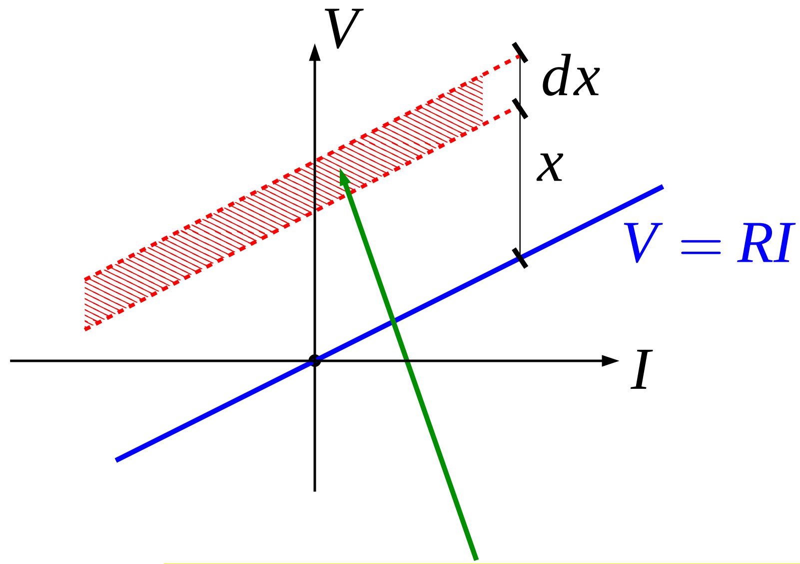
$\mathscr{E}_1 = (\emptyset, \mathscr{B}_1, \mathscr{B}_1^{\text{complement}}, \mathbb{R}^2)$,

$P_1(\mathscr{B}_1) = 1$.

Σ_1 is a deterministic system.

Noisy (or ‘hot’, or ‘Johnson-Nyquist’) **resistor**



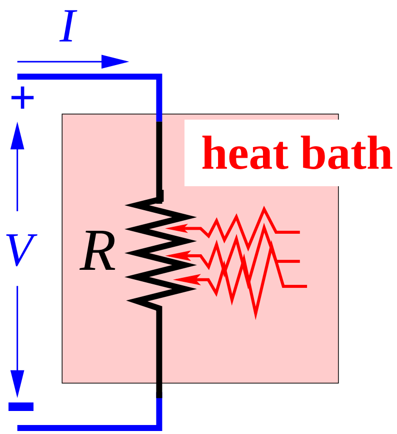


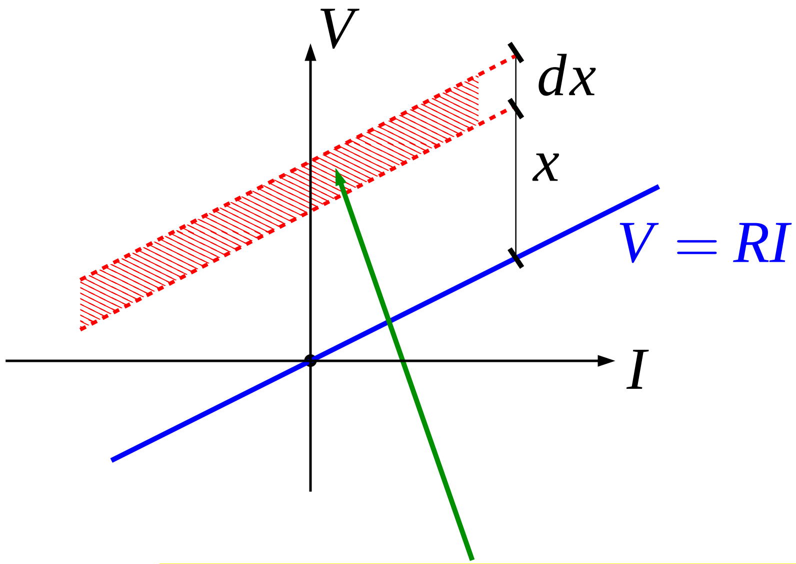
Probability = $\frac{1}{\sqrt{2\pi}\sigma}\, e^{-\frac{x^2}{2\sigma^2}}\, dx.$

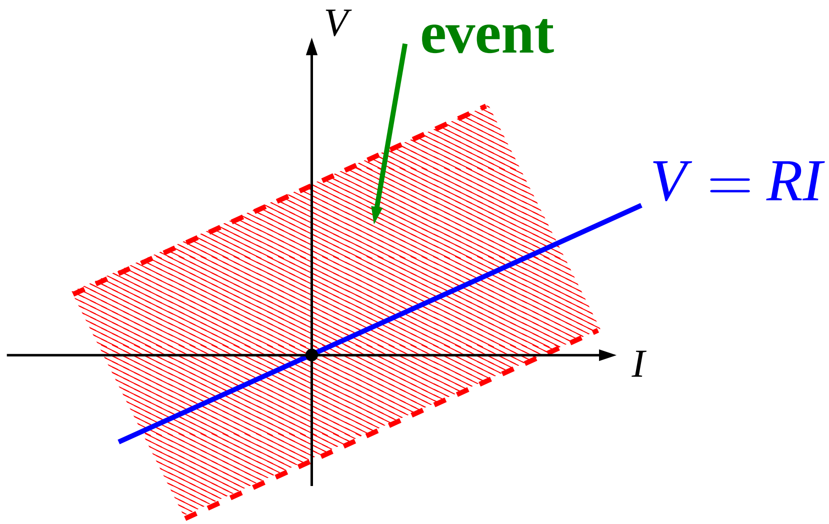
$\sigma \sim \sqrt{RT}$

T **= temperature**

Noisy (or ‘hot’, or ‘Johnson-Nyquist’) resistor







Probability = $\frac{1}{\sqrt{2\pi}\sigma}\, e^{-\frac{x^2}{2\sigma^2}}\, dx.$

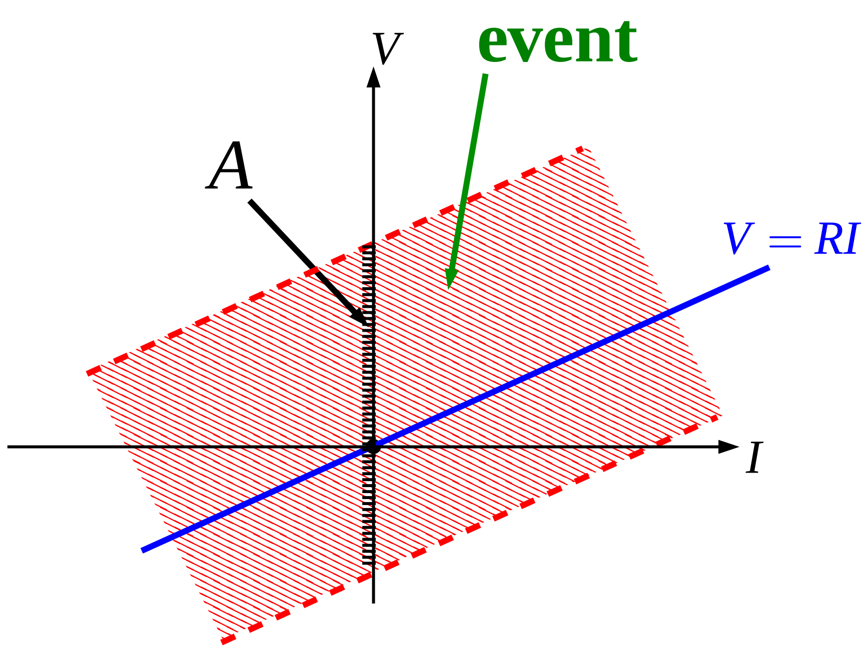
$\sigma \sim \sqrt{RT}$

T **= temperature**

Noisy (or 'hot', or 'Johnson-Nyquist') resistor

$\Sigma_2 = (\mathbb{R}^2, \mathscr{E}_2, P_2)$; **events in** $\mathscr{E}_2$ **= the subsets of** $\mathbb{R}^2$ **as**

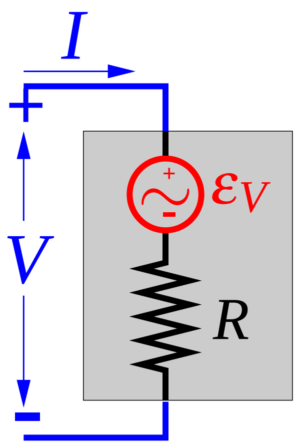
$$\left\{ \begin{bmatrix} V \\ I \end{bmatrix} \in \mathbb{R}^2 \mid V - RI \in A \text{ with } A \text{ a (Borel) subset of } \mathbb{R} \right\}$$



$$P_2(\textbf{event}) = \frac{1}{\sqrt{2\pi}\sigma} \int_A e^{-\frac{x^2}{2\sigma^2}}\, dx$$

Neither $\begin{bmatrix} V \\ I \end{bmatrix}$**,** I**, nor** V **possess a distribution or a pdf!**

Equivalent circuits



$$V = RI + \varepsilon_V$$

ε_V **gaussian**

zero mean

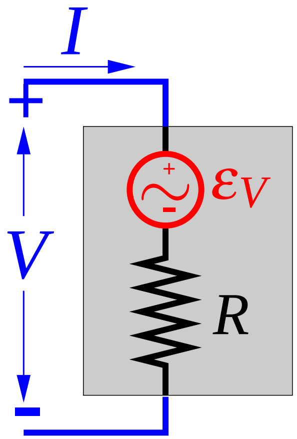
variance $\sim RT$

Note: $\{\varepsilon_V \in A \subseteq \mathbb{R},$ **Borel**$\} = \{V - RI \in A\}$

Shows that $\varepsilon_V \in \mathbb{R}$ σ**-algebra is Borel**

but $(V,I) \in \mathbb{R}^2$ σ**-algebra is coarse,** $\neq$ **Borel.**

Equivalent circuits



$$V = RI + \varepsilon_V$$

ε_V **gaussian**

zero mean

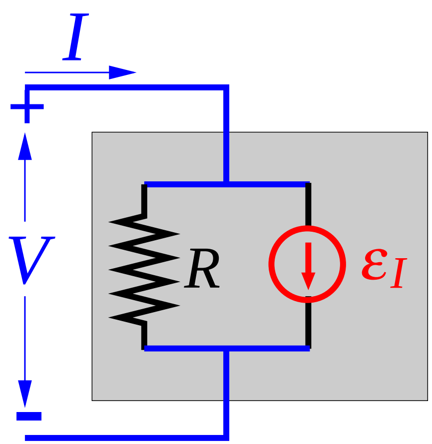
variance $\sim RT$

$$I = V/R + \varepsilon_I$$

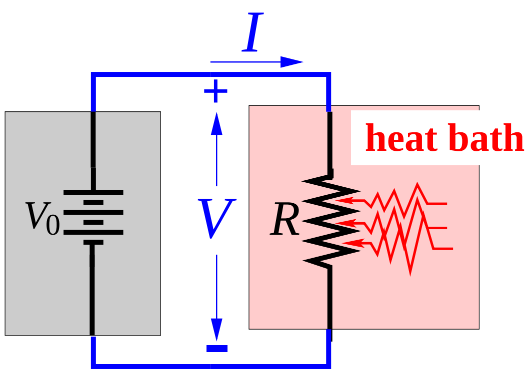
ε_I **gaussian**

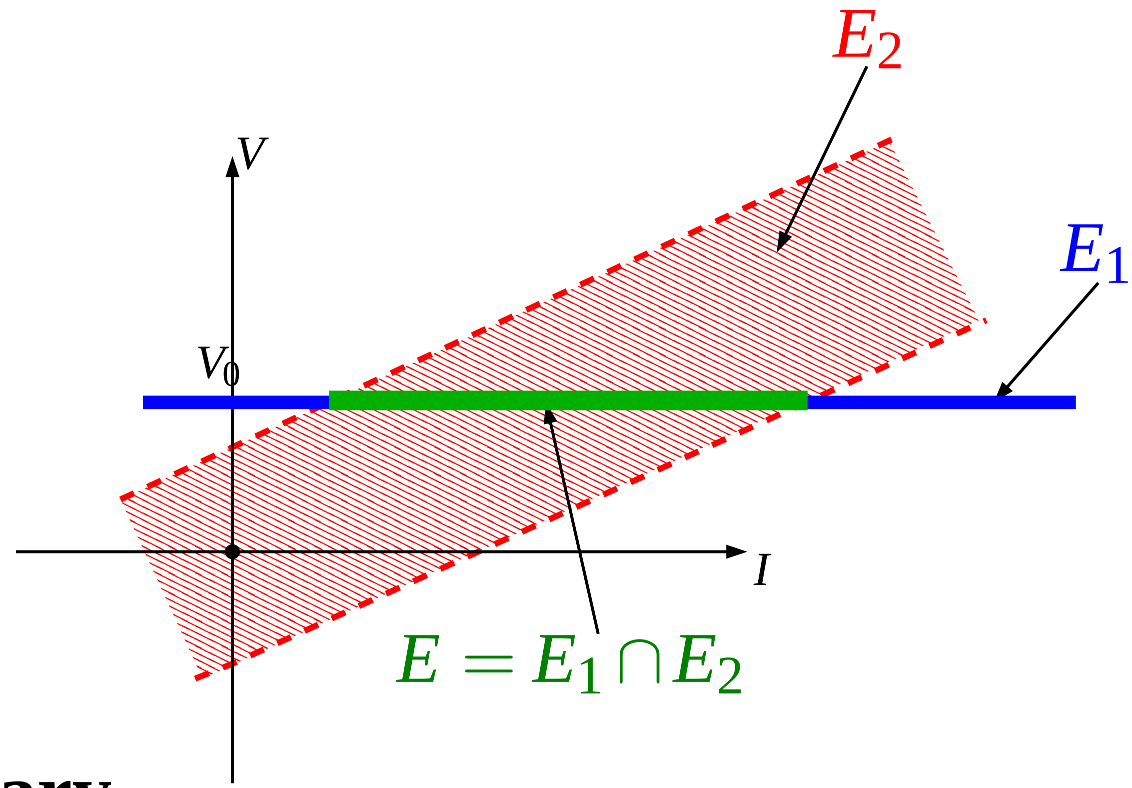
zero mean

variance $\sim T/R$



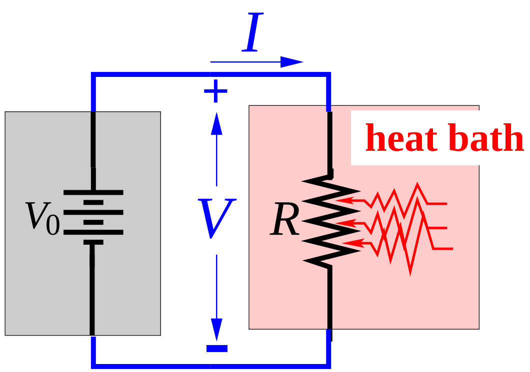
Noisy resistor terminated by a voltage source

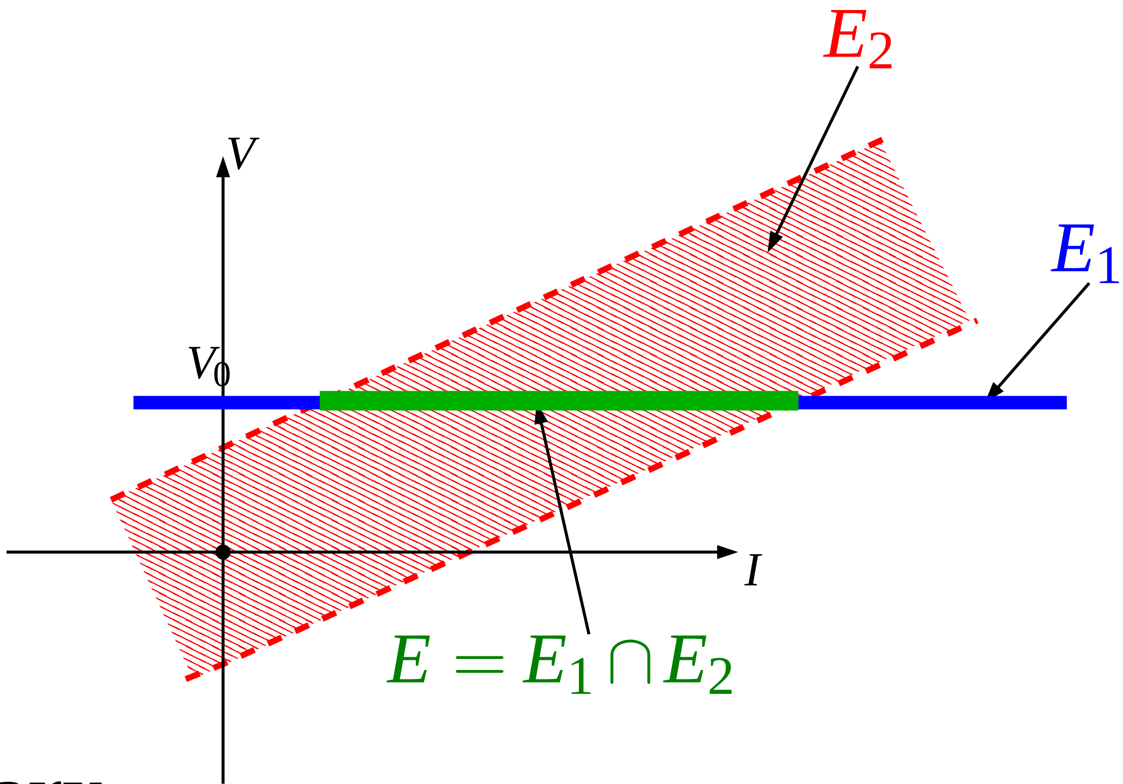




The σ-algebras are indeed complementary.

Noisy resistor terminated by a voltage source





The σ-algebras are indeed complementary.

$\Sigma_1 \wedge \Sigma_2 = (\mathbb{R}^2, \mathscr{E}, P)$, **with**

$P(E) = P_2(E_2)$ **(**P **is concentrated on** $V = V_0$**) and**

the completion of $\mathscr{E}$ = **the Borel** σ**-algebra on** $\mathbb{R}^2$**.**

Open stochastic systems

Open versus closed

Consider $\Sigma_1 = (\mathbb{R}^{\mathtt{n}}, \mathscr{E}_1, P_1)$**.**

If $\mathscr{E}_1$ **= the Borel** σ**-algebra, then** Σ_1 **is basically only interconnectable with the trivial stochastic system**

$$\Sigma_2 = (\mathbb{R}^{\mathtt{n}}, \{\emptyset, \mathbb{R}^{\mathtt{n}}\}, P_2).$$

Open versus closed

Consider $\Sigma_1 = (\mathbb{R}^{\mathtt{n}}, \mathscr{E}_1, P_1)$**.**

If $\mathscr{E}_1$ **= the Borel** σ**-algebra, then** Σ_1 **is basically only interconnectable with the trivial stochastic system**

$$\Sigma_2 = (\mathbb{R}^{\mathtt{n}}, \{\emptyset, \mathbb{R}^{\mathtt{n}}\}, P_2).$$

$\Rightarrow$ **classical** Σ_1 **= 'closed' system.**

Coarse $\mathscr{E}_1$

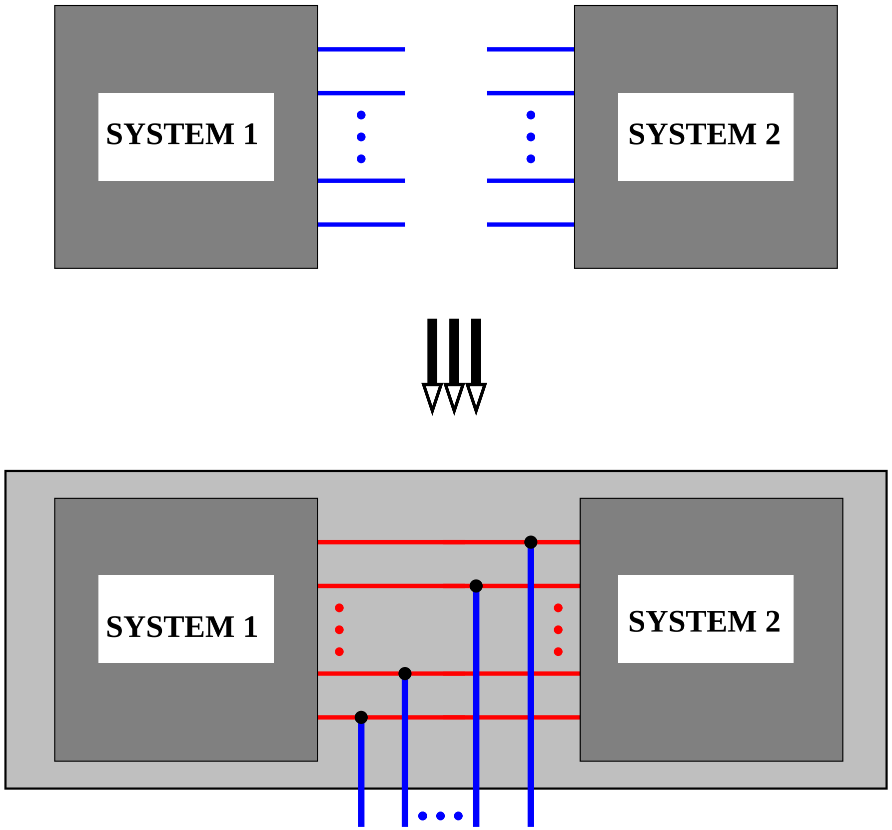
$\Rightarrow$ Σ_1 **is interconnectable.**

$\Rightarrow$ **'open' stochastic system.**

Conclusions

Stochastic systems

- **Complementary systems can be interconnected: two laws imposed on one set of variables.**



Stochastic systems

- **Open stochastic systems require a coarse σ-algebra.**

  Classical random vectors imply closed systems.

Stochastic systems

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- **Borel σ-algebra inadequate for applications.**

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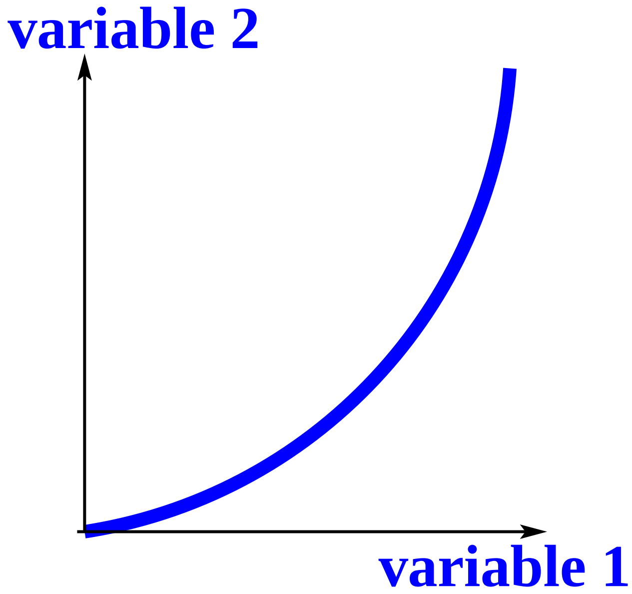
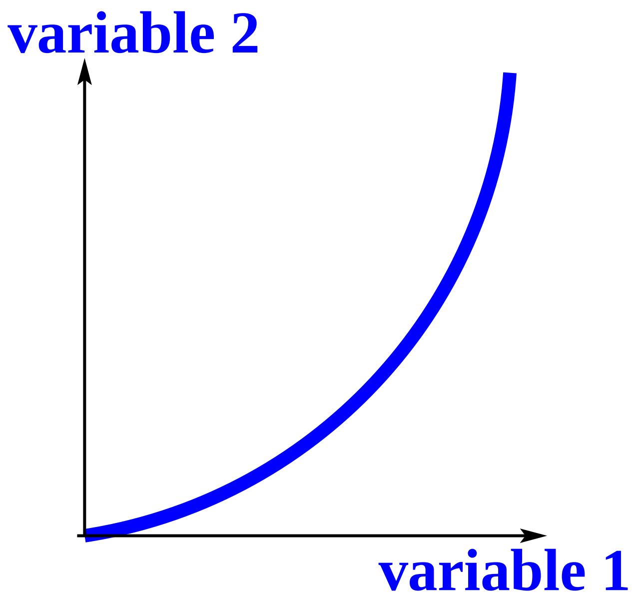
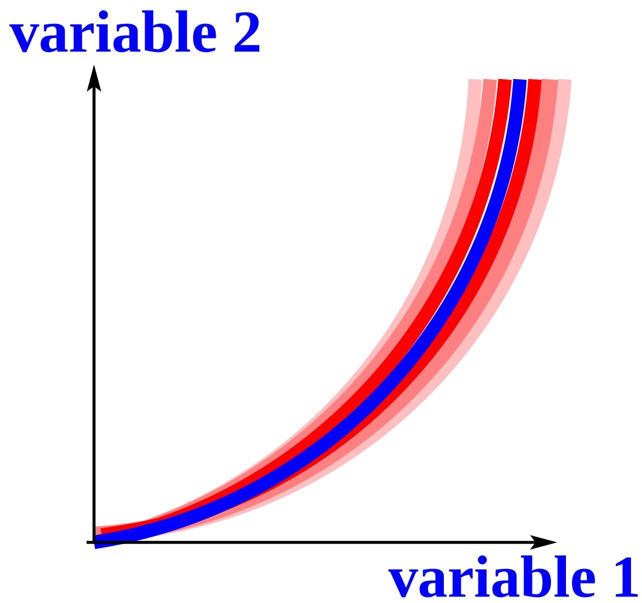


Deterministic

Stochastic systems

- Open stochastic systems require a coarse σ-algebra.
- Borel σ-algebra inadequate for applications.





Deterministic $\rightsquigarrow$ **stochastic.**

The presentation slides and an associated full article can be found on my website.

`http://homes.esat.kuleuven.be/~jwillems/`

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Thank you

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