OPEN STOCHASTIC SYSTEMS

&

THEIR INTERCONNECTION

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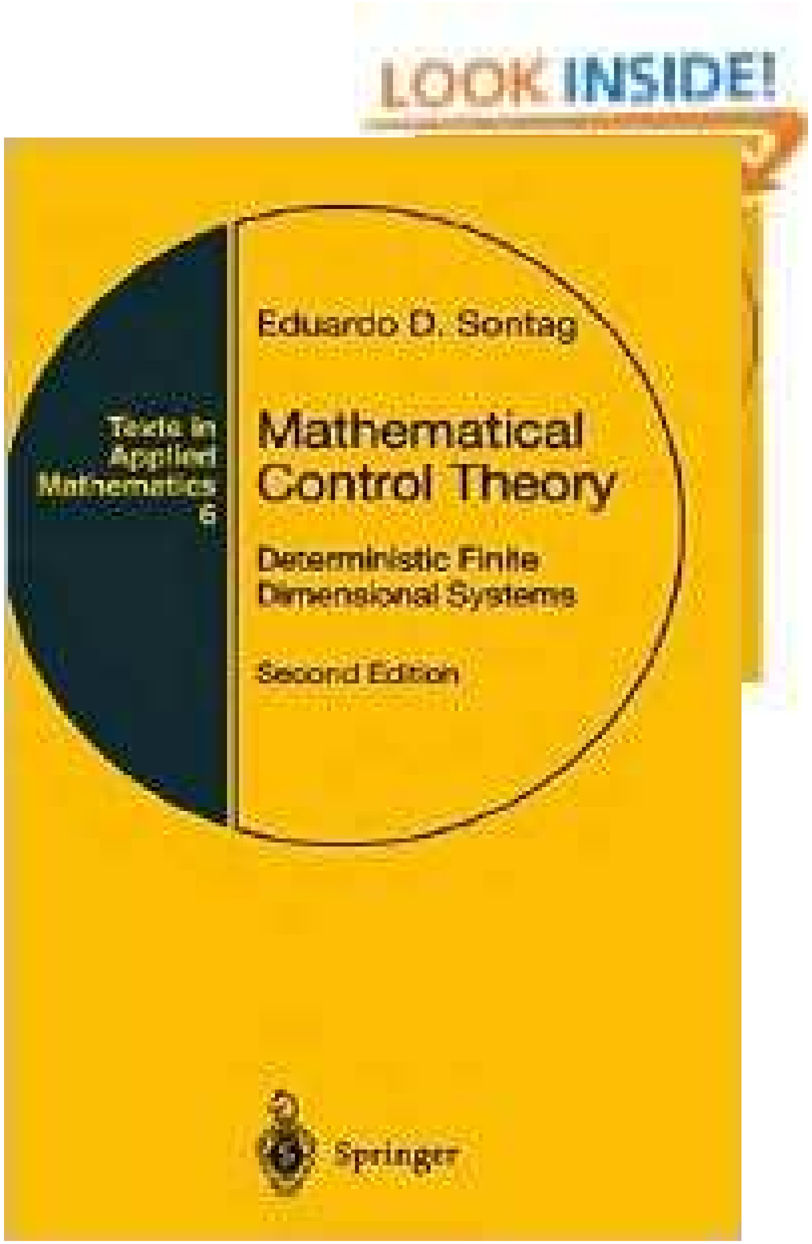
SontagFest **May 23, 2011**

In honor of Eduardo Sontag
on the occasion of his 60-th birthday.

When & where & how we first met


LOOK INSIDE!
Eduardo D. Sontag
Texts in Applied Mathematics 6
Mathematical Control Theory
Deterministic Finite Dimensional Systems
Second Edition
Springer

Stochastic systems

Outline

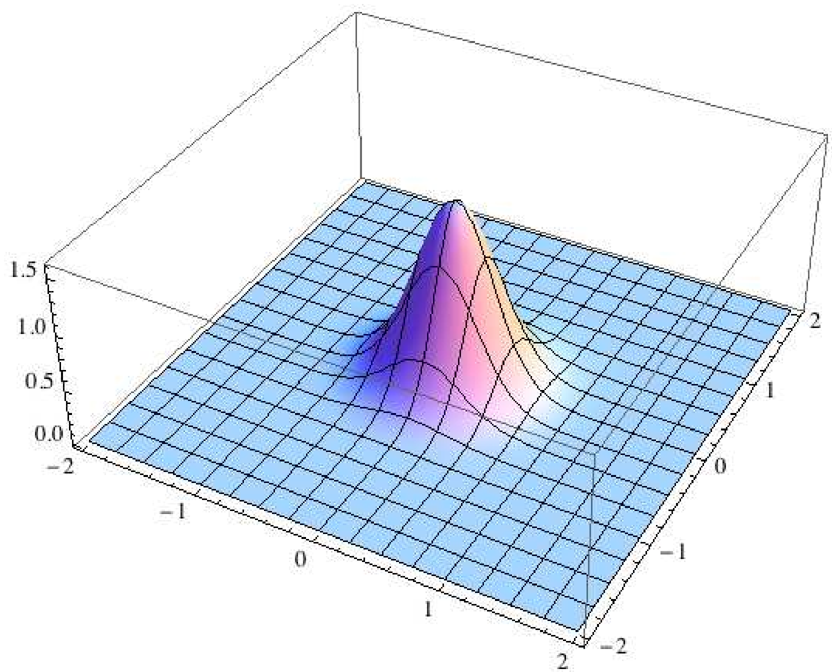
- Motivation
- Definitions
- Interconnection
- [Variable sharing versus input/output]
- [Identification]
- Conclusions

Theme

Model a phenomenon stochastically; outcomes in $\mathbb{R}^n$.

Usual framework:

- **probability distributions, probability density functions;**
- **means that the event σ-algebra consists of the Borel sets.**
  $\rightsquigarrow$ **'Every' subset of $\mathbb{R}^n$ is assigned a probability.**



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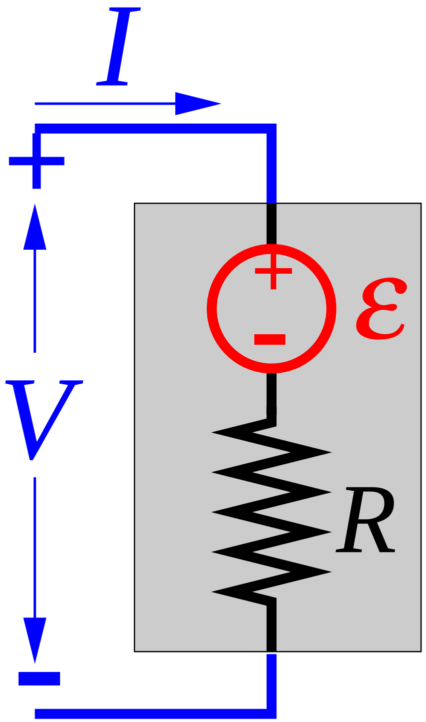
Thesis:

This is unduly restrictive,
even for elementary applications.

Motivating examples

Noisy resistor



$$\boxed{V = RI + \varepsilon}$$

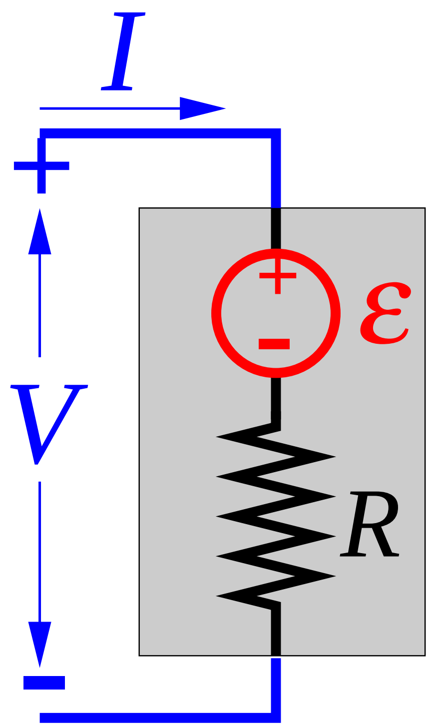
ε gaussian

zero mean

variance $\sigma \sim \sqrt{RT}$

‘Johnson-Nyquist resistor’

Noisy resistor



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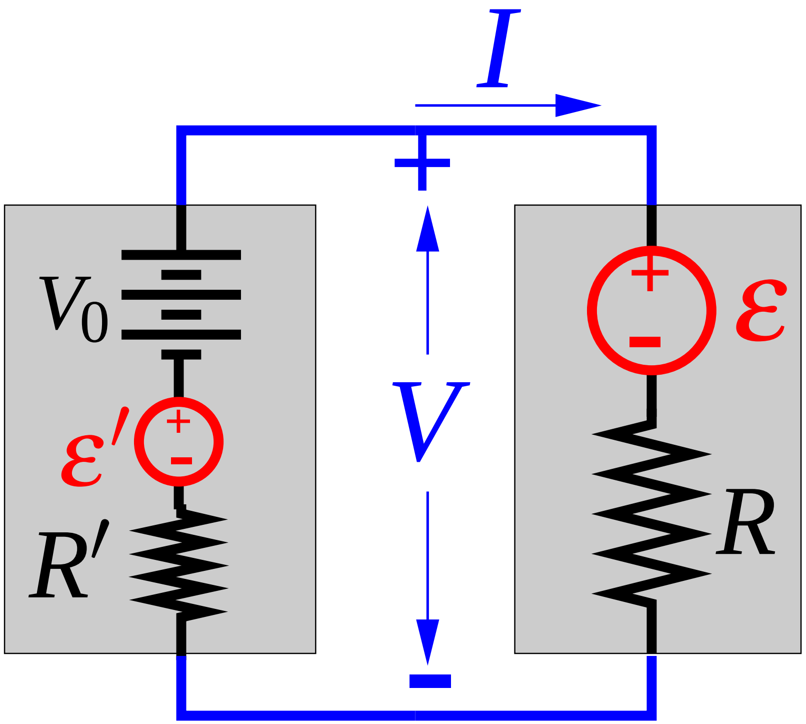
'Johnson-Nyquist resistor'

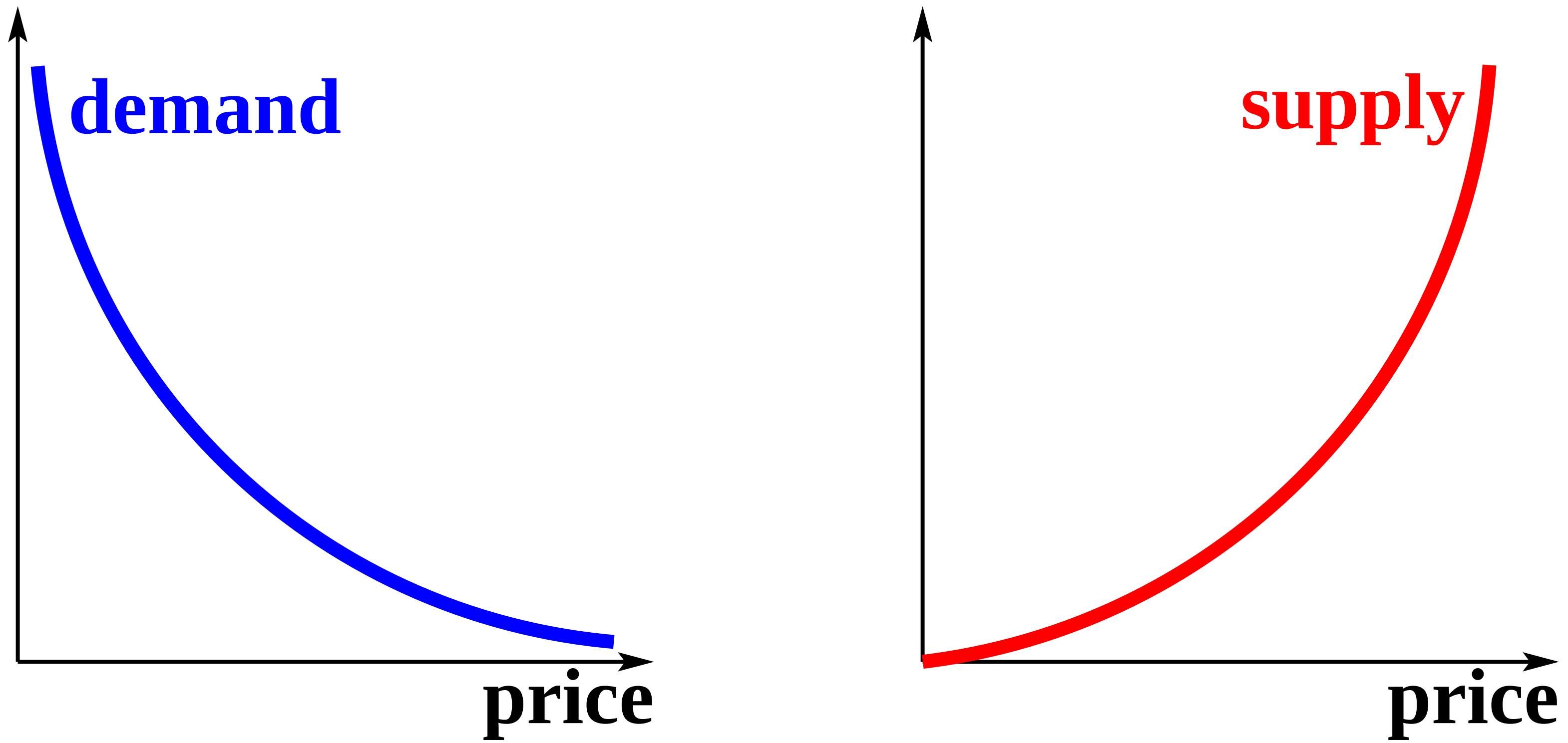
What is $\begin{bmatrix} V \\ I \end{bmatrix}$ as a mathematical object?

Noisy resistor



How do we deal with interconnection?

Deterministic price/demand/supply



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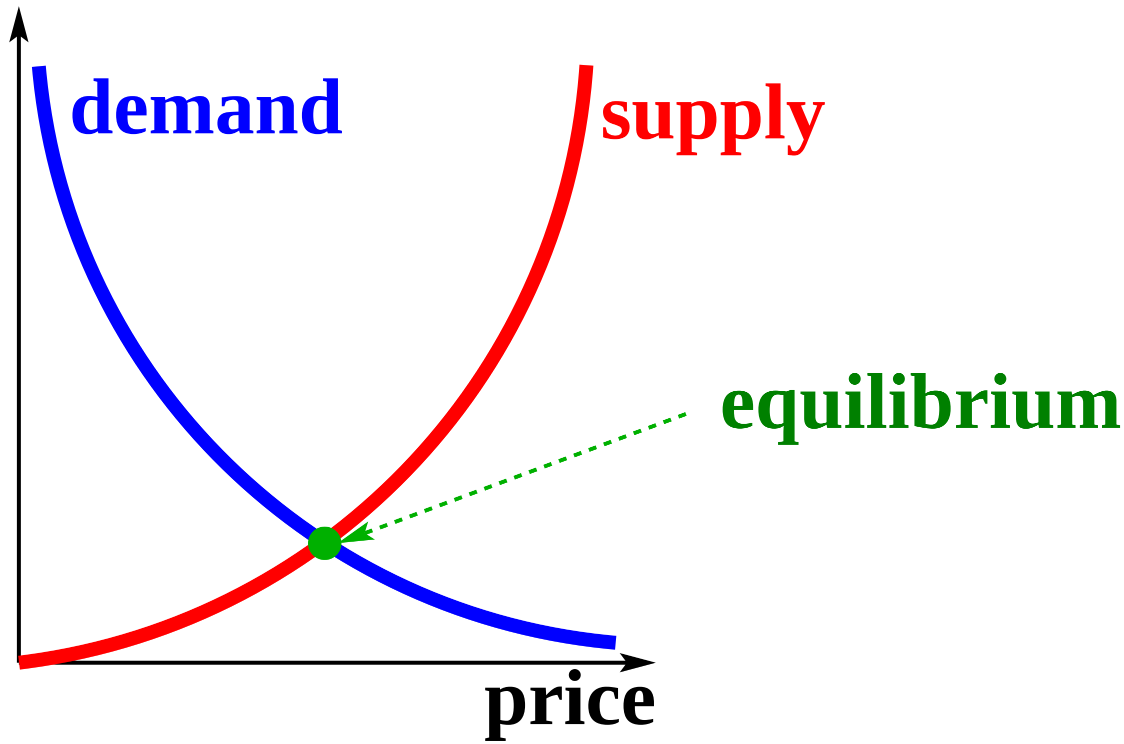
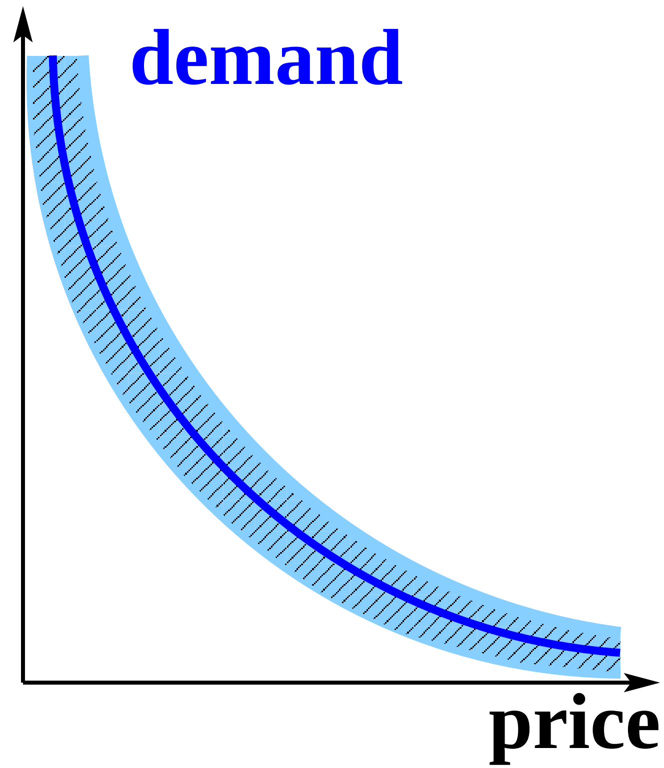
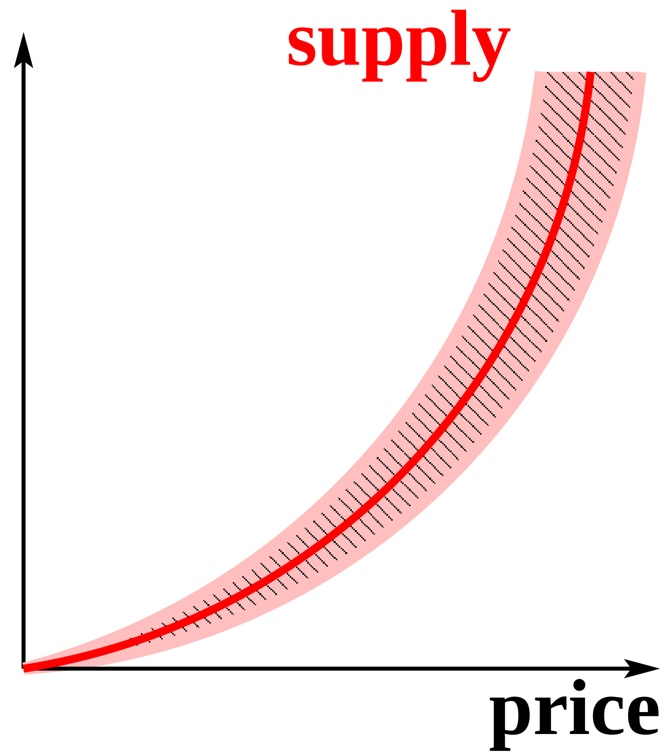


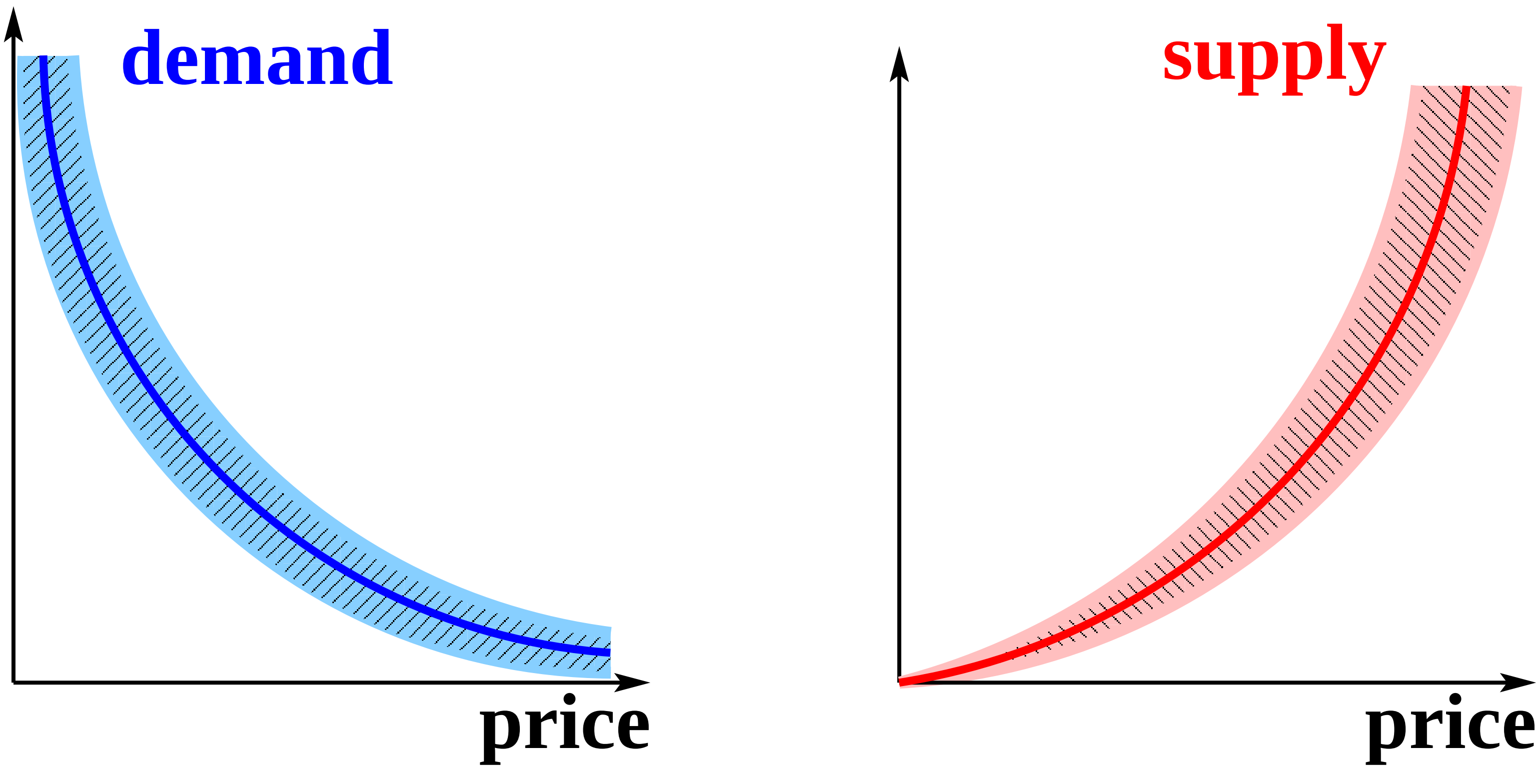
Stochastic price/demand/supply





Only certain regions of the $\begin{bmatrix} \text{price} \\ \text{demand} \end{bmatrix}$ and $\begin{bmatrix} \text{price} \\ \text{supply} \end{bmatrix}$ planes are assigned a probability.

Stochastic price/demand/supply



Only certain regions of the $\begin{bmatrix}\text{price}\\ \text{demand}\end{bmatrix}$ and $\begin{bmatrix}\text{price}\\ \text{supply}\end{bmatrix}$ planes are assigned a probability.

How do we deal with equilibrium supply = demand?

Formal definitions

Definition

A *stochastic system* is a probability triple $\boxed{(\mathbb{W}, \mathscr{E}, P)}$

- $\mathbb{W}$ **a non-empty set, the *outcome space*,**
- $\mathscr{E}$ **a σ-algebra of subsets of** $\mathbb{W}$**: the *events*,**
- $P : \mathscr{E} \to [0,1]$ **a *probability measure*.**

$\mathscr{E}$**: the subsets that are assigned a probability.**

Probability that outcomes $\in E$, $E \in \mathscr{E}$**, is** $P(E)$**.**

Model $\cong$ $\mathscr{E}$ **and** P**;** $\mathscr{E}$ **is an essential part.**

Definition

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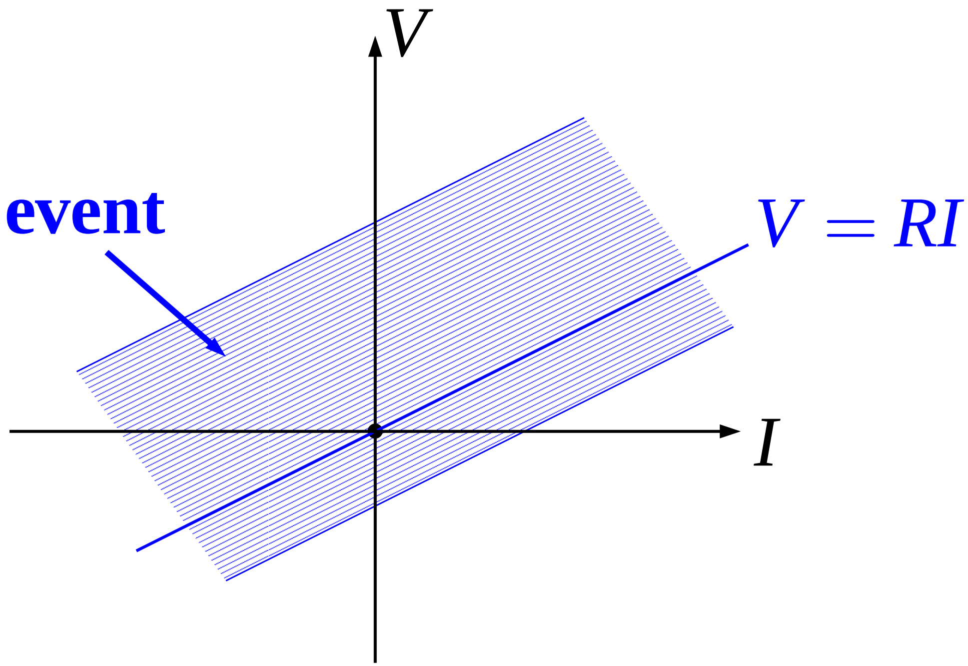
'Classical' stochastic system:

$\mathbb{W} = \mathbb{R}^{\mathtt{n}}$ **and $\mathscr{E}$ = the Borel subsets of $\mathbb{R}^{\mathtt{n}}$.**

$\mathscr{E}$ **is inherited from topology on $\mathbb{R}^{\mathtt{n}}$.**

P **can then be specified by a probability distribution.**

Noisy resistor



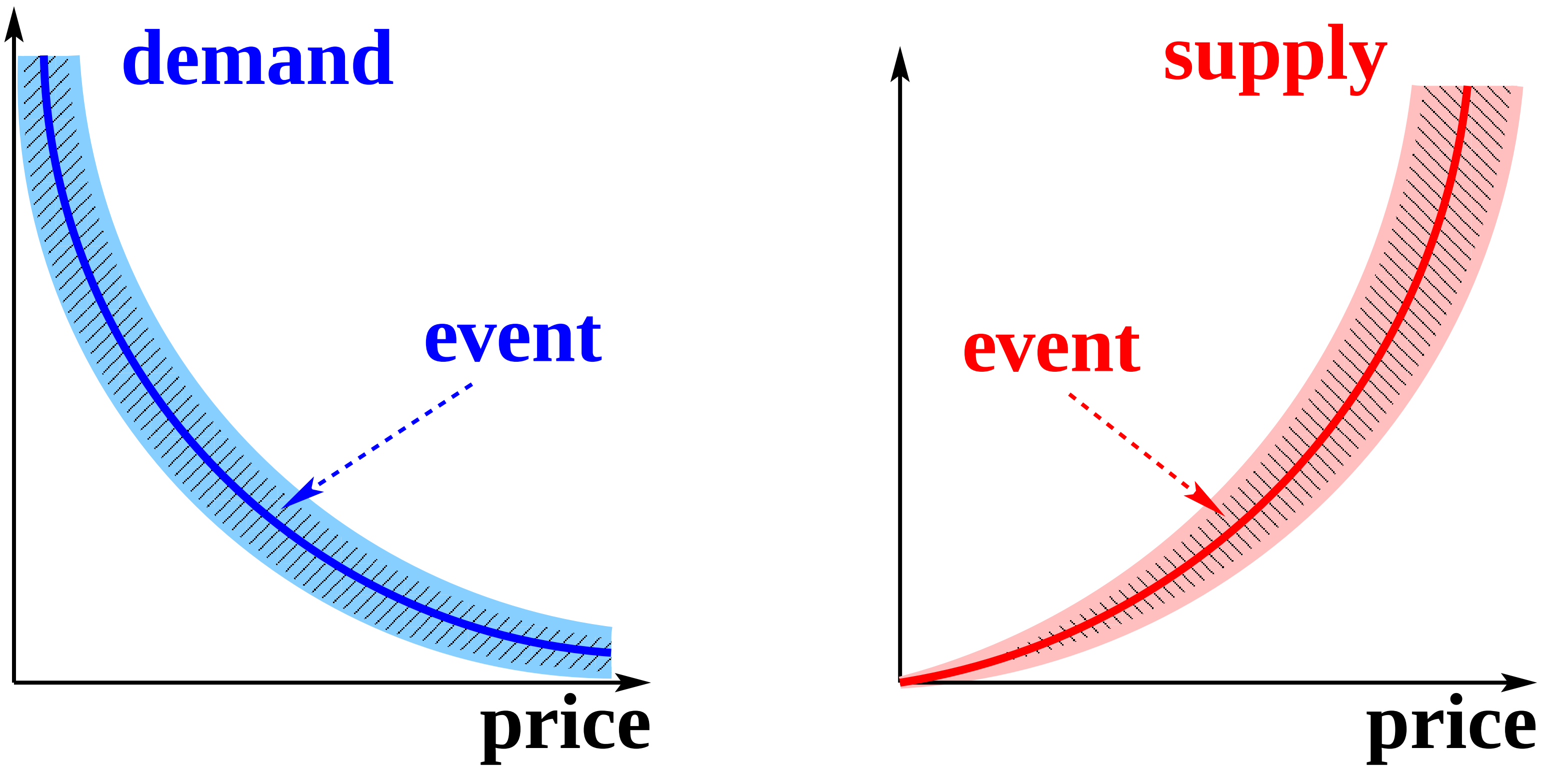
$V = RI + \varepsilon$: **stoch. system,** $\mathbb{W} = \mathbb{R}^2$**, outcomes** $\begin{bmatrix} V \\ I \end{bmatrix}$**.**

Events: $\left\{ \begin{bmatrix} V \\ I \end{bmatrix} \in \mathbb{R}^2 \mid V - RI \in A \text{ **with** } A \text{ **a Borel subset of** } \mathbb{R} \right\}$.

$P(\textbf{event}) =$ **gaussian measure of** A**.**

V **and** I **are not classical real random variables.**

Stochastic price/demand/supply

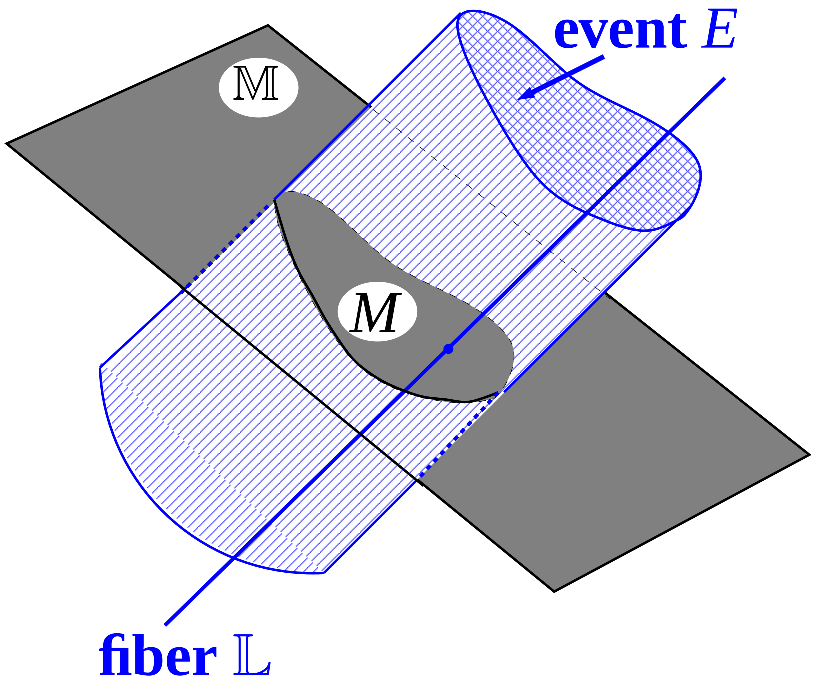


$\mathscr{E}$ = **the regions that are assigned a probability.**

p, d, **and** s **are not classical real random variables.**

Linearity

linear **:⇔ Borel probability on $\mathbb{R}^{\mathtt{n}}/\mathbb{L}$, $\mathbb{L}$ linear, 'fiber'.**



Borel probability on $\mathbb{M} \cong \mathbb{R}^{\mathtt{n}}/\mathbb{L}$.

gaussian **:⇔ linear, Borel probability gaussian.**

Classical ⇒ linear.

Deterministic

$(\mathbb{W}, \mathscr{E}, P)$ **is said to be *deterministic* if**

$$\mathscr{E} = \{\emptyset, \mathbb{B}, \mathbb{B}^{complement}, \mathbb{W}\} \text{ and } P(\mathbb{B}) = 1.$$

If $\mathbb{B} = \mathbb{W}$, the variables are *free*.

noisy resistor: linear, gaussian, fiber $V = RI$.

$w = V - RI$ **is a classical random variable.**

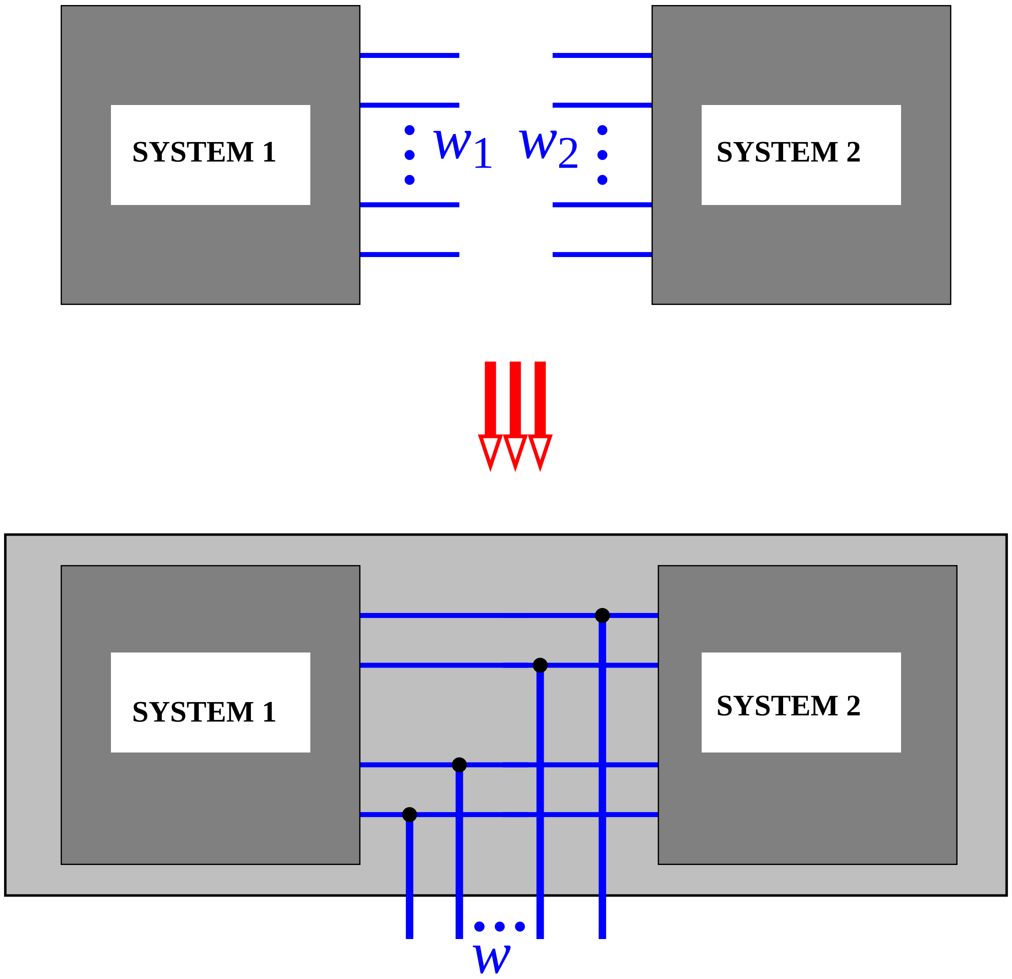
V **and** I **are free.**

Only statements $P(\{V \in \mathbb{R}\}) = 1$, $P(\{I \in \mathbb{R}\}) = 1$.

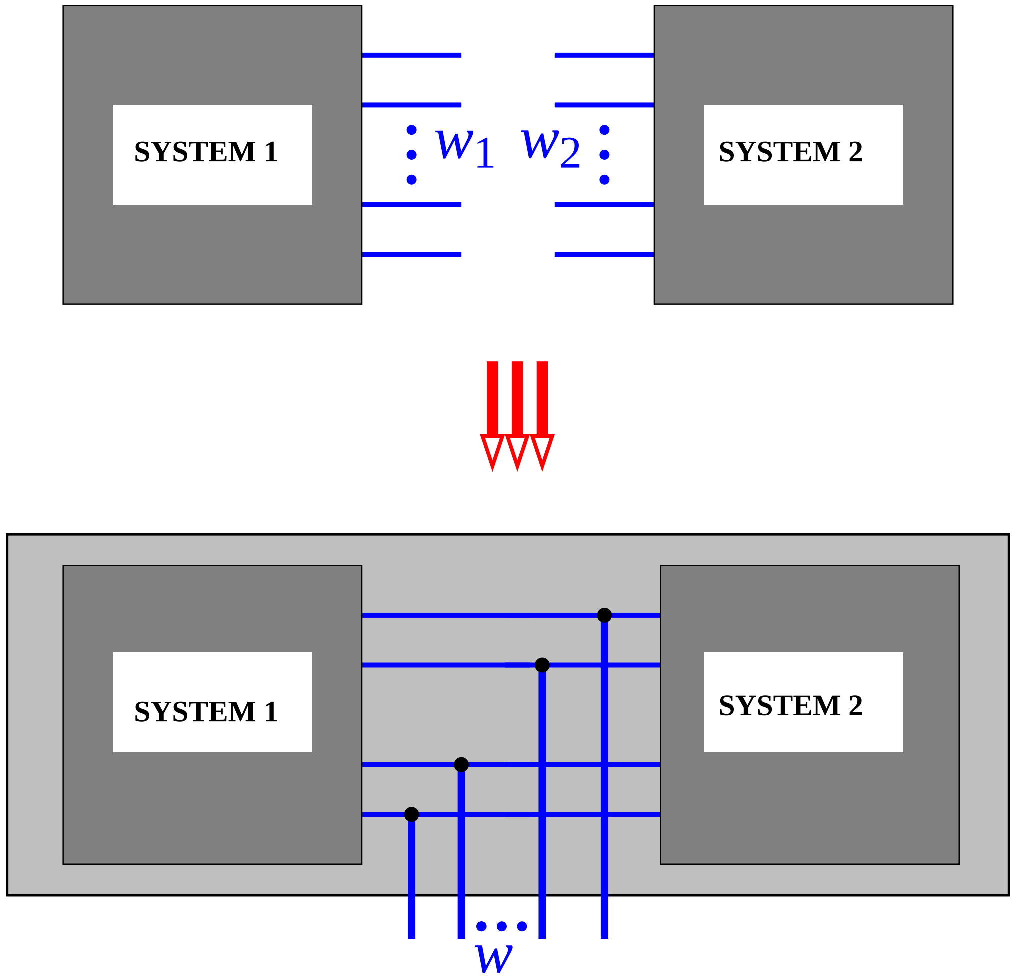
$\begin{bmatrix} V \\ I \end{bmatrix}$ **no pdf, no cumulative, no conditional distr'ions.**

Interconnection

Interconnection



Interconnection

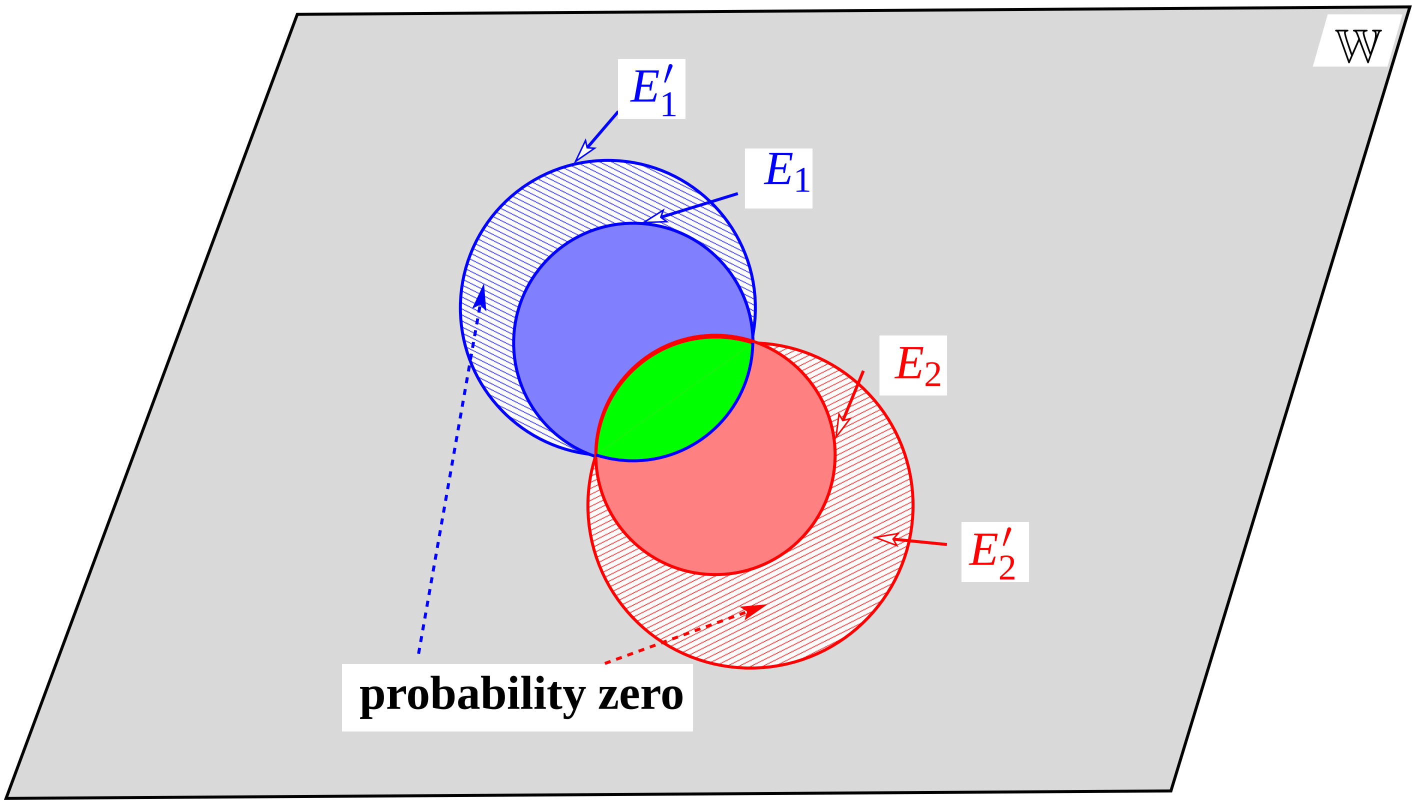


Can we impose two distinct probabilistic laws on the same set of variables?

Complementarity

$\Sigma_1 = (\mathbb{W}, \mathscr{E}_1, P_1)$ **and** $\Sigma_2 = (\mathbb{W}, \mathscr{E}_2, P_2)$ **are said to be** ***complementary*** $:\Leftrightarrow$ **for** $E_1, E_1' \in \mathscr{E}_1$ **and** $E_2, E_2' \in \mathscr{E}_2$**:**

$$\boxed{[\![E_1 \cap E_2 = E_1' \cap E_2']\!] \Rightarrow [\![P_1(E_1)P_2(E_2) = P_1(E_1')P_2(E_2')]\!].}$$



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Implied by $\mathscr{E}_1$ and $\mathscr{E}_2$ are *complementary* :$\Leftrightarrow$ for all nonempty sets $E_1, E_1' \in \mathscr{E}_1, E_2, E_2' \in \mathscr{E}_2$

$$[\![E_1 \cap E_2 = E_1' \cap E_2']\!] \Rightarrow [\![E_1 = E_1' \text{ and } E_2 = E_2']\!].$$

Interconnection of complementary systems

Let $\Sigma_1 = (\mathbb{W}, \mathscr{E}_1, P_1)$ **and** $\Sigma_2 = (\mathbb{W}, \mathscr{E}_2, P_2)$ **be complementary stochastic systems (assumed stochastically independent). Their** ***interconnection*** **is**

$$(\mathbb{W}, \mathscr{E}, P)$$

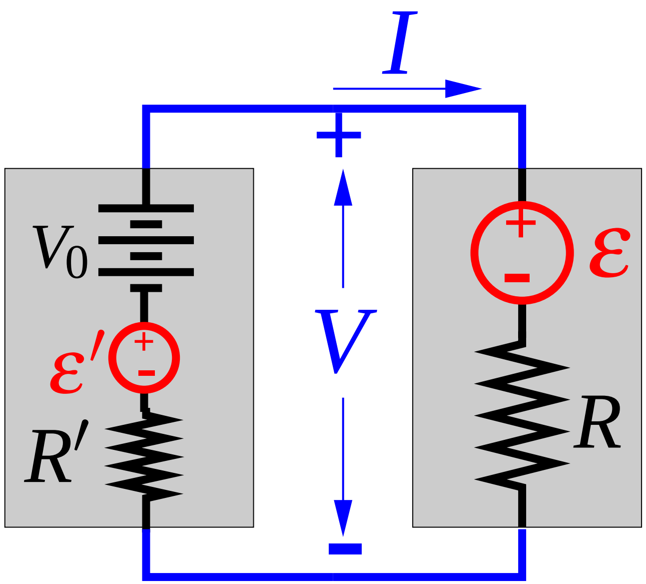
with $\mathscr{E}$:= **the** σ**-algebra generated by the 'rectangles'**

$$\{E_1 \cap E_2 \mid E_1 \in \mathscr{E}_1, E_2 \in \mathscr{E}_2\},$$

and P **defined through the rectangles by**

$$\boxed{P(E_1 \cap E_2) := P_1(E_1)P_2(E_2).}$$

Noisy resistor terminated by voltage source



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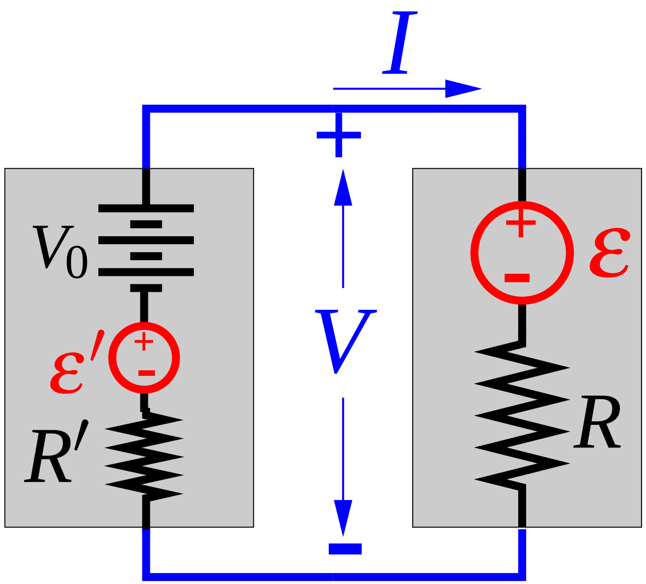
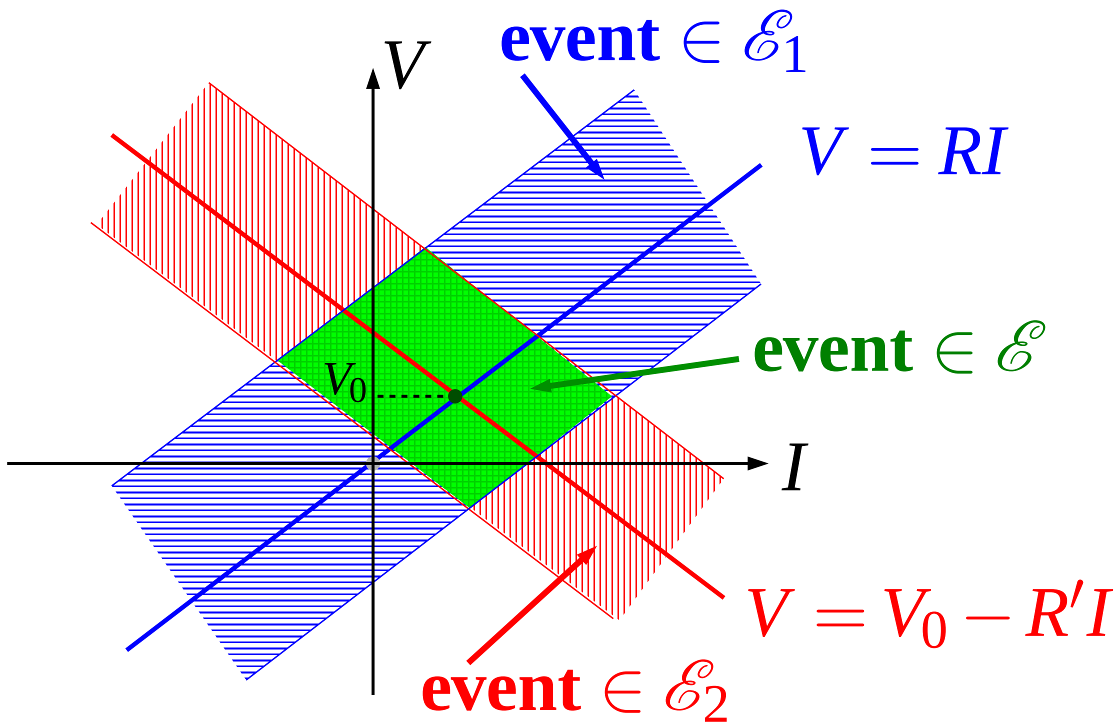
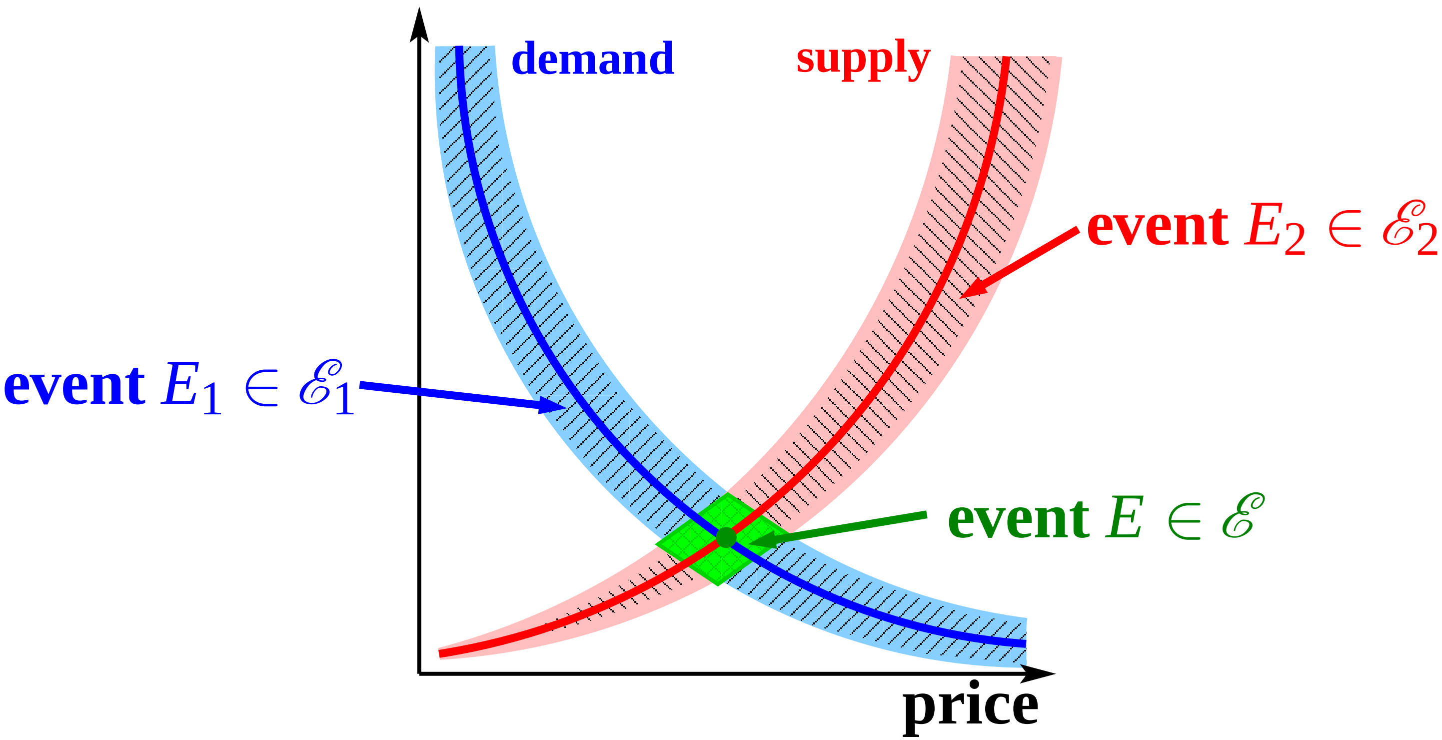


$$P(E) = P_1(E_1)P_2(E_2)$$



Equilibrium price/demand/supply



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Open versus closed

$$\Sigma_1 = (\mathbb{R}^{\mathtt{n}}, \mathscr{E}_1, P_1).$$

Parsimonious $\mathscr{E}_1 \Rightarrow \Sigma_1$ **is interconnectable.**

$\Rightarrow$ **'open' system.**

Open versus closed

$\Sigma_1 = (\mathbb{R}^{\mathtt{n}}, \mathscr{E}_1, P_1)$.

Parsimonious $\mathscr{E}_1 \Rightarrow \Sigma_1$ is interconnectable.

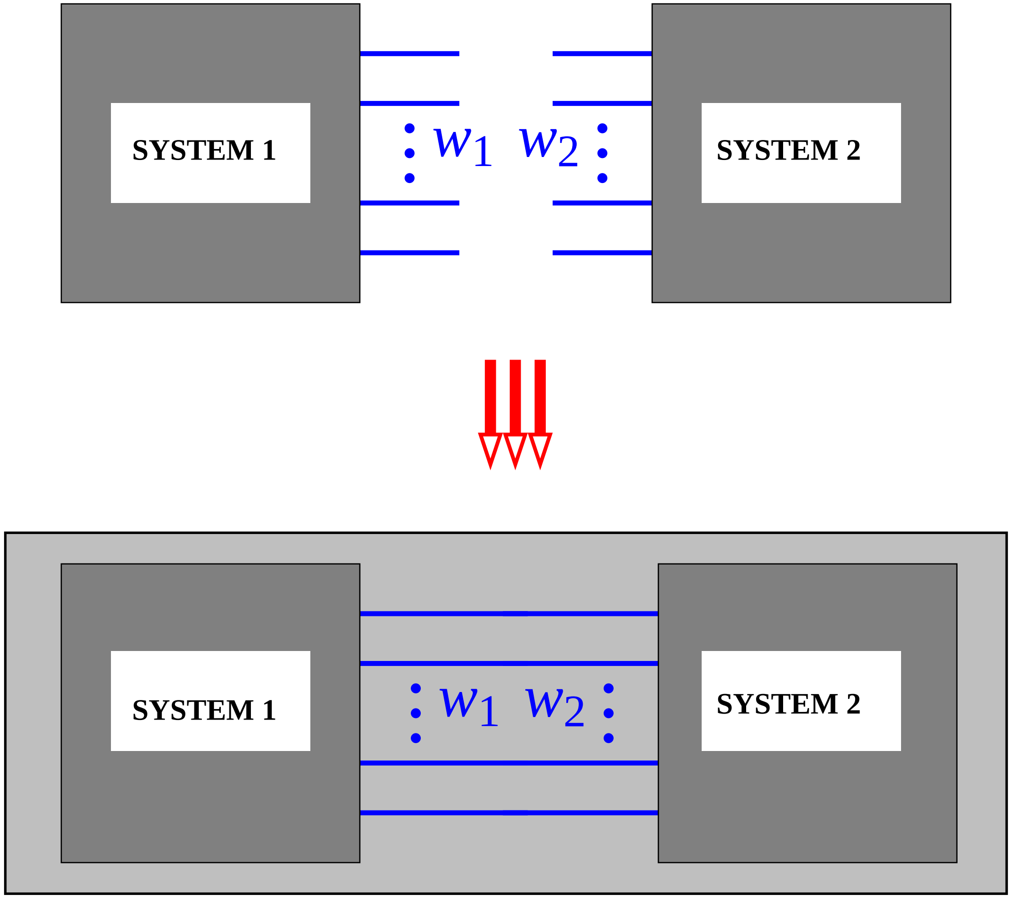
$\Rightarrow$ **'open' system.**

If $\mathscr{E}_1$ = the Borel σ-algebra, and $\texttt{support}(P_1) = \mathbb{R}^{\mathtt{n}}$, then Σ_1 interconnectable only with the free system $\Sigma_2 = (\mathbb{R}^{\mathtt{n}}, \mathscr{E}_2, P_2)$, $\mathscr{E}_2 = \{\emptyset, \mathbb{R}^{\mathtt{n}}\}$.

$\Rightarrow$ **classical = 'closed' system.**

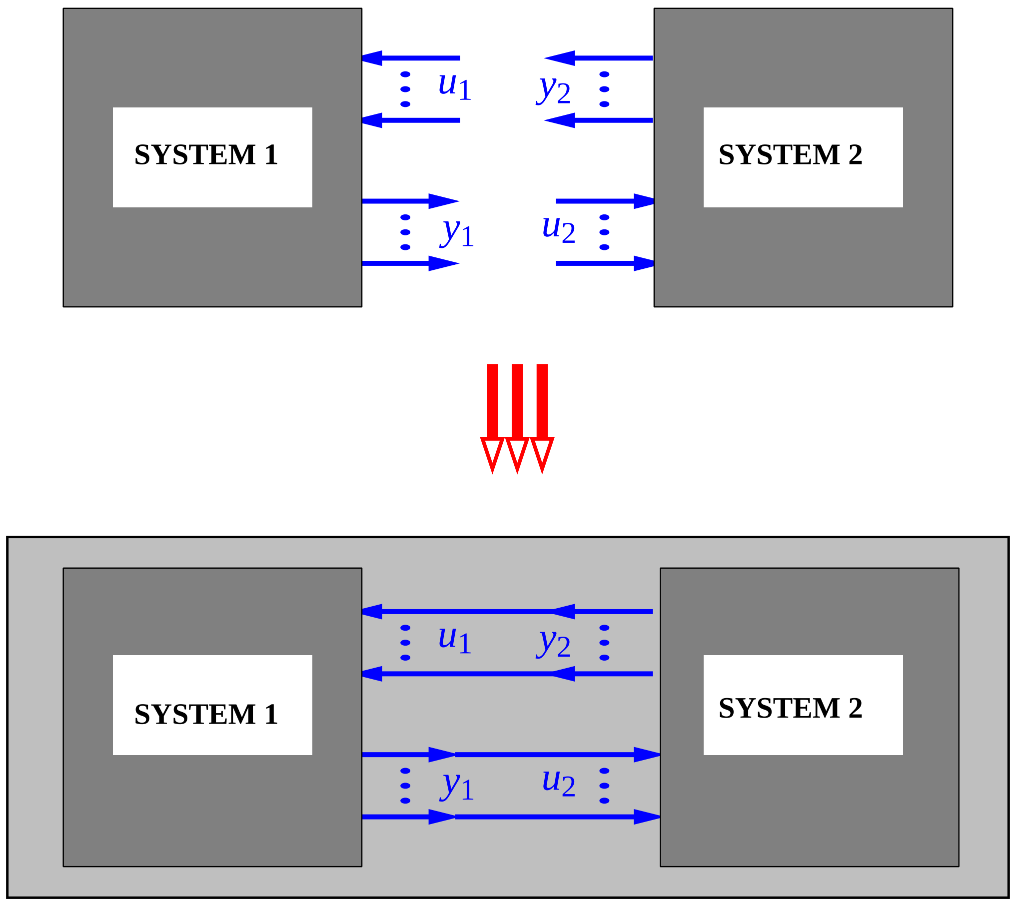
Interconnection $\Leftrightarrow$ variable sharing

Variable sharing



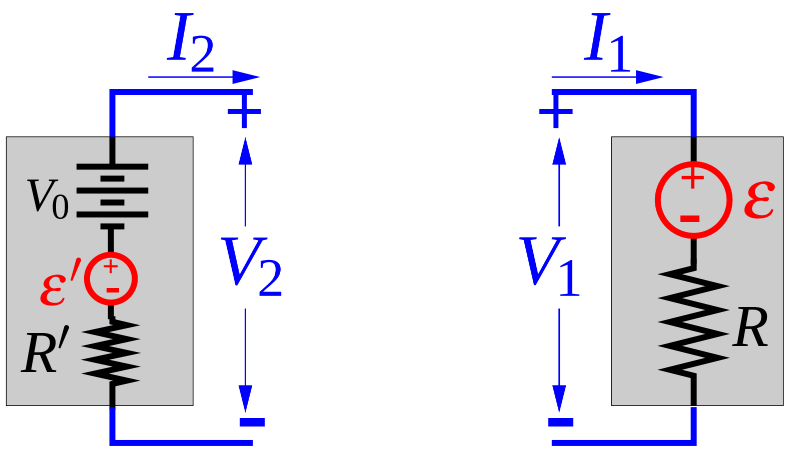
$$w_1 = w_2$$

Output-to-input assignment



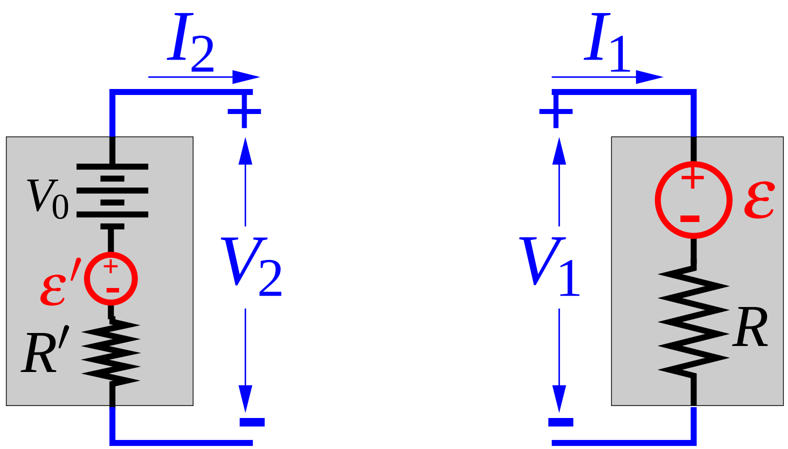
$$u_1 = y_2, \quad u_2 = y_1$$

Resistor interconnection



$$V_1 = V_2, \quad I_1 = I_2$$

Resistor interconnection



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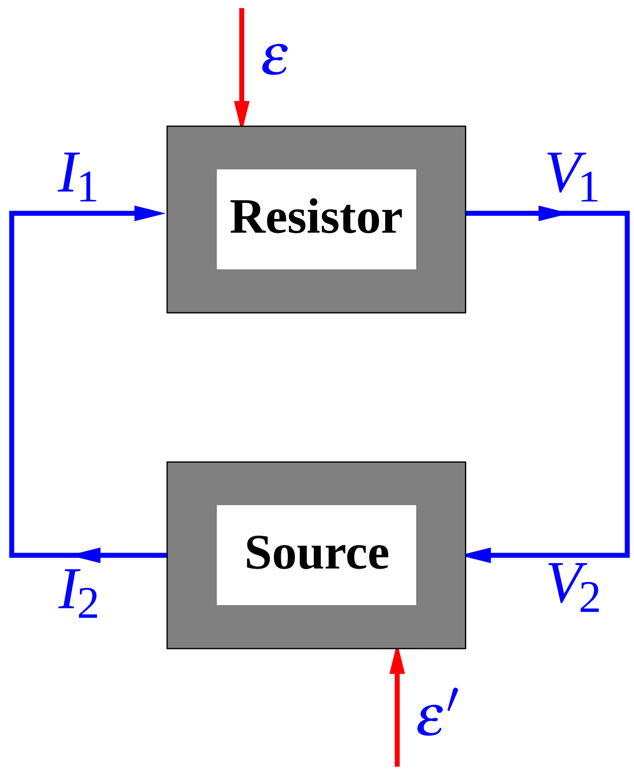
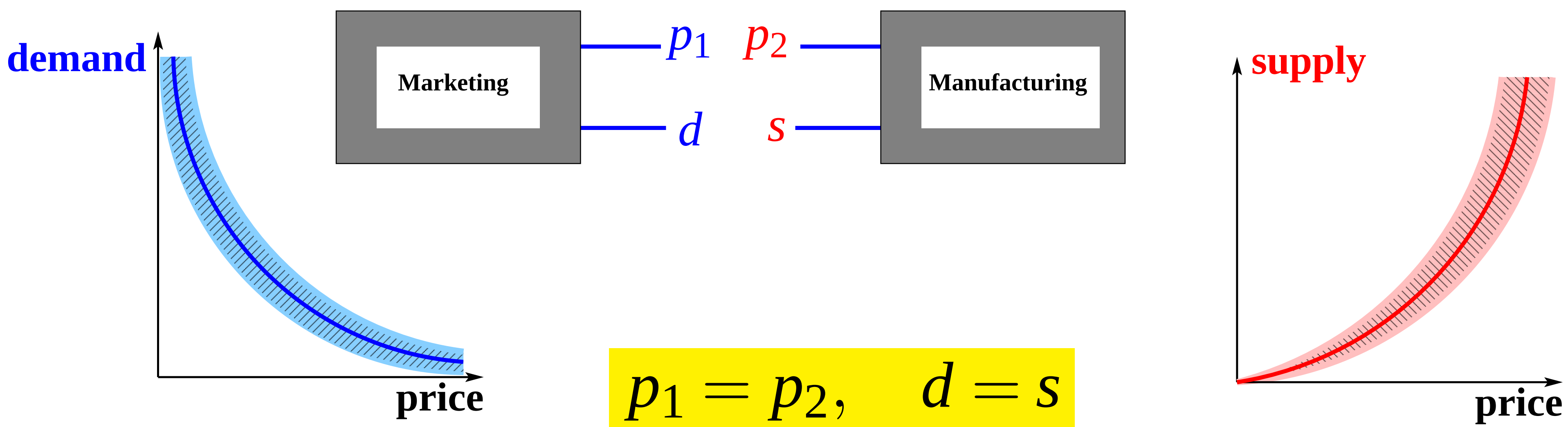
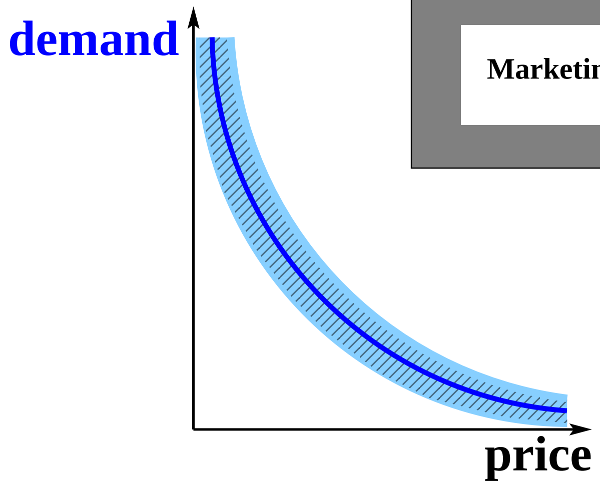


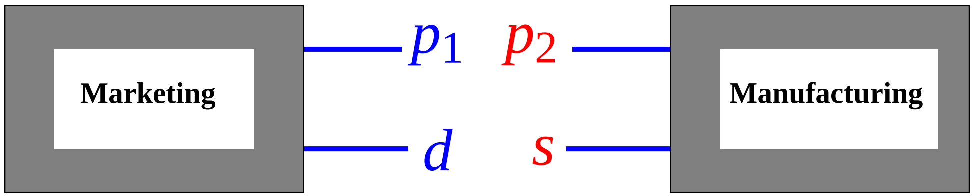
Price/demand/supply interconnection

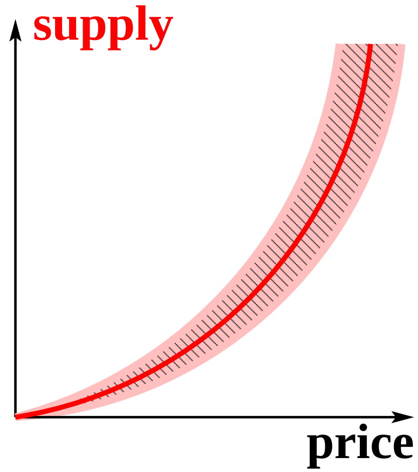


$$p_1 = p_2, \quad d = s$$

Price/demand/supply interconnection







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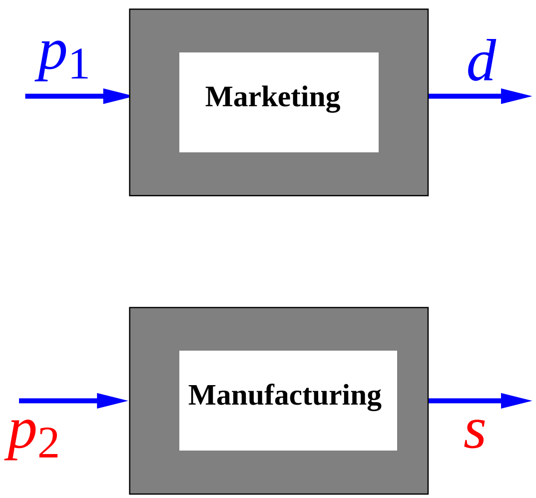
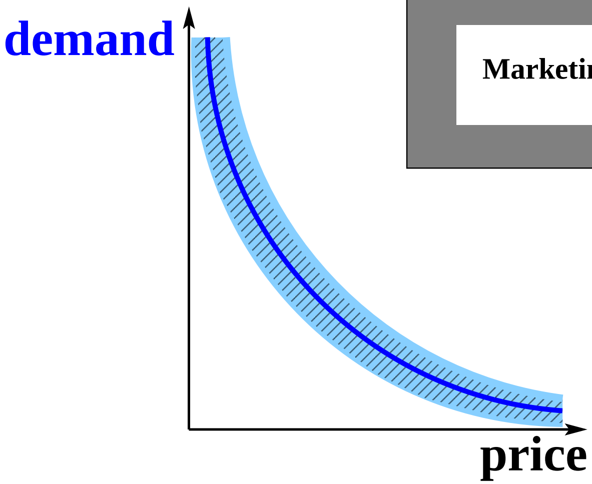
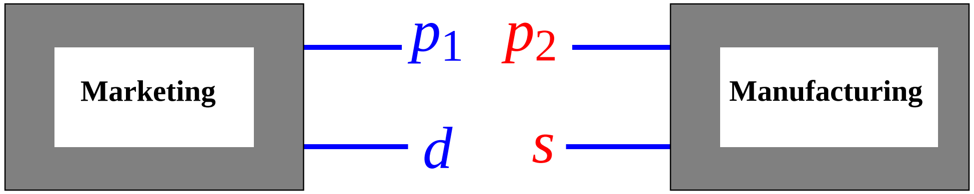
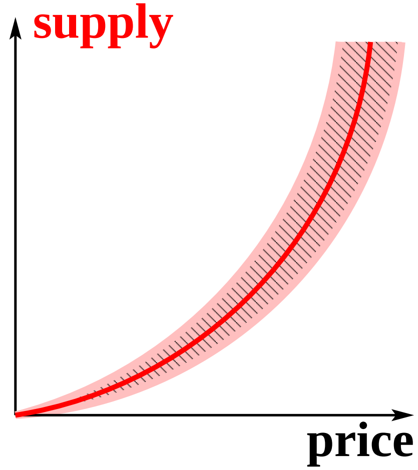


Price/demand/supply interconnection







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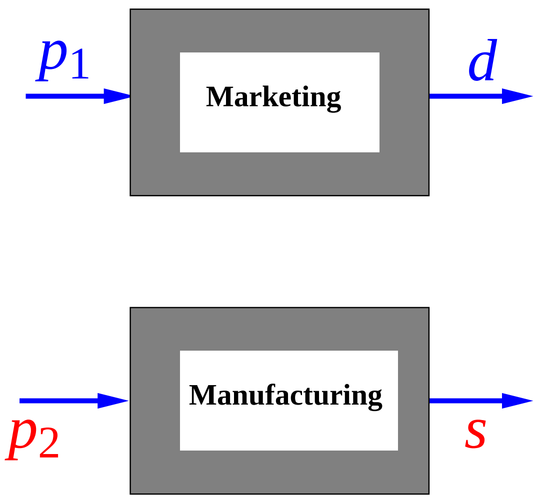
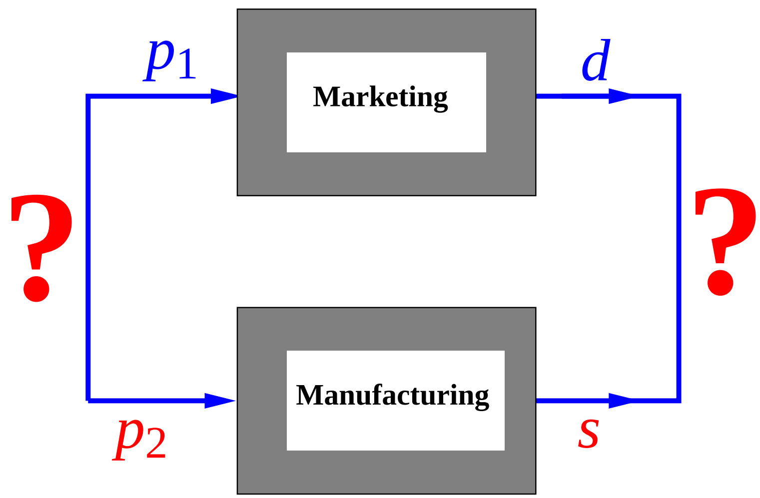




Identification

Measurements

Data collection requires observing a stochastic system *in interaction with an environment*.

Is it possible to disentangle the laws of a system from the laws of the environment?

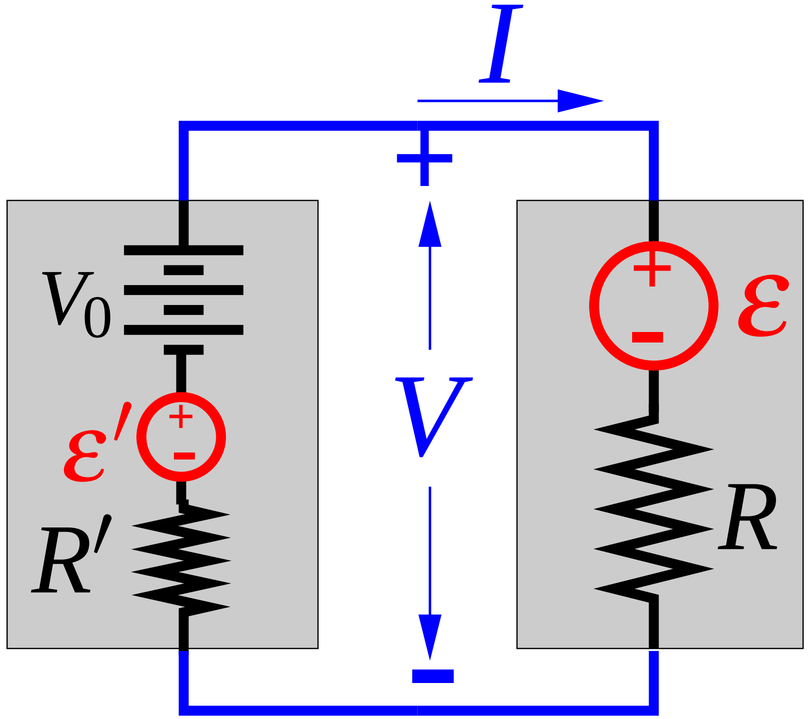
Measurements

Data collection requires observing a stochastic system *in interaction with an environment*.

Is it possible to disentangle the laws of a system from the laws of the environment?

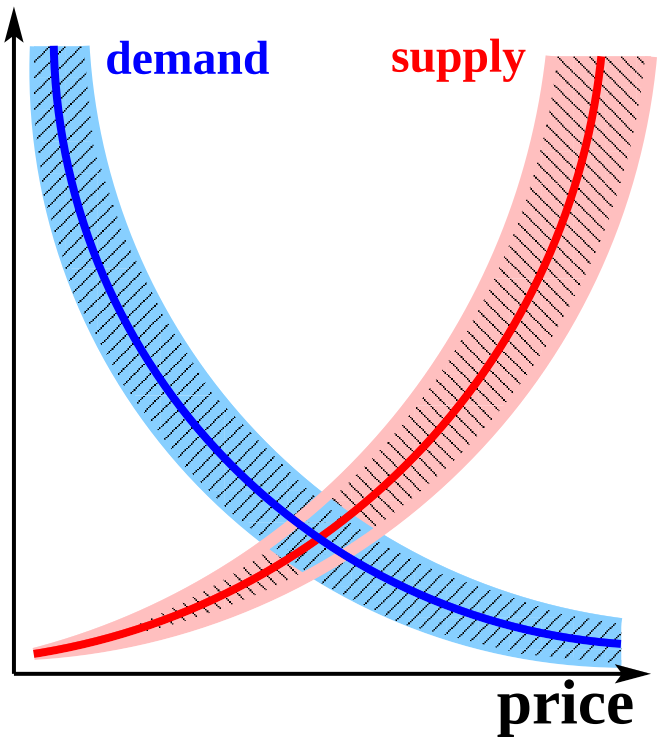
In engineering, it may be possible to set the experimental conditions.
In economics and the social sciences (and biology?), data often gathered passively 'in vivo'.

Disentangling



Can R and σ be deduced by sampling (V, I)?

Disentangling



Can the price/demand characteristic be deduced by sampling (p, d) in equilibrium?

SYSID for gaussian systems

Let Σ_1 and Σ_2 be complementary gaussian systems and assume that the interconnection $\Sigma_1 \wedge \Sigma_2$ is a classical random system.

Sampling $\rightsquigarrow$ the mean and covariance of $\Sigma_1 \wedge \Sigma_2$.

SYSID for gaussian systems

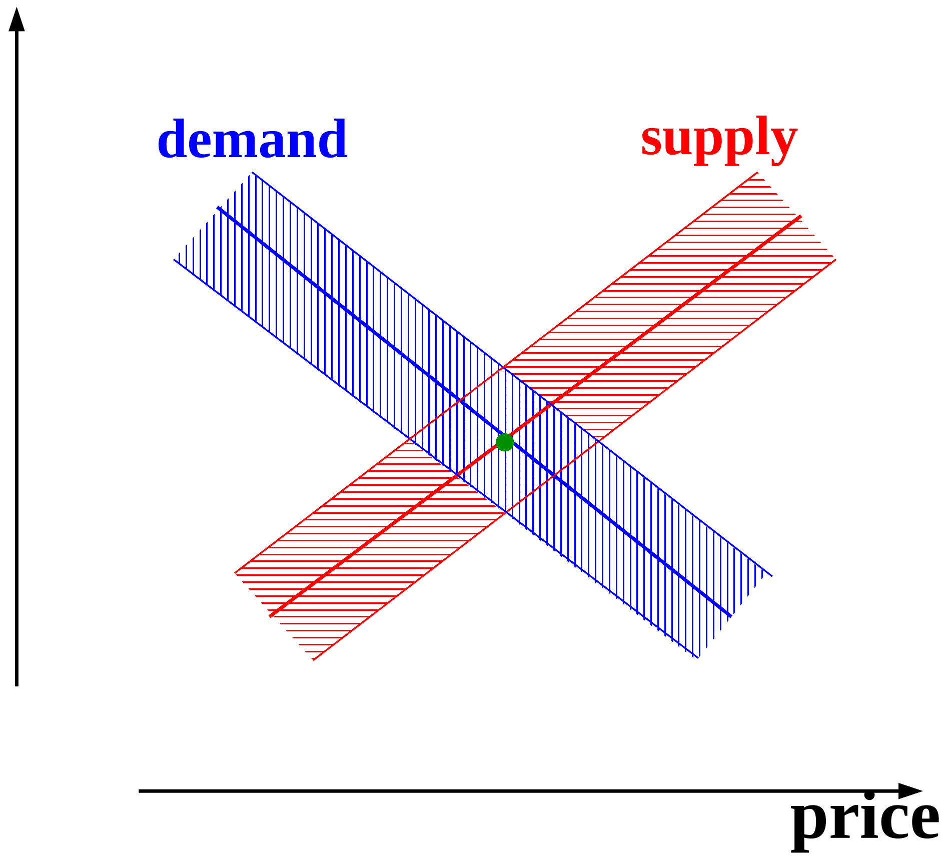
Let Σ_1 and Σ_2 be complementary gaussian systems and assume that the interconnection $\Sigma_1 \wedge \Sigma_2$ is a classical random system.

Sampling $\rightsquigarrow$ the mean and covariance of $\Sigma_1 \wedge \Sigma_2$.

Given the fiber of Σ_1 or Σ_2, all the other parameters of Σ_1 and Σ_2 can be deduced from $\Sigma_1 \wedge \Sigma_2$.

The fiber of Σ_1 or Σ_2 can be chosen freely.

Linearized gaussian price/demand/supply



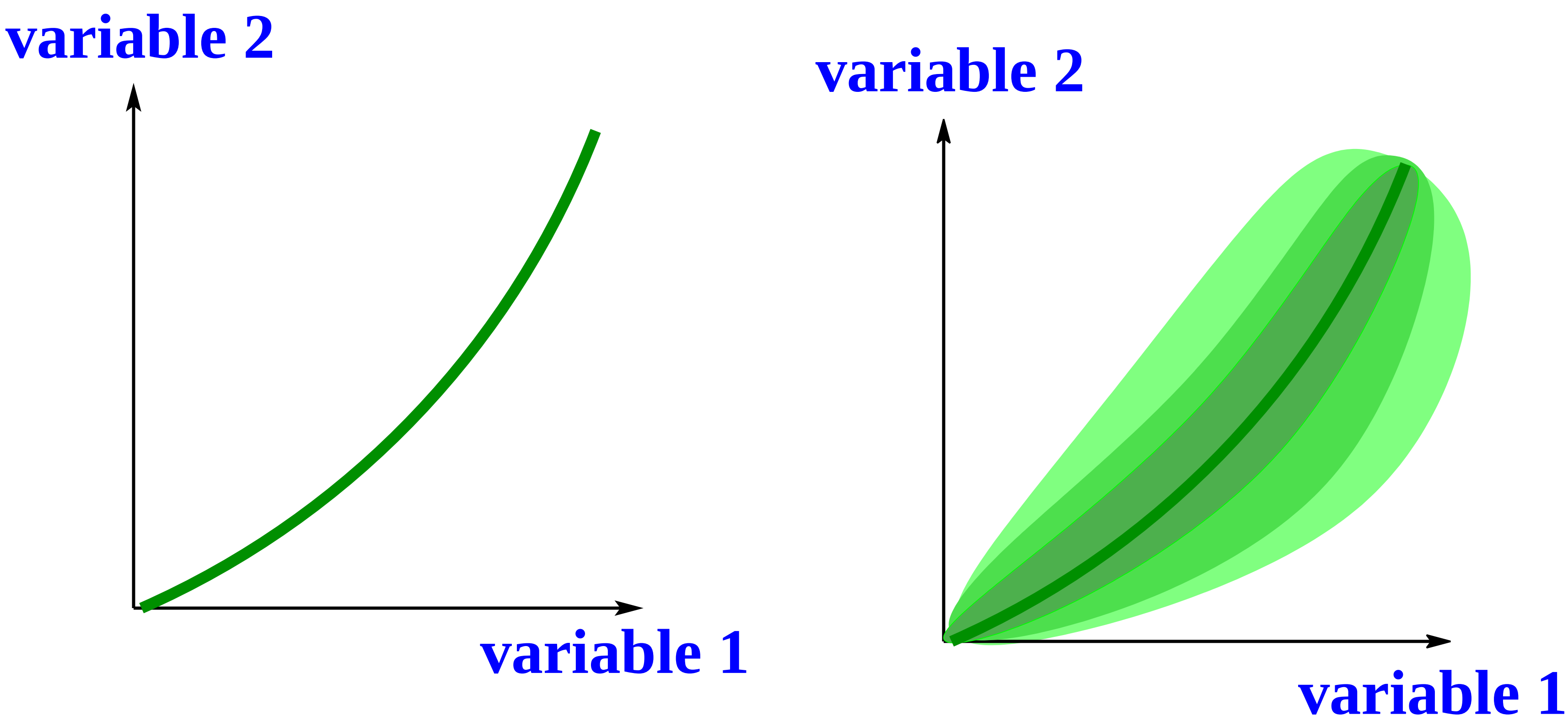
Identifiability provided one of the fibers is known.

Sampling alone does not give the elasticities.

Conclusions

Stochastic systems

- **The Borel σ-algebra is inadequate even for elementary applications.**

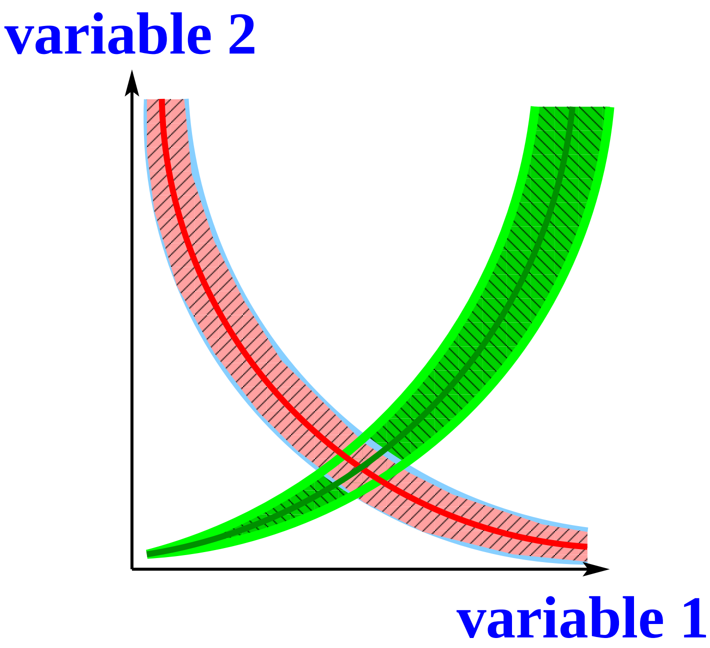


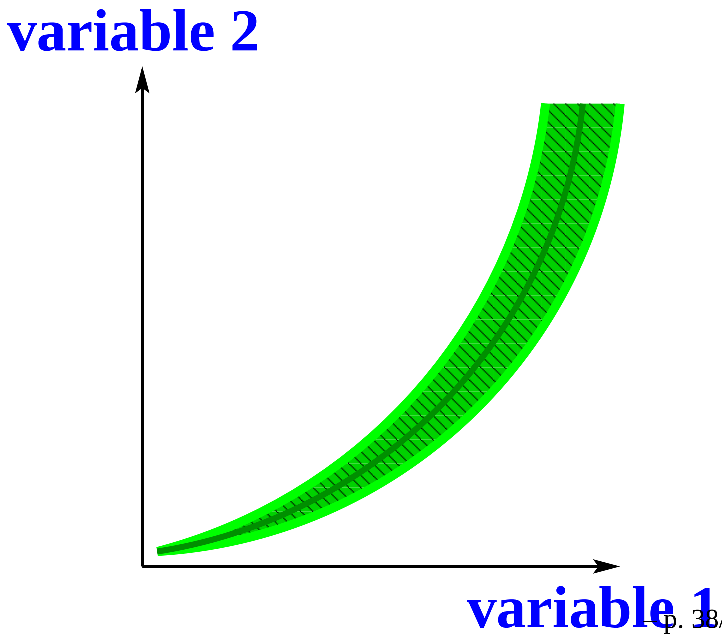
Stochastic systems

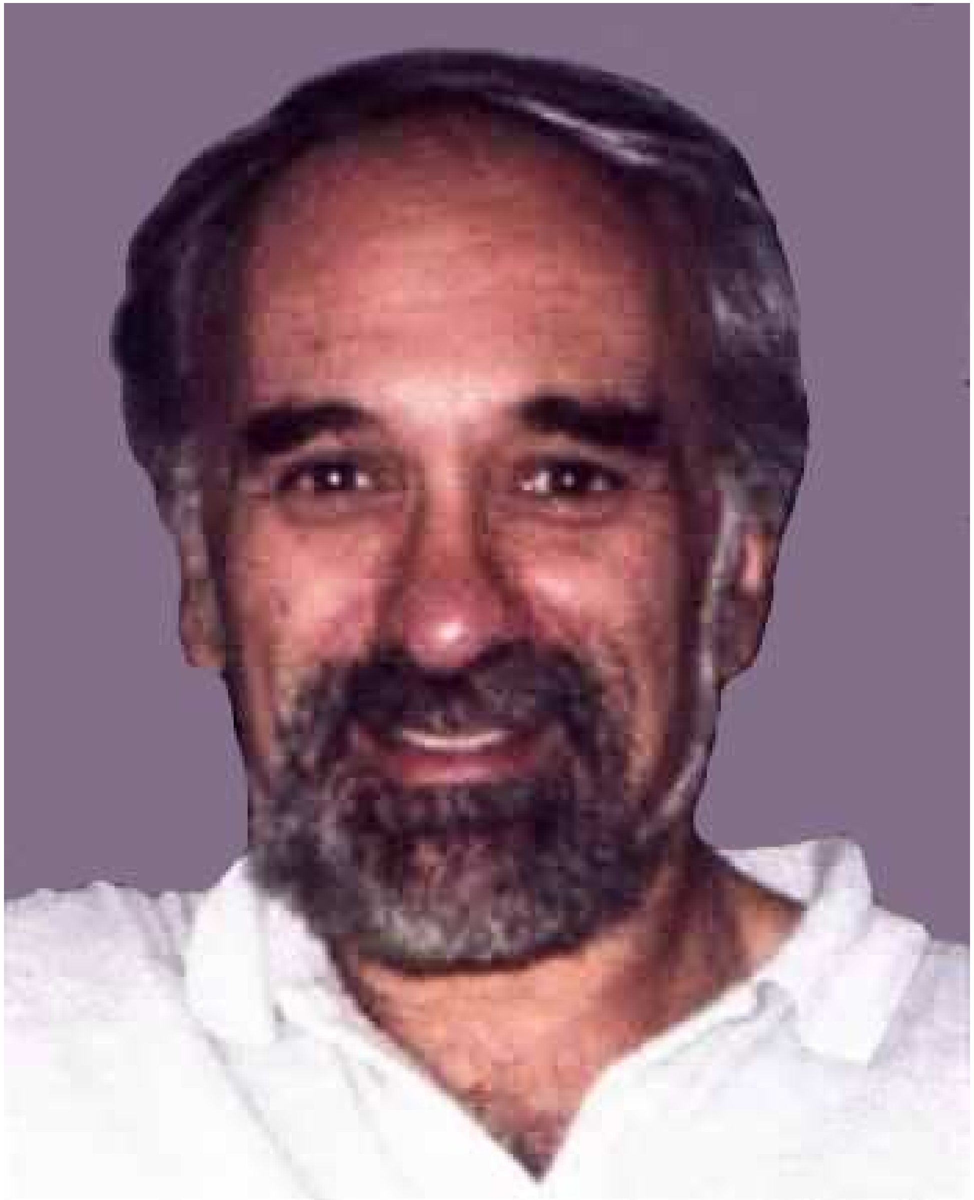
- **The Borel σ-algebra is inadequate even for elementary applications.**
- **Complementary stochastic systems can be interconnected:**
  two distinct laws imposed on one set of variables.

  Open stochastic systems require a parsimonious σ-algebra.
  Classical stochastic systems are closed systems.

- **Measurements are the result of interaction with an environment.**
  Modeling from data requires disentangling.
  The data alone are insufficient for identifiability.







Happy birthday, Eduardo!
Ad multos annos felices!

Reference: ***Open stochastic systems*, IEEE AC, submitted.**

Copies of the lecture frames available from/at

`http://www.esat.kuleuven.be/~jwillems`

Thank you

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