STOCHASTIC EVENTS

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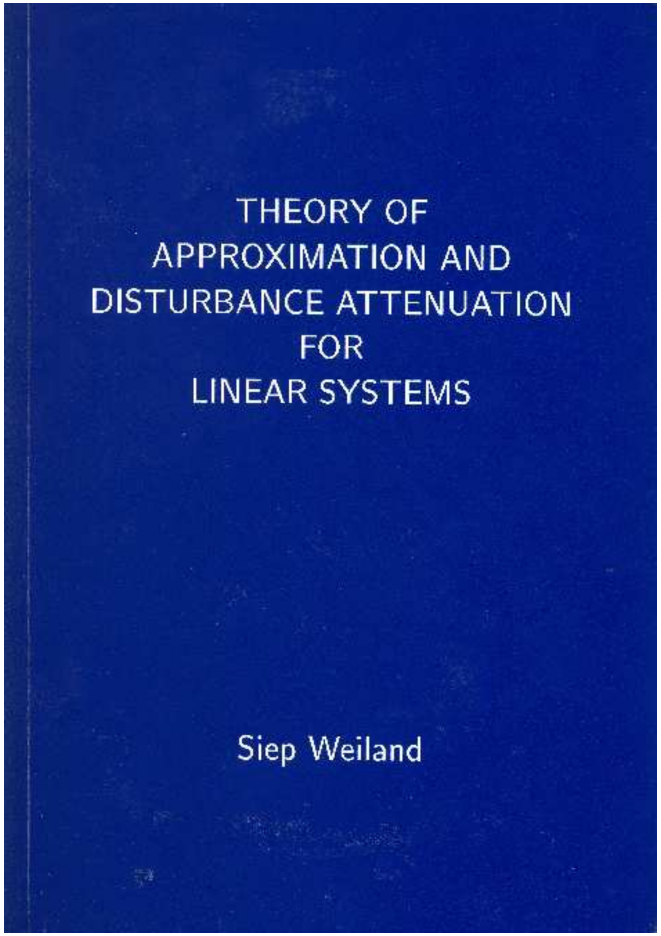
On the occasion of the inaugural lecture of Siep Weiland

Symposium *Complex Challenges* **Eindhoven University of Technology** **December 2, 2011**

In honor of Siep Weiland
on the occasion of his inauguration

Dissertation



Dept. of Mathematics, University of Groningen, Jan. 4, 1991.

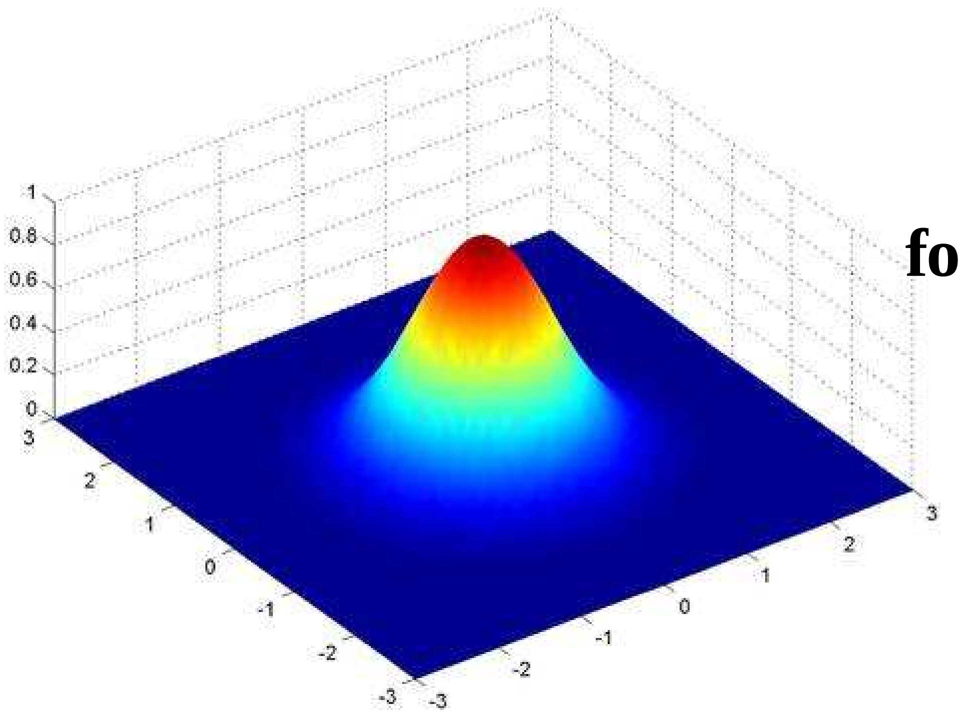
Stochastic events

Classical probability (as it is commonly taught)

Model a phenomenon stochastically; outcomes in $\mathbb{R}^n$.

Usual framework:

- **probability distributions, probability density functions;**
- $\rightsquigarrow$ **'Every' subset of $\mathbb{R}^n$ is assigned a probability.**



for $A \subseteq \mathbb{R}^n$ $$P(A) = \int_A p(x)\,dx$$

Classical probability (as it is commonly taught)

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- **probability distributions, probability density functions;**
- $\rightsquigarrow$ **'Every' subset of $\mathbb{R}^n$ is assigned a probability.**

Thesis

This is unduly restrictive, even for elementary applications.

What this lecture does/does not do

It tries to

- ► **explain some basic probability ideas that should be taught.**

What this lecture does/does not do

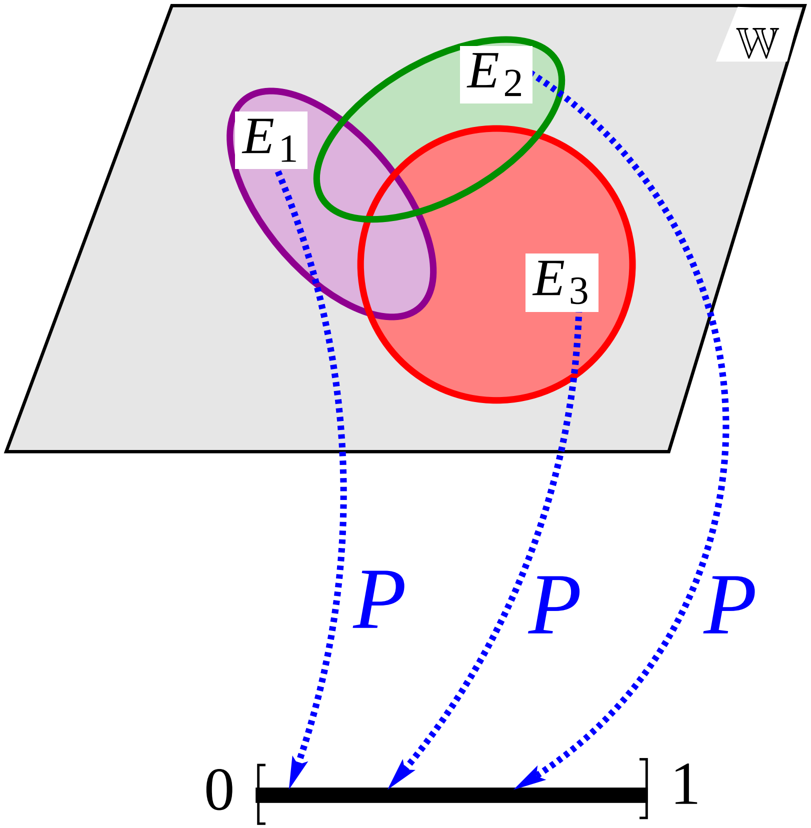
It tries to

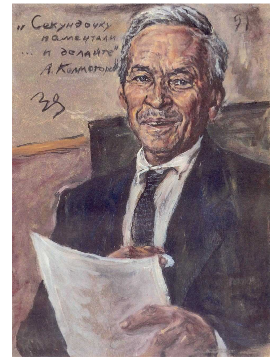
- **explain some basic probability ideas that should be taught.**

It does not address

- **mathematical foundations of probability,**
- **interpretation of probability.**

Events





A.N. Kolmogorov
1903 – 1987

A probability $P(E) \in [0,1]$ is assigned to certain subsets E (*'events'*) of the *outcome space* $\mathbb{W}$.

$\mathscr{E}$ = the sets that are assigned a probability,

:= the class of 'measurable' subsets of $\mathbb{W}$.

Main (not all) axioms

The events $\mathscr{E}$ **form a "σ-algebra" of subsets of** $\mathbb{W}$ $:\Rightarrow$

- $[\![E \in \mathscr{E}]\!] \Rightarrow [\![E^{\text{complement}} \in \mathscr{E}]\!]$
- $[\![E_1, E_2 \in \mathscr{E}]\!] \Rightarrow [\![E_1 \cap E_2 \in \mathscr{E},\ \ E_1 \cup E_2 \in \mathscr{E}]\!]$

$P : \mathscr{E} \to [0,1]$ **is a probability measure** $:\Rightarrow$

- $P(\mathbb{W}) = 1$,
- $[\![E_1, E_2 \in \mathscr{E}$ **and** $E_1 \cap E_2 = \emptyset]\!]$
  $\Rightarrow [\![P(E_1 \cup E_2) = P(E_1) + P(E_2)]\!]$ **(*P* is additive).**

Borel

Émile Borel
1871 – 1956

In applications the events often consist of the *Borel σ-algebra.*

Contains 'basically every' subset of $\mathbb{R}^n$.

Borel

In applications the events often consist of the *Borel σ-algebra.*

Émile Borel
1871 – 1956

$\mathscr{E}$ contains 'basically every' subset of $\mathbb{R}^n$.

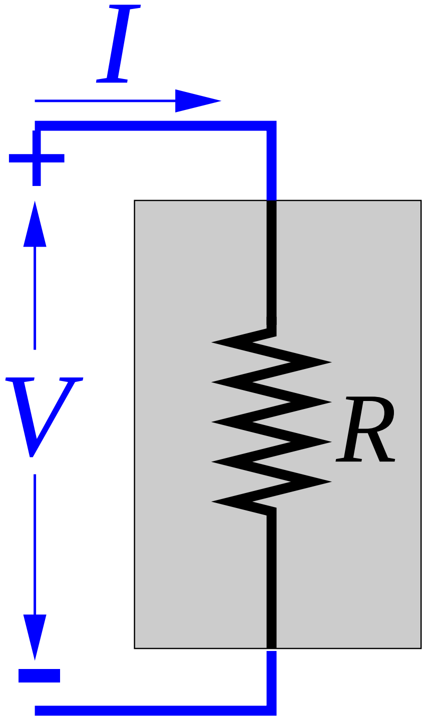
Allows to take probability distributions as the primitive concept, and avoids introducing $\mathscr{E}$.

Thesis

Borel is unduly restrictive for system theoretic applications.

Motivating examples

Ohmic resistor



$$\boxed{V = RI}$$

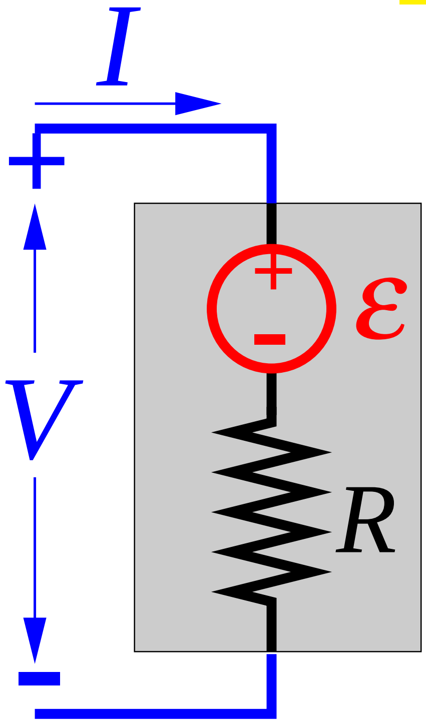
V**: voltage across**

I **current through**

R**: resistance (**≥ 0**)**

'Ohmic resistor'

Noisy (or ‘hot’) resistor



$$\boxed{V = RI + \varepsilon}$$

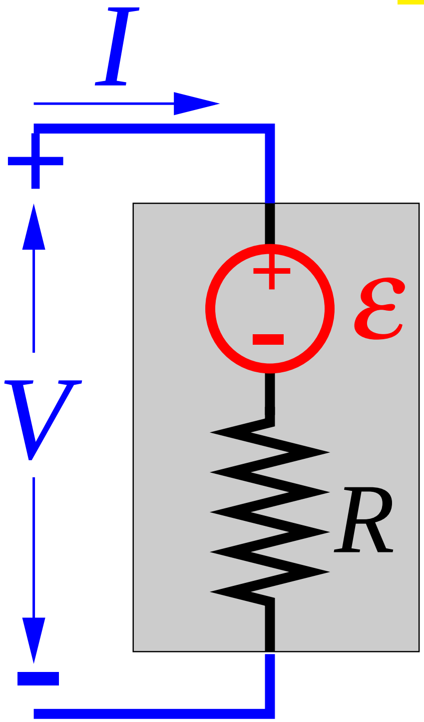
ε gaussian

zero mean

variance $\sim \sqrt{RT}$

‘Johnson-Nyquist resistor’

Noisy (or 'hot') resistor



$$\boxed{V = RI + \varepsilon}$$

ε gaussian

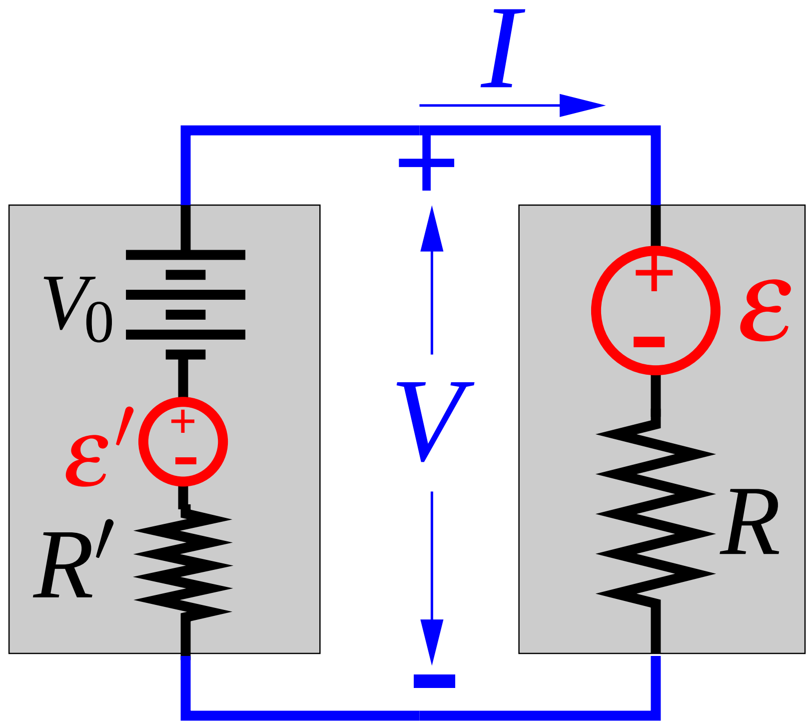
zero mean

variance $\sim \sqrt{RT}$

'Johnson-Nyquist resistor'

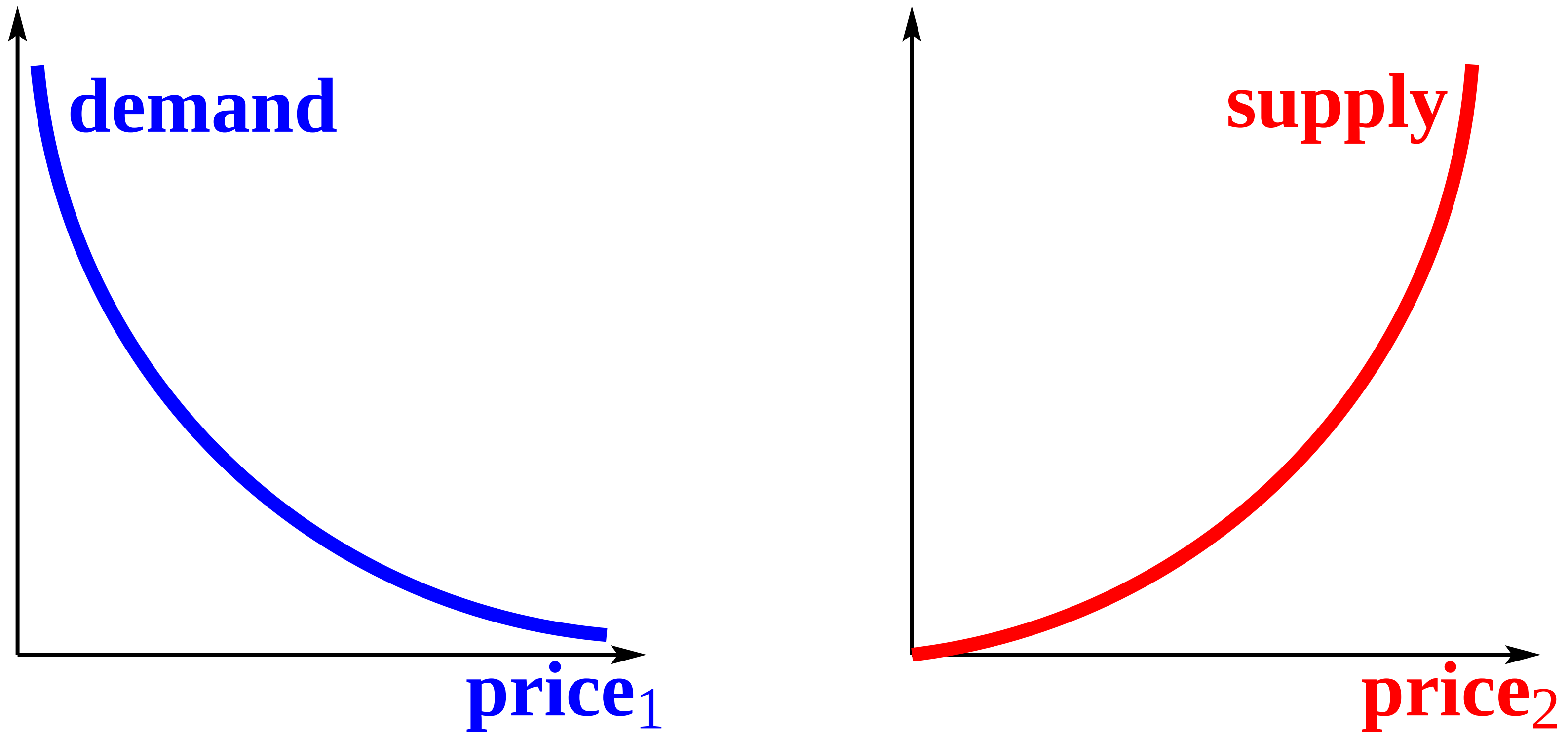
What is $\begin{bmatrix} V \\ I \end{bmatrix}$ ***as a mathematical entity?***

Noisy resistor terminated by a voltage source

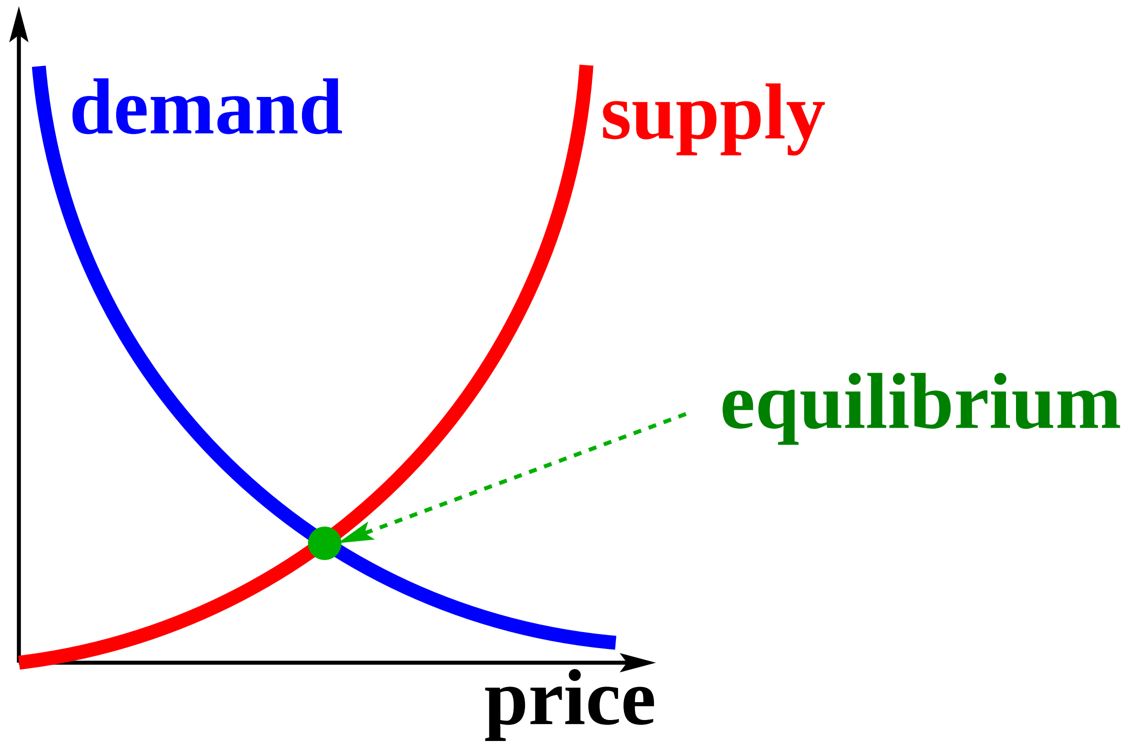


How do we deal with interconnection?

Deterministic price/demand/supply



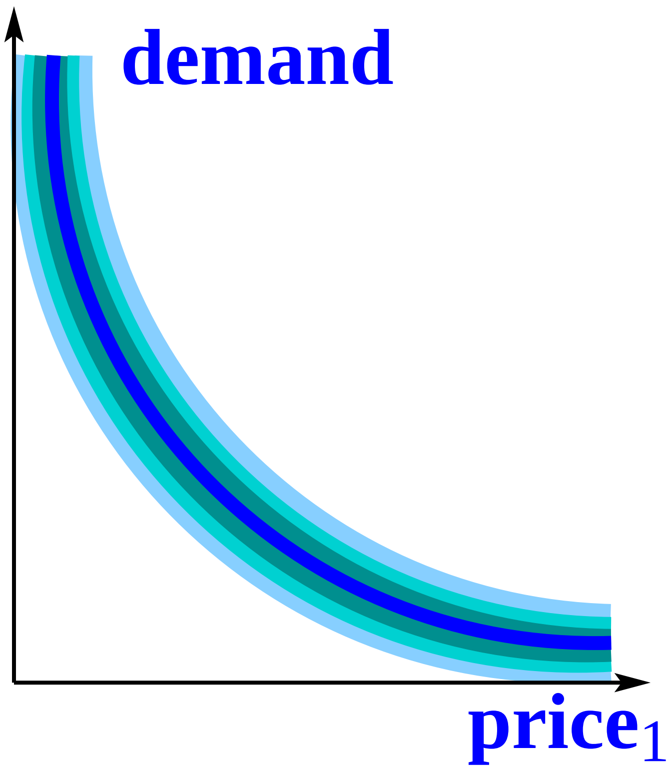
Deterministic price/demand/supply



‘Interconnection’

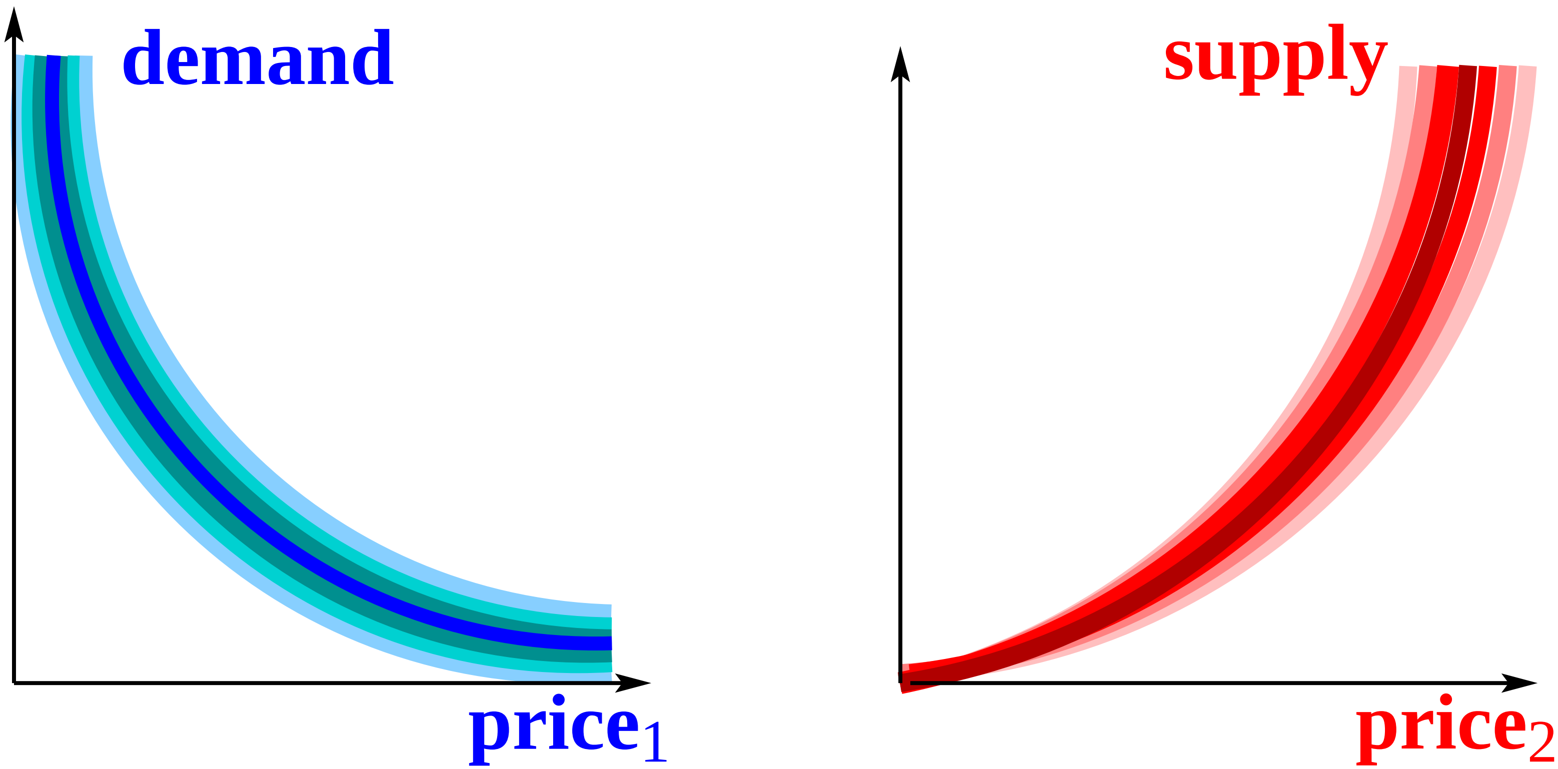
$\text{price}_1 = \text{price}_2$, **demand = supply.**

Stochastic price/demand/supply



(Only) certain regions of the $\begin{bmatrix} \text{price}_1 \\ \text{demand} \end{bmatrix}$ and $\begin{bmatrix} \text{price}_2 \\ \text{supply} \end{bmatrix}$ planes are assigned a probability.

Stochastic price/demand/supply



(Only) certain regions of the $\begin{bmatrix}\text{price}_1 \\ \text{demand}\end{bmatrix}$ and $\begin{bmatrix}\text{price}_2 \\ \text{supply}\end{bmatrix}$ planes are assigned a probability.

How do we deal with equilibrium: supply = demand?

Formal definitions

Definition

A *stochastic system* is a probability triple $(\mathbb{W}, \mathscr{E}, P)$

- $\mathbb{W}$ **a non-empty set, the *outcome space*,**
- $\mathscr{E}$ **a σ-algebra of subsets of** $\mathbb{W}$**: the *events*,**
- $P : \mathscr{E} \to [0, 1]$ **a *probability measure*.**

$\mathscr{E}$**: the subsets that are assigned a probability.**

Probability that outcomes $\in E$, $E \in \mathscr{E}$**, is** $P(E)$**.**

Model $\cong$ $\mathscr{E}$ **and** P**;**

$\mathscr{E}$ **is an essential part!**

$\mathscr{E}$ **should not be taken for granted.**

Definition

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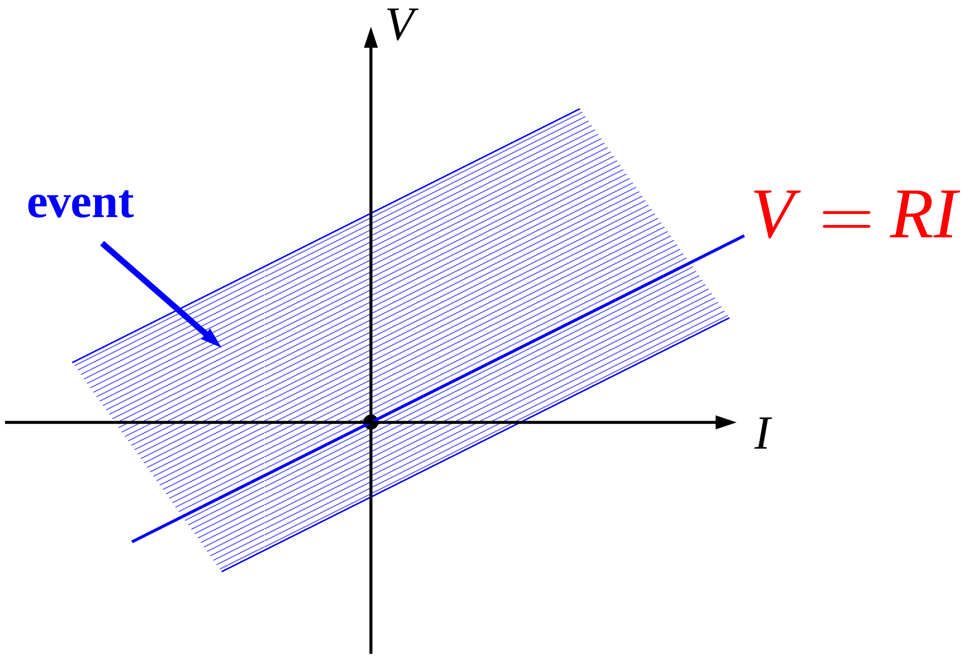
- $P : \mathscr{E} \to [0,1]$ **a *probability measure*.**

'Classical' stochastic system:

$\mathbb{W} = \mathbb{R}^{\mathrm{n}}$ **and** $\mathscr{E}$ **= 'all' subsets of** $\mathbb{R}^{\mathrm{n}}$**.**

P **specified by a probability distribution or a pdf.**

Noisy resistor

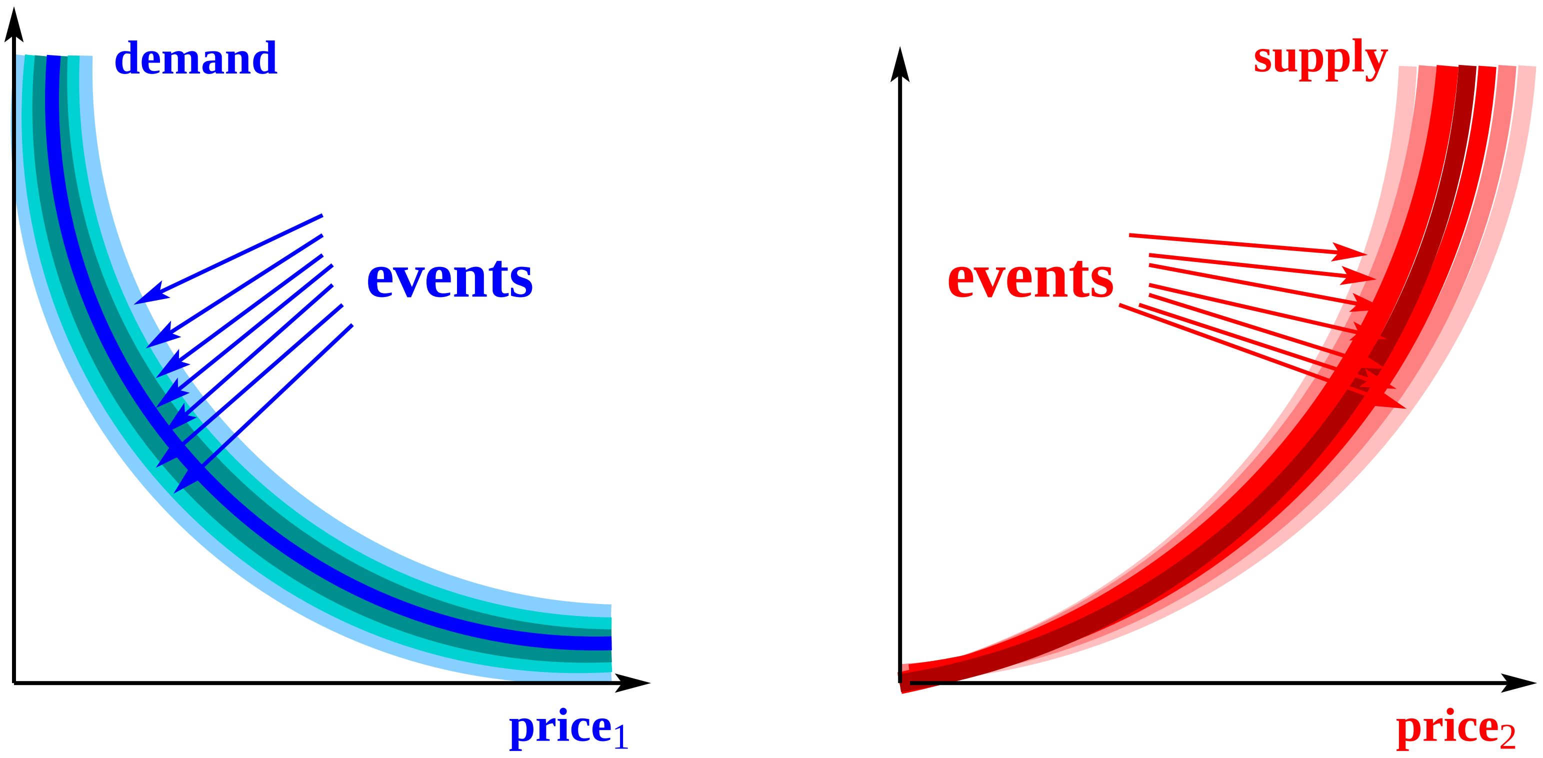


Events: $\left\{ \begin{bmatrix} V \\ I \end{bmatrix} \in \mathbb{R}^2 \mid V - RI \in A \right.$ **with** A **a Borel subset of** $\left. \mathbb{R} \right\}$**.**

$P(\textbf{event}) =$ **gaussian measure of** A**.**

Neither $\begin{bmatrix} V \\ I \end{bmatrix}$ **nor** I **nor** V **possess a pdf.**

Stochastic price/demand/supply



$\mathscr{E}$, $\mathscr{E}'$ = the regions that are assigned a probability.

***p*, *d*, nor *s* are not classical real random variables.**

Linearity

Linear stochastic system

linear stochastic system

$:\Leftrightarrow$ Borel probability on $\mathbb{R}^{\mathtt{n}}/\mathbb{L}$,

$\mathbb{L} \subseteq \mathbb{R}^{\mathtt{n}}$ a linear subspace, called the ‘fiber’.

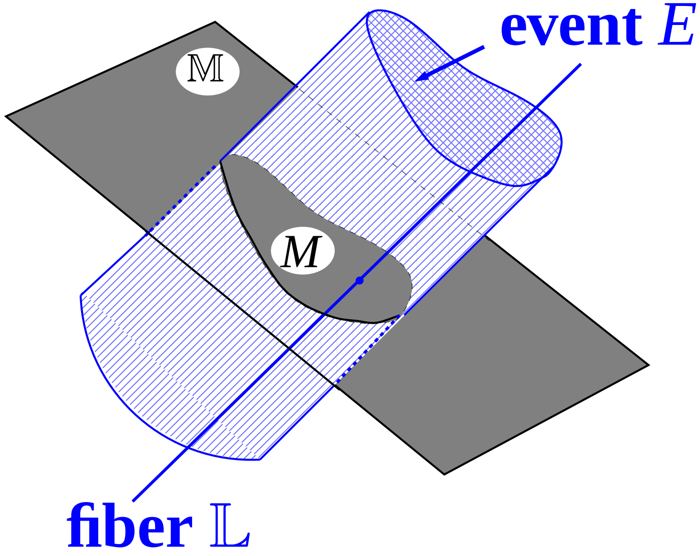
Note: $\mathbb{R}^{\mathtt{n}}/\mathbb{L}$ is a real vector space of dimension $\mathtt{n} - \mathtt{dimension}(\mathbb{L})$.

Events: **cylinders with sides parallel to $\mathbb{L}$.**

Subsets of $\mathbb{R}^{\mathtt{n}}$ as $A + \mathbb{L}$, $\mathbb{L}$ linear subspace, A Borel.

Linearity

linear stochastic system



Borel probability on $\mathbb{M}$.

Example: the noisy resistor.

Classical $\Rightarrow$ linear!

gaussian **:$\Leftrightarrow$ linear, probability on $\mathbb{M}$ gaussian.**

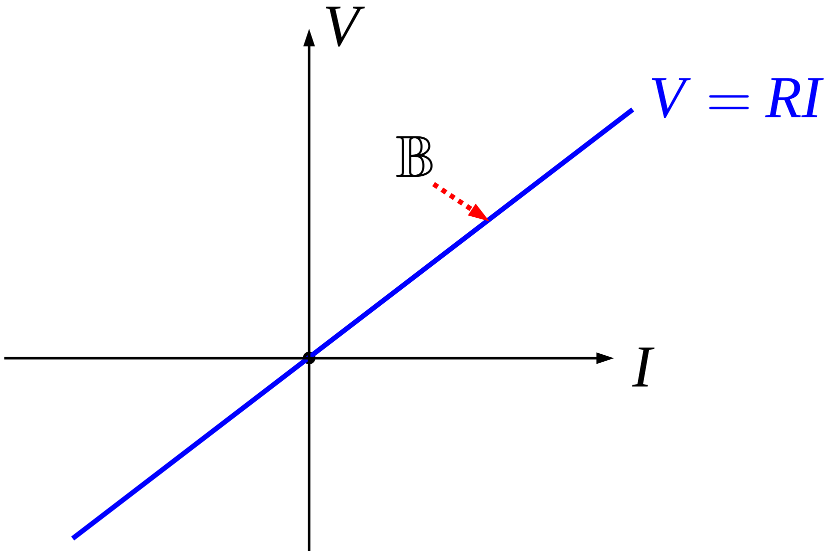
Deterministic system

$(\mathbb{W}, \mathscr{E}, P)$ **is said to be** ***deterministic*** **if**

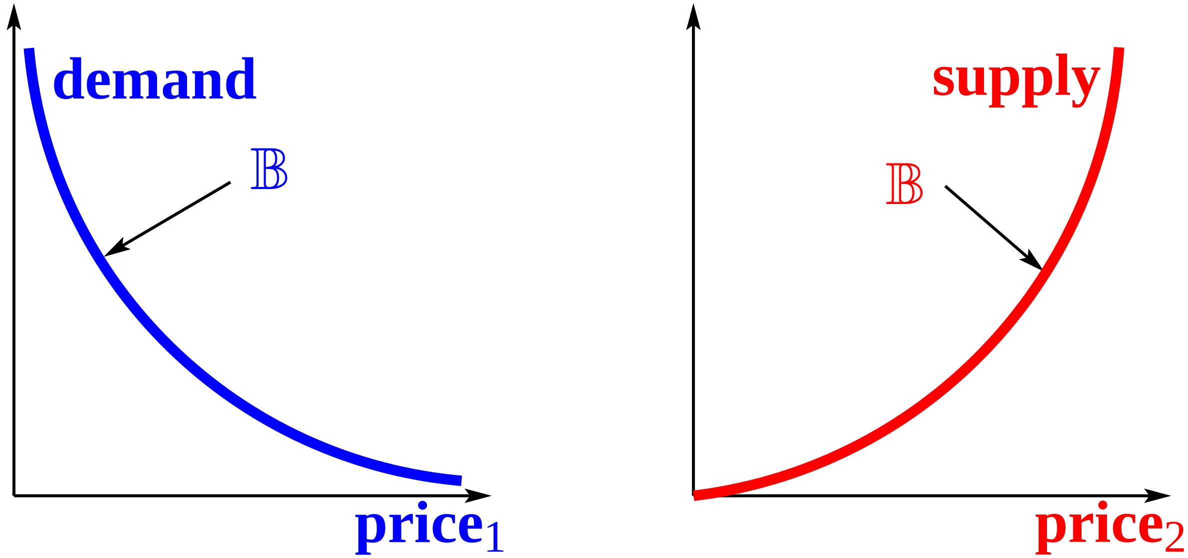
$$\mathscr{E} = \{\emptyset, \mathbb{B}, \mathbb{B}^{complement}, \mathbb{W}\} \text{ **and** } P(\mathbb{B}) = 1.$$

Deterministic examples

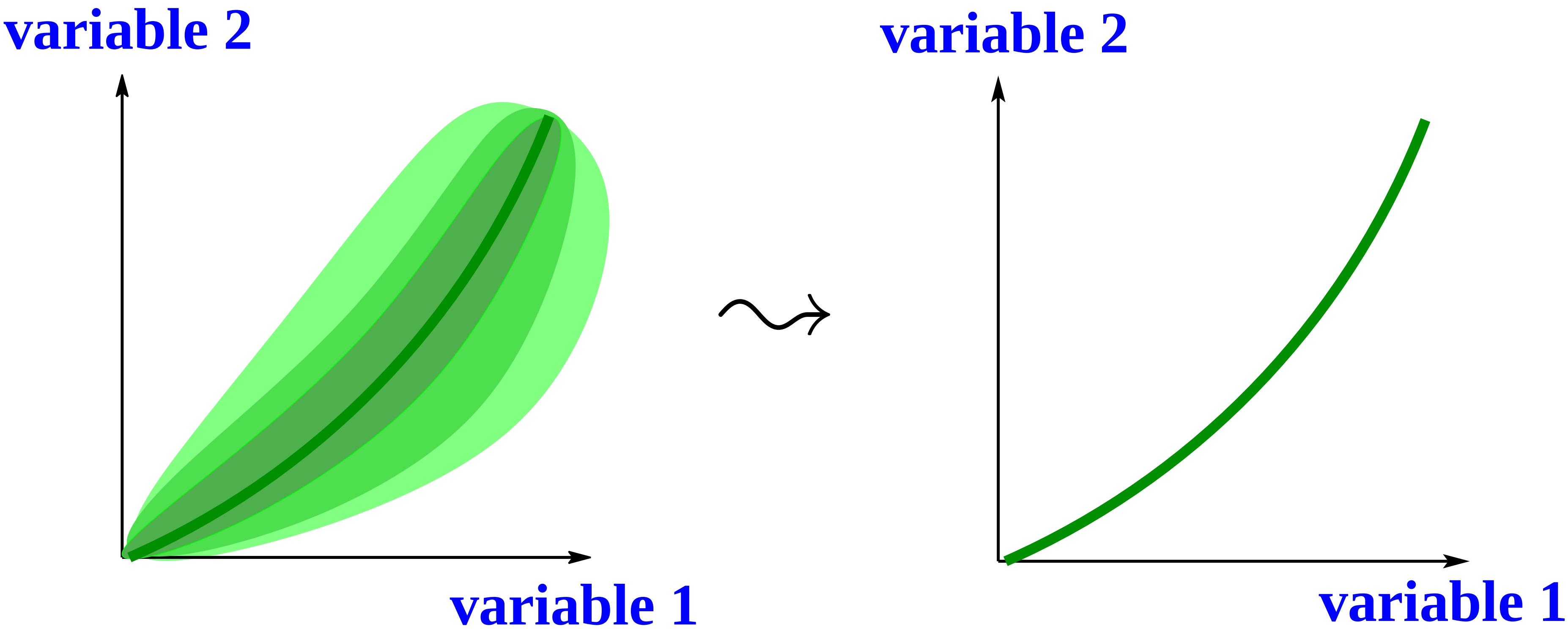
Ohmic resistor:



Economic example:



The need for ‘coarse’ σ-algebras



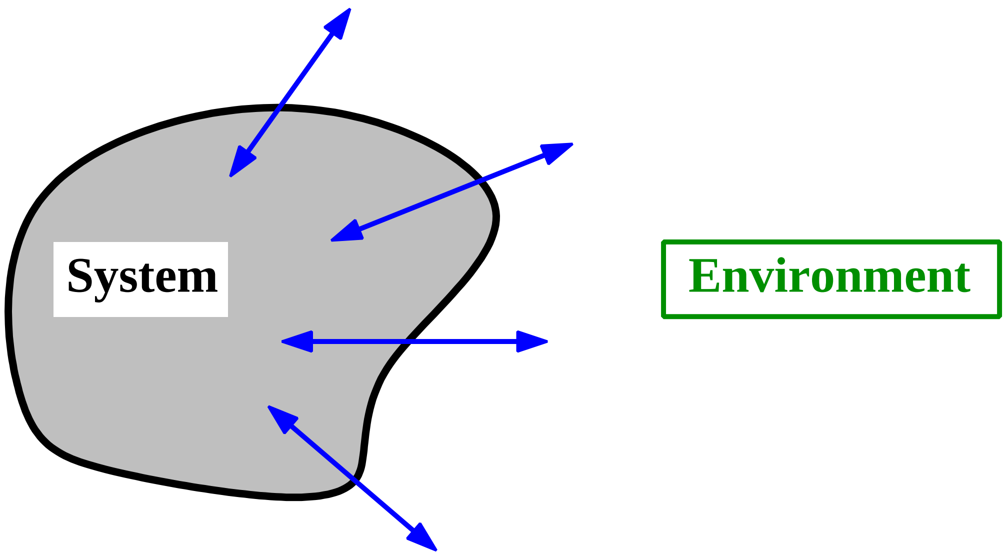
For a classical random vector, the deterministic limit

$\simeq$ a (singular) probability distribution.

Awkward from the modeling point of view.

Interconnection

Open and connected



Open

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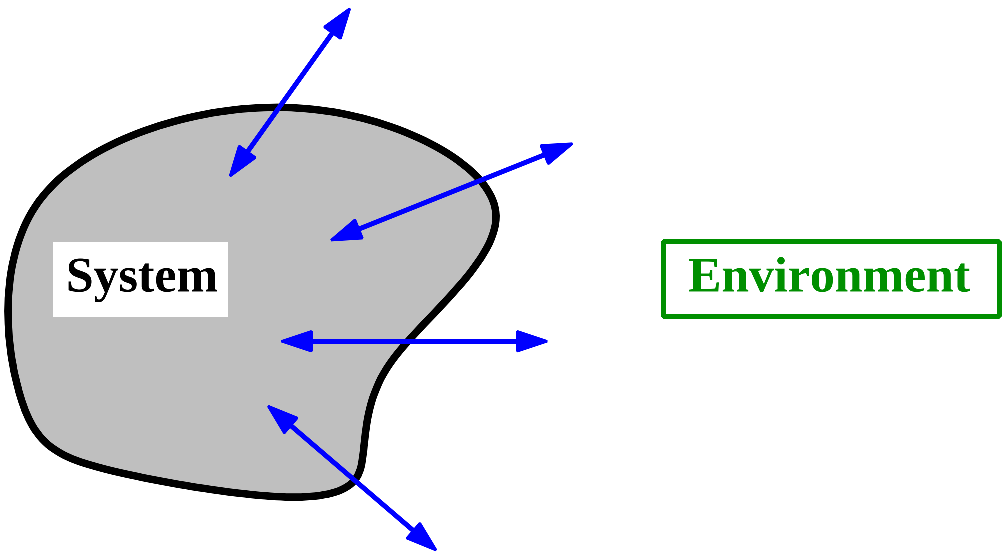
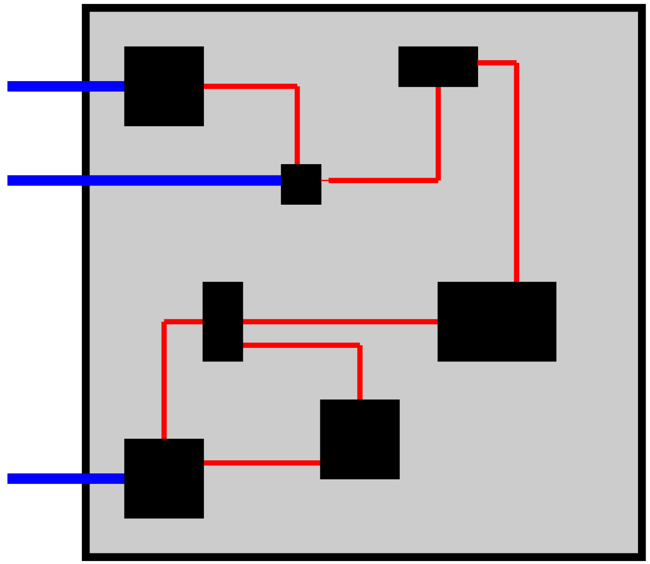
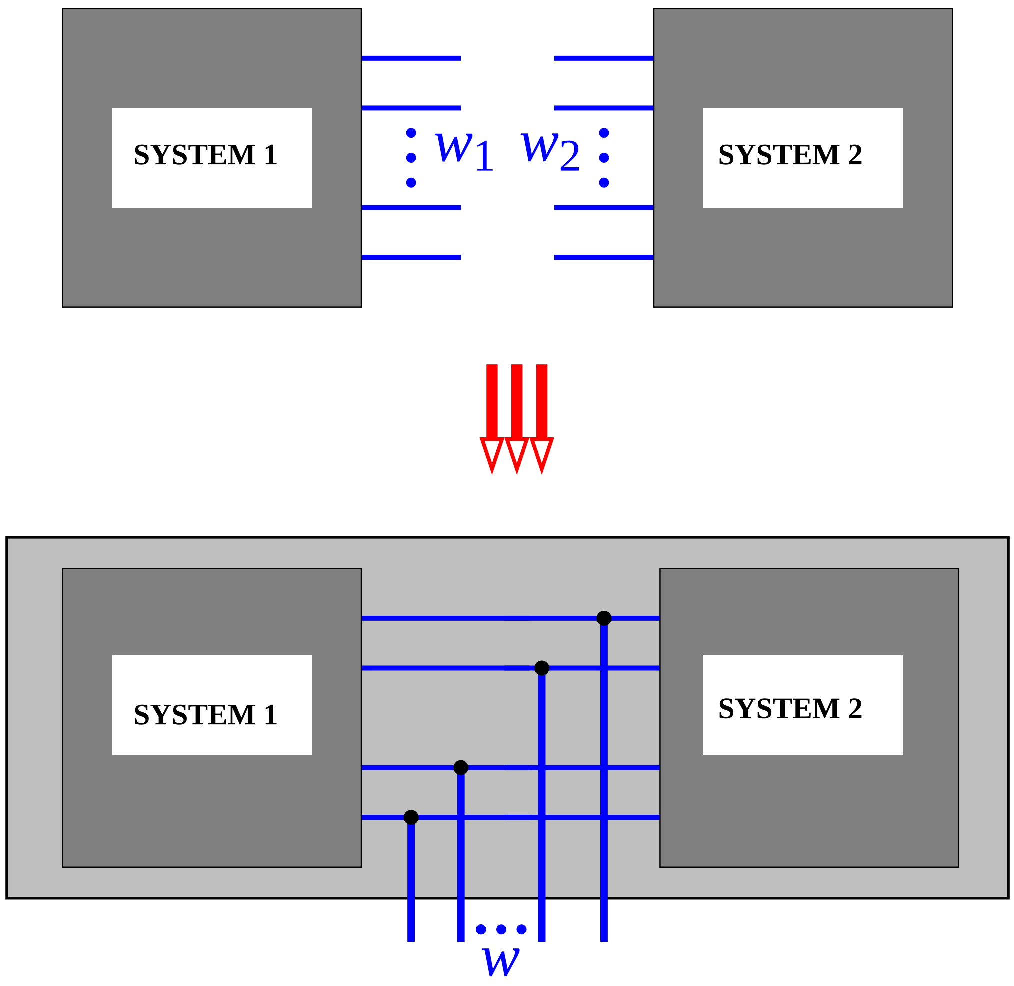


Open

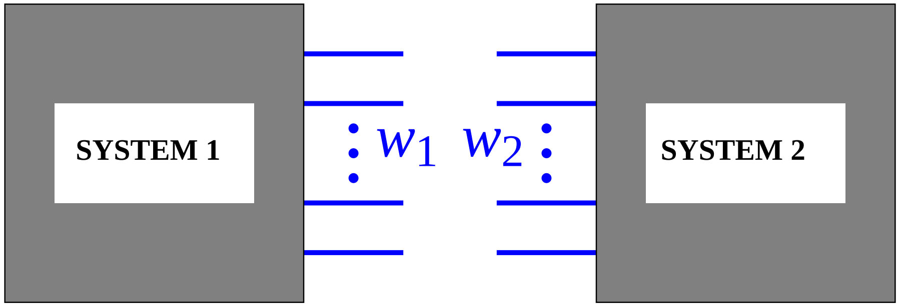


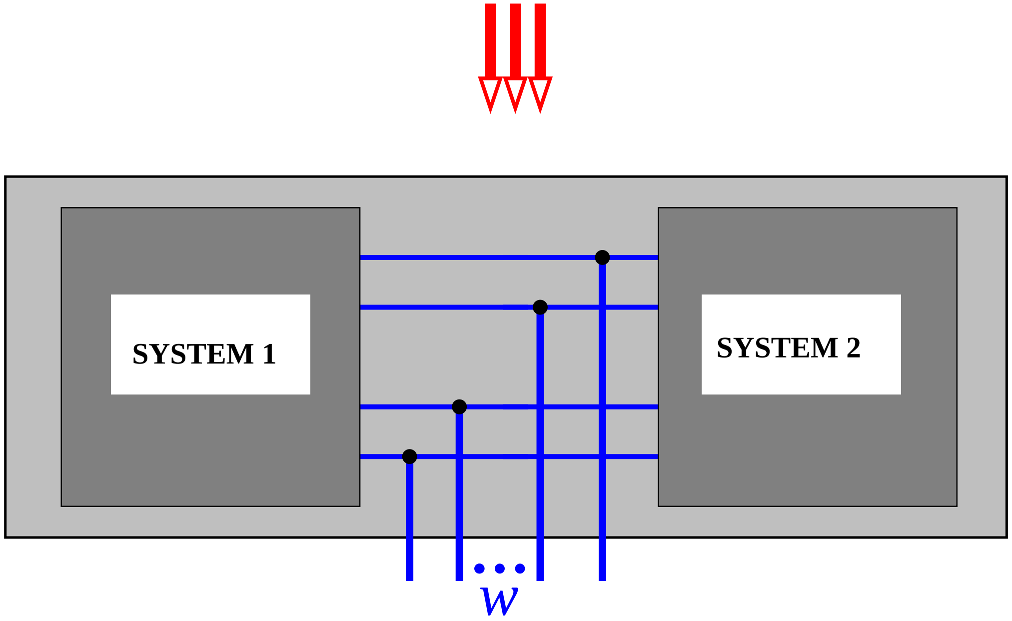
Connectable

Interconnection

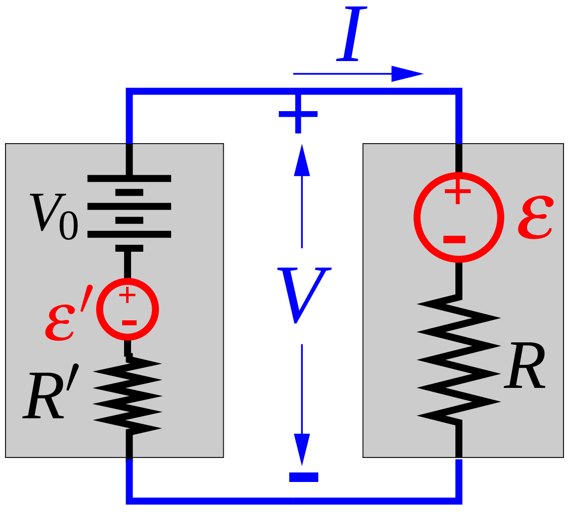


Interconnection

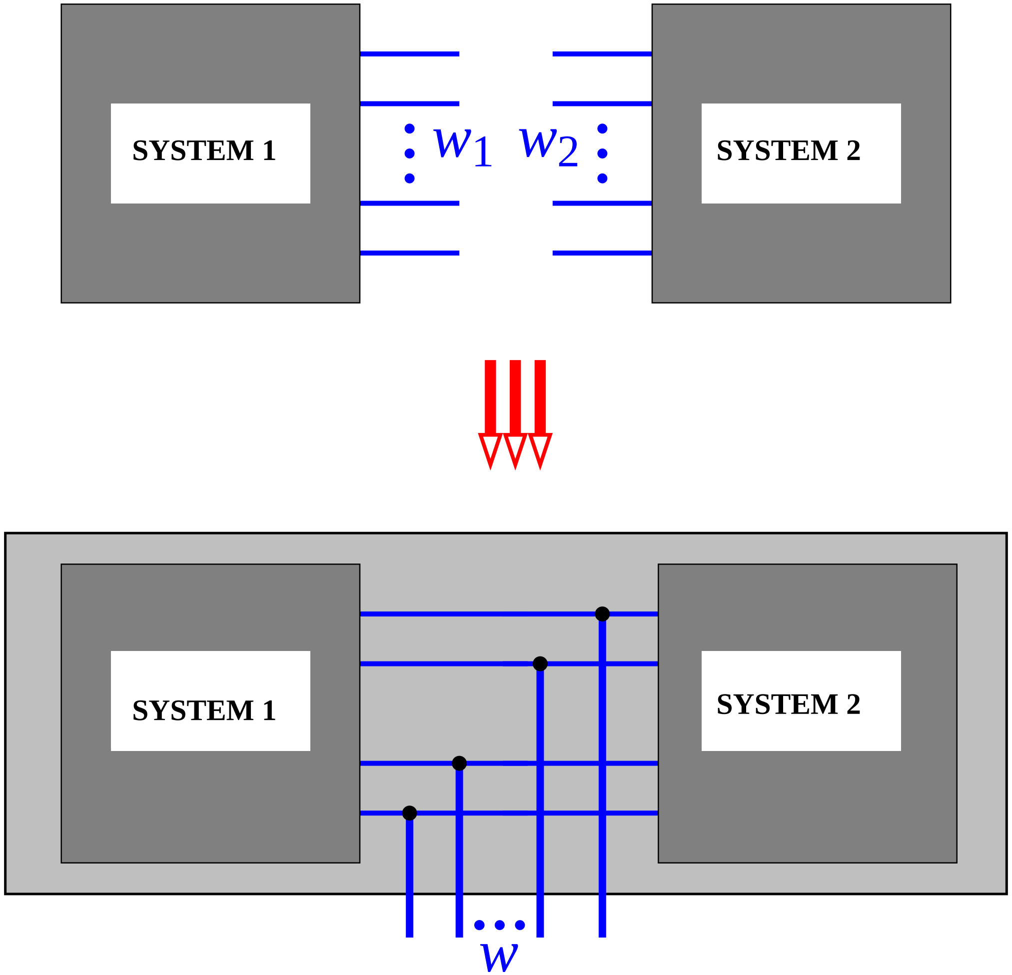




Example:



Interconnection

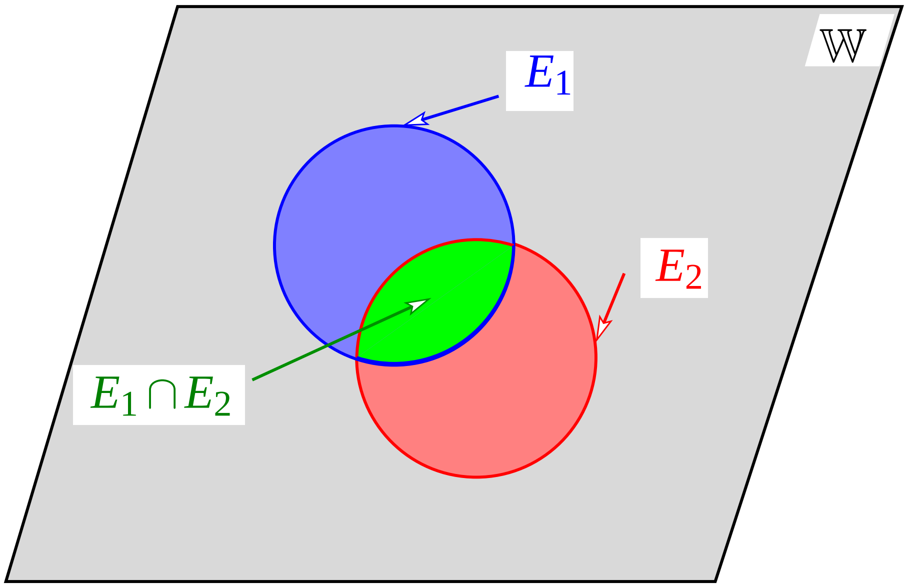


Can two distinct probabilistic laws be imposed on the same set of variables?

Complementarity of σ-algebras

$\mathscr{E}_1$ **and** $\mathscr{E}_2$ **are** ***complementary σ-algebras*** **:**$\Leftrightarrow$
for all nonempty sets $E_1, E_1' \in \mathscr{E}_1, E_2, E_2' \in \mathscr{E}_2$

$$[\![E_1 \cap E_2 = E_1' \cap E_2']\!] \Rightarrow [\![E_1 = E_1' \text{ and } E_2 = E_2']\!].$$



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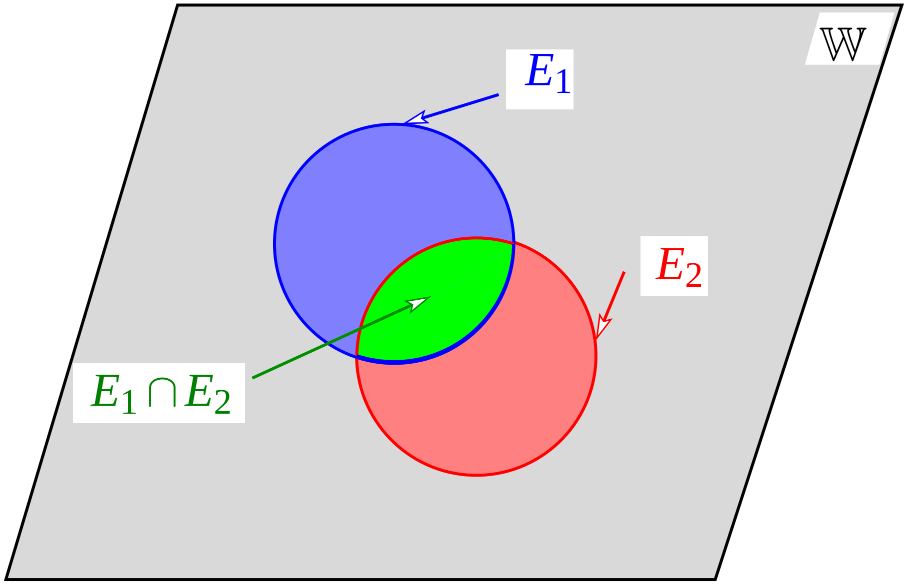
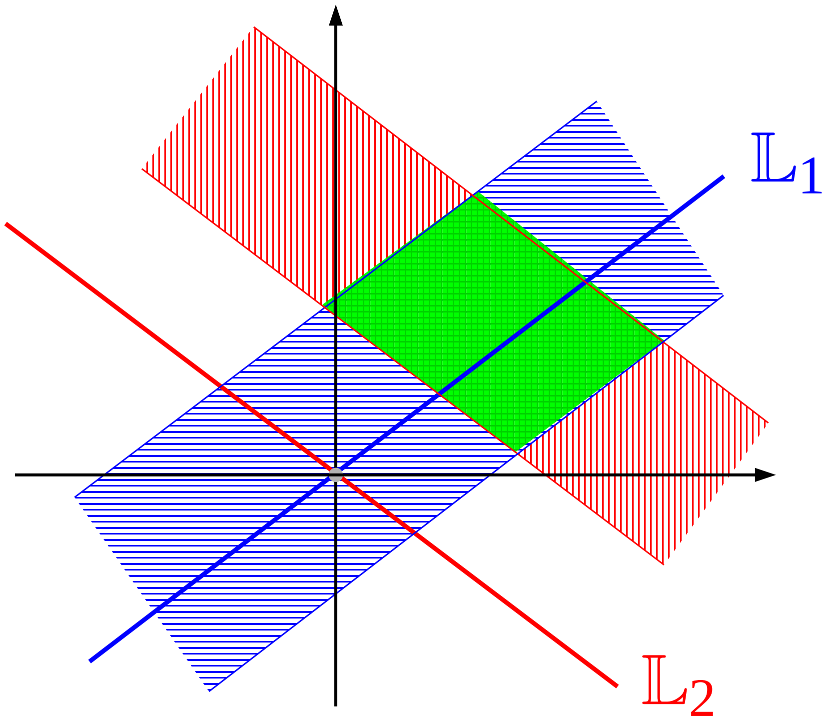


The intersection determines the intersectants.

Linear example



$$\mathbb{L}_1 + \mathbb{L}_2 = \mathbb{R}^{\mathtt{n}}$$

Interconnection of complementary systems

Let $(\mathbb{W}, \mathscr{E}_1, P_1)$ **and** $(\mathbb{W}, \mathscr{E}_2, P_2)$ **be stochastic systems (independent). Assume complementarity.**

Their ***interconnection*** **is defined as**

$$(\mathbb{W}, \mathscr{E}, P)$$

with $\mathscr{E} :=$ **the** σ**-algebra generated by 'rectangles'**

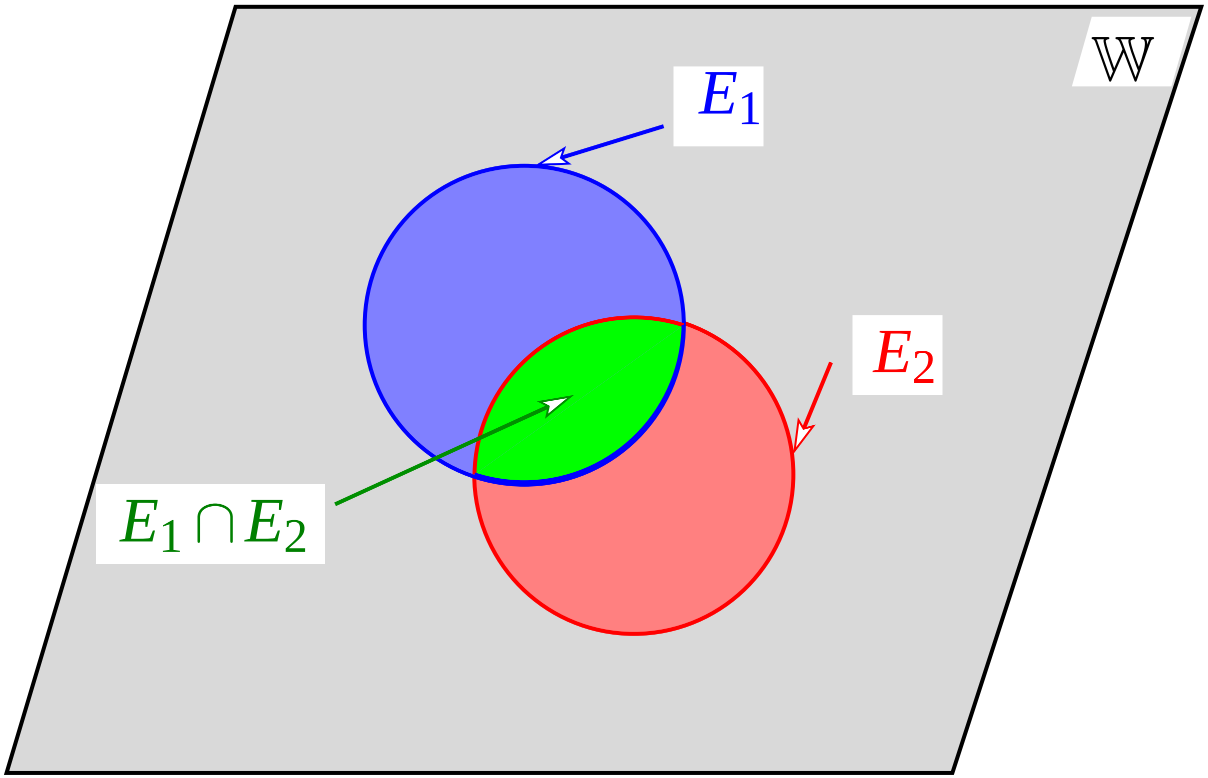
$$\{E_1 \cap E_2 \mid E_1 \in \mathscr{E}_1, E_2 \in \mathscr{E}_2\},$$

and P **defined through the rectangles by**

$$P(E_1 \cap E_2) := P_1(E_1) P_2(E_2).$$

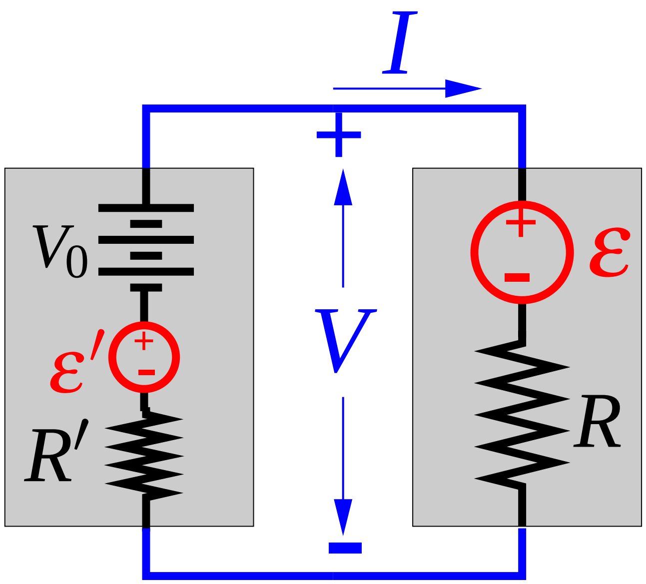
for $E_1 \in \mathscr{E}_1, E_2 \in \mathscr{E}_2$**.**

Interconnection of complementary systems



$$P(E_1 \cap E_2) := P_1(E_1)P_2(E_2).$$

Noisy resistor terminated by a voltage source



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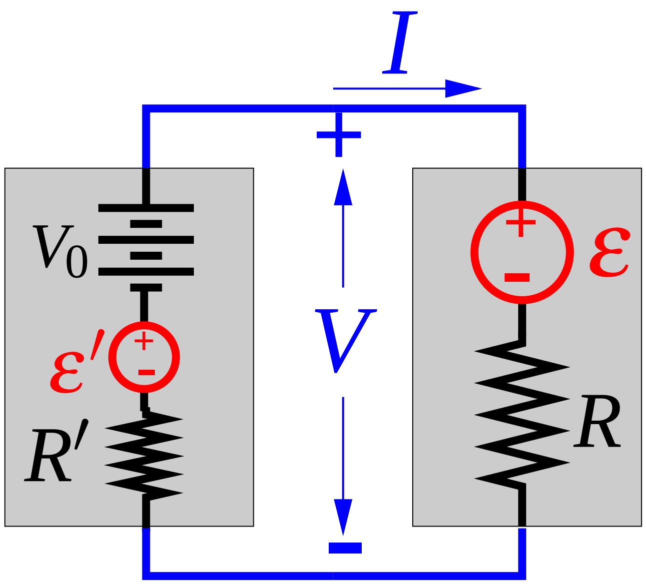
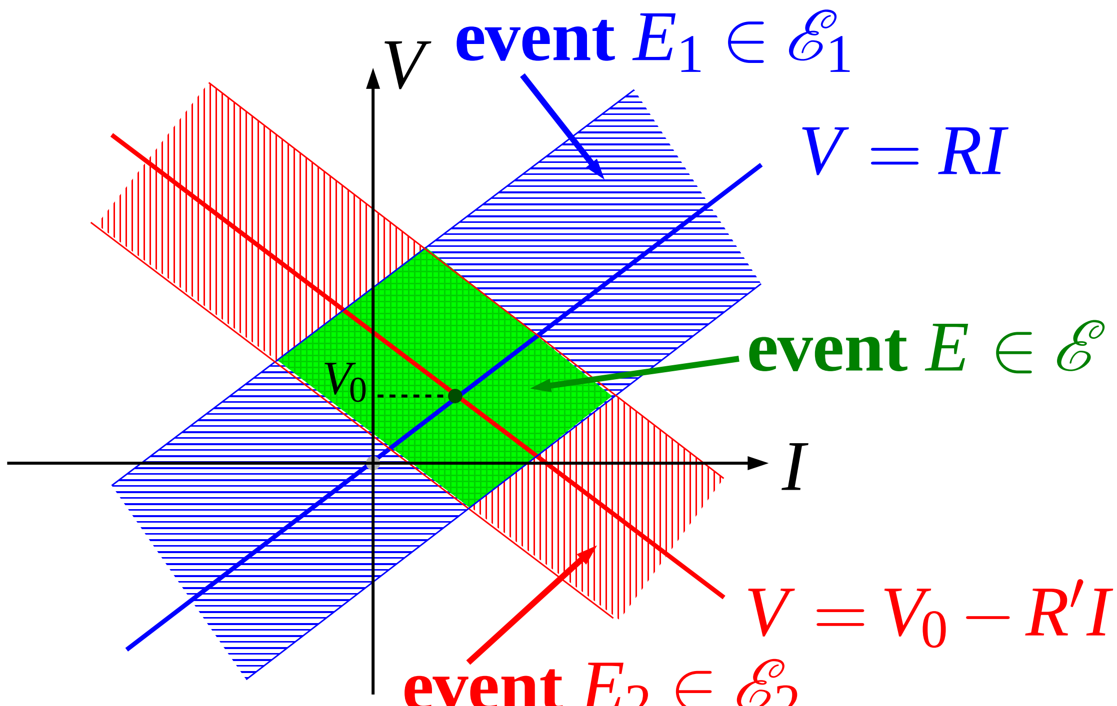
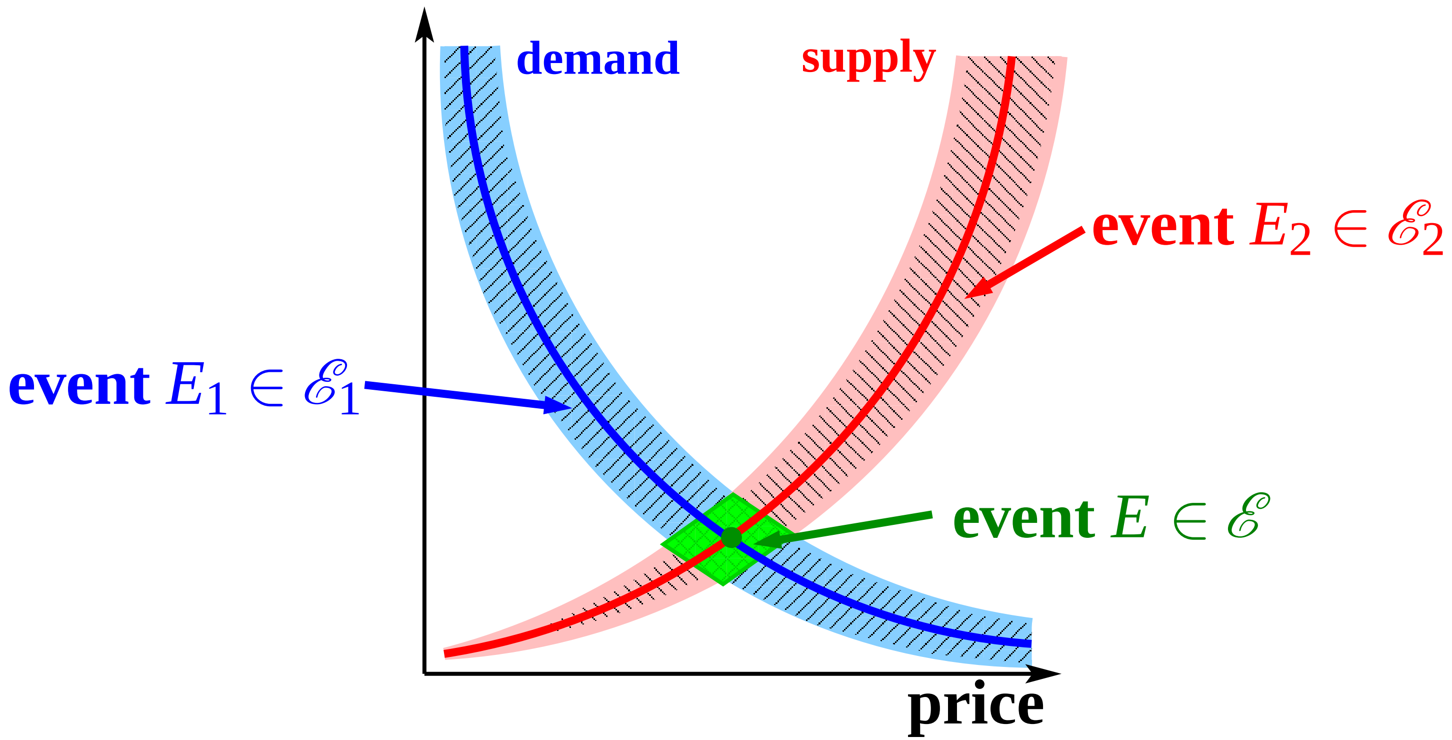


$$P(E) = P_1(E_1)P_2(E_2)$$



Equilibrium price/demand/supply



$$P(E) = P_1(E_1)P_2(E_2).$$

Open stochastic systems

Open versus closed

Consider $\Sigma_1 = (\mathbb{R}^{\mathtt{n}}, \mathscr{E}_1, P_1)$**.**

If $\mathscr{E}_1$ **= the Borel** σ**-algebra, and** $\mathtt{support}(P_1) = \mathbb{R}^{\mathtt{n}}$**, then** Σ_1 **is interconnectable only with the free system** $(\mathbb{R}^{\mathtt{n}}, \mathscr{E}_2, P_2)$**,** $\mathscr{E}_2 = \{\emptyset, \mathbb{R}^{\mathtt{n}}\}$**.**

$\Rightarrow$ **classical** Σ_1 **= 'closed' system.**

Open versus closed

Consider $\Sigma_1 = (\mathbb{R}^{\mathtt{n}}, \mathscr{E}_1, P_1)$**.**

If $\mathscr{E}_1$ **= the Borel** σ**-algebra, and** $\mathtt{support}(P_1) = \mathbb{R}^{\mathtt{n}}$**, then** Σ_1 **is interconnectable only with the free system** $(\mathbb{R}^{\mathtt{n}}, \mathscr{E}_2, P_2)$**,** $\mathscr{E}_2 = \{\emptyset, \mathbb{R}^{\mathtt{n}}\}$**.**

$\Rightarrow$ **classical** Σ_1 **= 'closed' system.**

Coarse $\mathscr{E}_1$

$\Rightarrow$ Σ_1 **is interconnectable.**

$\Rightarrow$ **'open' system.**

Open versus closed

In the Kolmogorov philosophy, random variables, random vectors, and random processes are (measurable) functions defined on the probability space $(\Omega, \mathscr{A}, P)$.
We view the randomness as 'internal' to the system.

So, once the Gods choose $\omega \in \Omega$, all the random variables are determined.
The environment has no influence on the outcomes.

$\Rightarrow$ **'closed' systems.**

Conditional and constrained probability

$$w \in \mathbb{S}$$

Conditional probability

Given $\Sigma = (\mathbb{W}, \mathscr{E}, P)$**.**

Look at outcomes $\boxed{w \in \mathbb{S}}$ **with** $\mathbb{S} \subseteq \mathbb{W}$**.**

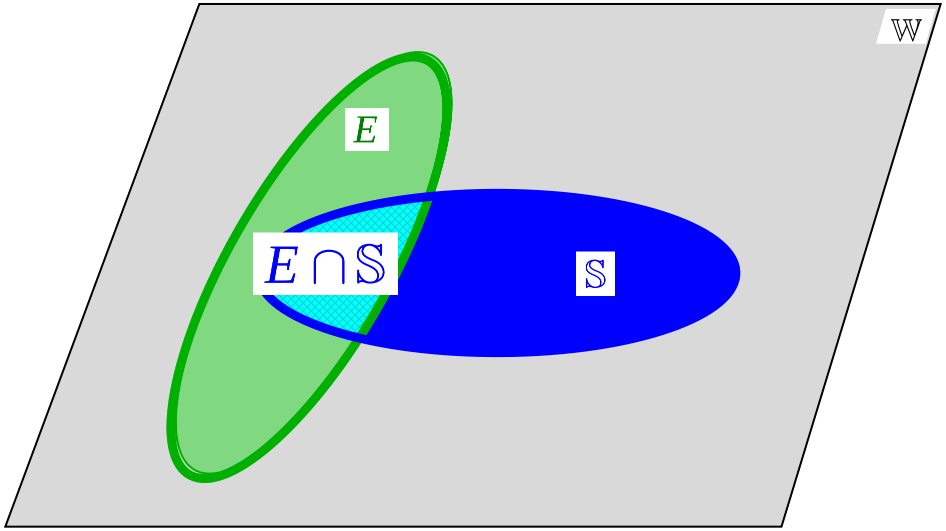
For $\mathbb{S}$ **an event,** $\mathbb{S} \in \mathscr{E}$, $\rightsquigarrow$ **conditional probability.**

Assume $P(S) > 0$**. Then**

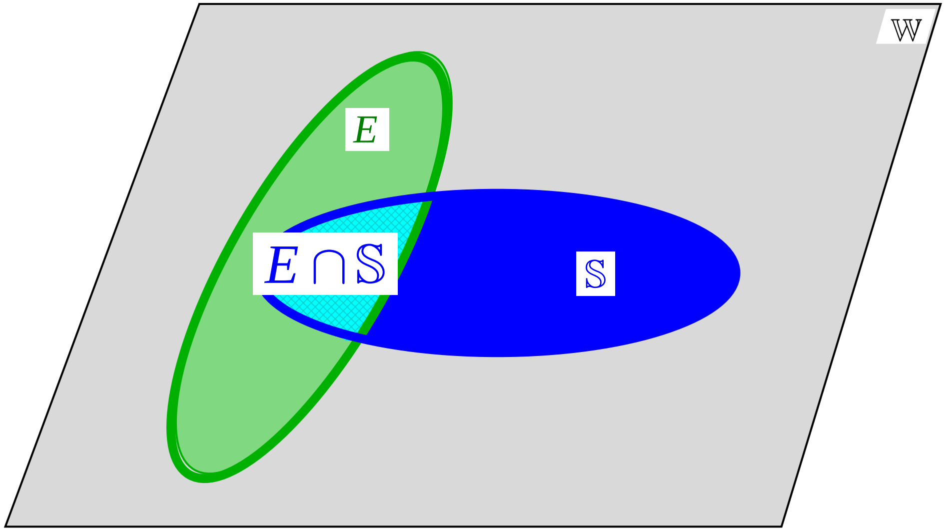
$$\Sigma_{|\mathbb{S}} := \left(\mathbb{S}, \mathscr{E} \cap \mathbb{S}, P_{|\mathbb{S}}\right), \textbf{ with } P_{|\mathbb{S}}(E \cap \mathbb{S}) := \frac{P(E \cap \mathbb{S})}{P(\mathbb{S})}.$$

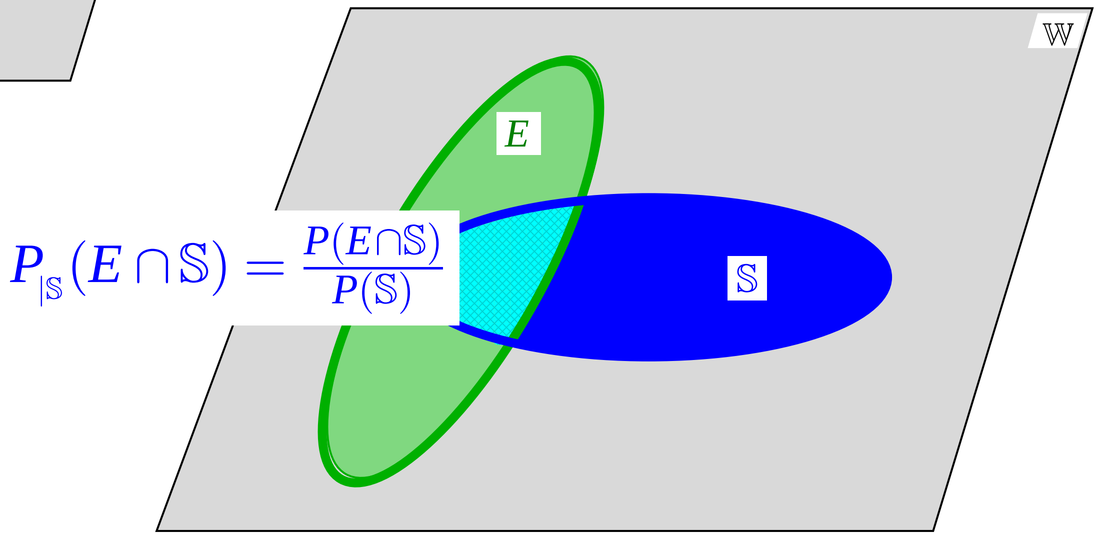
The construction of $P_{|\mathbb{S}}$ **is more complicated when** $P(\mathbb{S}) = 0$, **but well-known.**

Conditional probability



Conditional probability





Constrained probability

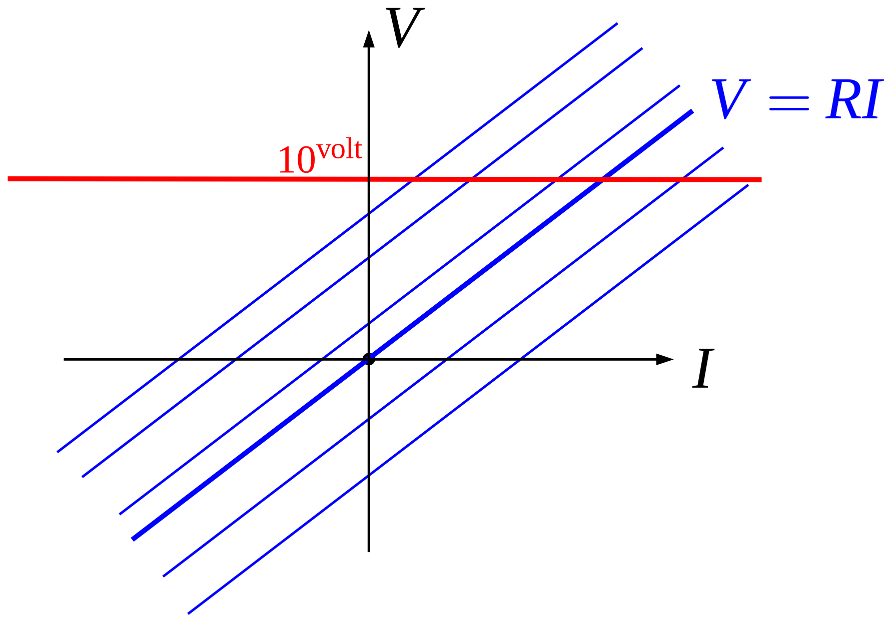
Let $\Sigma = (\mathbb{W}, \mathscr{E}, P)$**.**

Impose the constraint $\boxed{w \in \mathbb{S}}$ **with** $\mathbb{S} \subset \mathbb{W}$, $\boxed{\mathbb{S} \notin \mathscr{E}.}$

What is the stochastic nature of the outcomes in $\mathbb{S}$ ***?***

Is this a meaningful question?

Noisy resistor

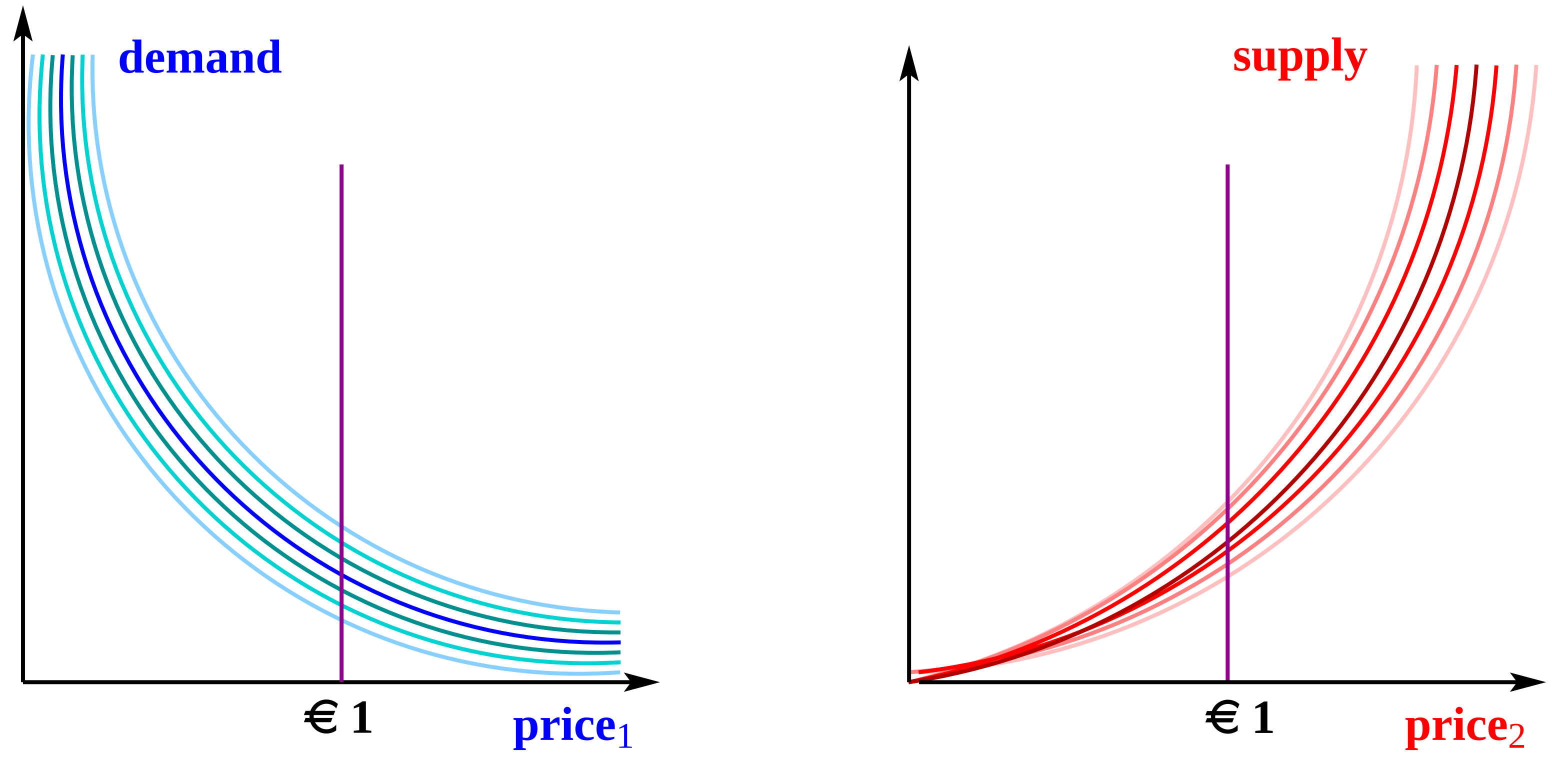


Impose $V = 10^{\text{volt}}$. What is the distribution of I?

$$V = RI + \varepsilon, V = 10^{\text{volt}} \Rightarrow I = \frac{V_0}{10} - \frac{\varepsilon}{10}.$$

I is a well-defined random variable!

Price/demand/supply example



Impose price = € 1. Probability of demand, supply?

Constrained probability

Let $\Sigma = (\mathbb{W}, \mathscr{E}, P)$.

Impose the constraint $\boxed{w \in \mathbb{S}}$ **with** $\mathbb{S} \subset \mathbb{W}$, $\boxed{\mathbb{S} \notin \mathscr{E}.}$

What is the stochastic nature of the outcomes in $\mathbb{S}$?

Is this a meaningful question? ***Yes, it is!***

Constrained probability

Constraining $\simeq$ **interconnection of** $\Sigma = (\mathbb{W}, \mathscr{E}, P)$ **and the deterministic system with behavior** $\mathbb{S}$**.**

Assume complementarity:

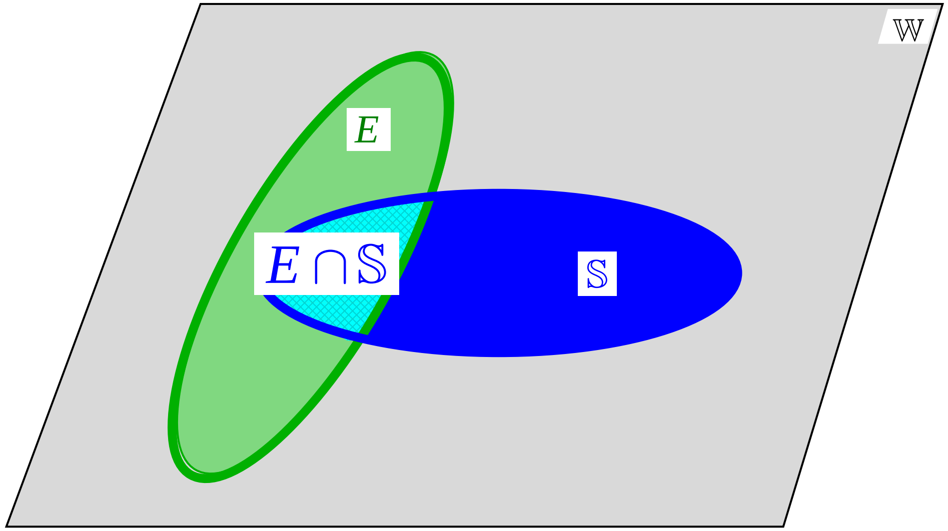
$[\![E_1, E_2 \in \mathscr{E}$ **and** $E_1 \cap \mathbb{S} = E_2 \cap \mathbb{S}]\!] \Rightarrow [\![E_1 = E_2]\!]$

Interconnection $\rightsquigarrow$

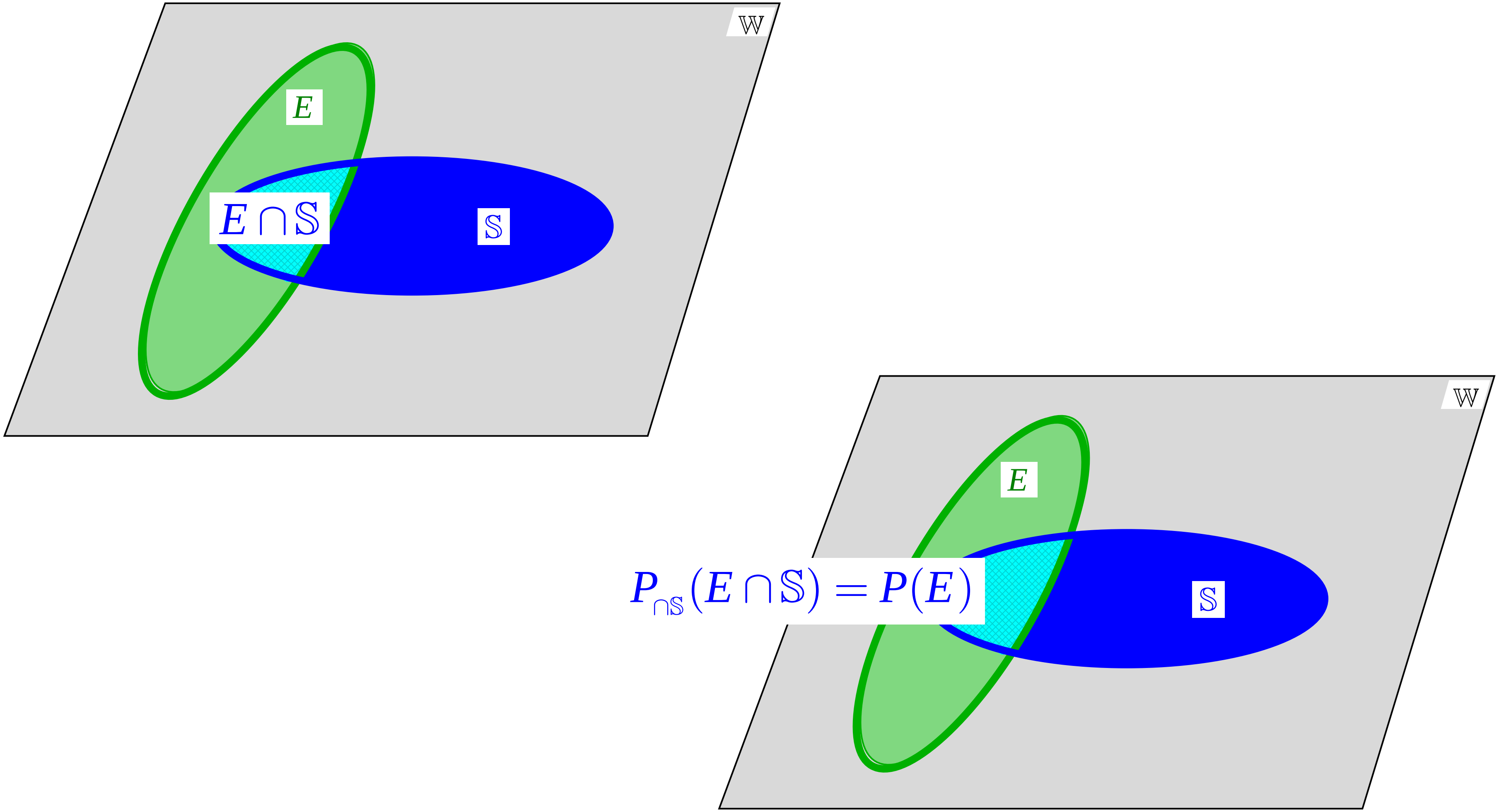
$$\Sigma_{\cap\mathbb{S}} = (\mathbb{S}, \mathscr{E} \cap \mathbb{S}, P_{\cap\mathbb{S}}) \quad \textbf{with} \quad P_{\cap\mathbb{S}}(E \cap \mathbb{S}) := P(E).$$

$P_{\cap\mathbb{S}}$ **= "probability of** w **constrained by** $w \in \mathbb{S}$**".**

Constrained probability



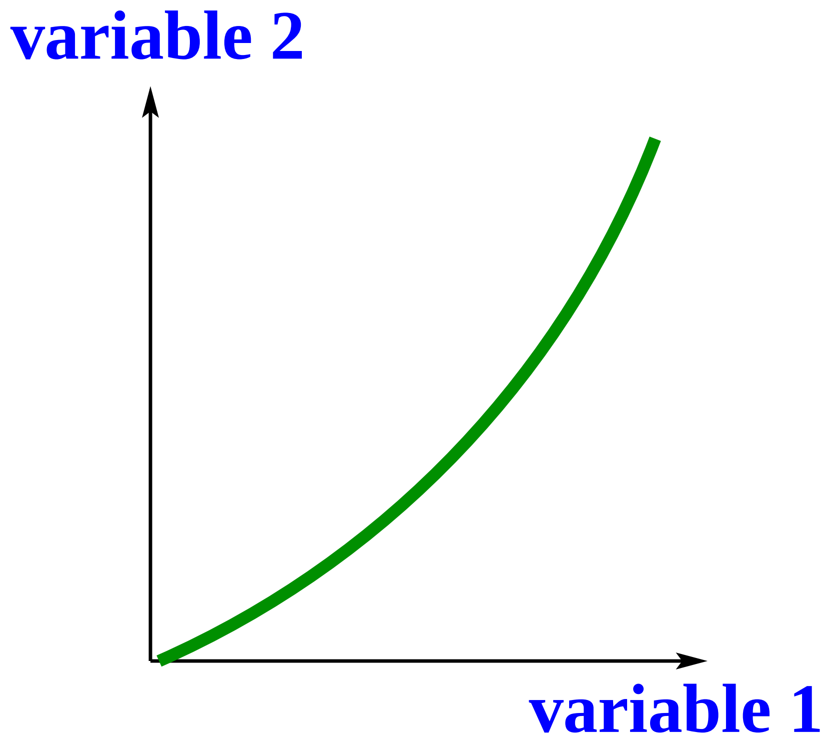
Constrained probability

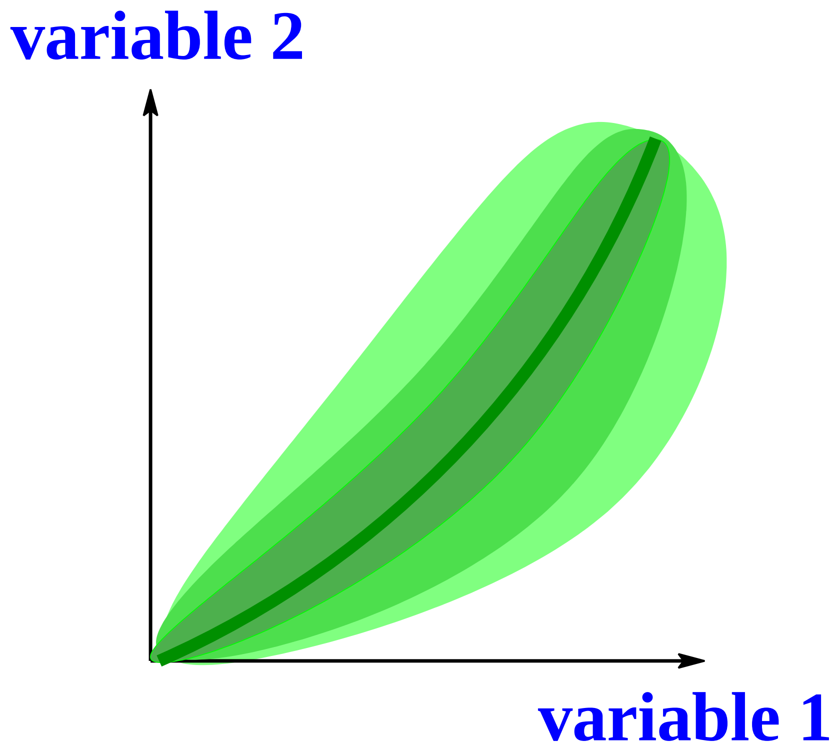


Conclusions

Stochastic systems

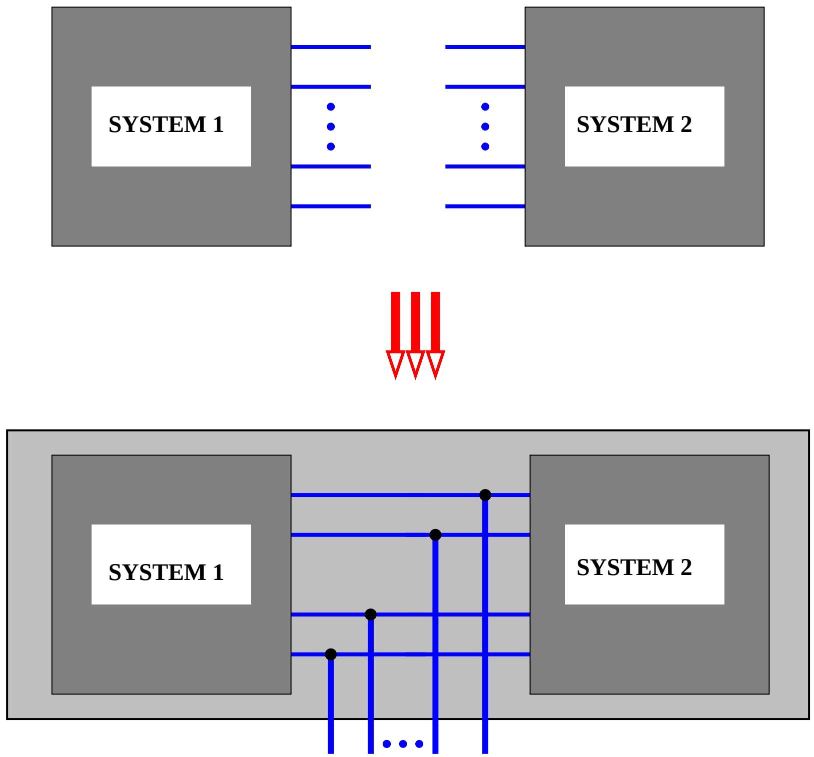
- The Borel σ-algebra is inadequate even for elementary applications.





Stochastic systems

- ► **Complementary stochastic systems can be interconnected: two distinct laws imposed on one set of variables.**



Stochastic systems

- **Open stochastic systems require a coarse σ-algebra.**

  Classical random vectors imply closed systems.

Stochastic systems

- **Open stochastic systems require a coarse σ-algebra.**

  Classical random vectors imply closed systems.

- $\rightsquigarrow$ **notion of 'constrained probability'.**

Future work

Urgent:

Generalization to stochastic processes.

Where to find more

Reference: ***Open stochastic systems*****, IEEE Tr. AC, submitted.**

Copies of the lecture frames available from/at

`http://www.esat.kuleuven.be/∼jwillems`

Professor Siep, het ga je goed!

Thank you

Thank you

Thank you

Thank you

Thank you

Thank you

Thank you

Thank you