OPEN STOCHASTIC SYSTEMS

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Workshop on

Control Theory: Mathematical Perspectives on Complex Networked Systems

Mathematisches Forschungsinstitut Oberwolfach

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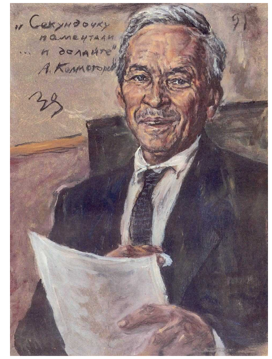
Orthodox probability

Mathematical probability

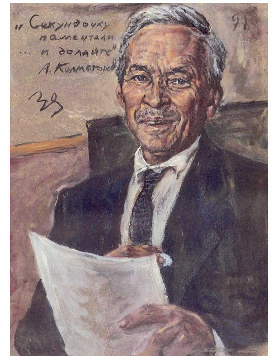


A.N. Kolmogorov
1903 – 1987

A ***stochastic system*** **is a triple** $(\mathbb{W}, \mathscr{E}, P)$

- $\mathbb{W}$ **the** ***outcome space*****,**
- $\mathscr{E}$ **a class of subsets of** $\mathbb{W}$**, elements called** ***events*****,**
- $P : \mathscr{E} \to [0, 1]$ **a** ***probability measure*****.**

Mathematical probability



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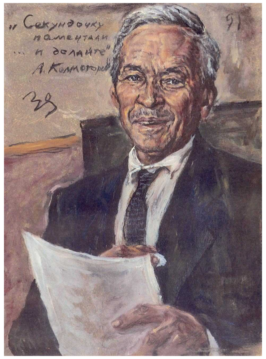
- $P : \mathscr{E} \to [0,1]$ **a *probability measure*.**

$\mathscr{E}$**: the subsets of $\mathbb{W}$ that are assigned a probability.**
Probability that outcome is in E, $E \in \mathscr{E}$, is $P(E)$.

Model $\cong$ $\mathscr{E}$ and P.

Events



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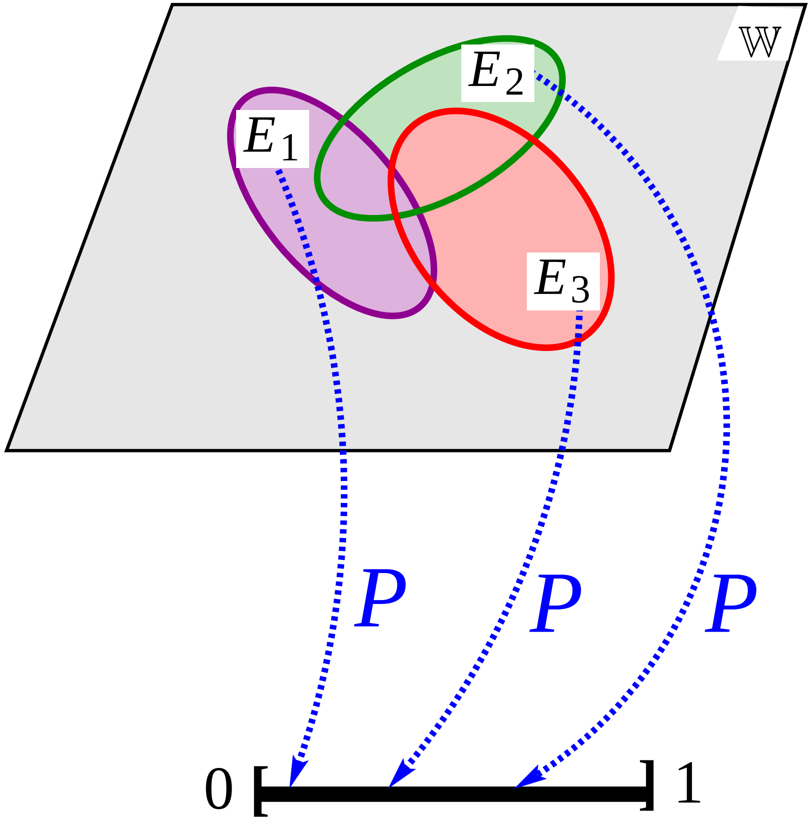

$\mathscr{E}$ = the subsets of $\mathbb{W}$ that are assigned a probability.

Axioms

The events $\mathscr{E}$ form a σ-algebra $:\Leftrightarrow$

- $\mathbb{W} \in \mathscr{E}$,
- $[\![E \in \mathscr{E}]\!] \Rightarrow [\![E^{\text{complement}} \in \mathscr{E}]\!]$,
- $[\![E_k \in \mathscr{E}, k \in \mathbb{N}]\!] \Rightarrow [\![\bigcap_{k \in \mathbb{N}} E_k \in \mathscr{E}, \bigcup_{k \in \mathbb{N}} E_k \in \mathscr{E}]\!]$.

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$P : \mathscr{E} \rightarrow [0,1]$ **is a probability measure** $:\Leftrightarrow$

- $P(\mathbb{W}) = 1$**,**
- P **is countably additive** $:\Leftrightarrow$

  $[\![E_k \in \mathscr{E}, k \in \mathbb{N},$ **disjoint**$]\!] \Rightarrow [\![P(\bigcup_{k\in\mathbb{N}} E_k) = \sum_{k\in\mathbb{N}} P(E_k)]\!]$**.**

Borel

Émile Borel
1871 – 1956

In expositions, introductory & advanced, with $\mathbb{W} = \mathbb{R}^n$, the events are often taken as the Borel σ-algebra.

$\mathscr{E}$ then contains 'basically every' subset of $\mathbb{R}^n$.

Borel

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In expositions, introductory & advanced, with $\mathbb{W} = \mathbb{R}^n$, the events are often taken as the Borel σ-algebra.

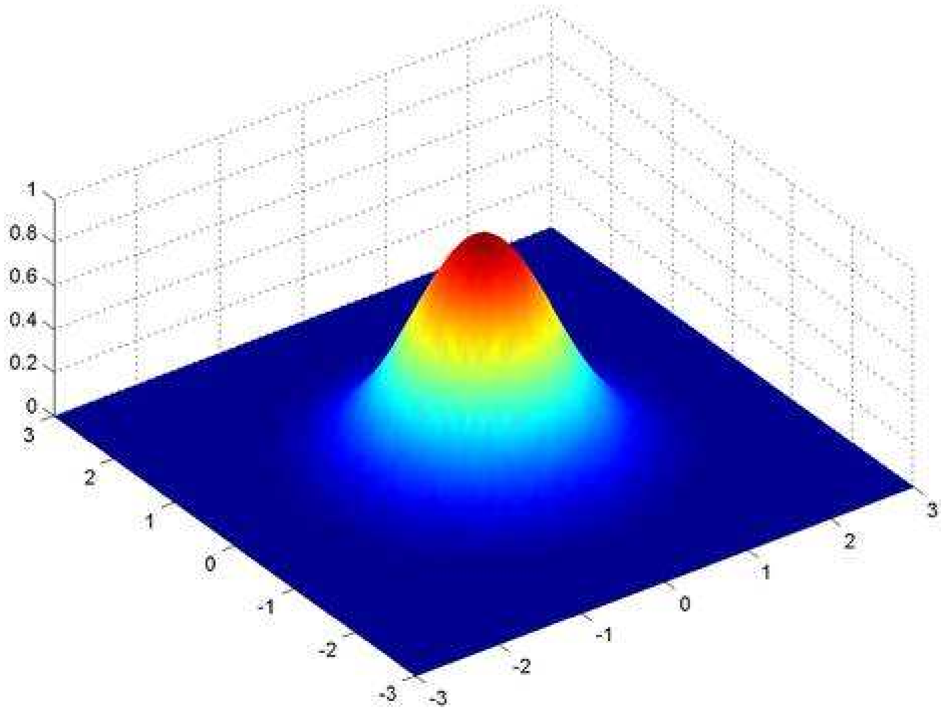
$\mathscr{E}$ then contains 'basically every' subset of $\mathbb{R}^n$. Allows to take probability distributions and pdf's as the primitive concepts.

$\mathscr{E}$ is inherited from the topological structure of the outcome space, avoids modeling of $\mathscr{E}$.

Probability (as commonly taught)

'Classical' stochastic system:

$\mathbb{W} = \mathbb{R}^n$, $\mathscr{E}$ = **Borel σ-algebra $\cong$ 'all' subsets of $\mathbb{R}^n$.**



for $A \subseteq \mathbb{R}^n$

$$P(A) = \int_A p(x)\,dx$$

Probability (as commonly taught)

'Classical' stochastic system:

$\mathbb{W} = \mathbb{R}^n$, $\mathscr{E}$ = **Borel σ-algebra $\cong$ 'all' subsets of $\mathbb{R}^n$.**

Borel is assumed for many basic concepts, such as

- **mean, variance, moments, correlation,**
- **random variable, random vector,**
- **marginal probability,**
- **random process, autocorrelation,**
- **entropy, mutual information,**
- **Brownian motion, Markov process, etc.**

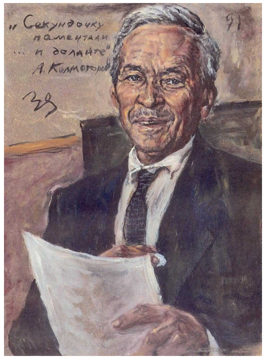
Theme

Émile Borel
1871 – 1956

Borel is unduly restrictive, even for elementary applications.

$\mathscr{E}$ is an essential part of the model!

In the Kolmogorov setting



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$$(\Omega, \mathscr{A}, P) \xrightarrow{\ f\ } (\mathbb{R}^{n}, \mathscr{B}(\mathbb{R}^{n}), P')$$

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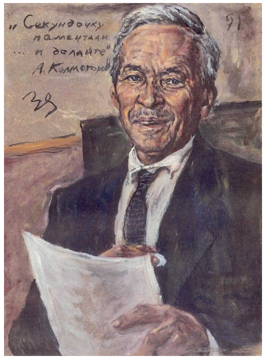

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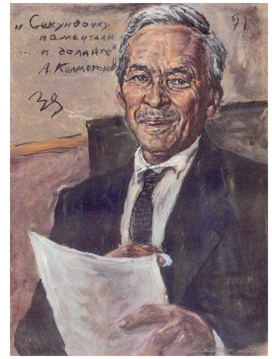

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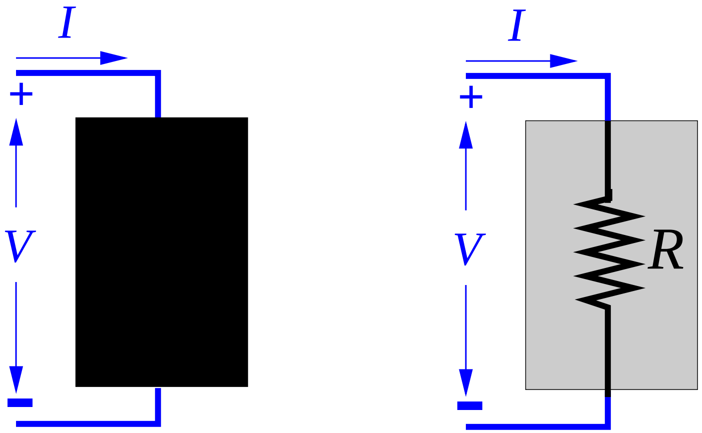
$$\left(\Omega, \mathscr{A}, P\right) \xrightarrow{\ f\ } \left(\mathbb{R}^n, \mathscr{B}\left(\mathbb{R}^n\right), P'\right)$$

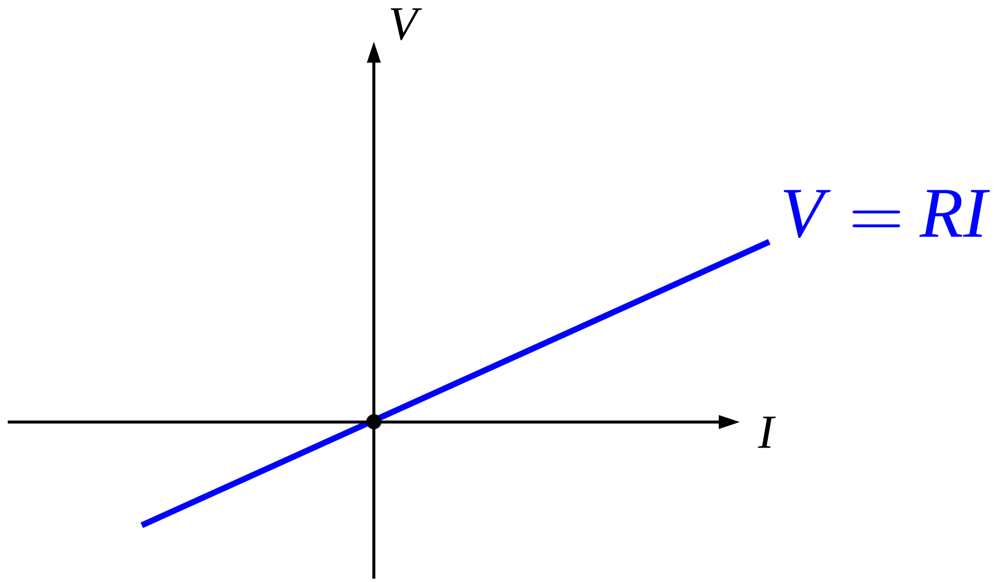
$$\left(\Omega, \mathscr{A}, P\right) \xrightarrow{\ f\ } \left(\mathbb{R}^n, \mathscr{E}, P'\right)$$

Requires modeling $\mathscr{E}$ anyway.

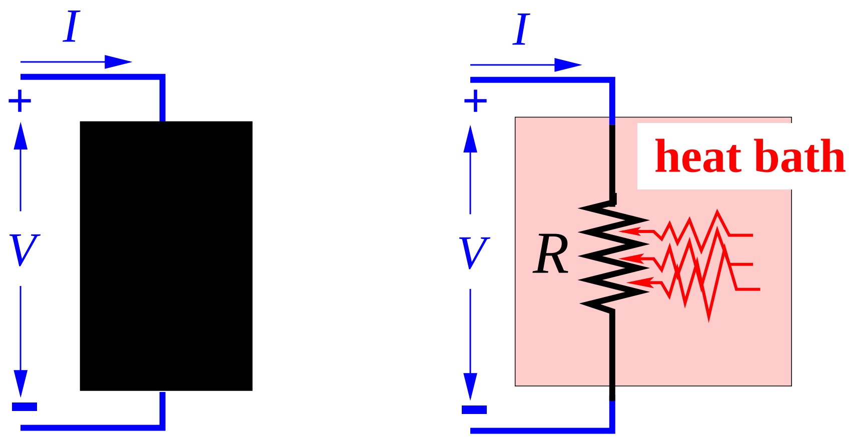
Examples

Ohm resistor





Noisy (or ‘hot’, or ‘Johnson-Nyquist’) resistor



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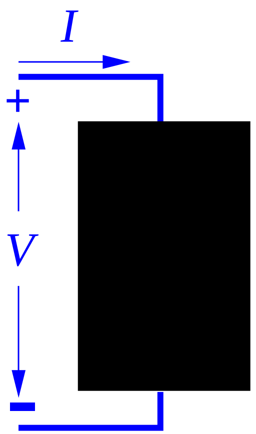

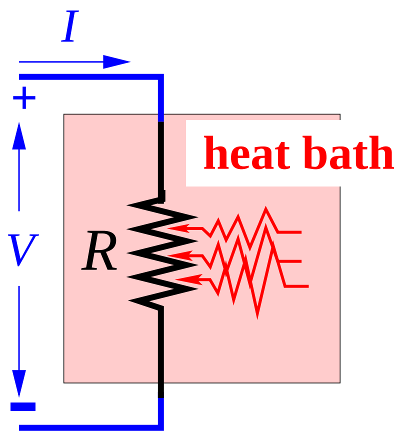

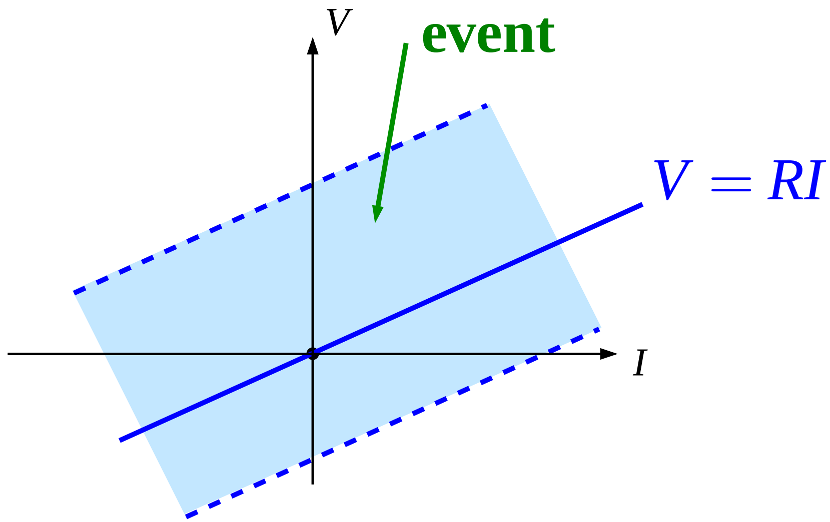

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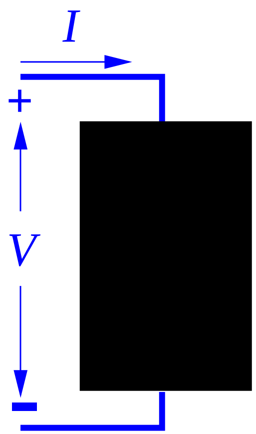
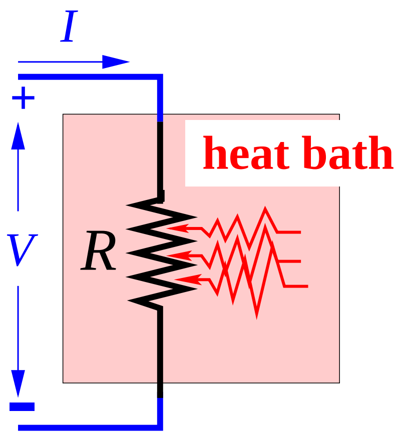
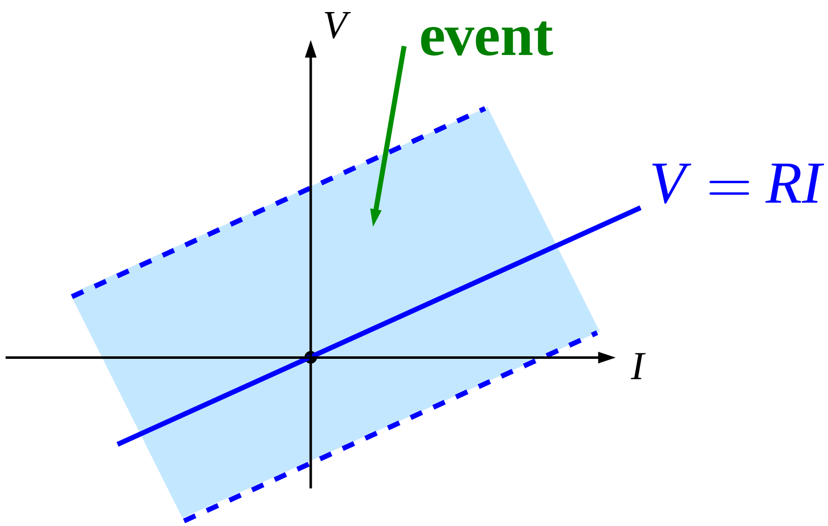
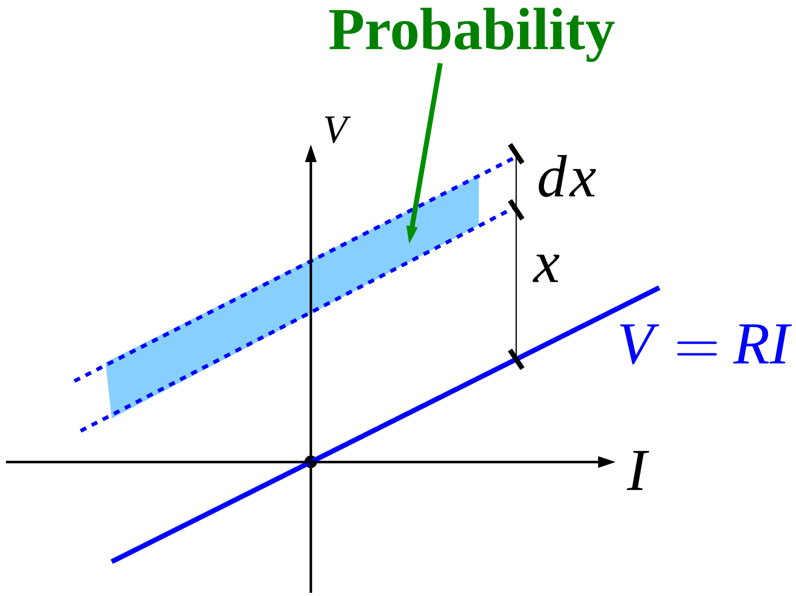








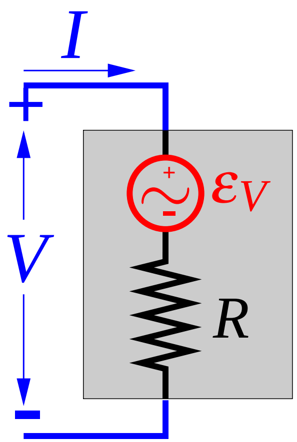
$$\textbf{Probability} = \frac{1}{\sqrt{2\pi}\sigma}\, e^{-\frac{x^2}{2\sigma^2}}\, dx.$$

$$\sigma \sim \sqrt{RT}$$

T = **temperature**

b

Equivalent circuits



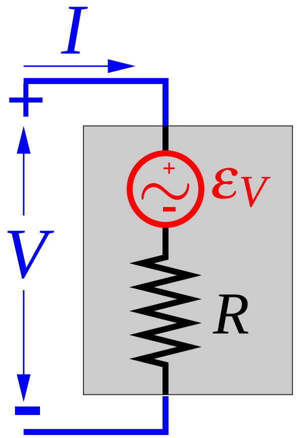
$$V = RI + \varepsilon_V$$

ε_V gaussian

zero mean

variance $\sim RT$

Equivalent circuits



$$V = RI + \varepsilon_V$$

ε_V gaussian

zero mean

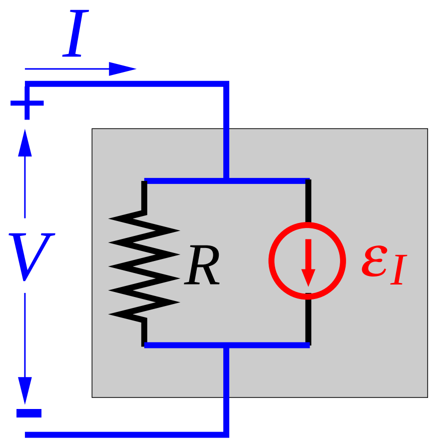
variance $\sim RT$

$$I = \frac{1}{R}V + \varepsilon_I$$

ε_I gaussian

zero mean

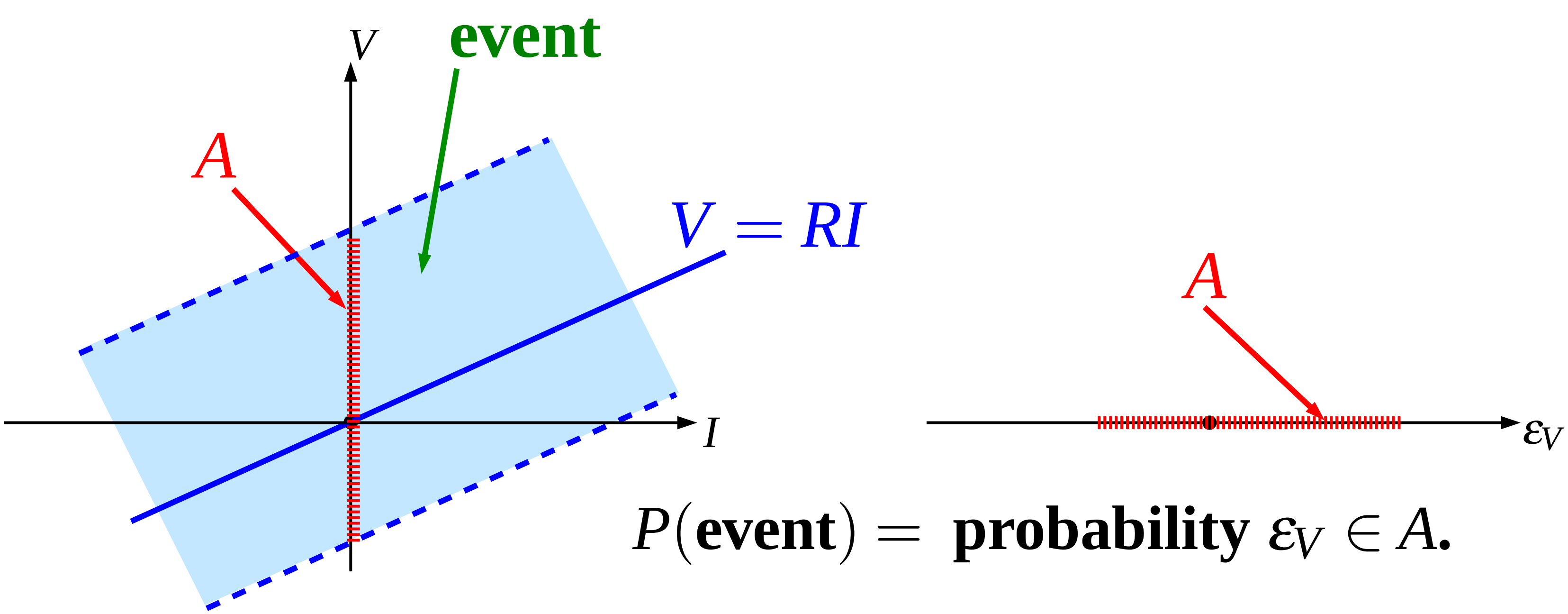
variance $\sim \frac{T}{R}$



$V = RI + \varepsilon_V$

Outcomes $\begin{bmatrix} V \\ I \end{bmatrix}$, $\mathbb{W} = \mathbb{R}^2$; **events: subsets of** $\mathbb{R}^2$ **as**

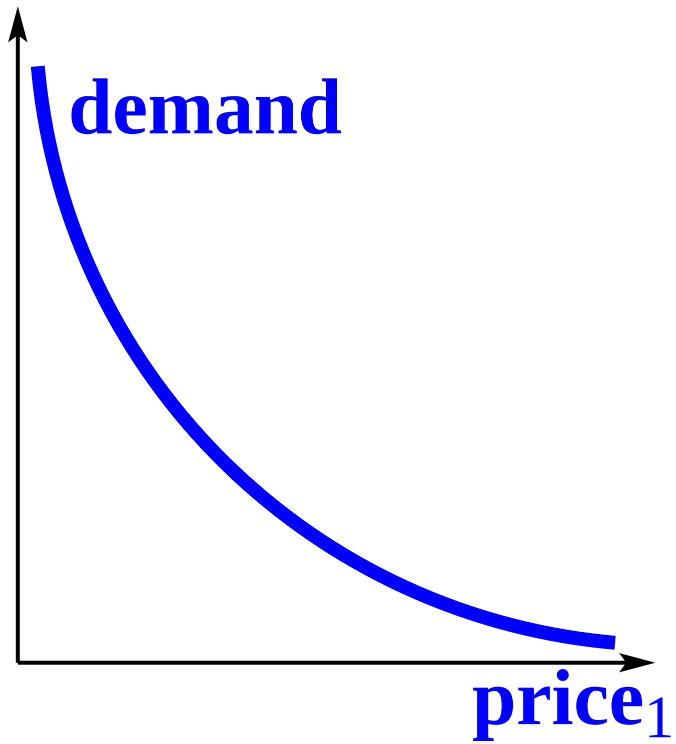
$\left\{ \begin{bmatrix} V \\ I \end{bmatrix} \in \mathbb{R}^2 \mid V - RI \in A \text{ with } A \text{ a Borel subset of } \mathbb{R} \right\}$.

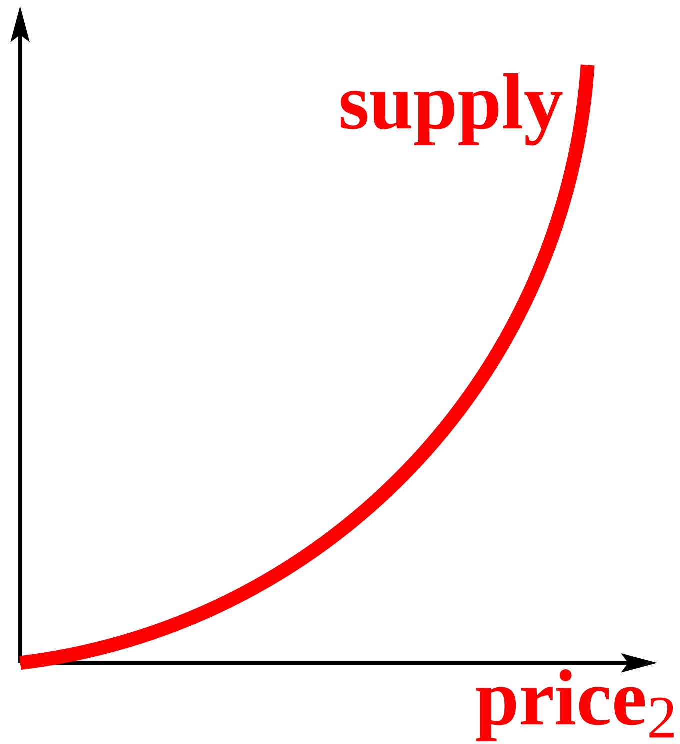


$P(\textbf{event}) =$ **probability** $\varepsilon_V \in A$.

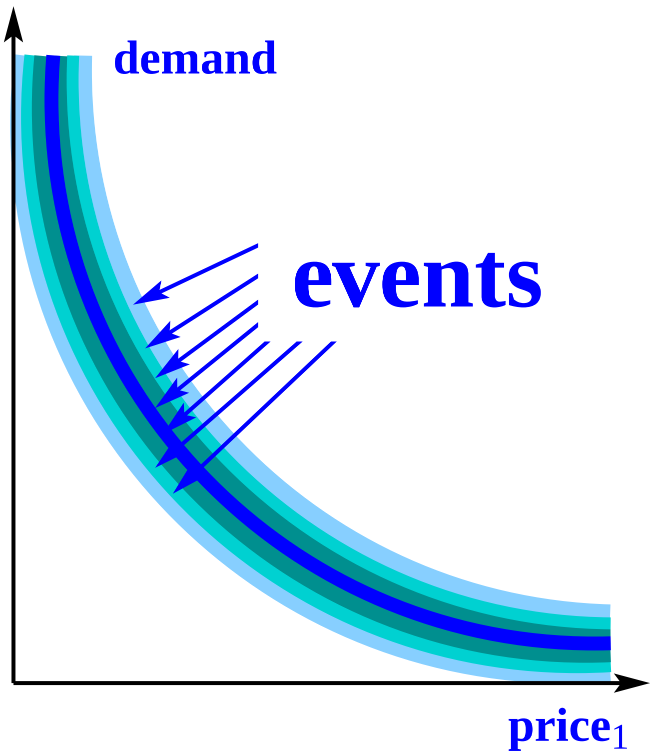
Neither $\begin{bmatrix} V \\ I \end{bmatrix}$, **nor** I, **nor** V **possess a pdf.**

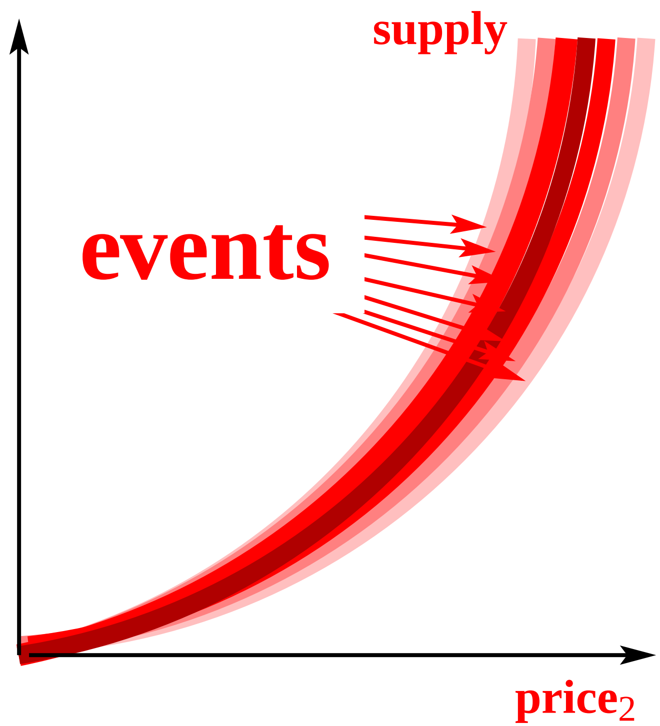
Deterministic price/demand/supply





Stochastic price/demand/supply





$\mathscr{E}, \mathscr{E}'$ **= the regions that are assigned a probability.**

p_1, p_2, d, s **are not classical real random variables.**

1. **Linearity**
2. **Interconnection**
3. **Constrained probability**

Linearity

Linearity

linear stochastic system $:\Leftrightarrow$

Borel probability on $\mathbb{R}^{\mathtt{n}}/\mathbb{L}$,

with $\mathbb{L} \subseteq \mathbb{R}^{\mathtt{n}}$ a linear subspace, the ‘fiber’.

$\mathbb{R}^{\mathtt{n}}/\mathbb{L}$ **real vector space of dimension** $\mathtt{n} - \mathtt{dim}(\mathbb{L})$.

Linearity

linear stochastic system $:\Leftrightarrow$

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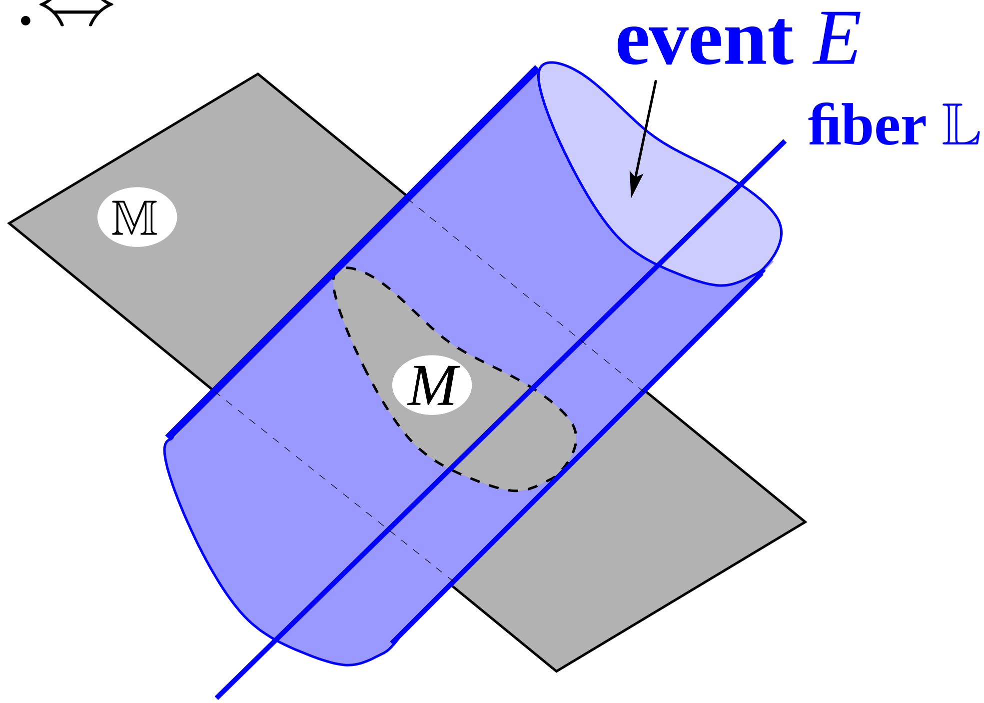
with $\mathbb{L} \subseteq \mathbb{R}^{\mathtt{n}}$ **a linear subspace, the 'fiber'.**

$\mathbb{R}^{\mathtt{n}}/\mathbb{L}$ **real vector space of dimension** $\mathtt{n} - \mathtt{dim}(\mathbb{L})$**.**

Events: cylinders with sides parallel to $\mathbb{L}$**.**
Subsets of $\mathbb{R}^{\mathtt{n}}$ **as** $A + \mathbb{L}, \;\; A \subseteq \mathbb{R}^{\mathtt{n}}$ **Borel.**

Linearity

linear stochastic system $:\Leftrightarrow$



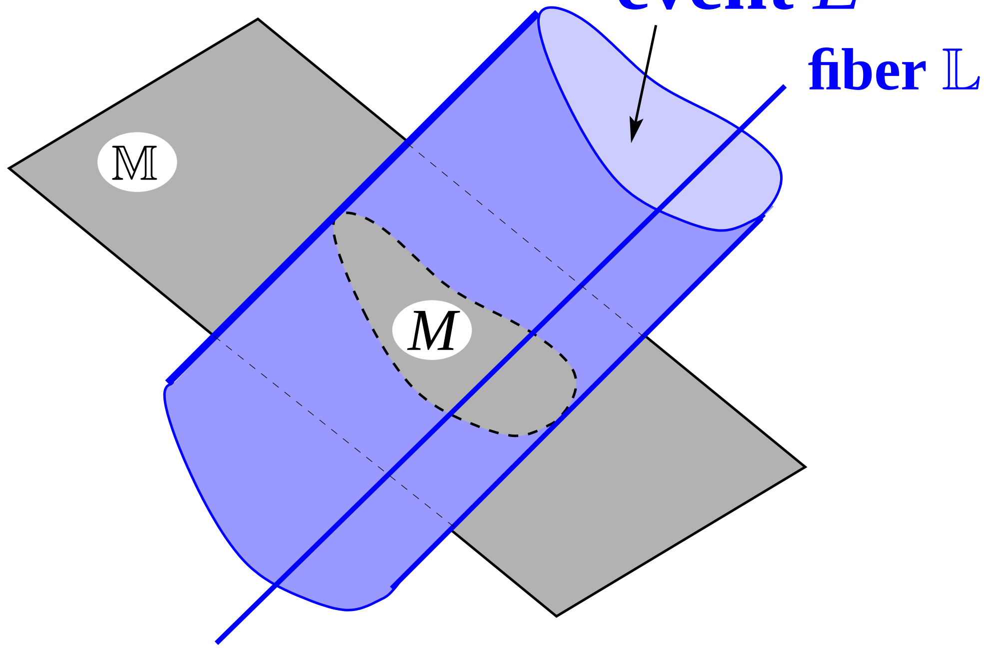
$\mathbb{L} \oplus \mathbb{M} = \mathbb{R}^n, \mathbb{M} \cong \mathbb{R}^n/\mathbb{L}$.

Borel probability on $\mathbb{M}$**.**

$P(E) = P_{\mathbb{M}}(M)$.

Linearity

linear stochastic system $:\Leftrightarrow$



$\mathbb{L} \oplus \mathbb{M} = \mathbb{R}^n, \mathbb{M} \cong \mathbb{R}^n/\mathbb{L}$.

Borel probability on $\mathbb{M}$.

$P(E) = P_{\mathbb{M}}(M)$.

Example: the noisy resistor.

Classical $\Rightarrow$ linear!

gaussian $:\Leftrightarrow$ **linear, probability on $\mathbb{R}^n/\mathbb{L}$ gaussian.**

Deterministic system

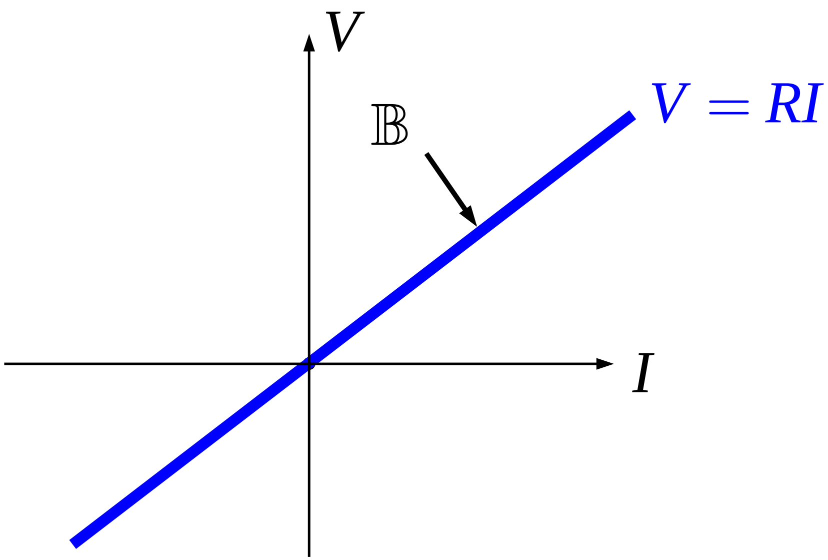
$(\mathbb{W}, \mathscr{E}, P)$ **is said to be deterministic if**

$$\mathscr{E} = \{\emptyset, \mathbb{B}, \mathbb{B}^{complement}, \mathbb{W}\} \textbf{ and } P(\mathbb{B}) = 1.$$

$\mathbb{B}$ **= the ‘behavior’.**

Deterministic examples

Ohmic resistor:



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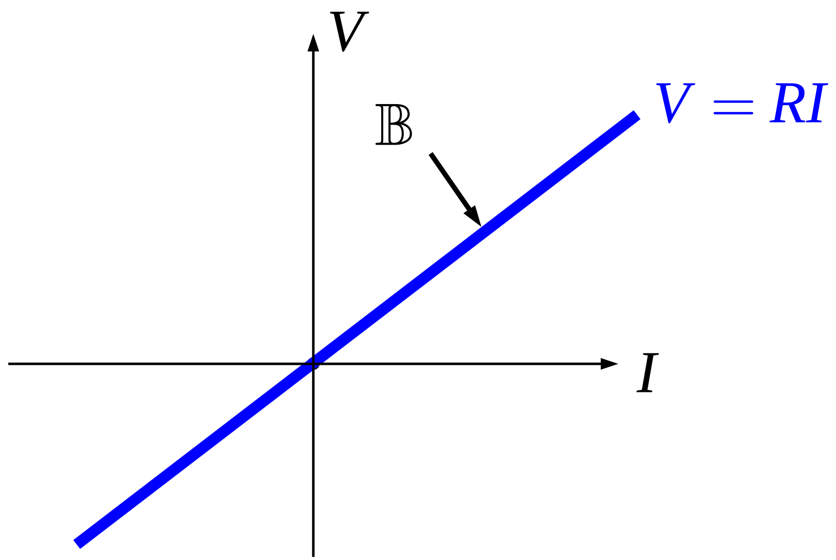
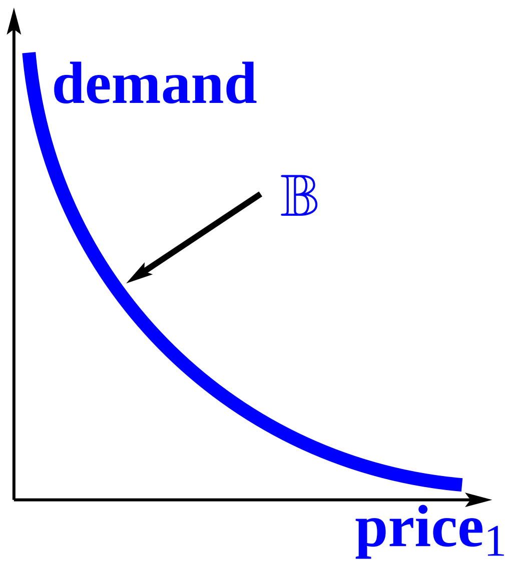
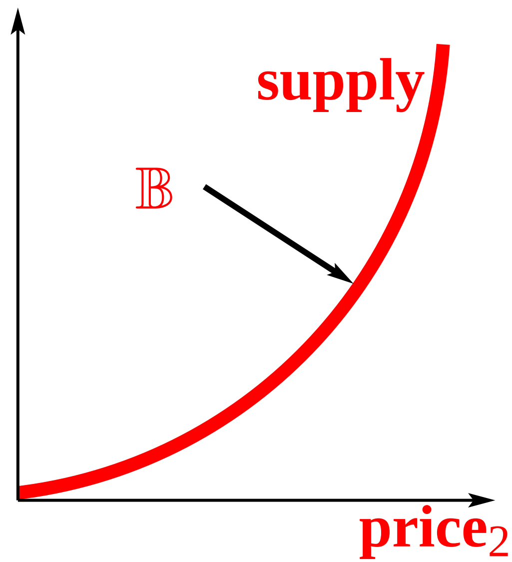


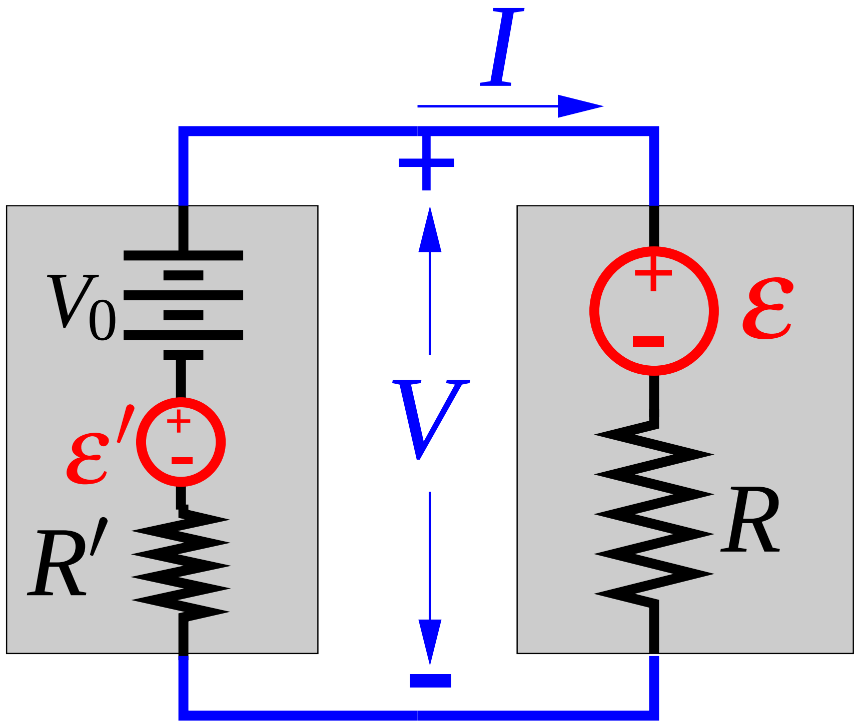
Economic example:





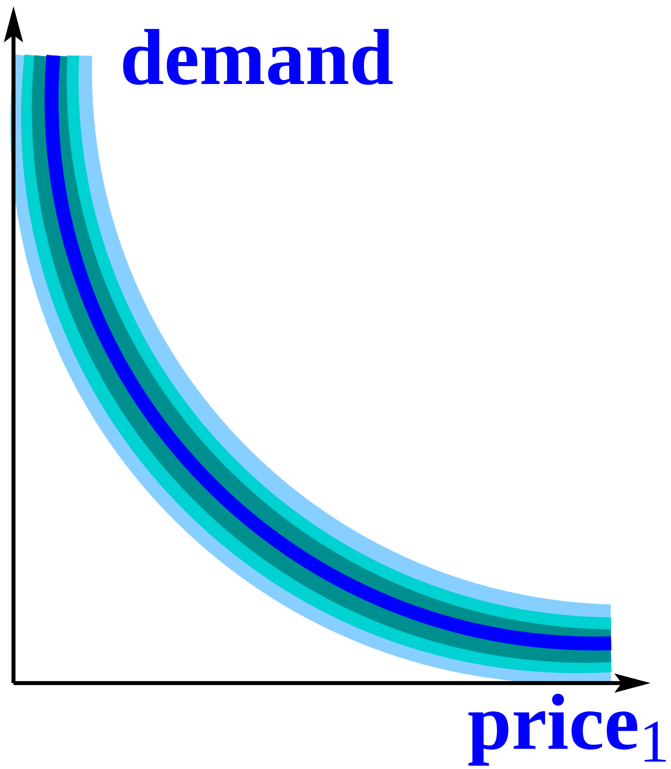
Interconnection

Noisy resistor terminated by a voltage source



How do we deal with interconnection?

Stochastic price/demand/supply

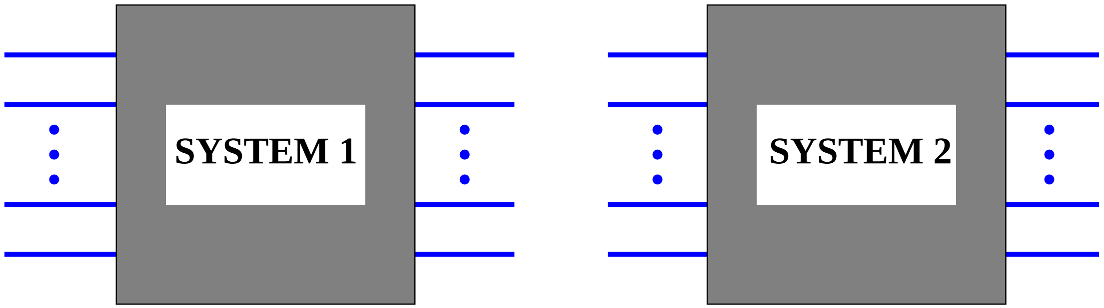


How do we deal with equilibrium?

Equilibrium: $\text{price}_1 = \text{price}_2$, **demand** = **supply**.

Interconnection

a



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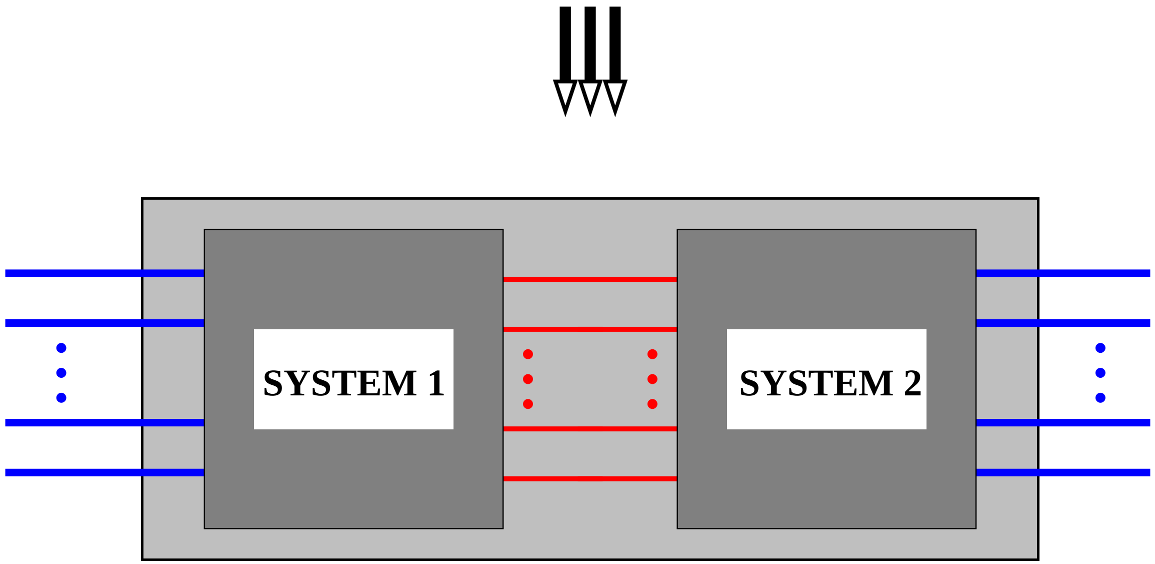

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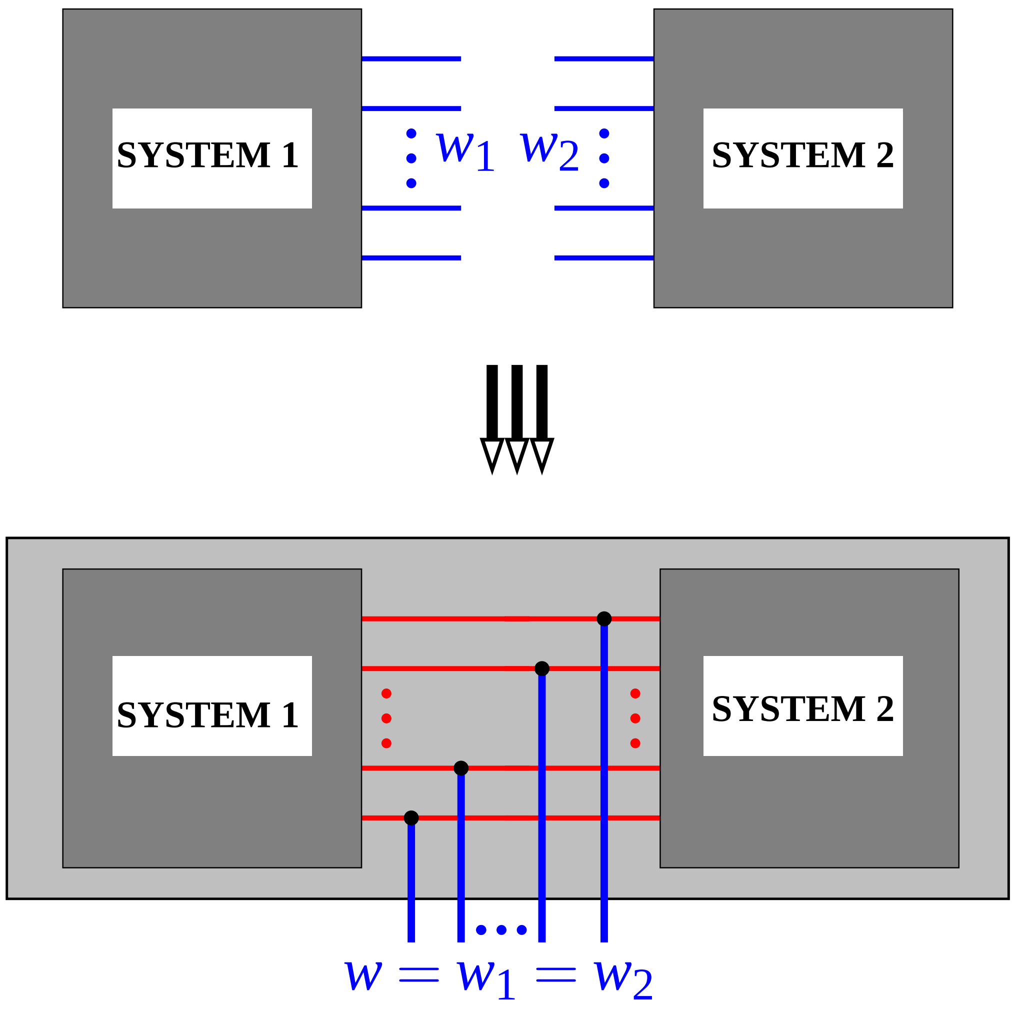

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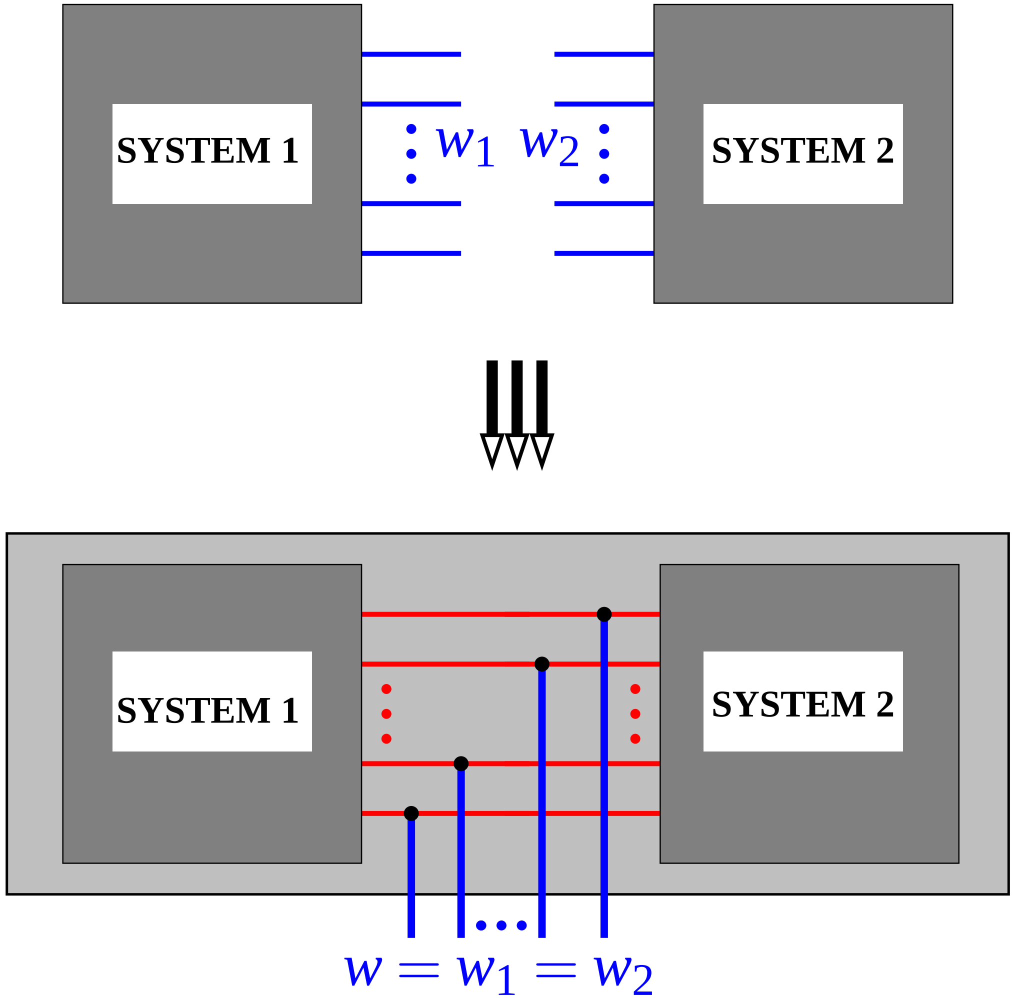
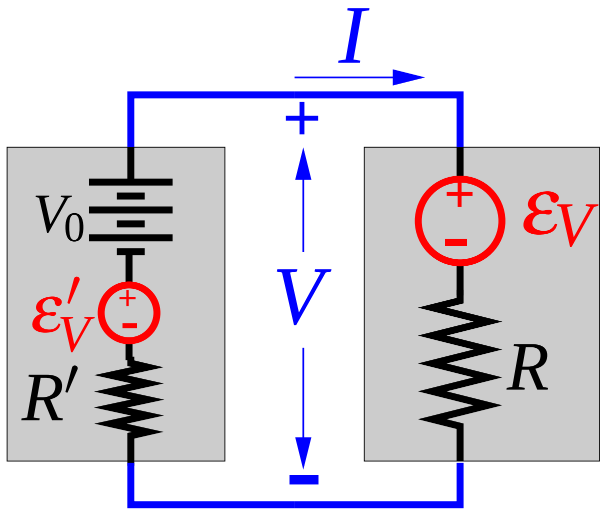


Example:



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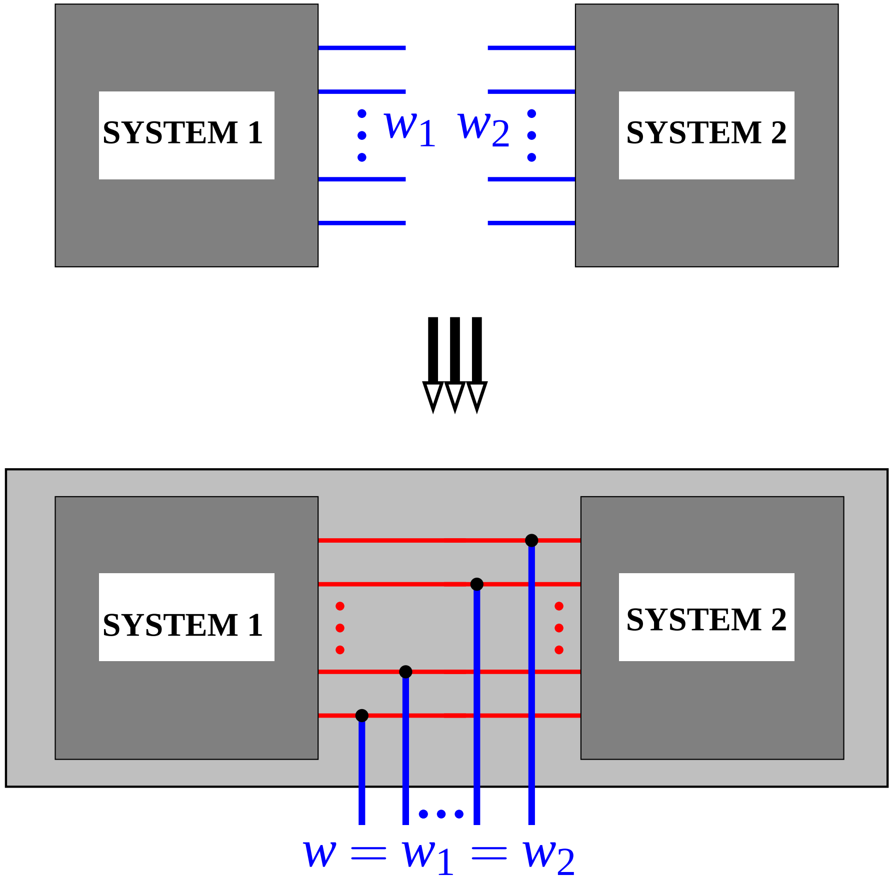


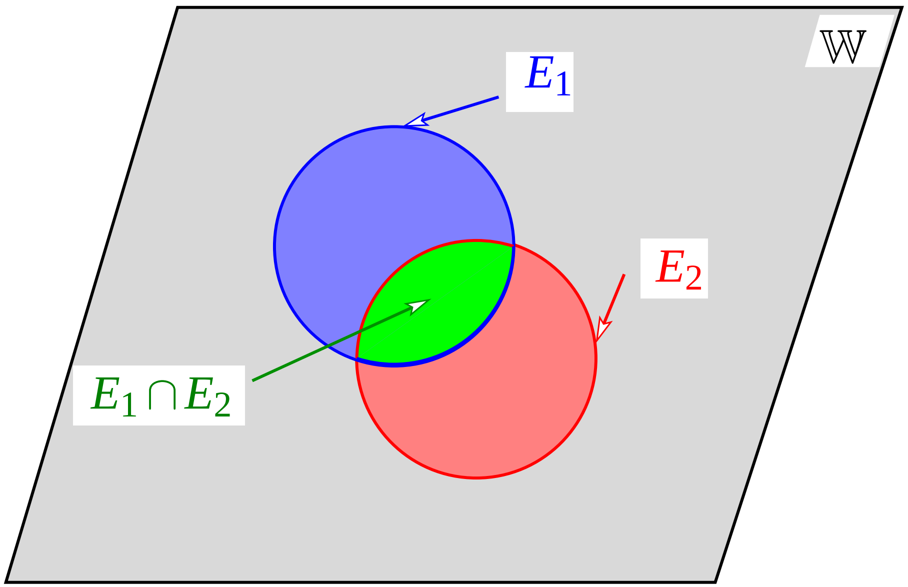
Can two distinct probabilistic laws be imposed on the same set of variables?

Complementarity of σ-algebras

$\mathscr{E}_1$ **and** $\mathscr{E}_2$ **are complementary** ***σ-algebras***

$:\Leftrightarrow$ **for all nonempty** $E_1, E_1' \in \mathscr{E}_1, E_2, E_2' \in \mathscr{E}_2$

$$[\![E_1 \cap E_2 = E_1' \cap E_2']\!] \Rightarrow [\![E_1 = E_1' \text{ and } E_2 = E_2']\!].$$



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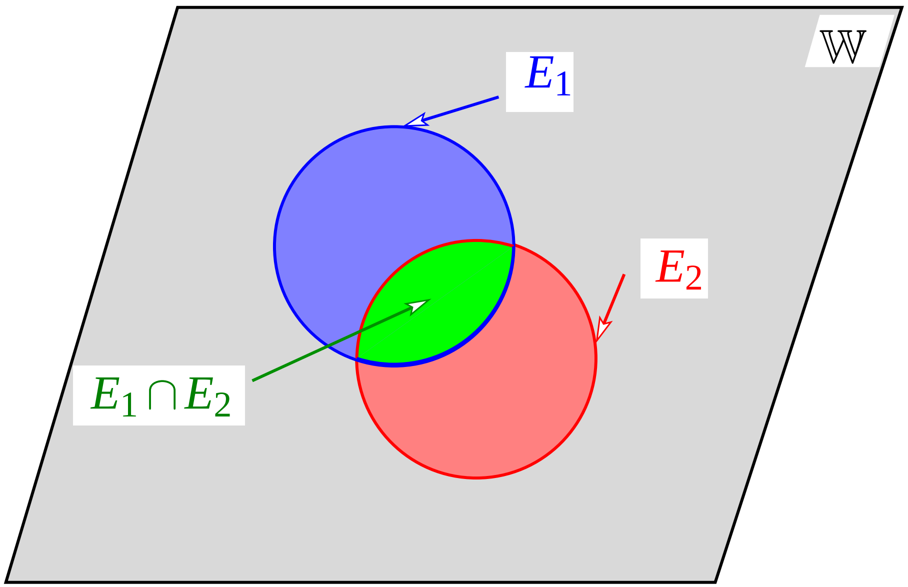

Intersection $\Rightarrow$ intersectants.

Complementarity of stochastic systems

$(\mathbb{W}, \mathscr{E}_1, P_1)$ **and** $(\mathbb{W}, \mathscr{E}_2, P_2)$ **are complementary systems**

$$:\Leftrightarrow \textbf{ for all } E_1, E_1' \in \mathscr{E}_1, E_2, E_2' \in \mathscr{E}_2$$

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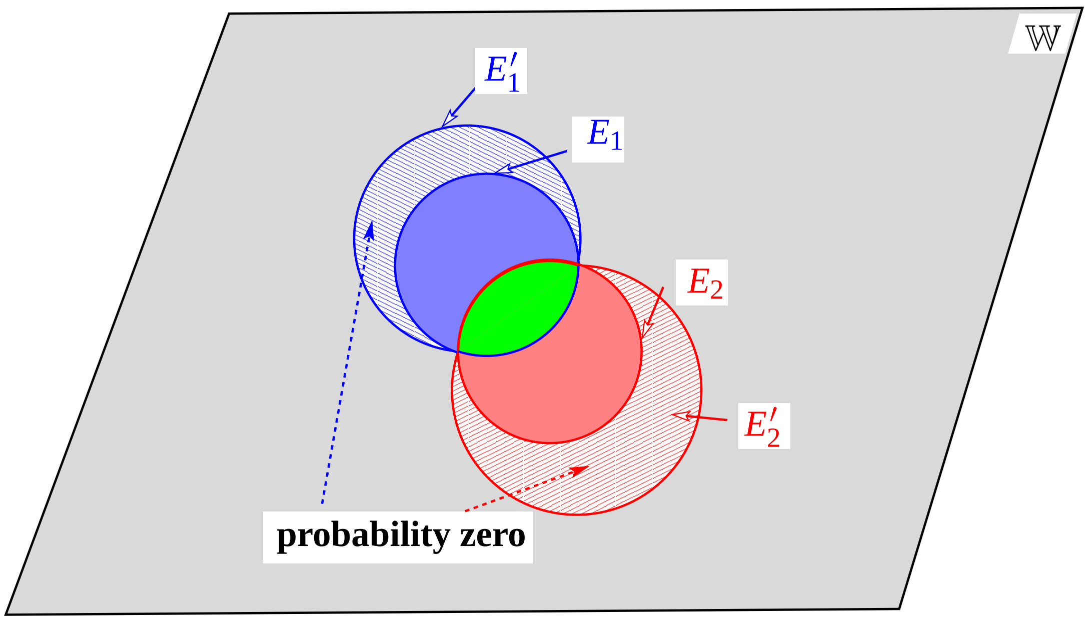

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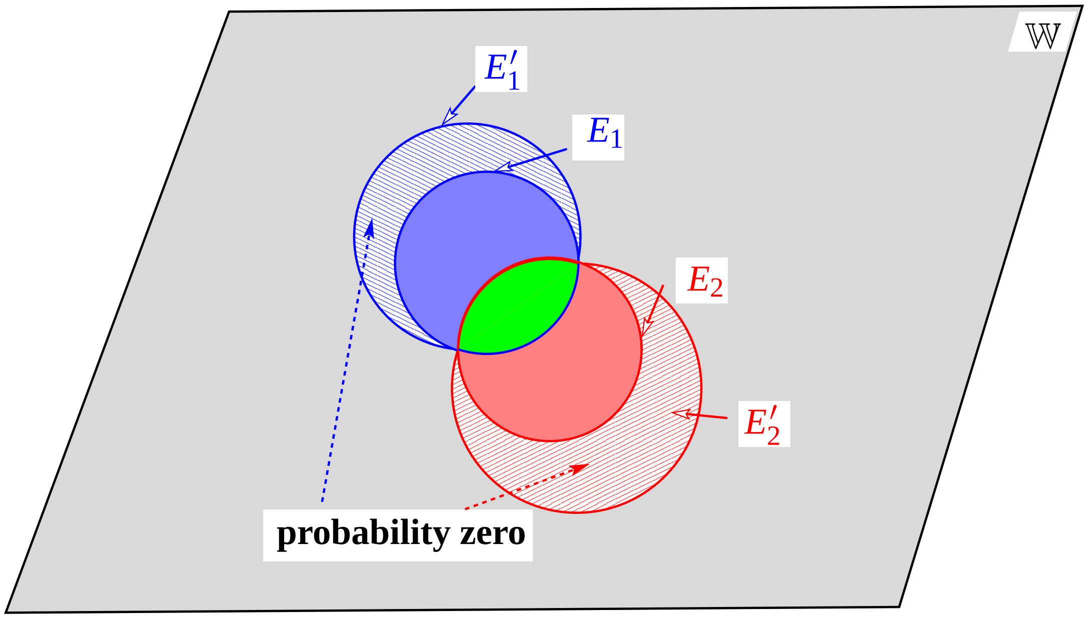
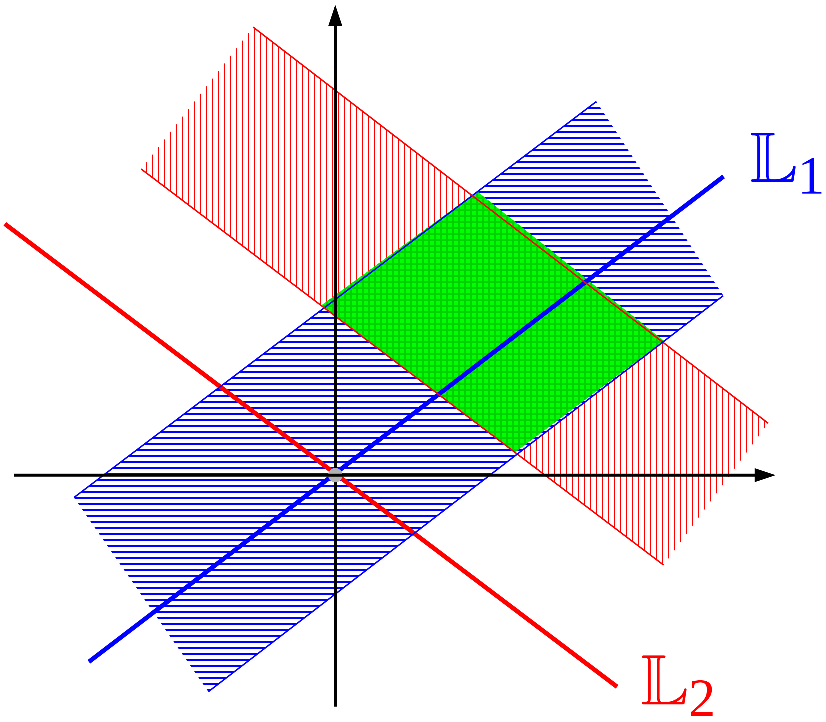


Intersection $\Rightarrow$ product of probabilities of intersectants.

Linear example



complementarity $\Leftrightarrow$ $\mathbb{L}_1 + \mathbb{L}_2 = \mathbb{R}^{\mathtt{n}}$

Interconnection of complementary systems

Let $(\mathbb{W}, \mathscr{E}_1, P_1)$ **and** $(\mathbb{W}, \mathscr{E}_2, P_2)$ **be stochastic systems (independent). Assume complementarity.**

Their *interconnection* is defined as

$$(\mathbb{W}, \mathscr{E}, P)$$

with $\mathscr{E} :=$ **the** σ**-algebra generated by 'rectangles'**

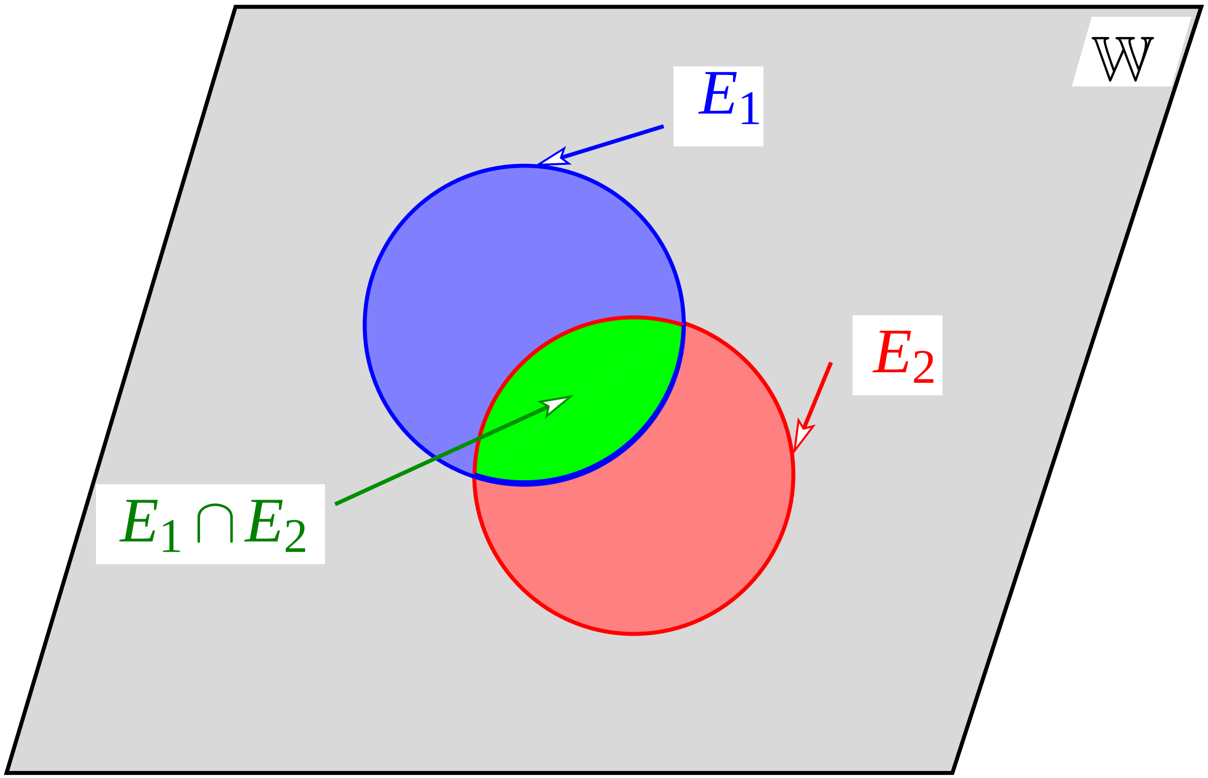
$$\{E_1 \cap E_2 \mid E_1 \in \mathscr{E}_1, E_2 \in \mathscr{E}_2\},$$

and P **defined through the rectangles by**

$$P(E_1 \cap E_2) := P_1(E_1) P_2(E_2).$$

$\mathscr{E}$ **and** P **via Hahn-Kolmogorov extension theorem.**

Interconnection of complementary systems



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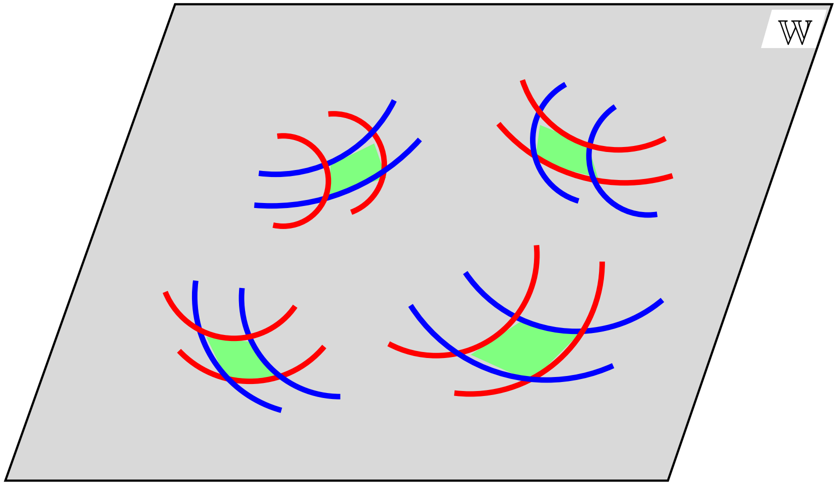
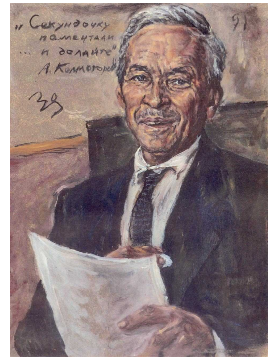
Hahn-Kolmogorov construction of P



H. Hahn
1879 – 1934

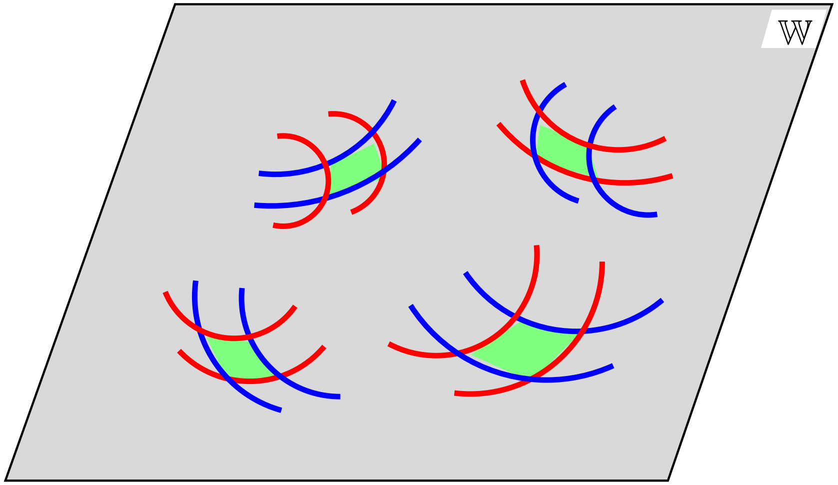


A.N. Kolmogorov
1903 – 1987

Union of finite # of disjoint rectangles is closed under complementation, intersection, and union.

Hence forms an algebra of subsets of $\mathbb{W}$.

Hahn-Kolmogorov construction of P



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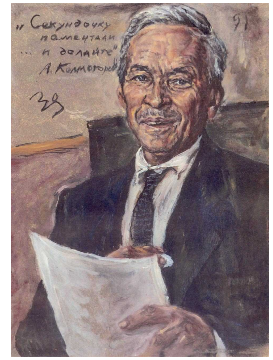


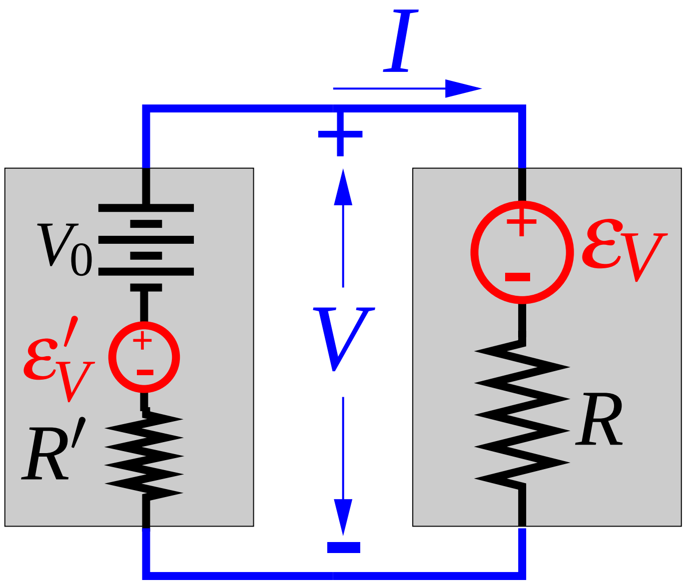
A.N. Kolmogorov
1903 – 1987

P is defined on this algebra by finite additivity.

Hahn-Kolmogorov extension thm

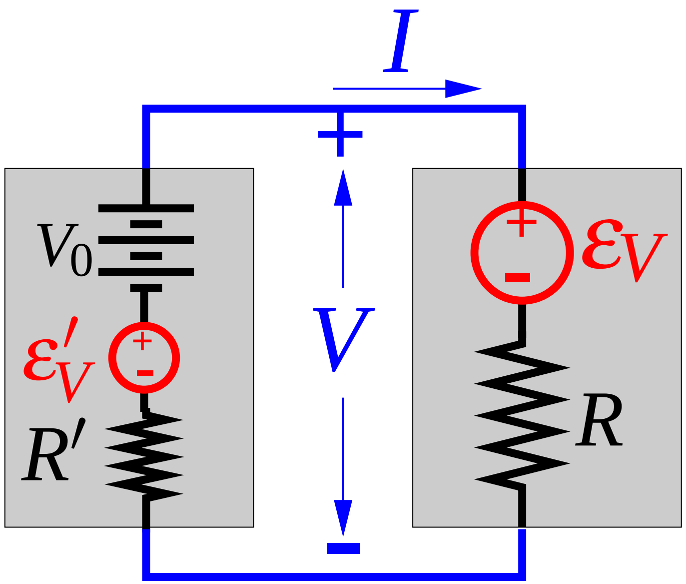
$\Rightarrow \exists(!)$ extension to σ-algebra generated by $\mathscr{E}_1 \cup \mathscr{E}_2$.

Noisy resistor terminated by a voltage source



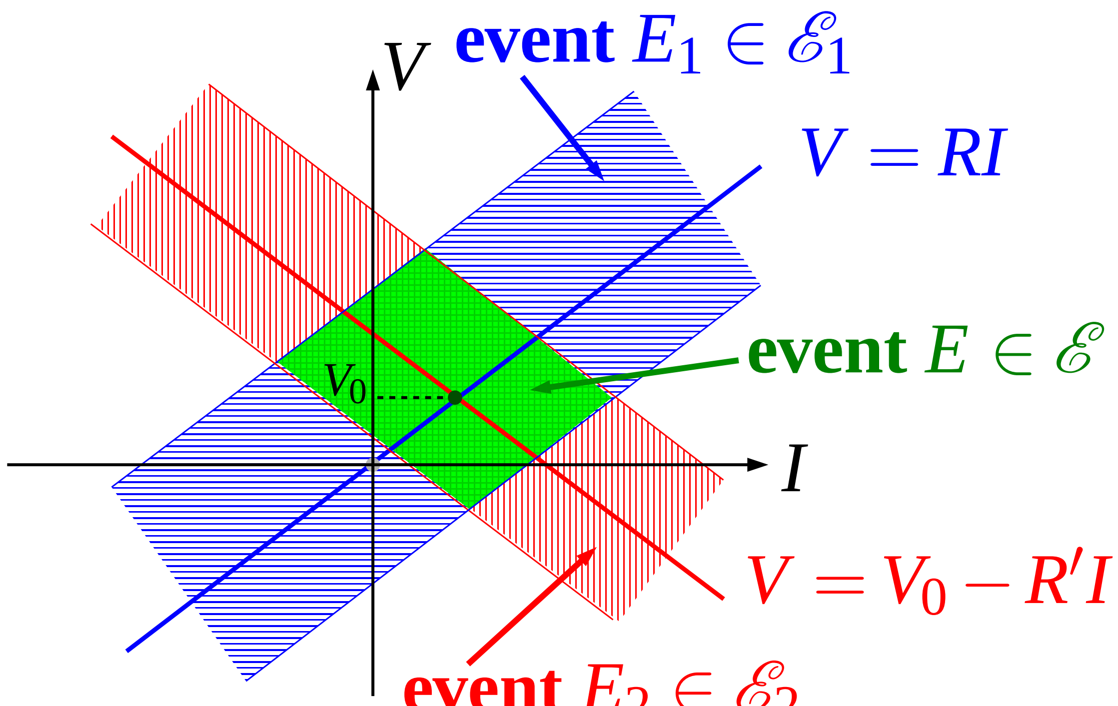
Probability of $\begin{bmatrix} V \\ I \end{bmatrix}$**?**

Noisy resistor terminated by a voltage source

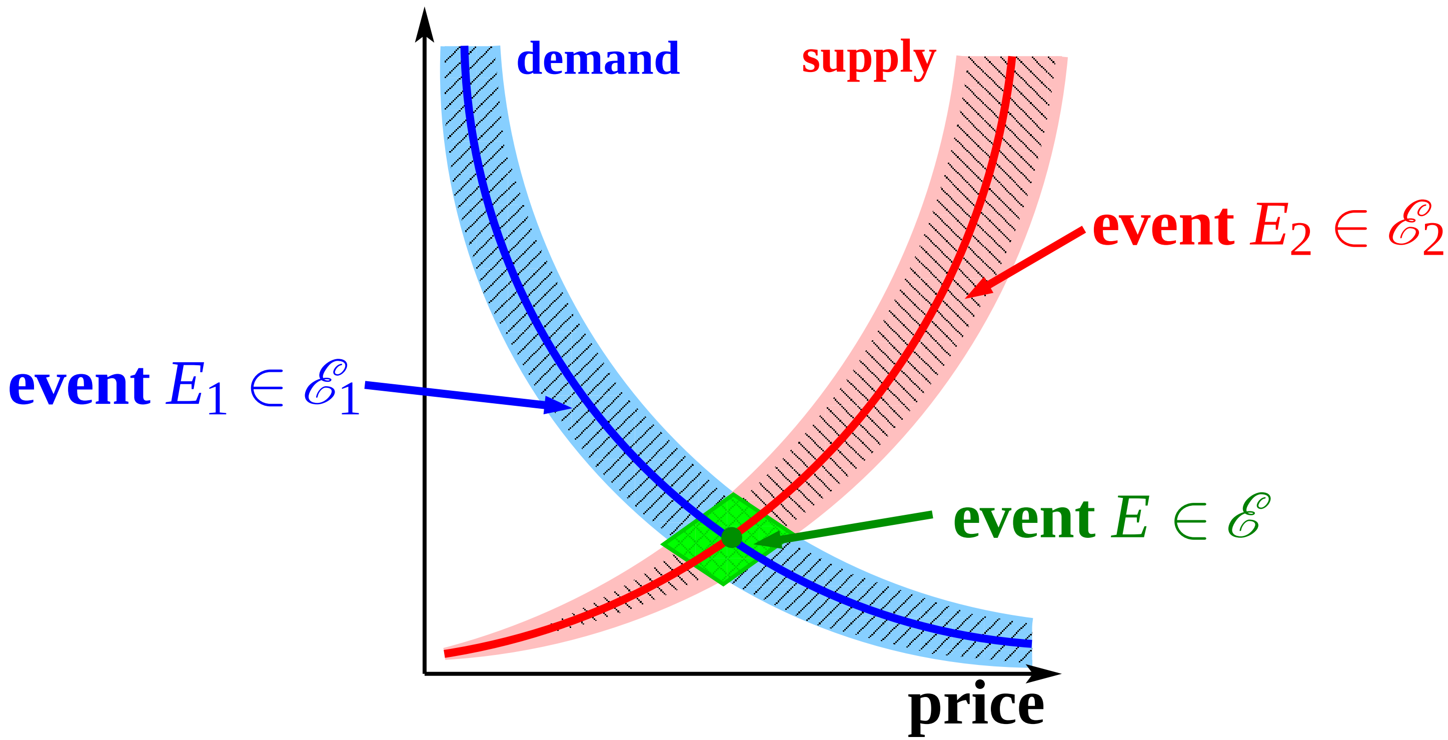


Probability of $\begin{bmatrix} V \\ I \end{bmatrix}$**?**

$$P(E) = P_1(E_1)P_2(E_2)$$



Equilibrium price/demand/supply



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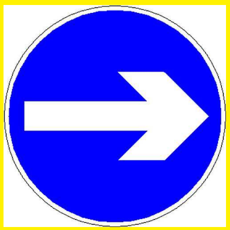
Constrained probability



Impose $w \in \mathbb{S}$

Constrained probability

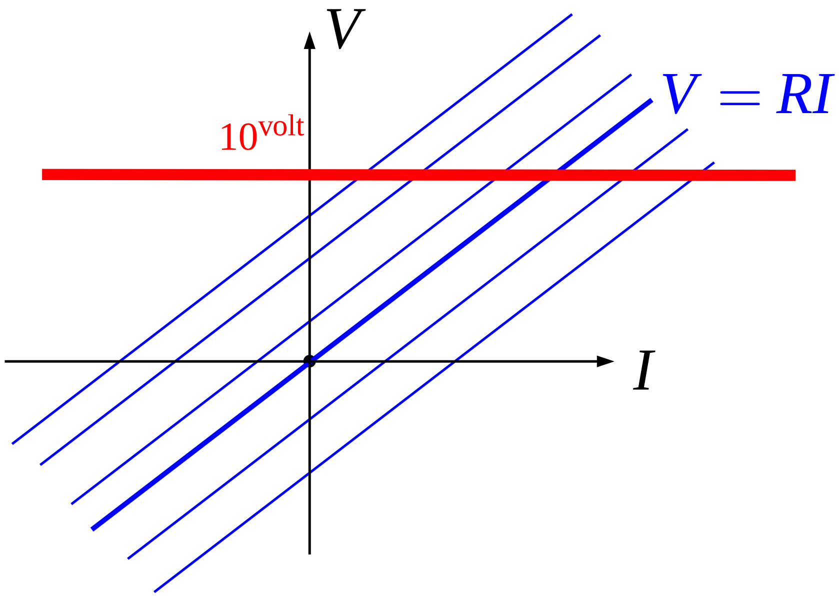
Let $\Sigma = (\mathbb{W}, \mathscr{E}, P)$ be a stochastic system.

Impose the constraint $\boxed{w \in \mathbb{S}}$ **with** $\mathbb{S} \subset \mathbb{W}$.

What is the stochastic nature of the outcomes in $\mathbb{S}$?

Is this a meaningful question?

Noisy resistor

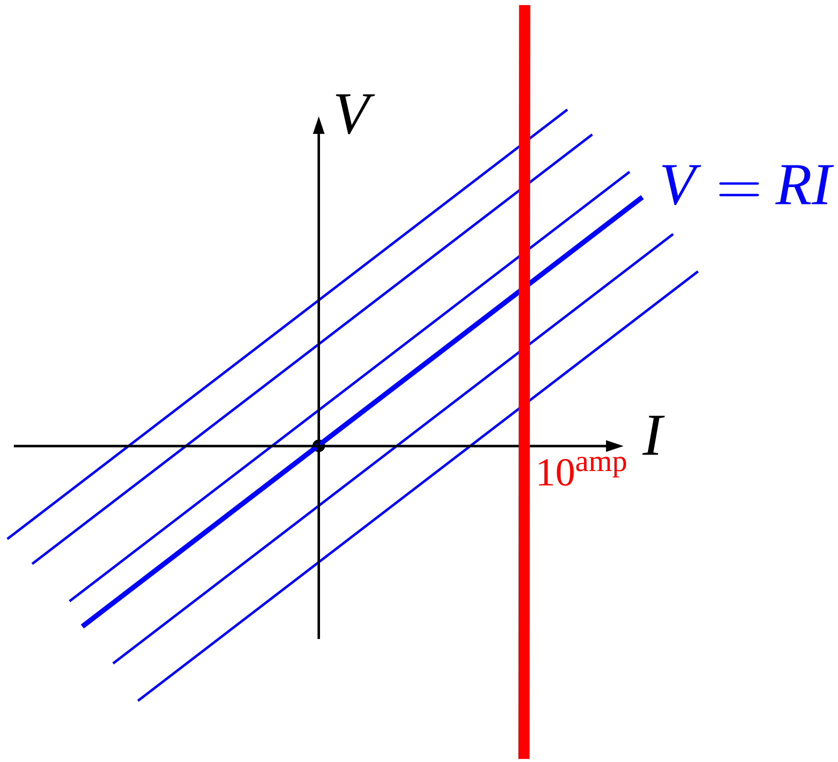


Impose $V = 10^{\text{volt}}$. What is the distribution of I?

$$V = RI + \varepsilon, V = 10^{\text{volt}} \;\Rightarrow\; I = \frac{10}{R} - \frac{\varepsilon}{R}.$$

I is a well-defined gaussian random variable!

Noisy resistor

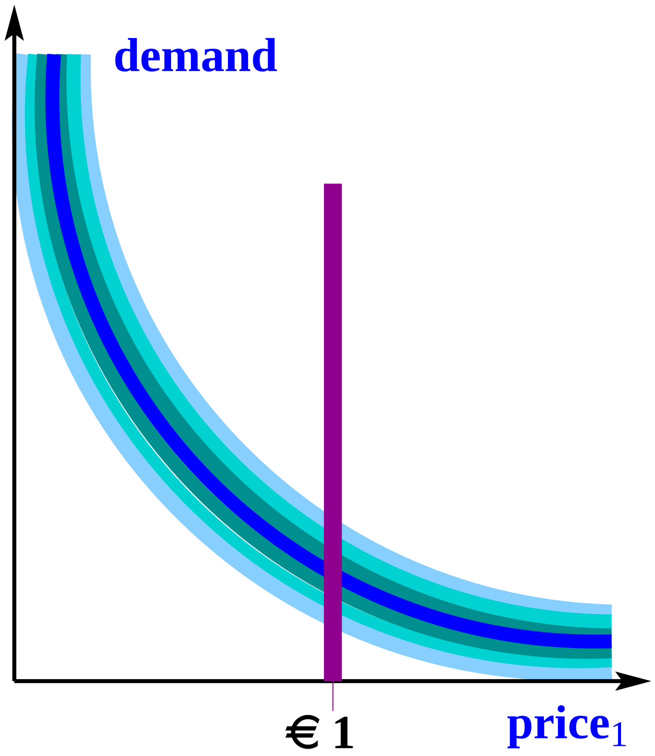


Impose $I = 10^{\text{amp}}$. What is the distribution of V?

$$V = RI + \varepsilon, I = 10^{\text{amp}} \Rightarrow V = 10R - \varepsilon.$$

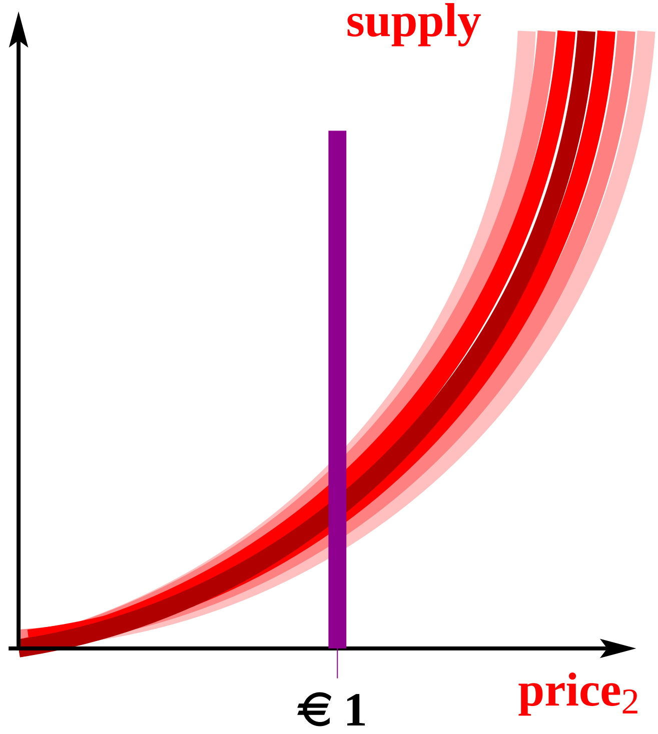
V is a well-defined gaussian random variable!

Price/demand/supply example



Impose price = € 1. Probability of demand?

Price/demand/supply example



Impose price = € 1. Probability of supply?

Constrained probability

Constraining $\simeq$ **interconnection of** $\Sigma = (\mathbb{W}, \mathscr{E}, P)$ **with the deterministic system with behavior** $\mathbb{S}$**.**

Complementarity:$\Leftrightarrow$

$$[\![E_1, E_2 \in \mathscr{E} \text{ and } E_1 \cap \mathbb{S} = E_2 \cap \mathbb{S}]\!] \Rightarrow [\![P(E_1) = P(E_2)]\!].$$

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Complementarity basically implies $\boxed{\mathbb{S} \notin \mathscr{E}!}$

$[\![$**complementarity and** $\mathbb{S} \in \mathscr{E}]\!] \Leftrightarrow [\![P(\mathbb{S}) = 1]\!].$

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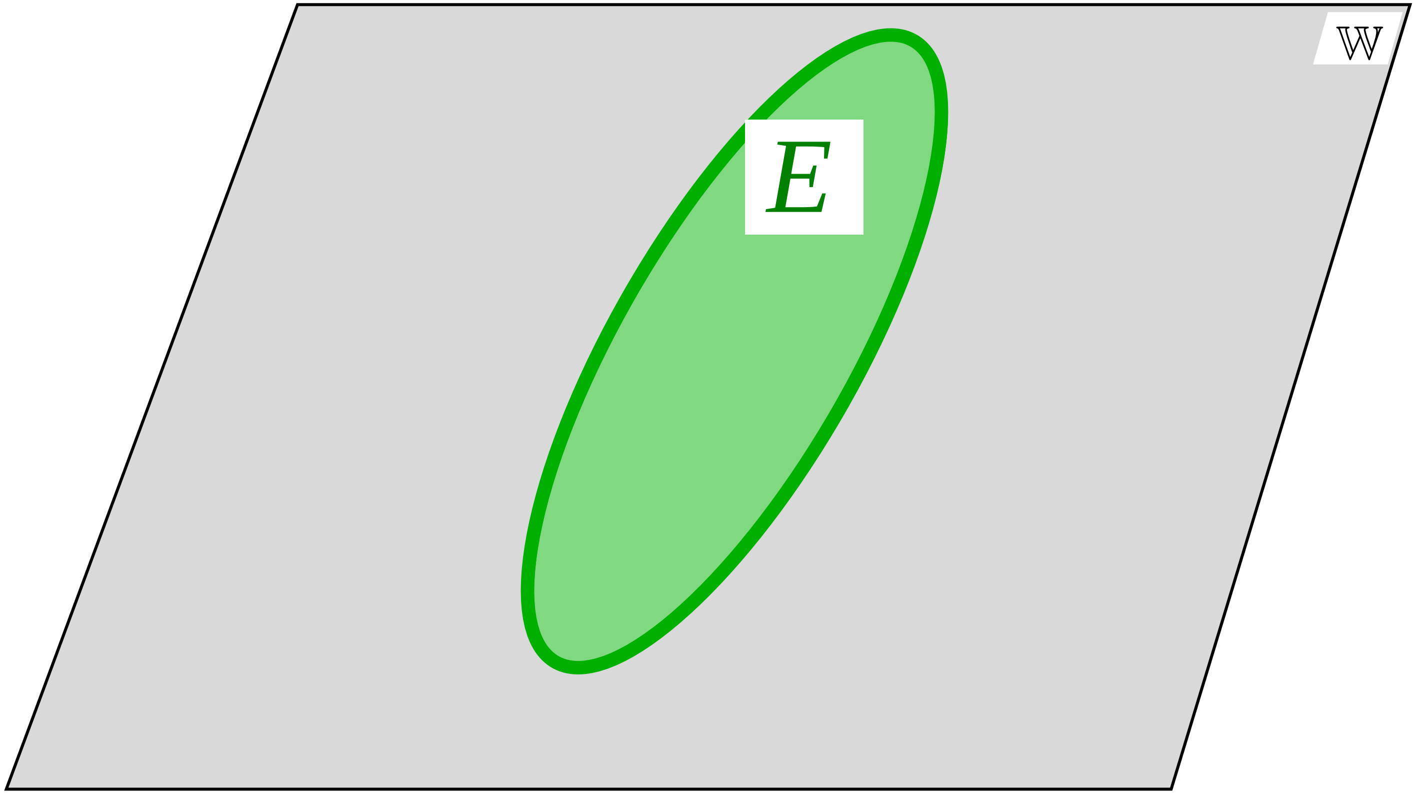
Constraining $\rightsquigarrow$

$$\Sigma_{|\mathbb{S}} = \left(\mathbb{S}, \mathscr{E} \cap \mathbb{S}, P_{|\mathbb{S}}\right) \quad \textbf{with} \quad P_{|\mathbb{S}}(E \cap \mathbb{S}) := P(E).$$

$P_{|\mathbb{S}}$ **= "probability of** w **constrained by** $w \in \mathbb{S}$**".**

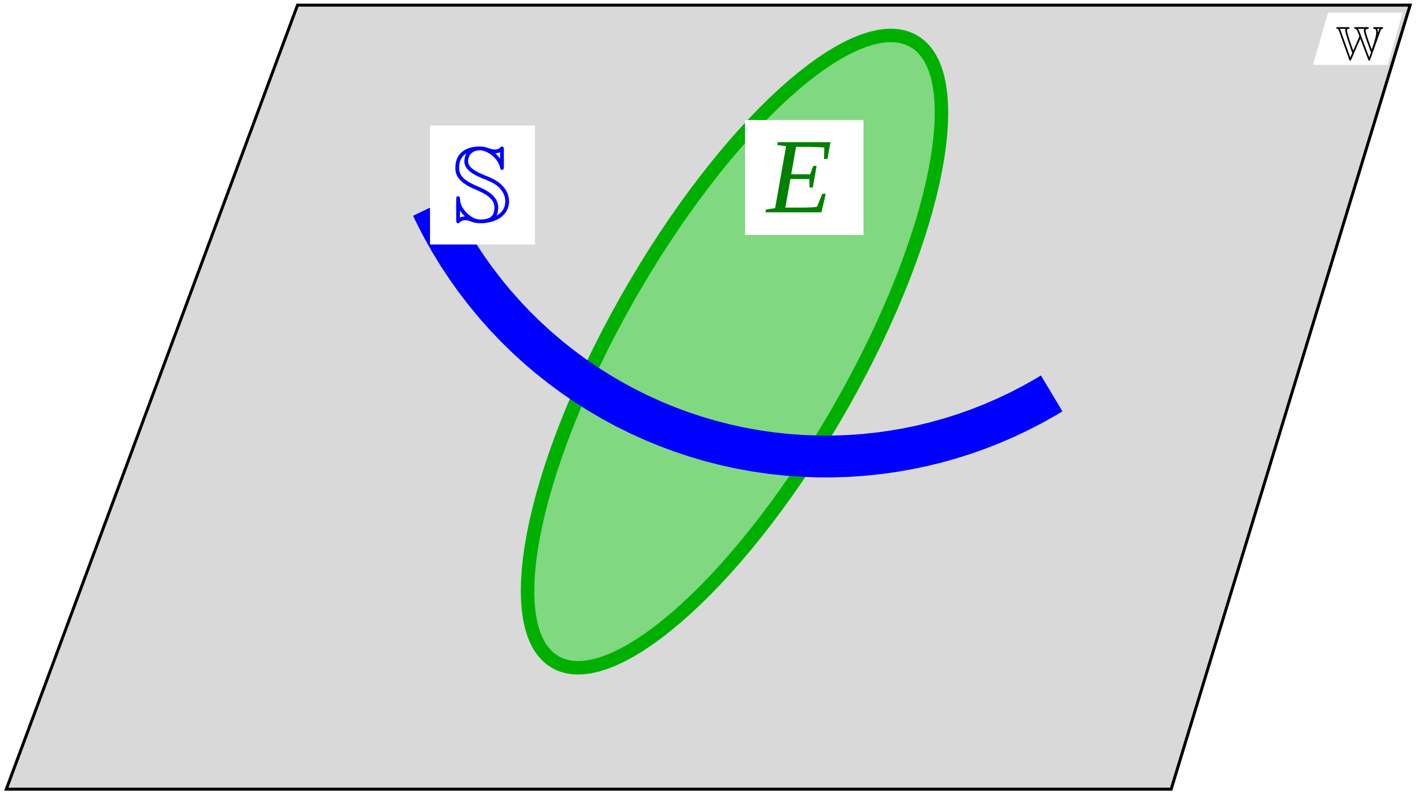
Constraining versus conditioning

Constrained and conditional probability



$E \in \mathscr{E}$ **event**

Constrained and conditional probability



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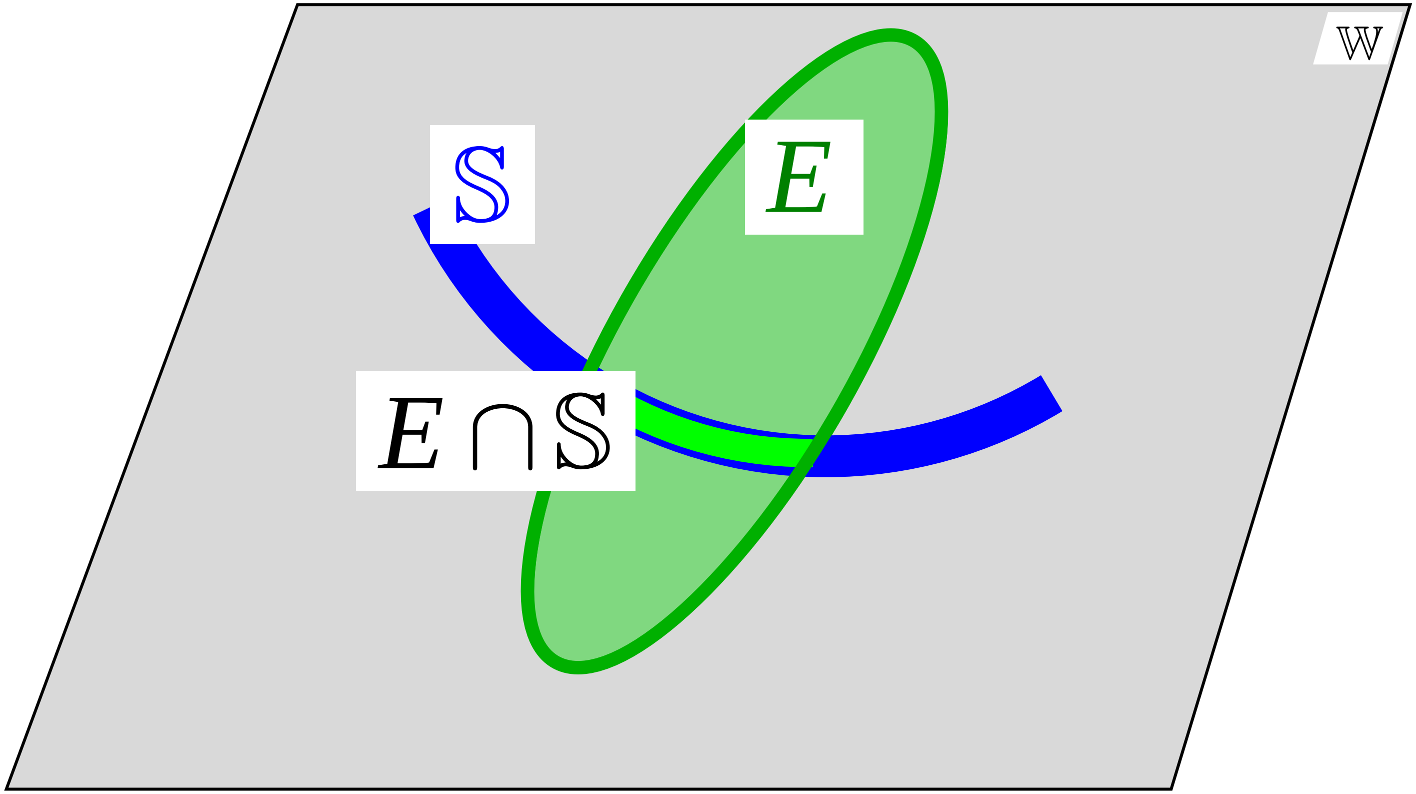

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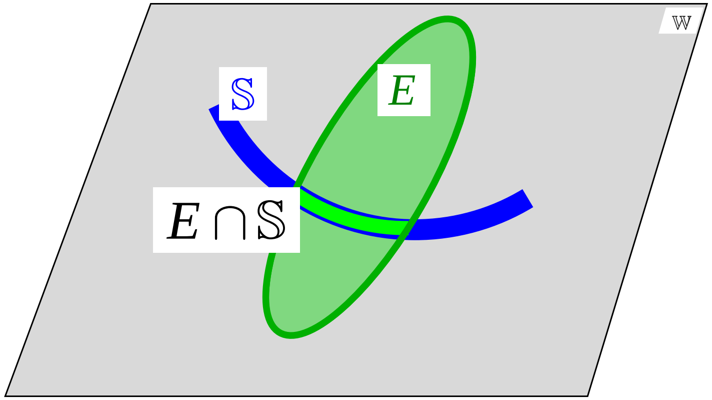
Constrained and conditional probability

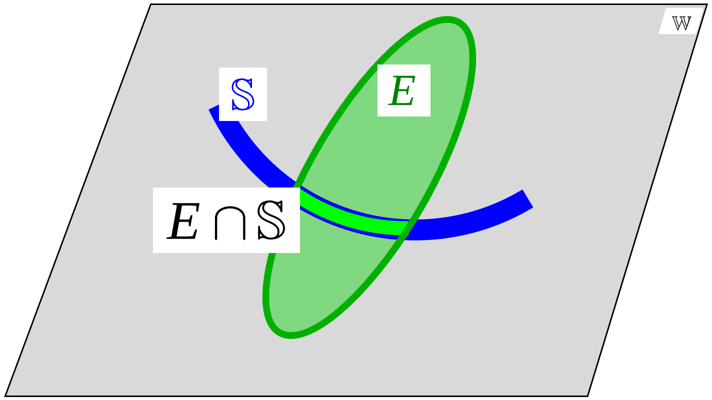


E **= event**

$\mathbb{S}$ **= constraining/conditioning set**

$E \cap \mathbb{S}$ **new event**

Constrained and conditional probability



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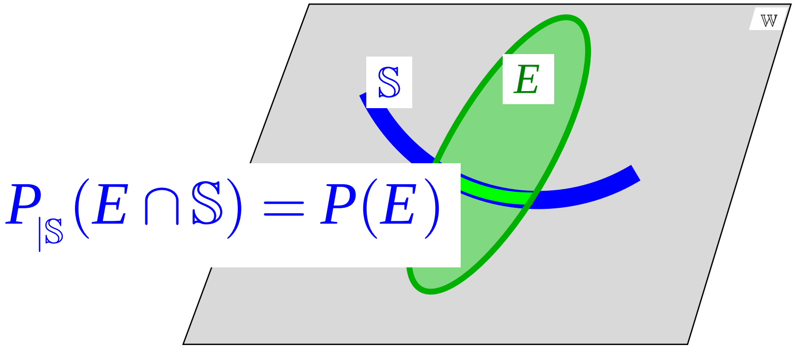
Constraining

$$\mathbb{S} \notin \mathscr{E},$$

$$\left(\mathbb{S}, \mathscr{E}_{|\mathbb{S}}, P_{|\mathbb{S}}\right),$$

$$P_{|\mathbb{S}}(E \cap \mathbb{S}) = P(E).$$



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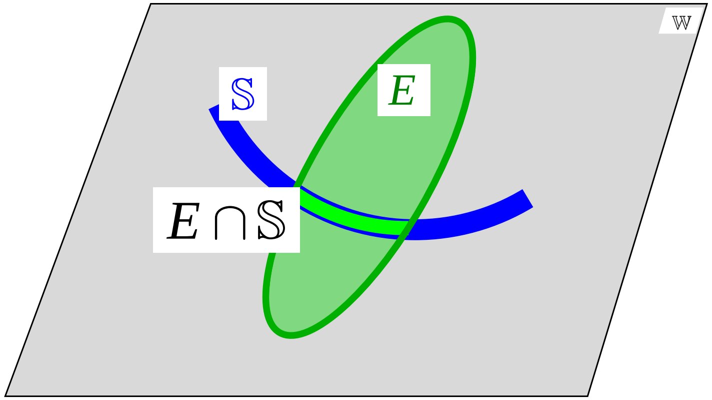

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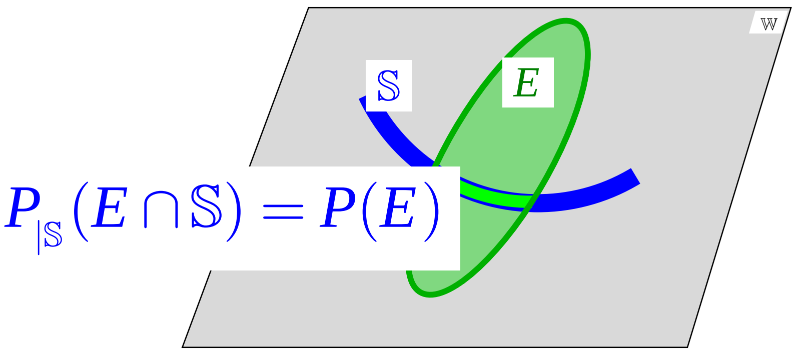
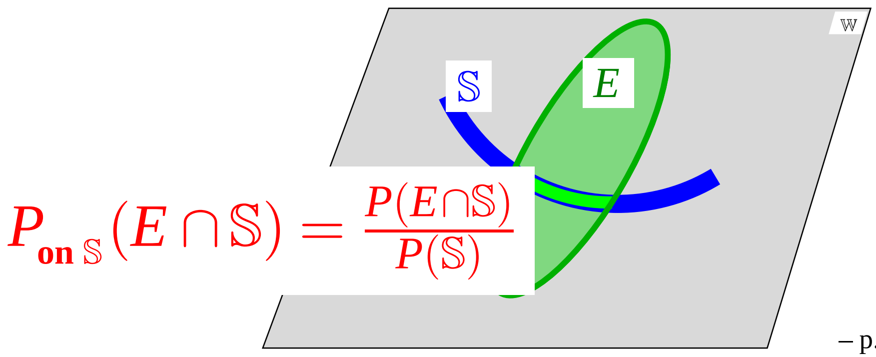


Conditioning

$$\mathbb{S} \in \mathscr{E},$$

$$\left(\mathbb{S}, \mathscr{E}_{|\mathbb{S}}, P_{\textbf{on}\ \mathbb{S}}\right),$$

$$P_{\textbf{on}\ \mathbb{S}}(E \cap \mathbb{S}) = \frac{P(E \cap \mathbb{S})}{P(\mathbb{S})}.$$



Open stochastic systems

Open versus closed

Consider $\Sigma_1 = (\mathbb{R}^{\mathtt{n}}, \mathscr{E}_1, P_1)$**.**

If $\mathscr{E}_1$ **= the Borel** σ**-algebra, then** Σ_1 **is basically only interconnectable with the trivial stochastic system**

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It don't mean a thing, if it ain't interconnecting!

Open versus closed

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Coarse $\mathscr{E}_1$

$\Rightarrow$ Σ_1 **is interconnectable.**

$\Rightarrow$ **'open' stochastic system.**

Conclusions

Stochastic systems

- Borel σ-algebra inadequate for elementary applications.

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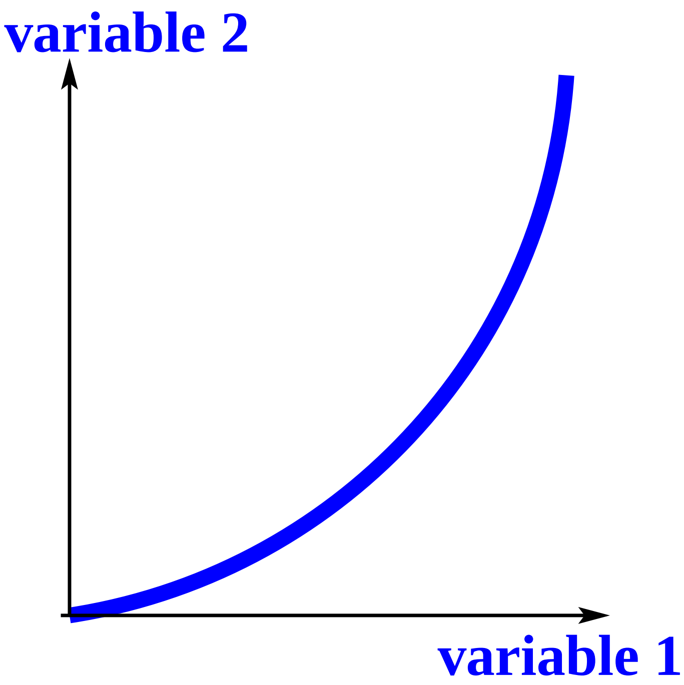

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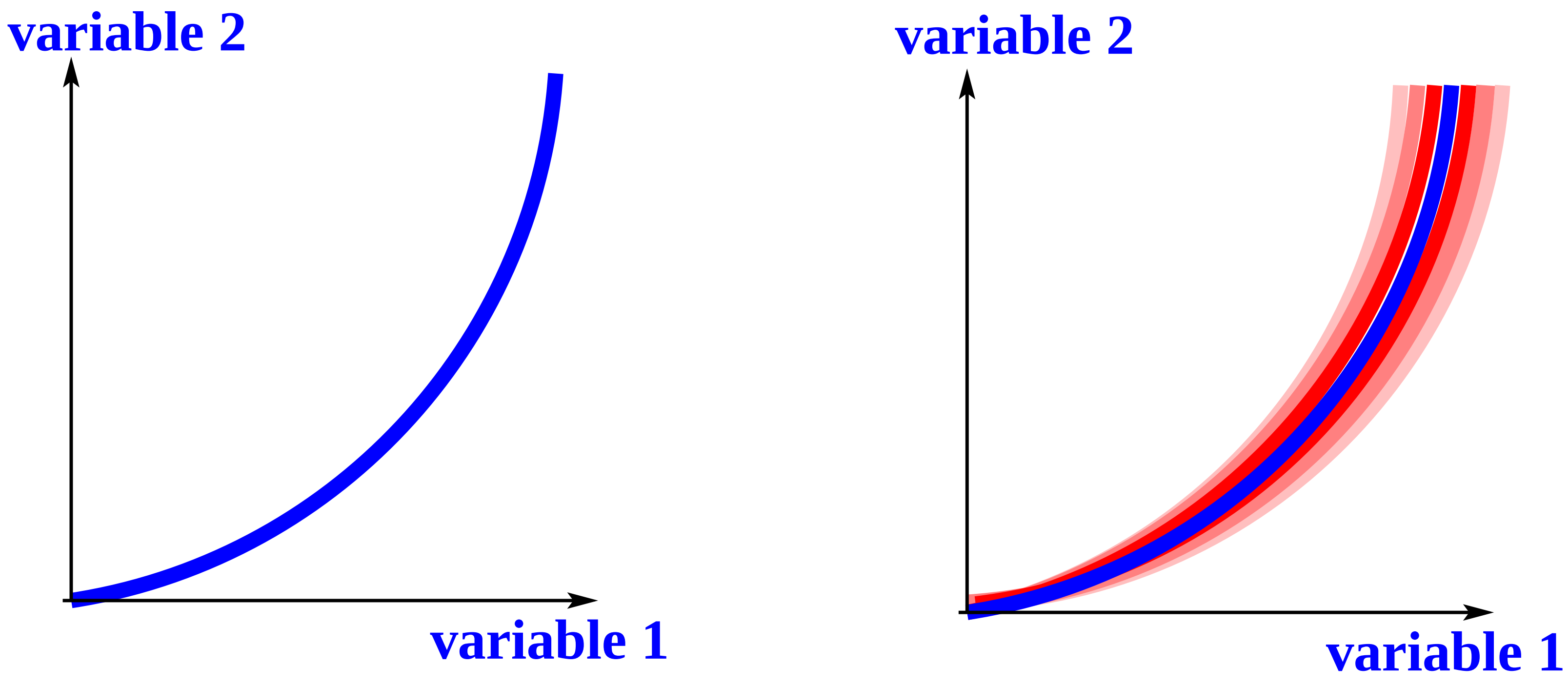
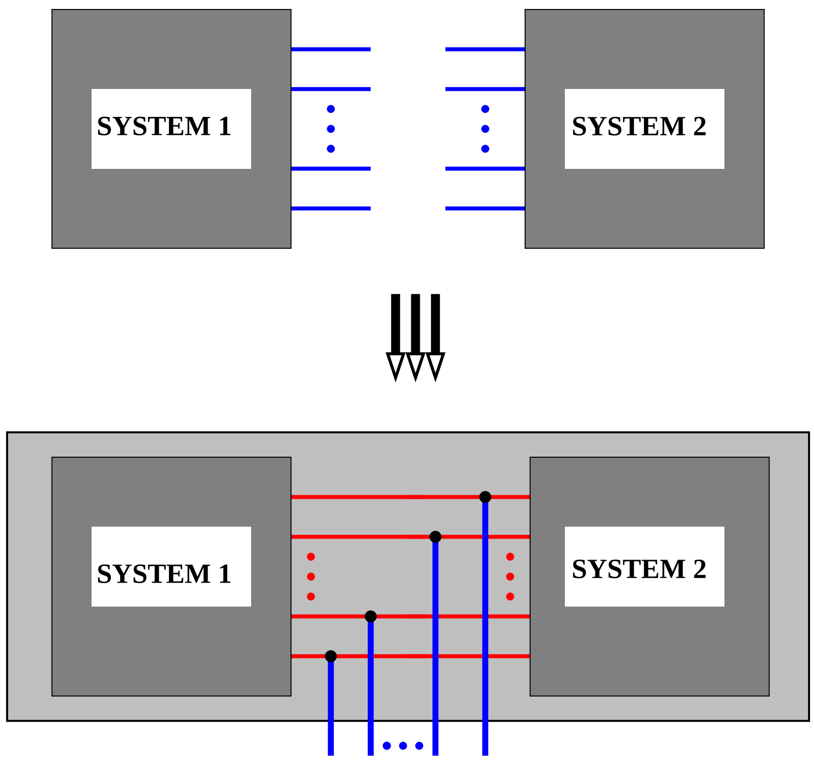


Deterministic $\rightsquigarrow$ **stochastic.**

Stochastic systems

- **Borel σ-algebra inadequate for elementary applications.**
- **Complementary stochastic systems can be interconnected: two distinct laws imposed on one set of variables.**



Stochastic systems

- Borel σ-algebra inadequate for elementary applications.
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- Open stochastic systems require a coarse σ-algebra. Classical random vectors imply closed systems.

Stochastic systems

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- **Open stochastic systems require a coarse σ-algebra.**
  Classical random vectors imply closed systems.
- $\rightsquigarrow$ **Notion of 'constrained probability'.**

Presentation slides and associated article are on my website.
http://homes.esat.kuleuven.be/∼jwillems/

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Thank you

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