FIRST ORDER REPRESENTATIONS

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Open Problems Session of the Workshop on
Control Theory: Mathematical Perspectives on Complex Networked Systems

Mathematisches Forschungsinstitut Oberwolfach

February 28, 2012

PDEs

$x = (x_1, x_2, \ldots, x_{\mathtt{n}})$ **independent variables,**

$w = (w_1, w_2, \ldots, w_{\mathtt{w}})$ **dependent variables.**

$$P\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_{\mathtt{n}}}\right) w = 0,$$

$$P \in \mathbb{R}^{\bullet \times \mathtt{w}}[\xi_1, \xi_2, \ldots, \xi_{\mathtt{n}}],$$

$$\mathscr{B}_P := \{w \in \mathbb{C}^0(\mathbb{R}^{\mathtt{n}}, \mathbb{R}^{\mathtt{w}}) \mid P\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_{\mathtt{n}}}\right) w = 0\}.$$

$[\![P_1 = P_2]\!] \Rightarrow [\![B_{P_1} = B_{P_2}]\!]$, **but not converse.**

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$$\mathscr{L} := \{\mathscr{B} \mid \exists\, P \text{ such that } \mathscr{B} = \mathscr{B}_P\}.$$

First order PDEs

$$P_0 w + P_1 \frac{\partial}{\partial x_1} w + P_2 \frac{\partial}{\partial x_2} w + \cdots + P_{\mathtt{n}} \frac{\partial}{\partial x_{\mathtt{n}}} w = 0.$$

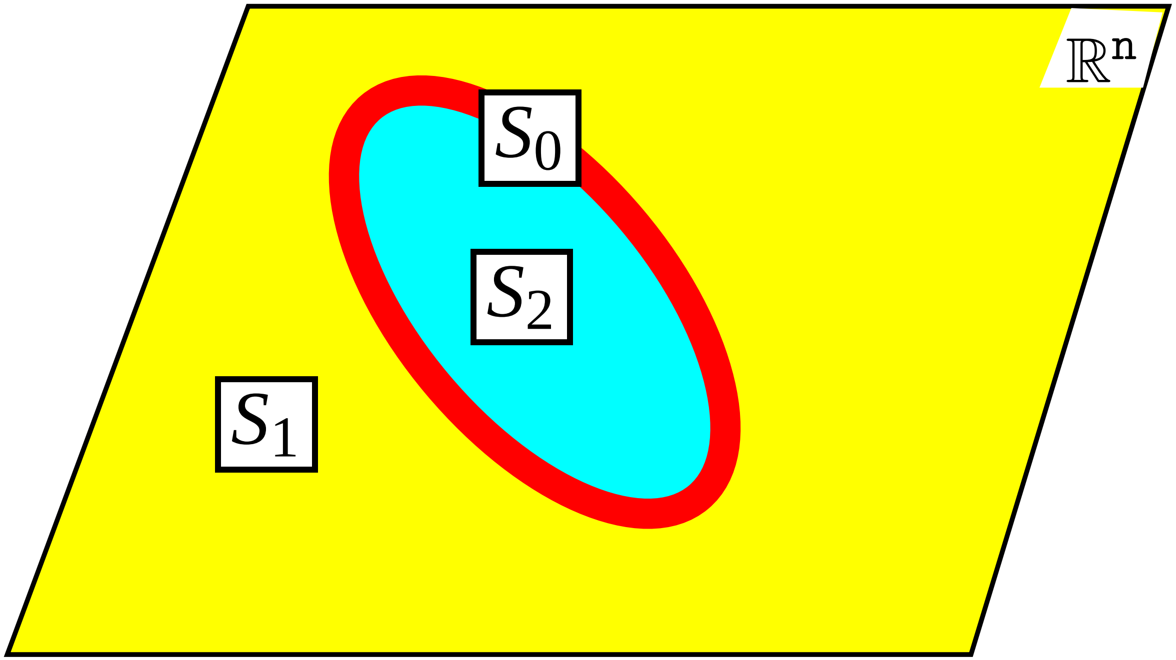
Examples:

- **Maxwell's equations,**
- $E \frac{d}{dt} w + F w = 0,$
- $\frac{d}{dt} f = A f + B u, y = C f + D u, w = (u, f, y).$

OPEN PROBLEM

What does first order say about $\mathscr{B} \in \mathscr{L}$?

Partition

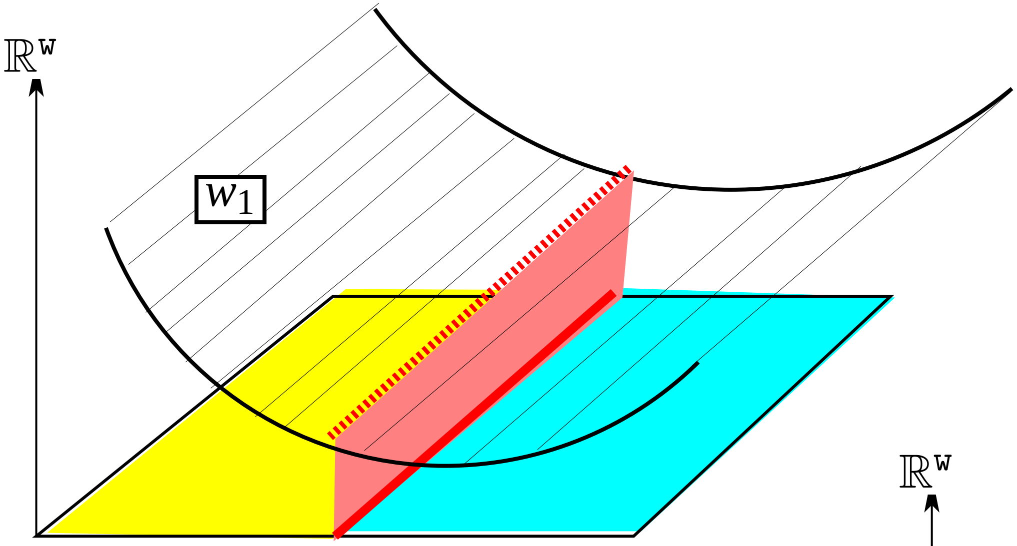


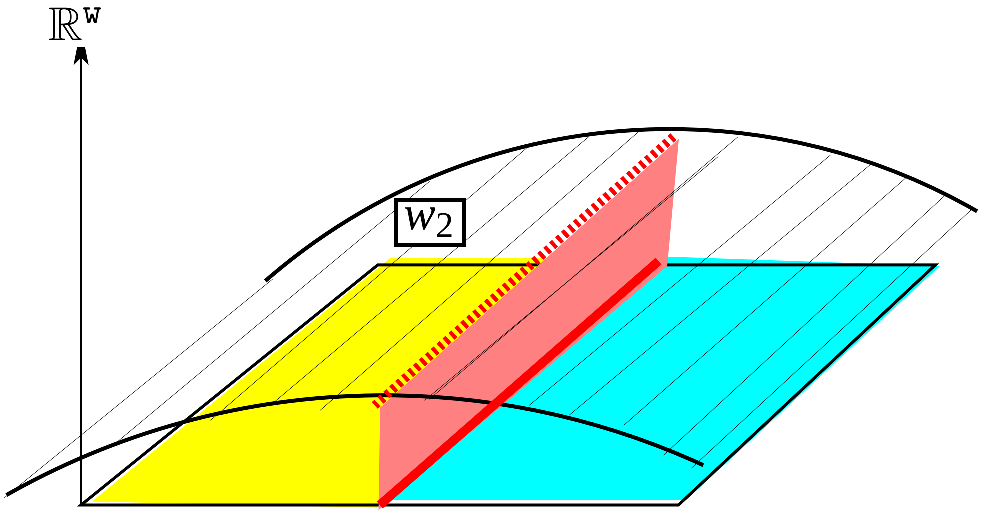
S_0, S_1, S_2 **disjoint,** $\quad S_0 \cup S_1 \cup S_2 = \mathbb{R}^n$**,**

S_1, S_2 **open,** S_0 **closed.**

$$\pi = (S_0, S_1, S_2).$$

Concatenation





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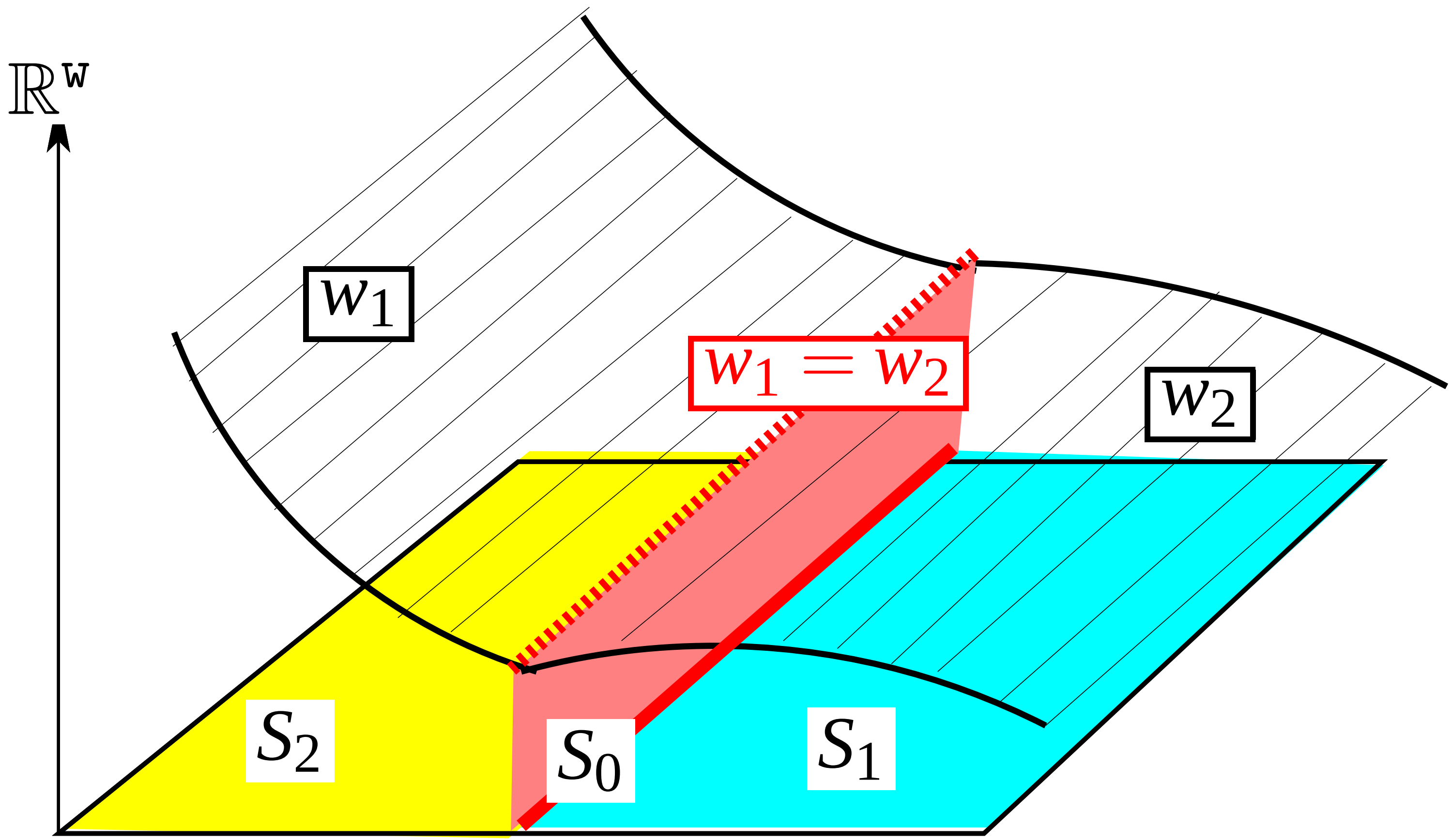


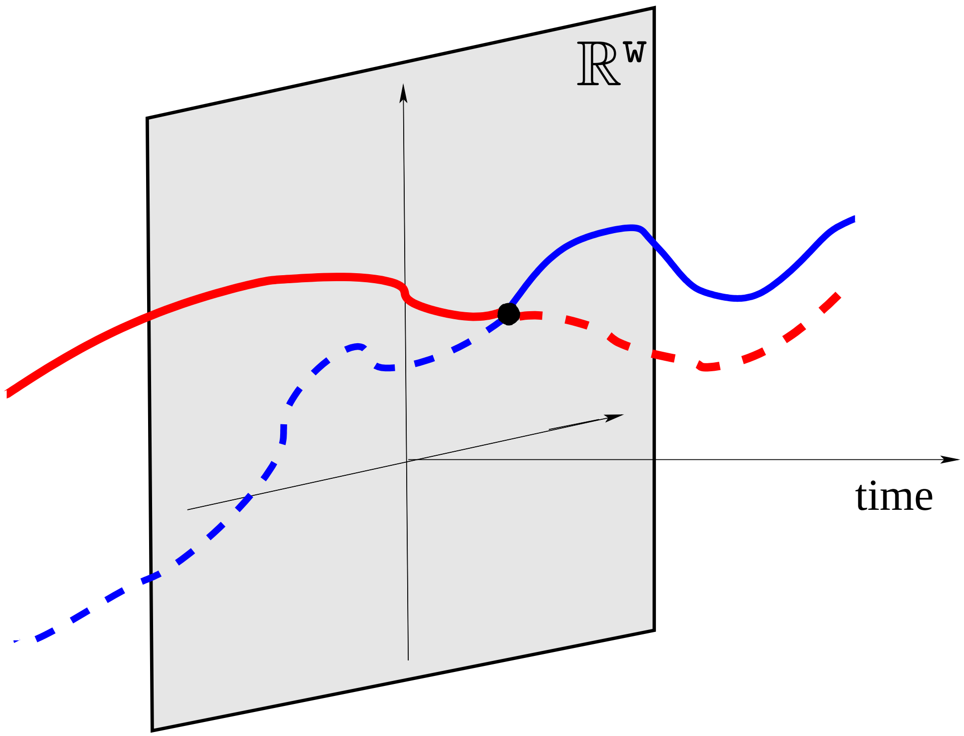
$$\pi = (S_0, S_1, S_2), \qquad w_1 \underset{\pi}{\wedge} w_2$$

Markov property

$\mathscr{B} \in \mathscr{L}$ **is** **Markov** $:\Leftrightarrow$

$$[\![w_1, w_2 \in \mathscr{B}]\!] \Rightarrow [\![\forall\, \pi : \quad w_1 \underset{\pi}{\wedge} w_2 \in \mathscr{B}]\!]$$

Markov property for $n = 1$



Partial results

- **First order $\Rightarrow$ Markov,**
- **for $\mathtt{n} = 1$, first order $\Leftrightarrow$ Markov (P. Rapisarda),**
- **Markov $\not\Rightarrow$ first order (P. Rocha),**
- **partial results for $\mathtt{n} = 2, \mathtt{w} = 1$ (P. Rocha).**

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What more than Markov is needed for first order?