OPEN STOCHASTIC SYSTEMS

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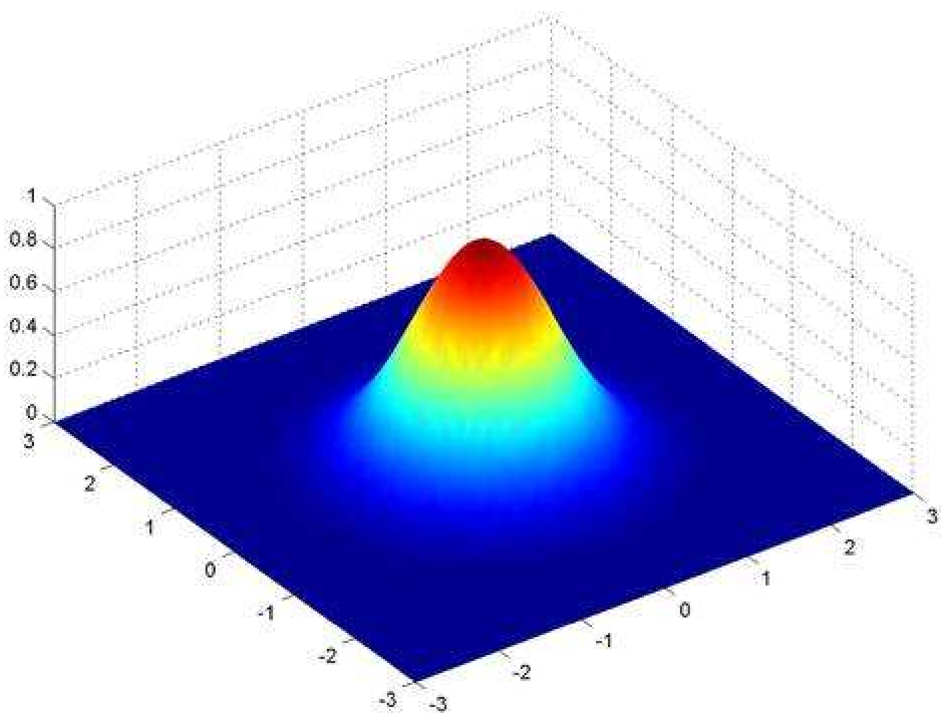
The idea

Theme

Model a phenomenon stochastically; outcomes in $\mathbb{R}^n$.

Usual framework:

- **probability distributions, probability density functions;**
- **means that the event σ-algebra consists of the Borel sets.**
  $\rightsquigarrow$ **'Every' subset of $\mathbb{R}^n$ is assigned a probability.**



for $A \subseteq \mathbb{R}^n$ Borel,

$$P(A) = \int_A p(x)\,dx$$

Theme

Model a phenomenon stochastically; outcomes in $\mathbb{R}^n$.

Usual framework:

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- **means that the event σ-algebra consists of the Borel sets.**
  $\rightsquigarrow$ **'Every' subset of $\mathbb{R}^n$ is assigned a probability.**

Thesis

This is unduly restrictive, even for elementary applications.

What this lecture does/does not

It tries to

- **explain some probability ideas that should be taught,**
- **in the setting of orthodox mathematical probability theory.**

What this lecture does/does not

It tries to

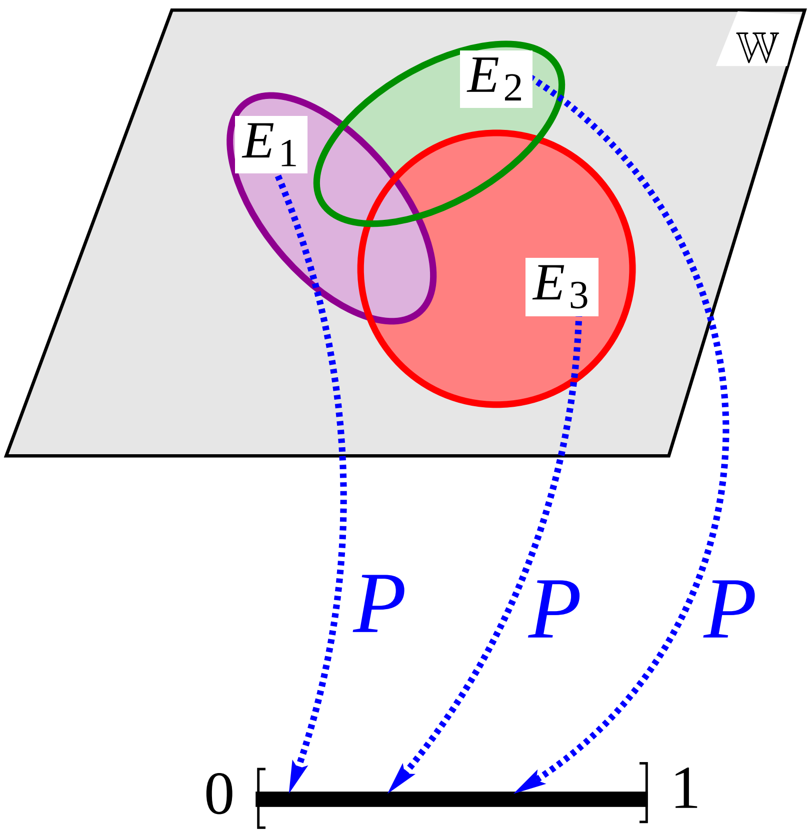
- ► **explain some probability ideas that should be taught,**
- ► **in the setting of orthodox mathematical probability theory.**

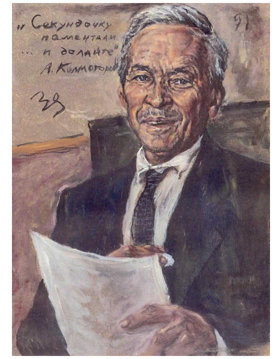
It does not address

- ► **mathematical foundations of probability,**
- ► **interpretation of probability.**

Basic probability

Events





A.N. Kolmogorov
1903 – 1987

A **probability** $P(E) \in [0,1]$ **is assigned to certain subsets** E **(*‘events’*) of the *outcome space*** $\mathbb{W}$**.**

$\mathscr{E}$ **= the sets that are assigned a probability,**

:= the class of ‘measurable’ subsets of $\mathbb{W}$**.**

Main (not all) axioms

The events $\mathscr{E}$ form a σ-algebra of subsets of $\mathbb{W}$:⇒

- $[\![E \in \mathscr{E}]\!] \Rightarrow [\![E^{\text{complement}} \in \mathscr{E}]\!]$
- $[\![E_1, E_2 \in \mathscr{E}]\!] \Rightarrow [\![E_1 \cap E_2 \in \mathscr{E}, \;\; E_1 \cup E_2 \in \mathscr{E}]\!]$

$P : \mathscr{E} \to [0,1]$ **is a probability measure :⇒**

- $P(\mathbb{W}) = 1$,
- $[\![E_1, E_2 \in \mathscr{E}$ **and** $E_1 \cap E_2 = \emptyset]\!]$
  $\Rightarrow [\![P(E_1 \cup E_2) = P(E_1) + P(E_2)]\!]$ **(*P* is additive).**

Borel

Émile Borel
1871 – 1956

In applications the measurable sets often consist of the ***Borel σ-algebra.***

$\mathscr{B}(\mathbb{R}^n)$ **:= the Borel σ-algebra on** $\mathbb{R}^n$**;**

random variable: $\mathbb{W} = \mathbb{R}$ **(or** $\mathbb{C}$**), and** $\mathscr{E} = \mathscr{B}(\mathbb{R})$

random vector: $\mathbb{W} = \mathbb{R}^n$**, and** $\mathscr{E} = \mathscr{B}(\mathbb{R}^n)$

random process: **a family of random variables, etc.**

$\mathscr{B}(\mathbb{R}^n)$ **contains 'basically every' subset of** $\mathbb{R}^n$**.**

Borel

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In applications the measurable sets often consist of the *Borel σ-algebra.*

$\mathscr{B}(\mathbb{R}^n)$ = the Borel σ-algebra on $\mathbb{R}^n$;

$\mathscr{B}(\mathbb{R}^n)$ contains ‘basically every’ subset of $\mathbb{R}^n$.

Allows to take probability distributions as the primitive concept, avoids introducing $\mathscr{E}$ *ab initio*.

<u>Thesis</u>

Borel is unduly restrictive for system theoretic applications.

Borel

Émile Borel
1871 – 1956

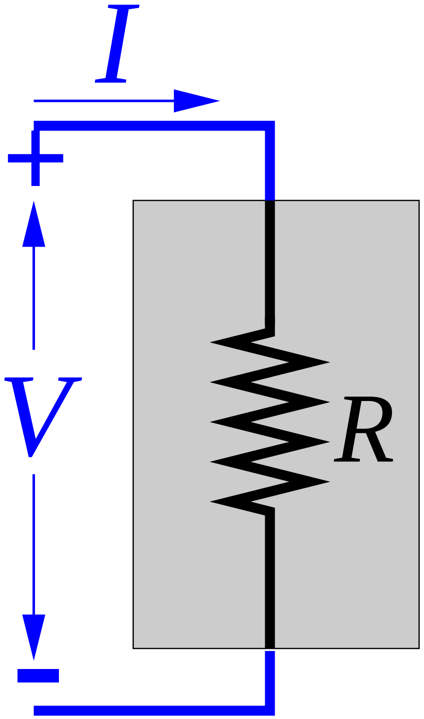
In applications the measurable sets often consist of the *Borel σ-algebra.*

Borel is usually assumed for many basic concepts, as

- ► **random variable, random vector,**
- ► **independence of random variables,**
- ► **marginal measure, conditioning,**
- ► **random process,**
- ► **Brownian motion, Markov process, etc.**

Motivating examples

Ohmic resistor



$$\boxed{V = RI}$$

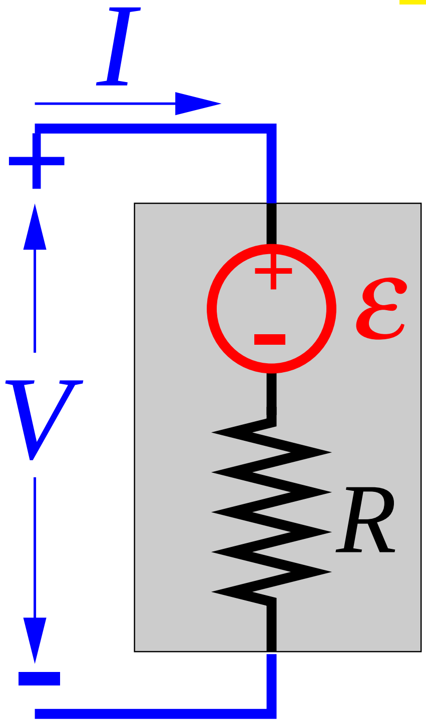
V: **voltage across**

I **current through**

R: **resistance (≥ 0)**

'Ohmic resistor'

Noisy (or ‘hot’) resistor



$$\boxed{V = RI + \varepsilon}$$

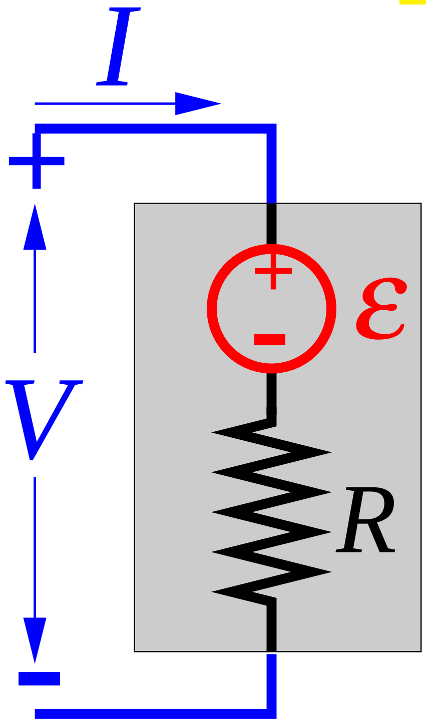
ε gaussian

zero mean

variance $\sim \sqrt{RT}$

‘Johnson-Nyquist resistor’

Noisy (or 'hot') resistor



$$\boxed{V = RI + \varepsilon}$$

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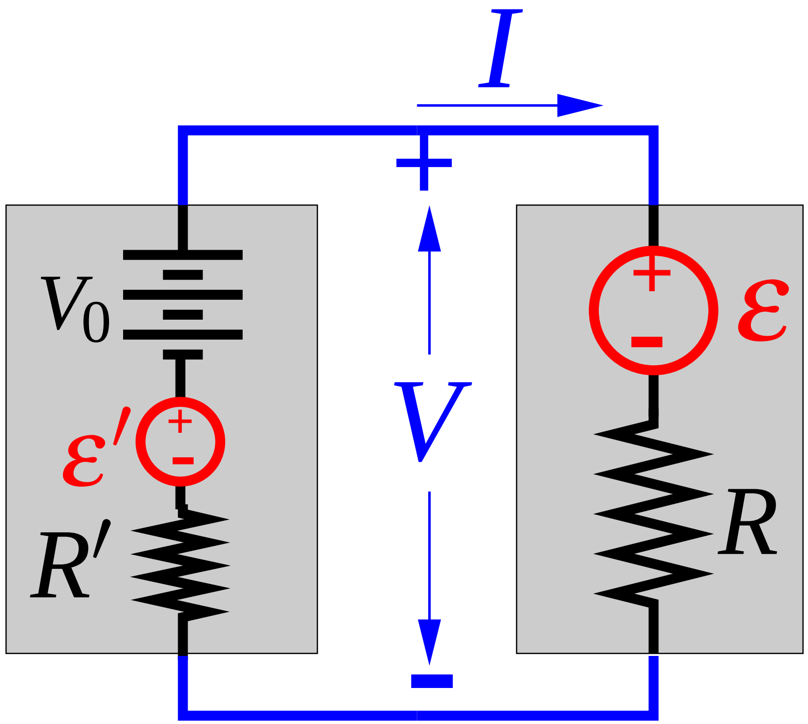
zero mean

variance $\sim \sqrt{RT}$

'Johnson-Nyquist resistor'

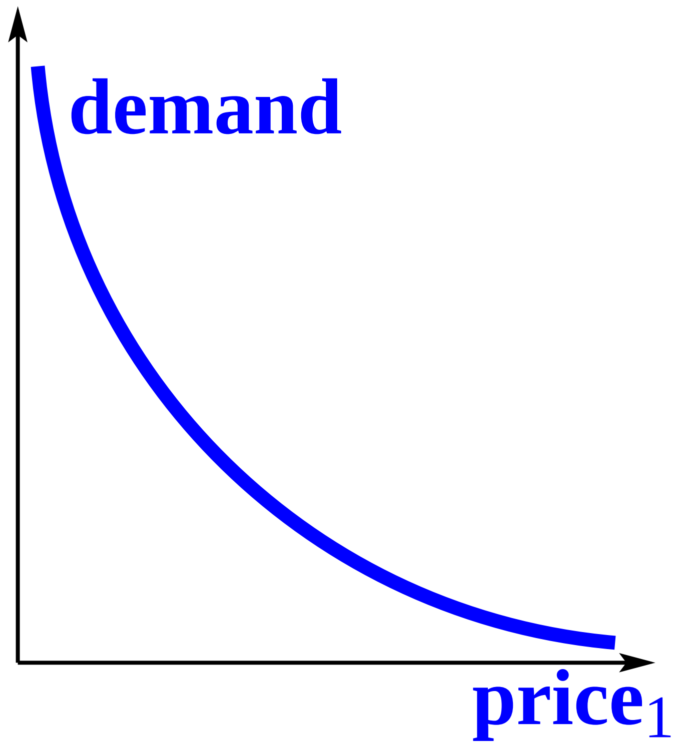
What is $\begin{bmatrix} V \\ I \end{bmatrix}$ as a mathematical entity?

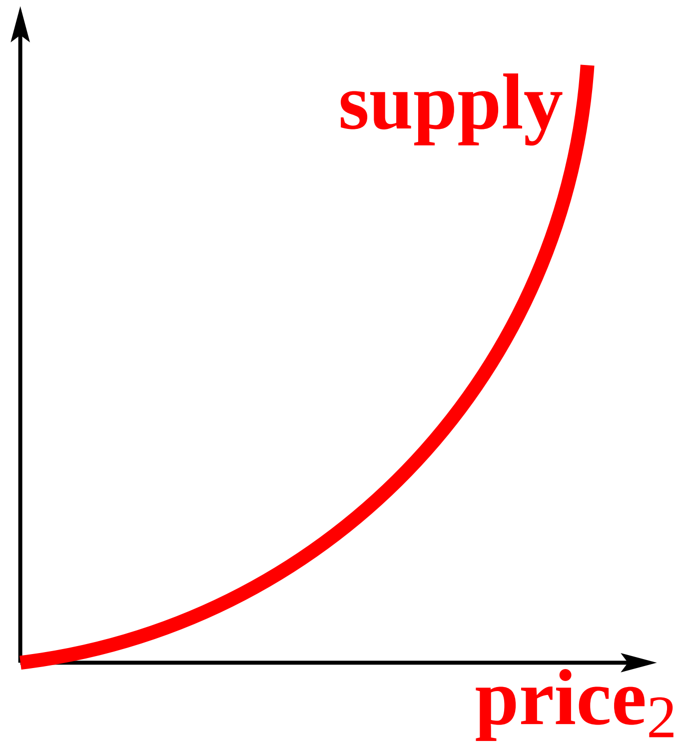
Noisy resistor terminated by a voltage source



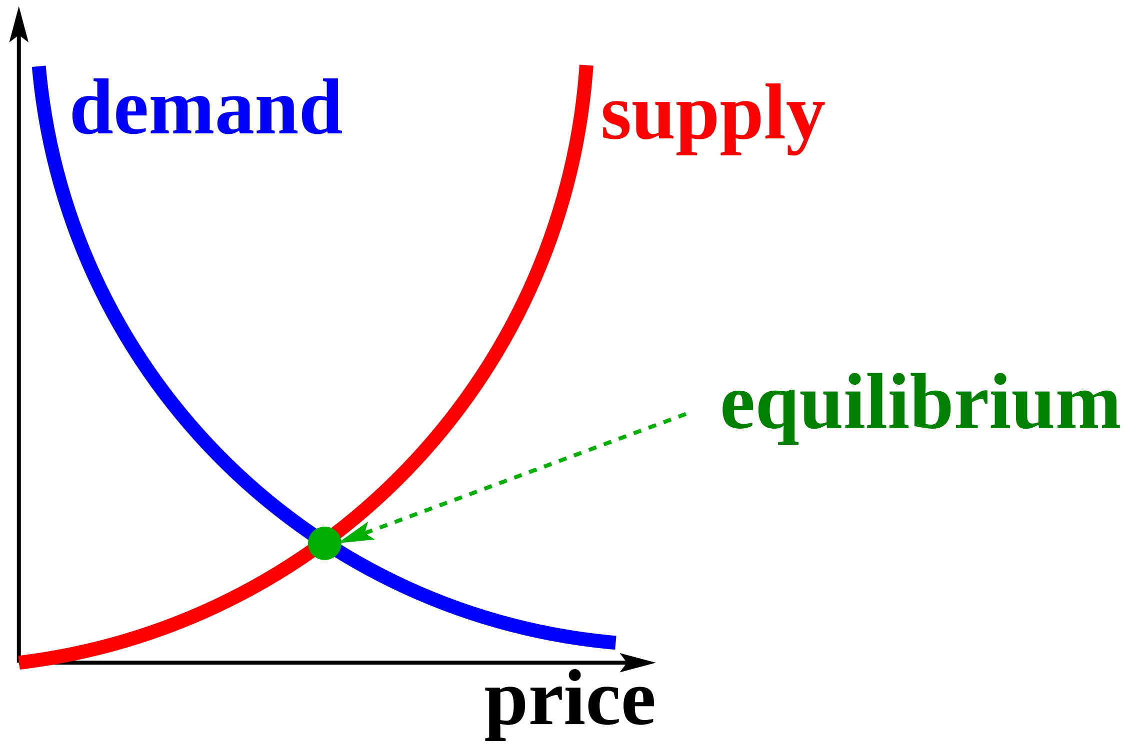
How do we deal with interconnection?

Deterministic price/demand/supply





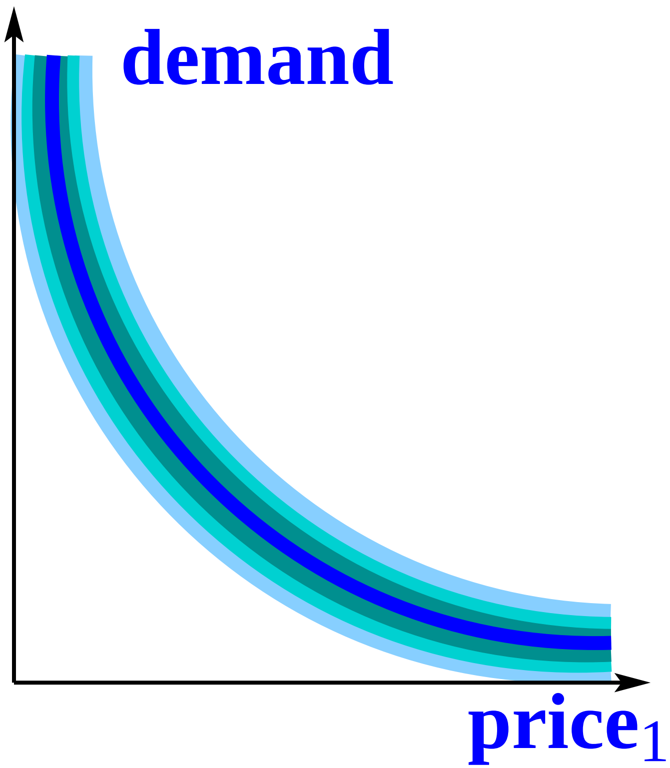
Deterministic price/demand/supply



‘Interconnection’

$\text{price}_1 = \text{price}_2$, **demand = supply**.

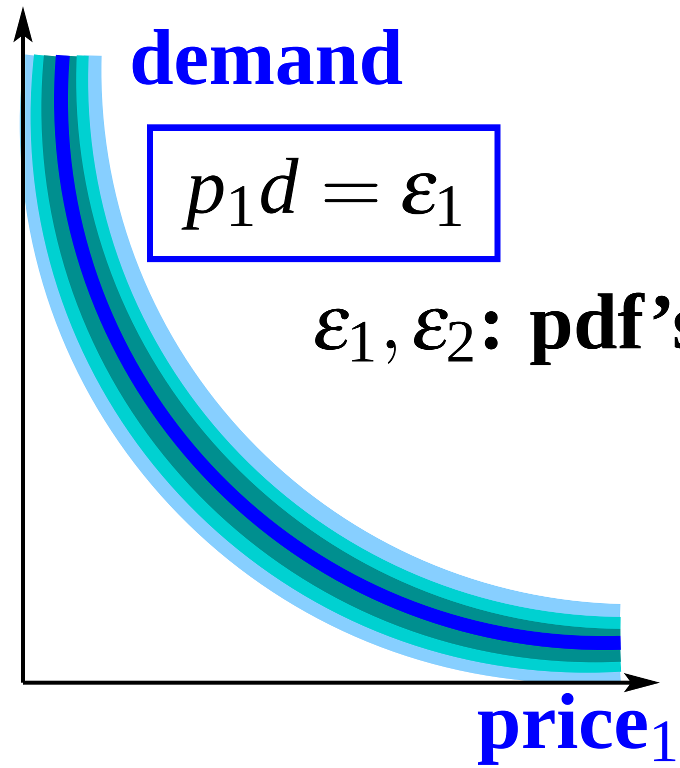
Stochastic price/demand/supply



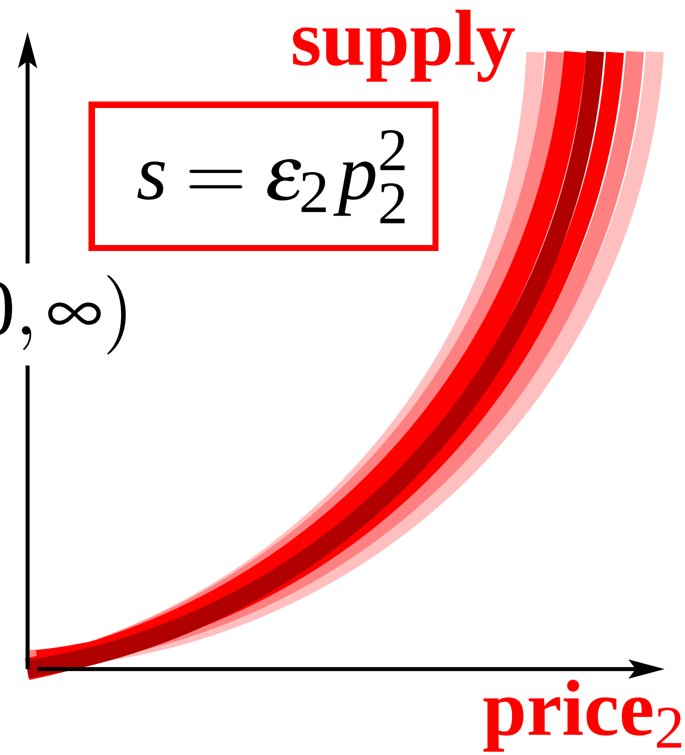
(Only) certain regions of the $\begin{bmatrix} \text{price}_1 \\ \text{demand} \end{bmatrix}$ and $\begin{bmatrix} \text{price}_2 \\ \text{supply} \end{bmatrix}$ planes are assigned a probability.

Stochastic price/demand/supply

e.g.



$$p_1 d = \varepsilon_1$$



$$s = \varepsilon_2 p_2^2$$

$\varepsilon_1, \varepsilon_2$: **pdf's on** $(0, \infty)$

(Only) certain regions of the $\begin{bmatrix} \textbf{price}_1 \\ \textbf{demand} \end{bmatrix}$ **and** $\begin{bmatrix} \textbf{price}_2 \\ \textbf{supply} \end{bmatrix}$ **planes are assigned a probability.**

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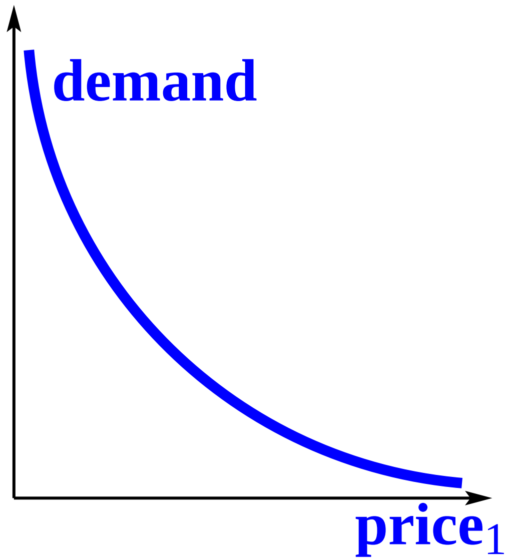
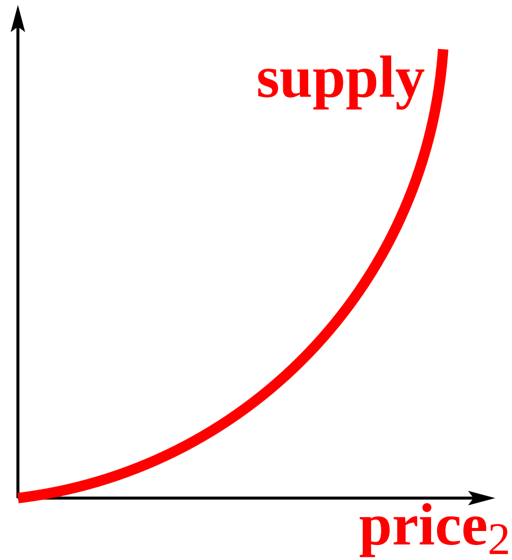
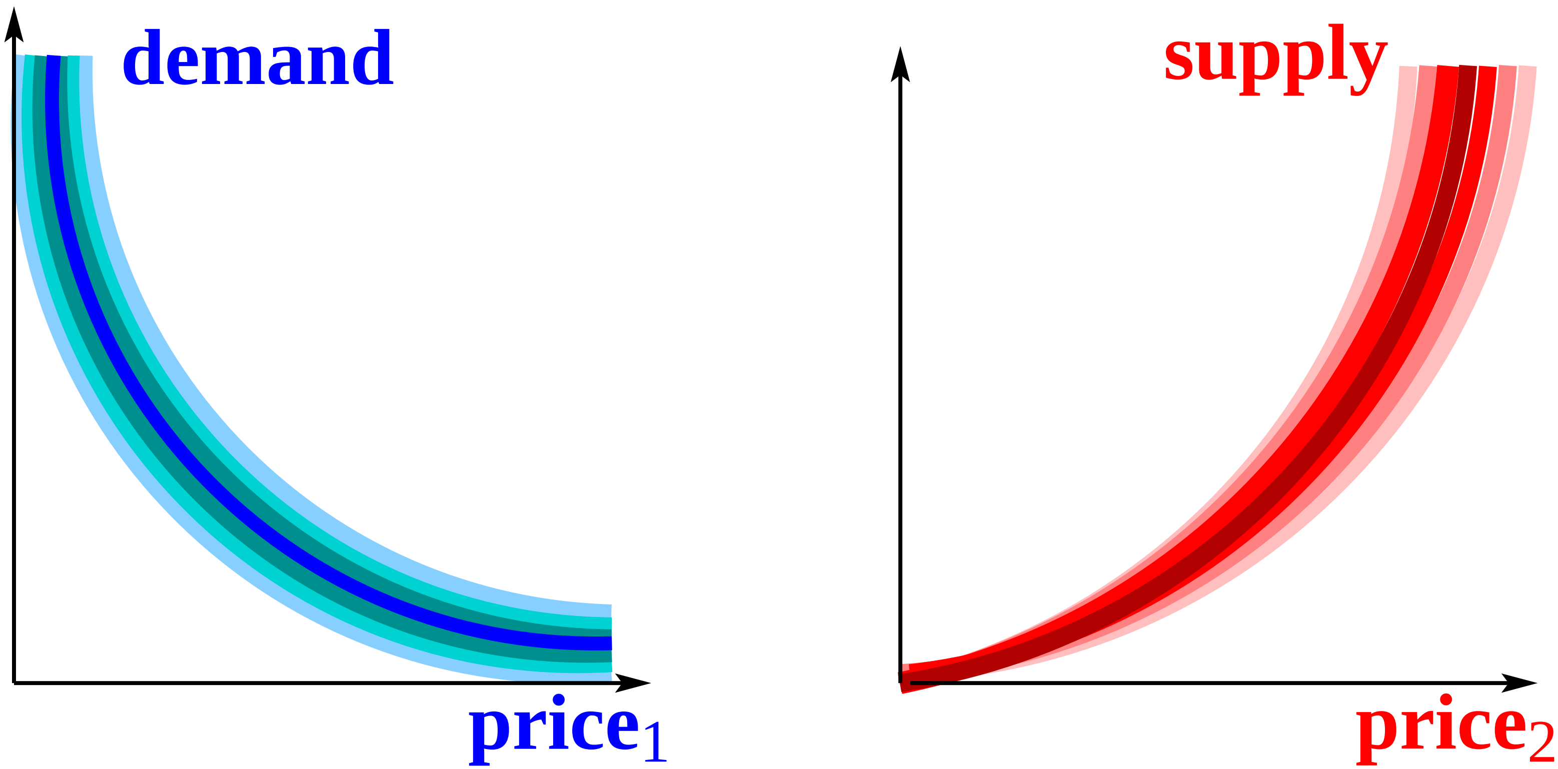




$(p_1, d) \in$ **characteristic w.p.** 1.

$(p_2, s) \in$ **characteristic w.p.** 1.

Stochastic price/demand/supply



(Only) certain regions of the $\begin{bmatrix} \text{price}_1 \\ \text{demand} \end{bmatrix}$ **and** $\begin{bmatrix} \text{price}_2 \\ \text{supply} \end{bmatrix}$ **planes are assigned a probability.**

How do we deal with equilibrium: supply = demand?

Formal definitions

Definition

A *stochastic system* is a probability triple $(\mathbb{W}, \mathscr{E}, P)$

- $\mathbb{W}$ **a non-empty set, the *outcome space*,**
- $\mathscr{E}$ **a σ-algebra of subsets of $\mathbb{W}$: the *events*,**
- $P : \mathscr{E} \to [0,1]$ **a *probability measure*.**

$\mathscr{E}$**: the subsets that are assigned a probability.**

Probability that outcomes $\in E$, $E \in \mathscr{E}$, **is** $P(E)$**.**

Model $\cong$ $\mathscr{E}$ **and** P**;** $\mathscr{E}$ **is an essential part!**

$\mathscr{E}$ **should not be taken for granted.**

Definition

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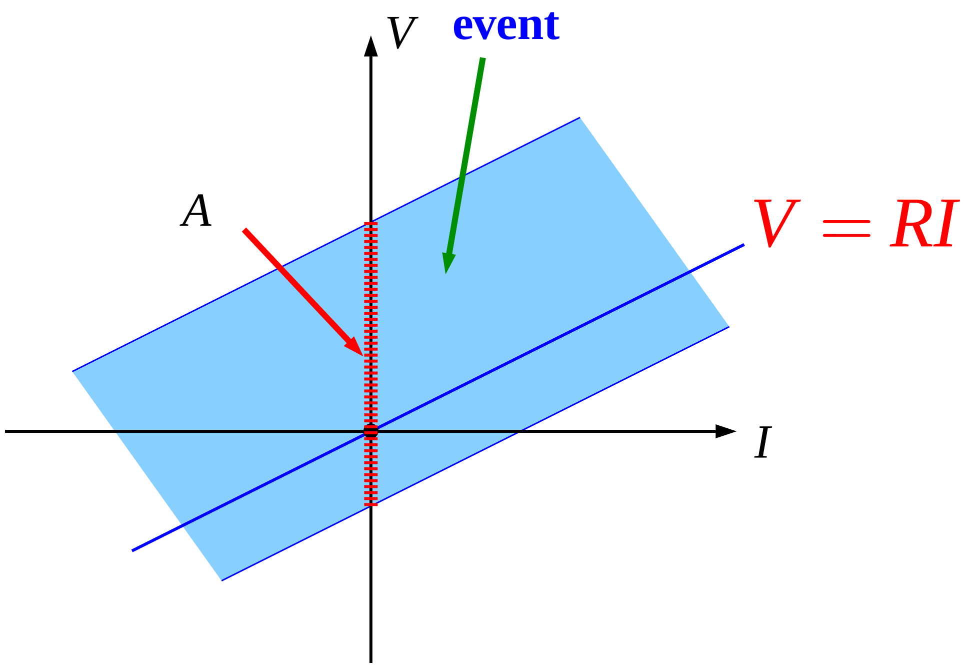
'Classical' stochastic system:

$\mathbb{W} = \mathbb{R}^{\mathrm{n}}$ **and $\mathscr{E}$ = the Borel subsets of $\mathbb{R}^{\mathrm{n}}$.**

P **specified by a probability distribution or a pdf.**

$\mathscr{E}$ is inherited from the topology of the outcome space, it does not involve the randomness.

Noisy resistor



$V = RI + \varepsilon$: stoch. system, outcomes $\begin{bmatrix} V \\ I \end{bmatrix}$, $\mathbb{W} = \mathbb{R}^2$.

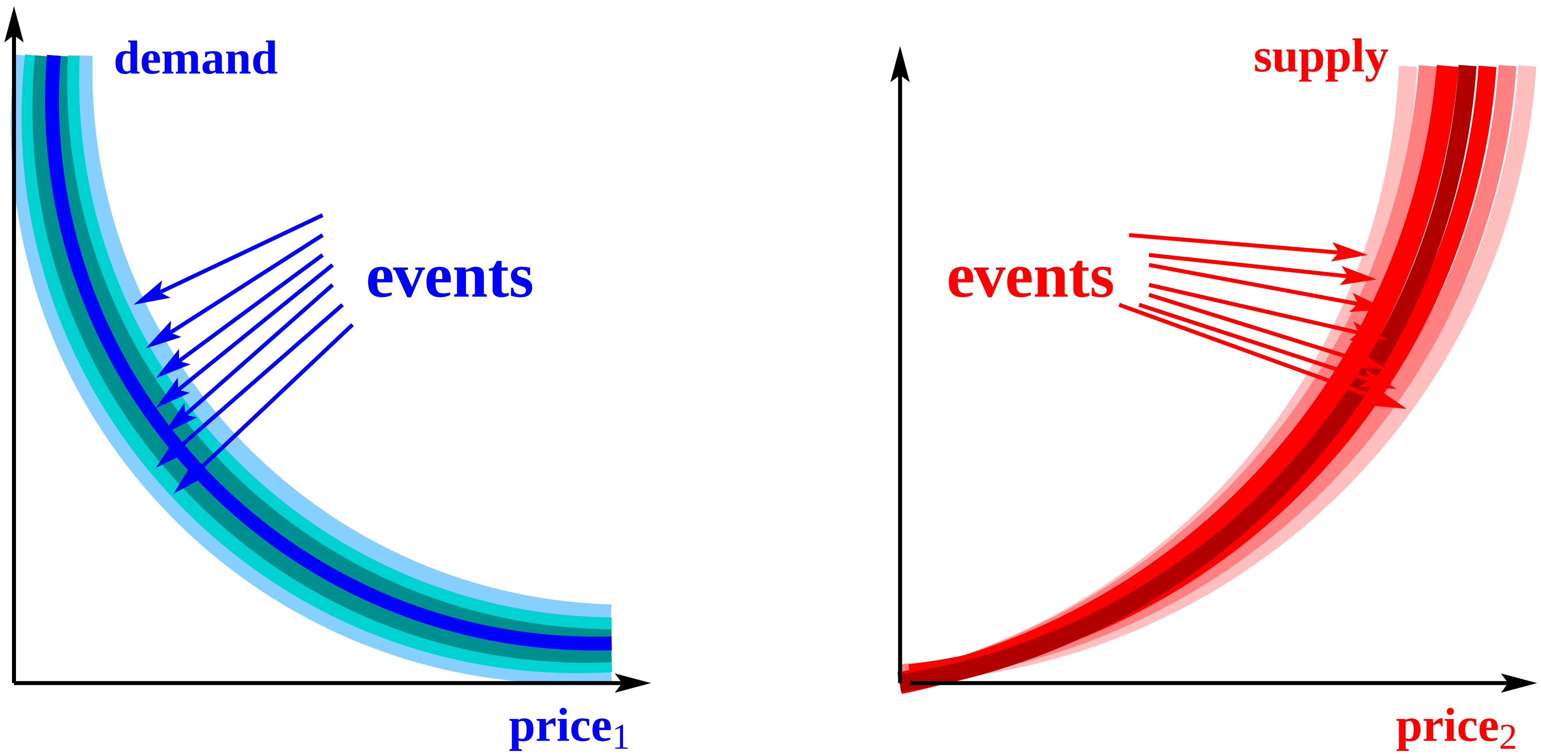
Events: $\left\{ \begin{bmatrix} V \\ I \end{bmatrix} \in \mathbb{R}^2 \mid V - RI \in A \text{ with } A \text{ a Borel subset of } \mathbb{R} \right\}$.

$P(\text{event}) =$ gaussian measure of A.

V nor I are *not* classical real random variables.

Neither $\begin{bmatrix} V \\ I \end{bmatrix}$ nor I nor V possess a pdf.

Stochastic price/demand/supply

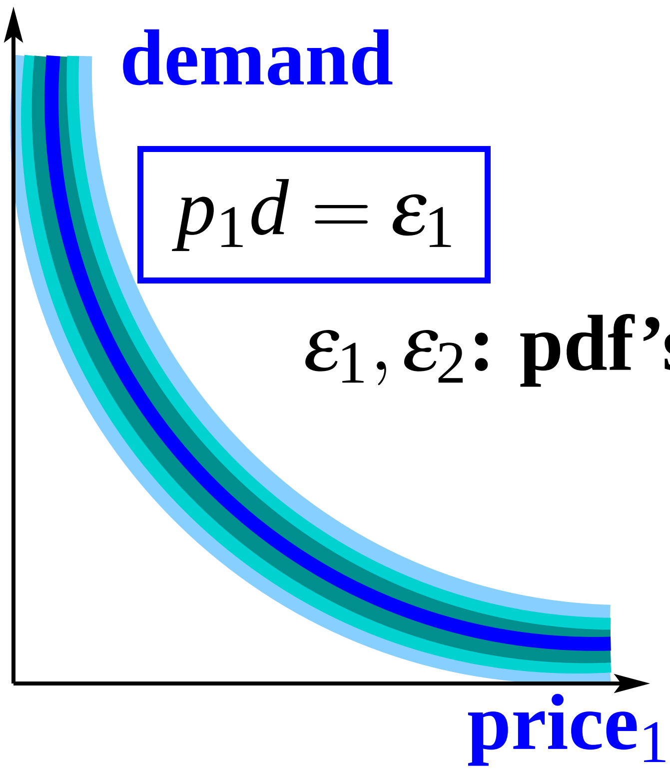


$\mathscr{E}$, $\mathscr{E}'$ = the regions that are assigned a probability.

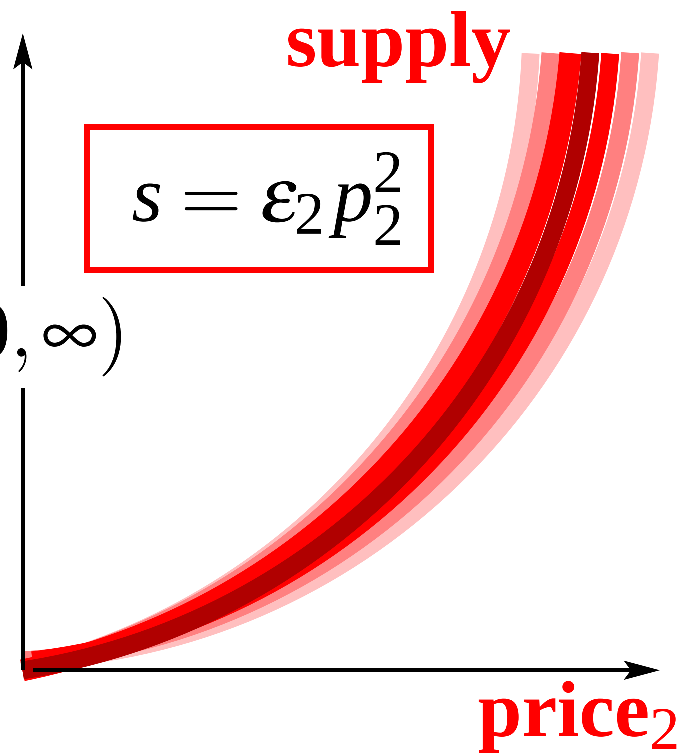
p, d, nor s are not classical real random variables.

Stochastic price/demand/supply

e.g.



$\varepsilon_1, \varepsilon_2$**: pdf's on** $(0, \infty)$



Often the events can be parameterized (as here by $\varepsilon_1, \varepsilon_2$**).**

σ**-algebra definition is more elementary/desirable/general.**

Linearity

Atoms

Let $(\mathbb{W}, \mathscr{E})$ **be a measurable space.**

$E \in \mathscr{E}$ **is said to be atomic** $:\Leftrightarrow$

$$[\![E' \in \mathscr{E}, E' \subseteq E]\!] \Rightarrow [\![E' = E \text{ or } E' = \emptyset]\!].$$

Examples:

- **For** $(\mathbb{R}^{\mathtt{n}}, \mathscr{B}(\mathbb{R}^{\mathtt{n}}))$ **the atoms are the singletons.**

Atoms

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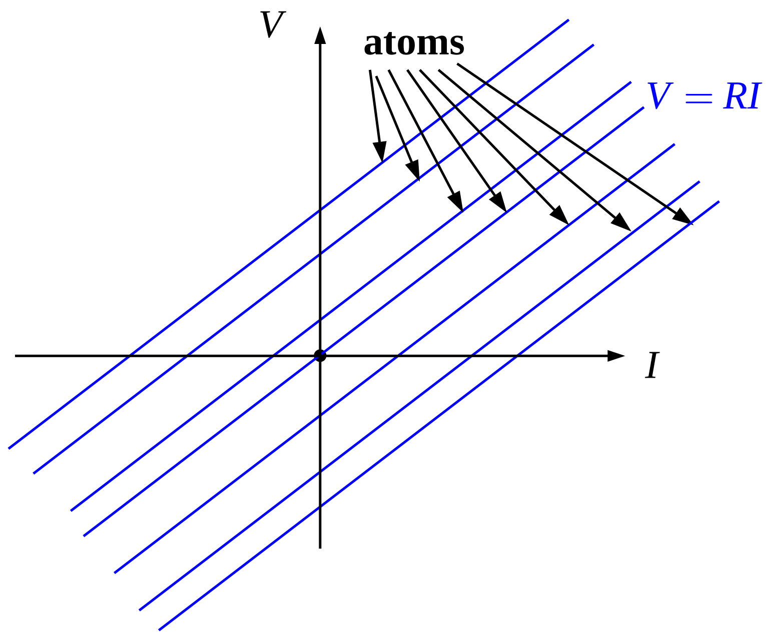
- **For the noisy resistor the atoms are the lines parallel to** $V = RI$**.**



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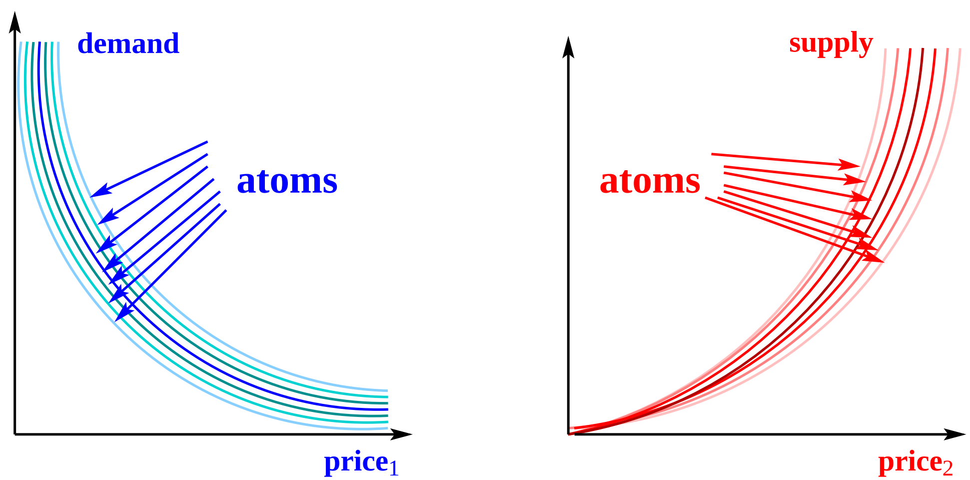
- **Economic example: atoms are** (p_1, d) **hyperbolas;** (p_2, s) **parabolas.**



Linear stochastic system

$(\mathbb{R}^{\mathtt{n}}, \mathscr{E}, P)$ **is said to be a** ***linear stochastic system***

$:\Leftrightarrow \exists\, \mathbb{L}$**, a linear subspace of** $\mathbb{R}^{\mathtt{n}}$**, such that**

$\mathbb{L}$ **is an atomic event, and**

$$[\![E \in \mathscr{E} \text{ is atomic}]\!] \Leftrightarrow [\![E = E + \mathbb{L}]\!].$$

I.e., all atoms are cylinders with sides parallel to $\mathbb{L}$**.**

However, in the remainder of this lecture we will use the following more restrictive definition.

Linear stochastic system

linear stochastic system

:$\Leftrightarrow$ **Borel probability on $\mathbb{R}^{\mathtt{n}}/\mathbb{L}$,**

$\mathbb{L} \subseteq \mathbb{R}^{\mathtt{n}}$ **a linear subspace, called the *'fiber'*.**

Note: $\mathbb{R}^{\mathtt{n}}/\mathbb{L}$ **is a real vector space of dimension** $\mathtt{n} - \mathtt{dimension}(\mathbb{L})$**.**

Events: **cylinders with sides parallel to $\mathbb{L}$.**

Subsets of $\mathbb{R}^{\mathtt{n}}$ as $A + \mathbb{L}$, $\mathbb{L}$ linear subspace, A Borel.

Linearity

linear stochastic system

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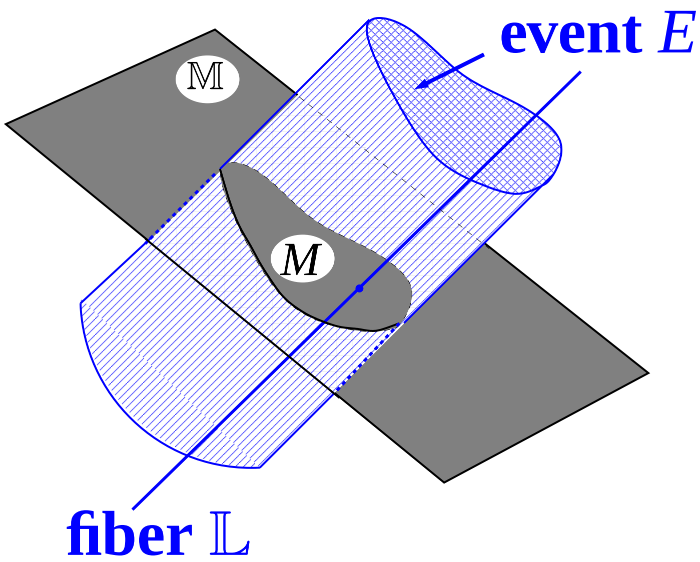

Borel probability on $\mathbb{M} \cong \mathbb{R}^{\mathtt{n}}/\mathbb{L}, \quad (\mathbb{M} \oplus \mathbb{L} = \mathbb{R}^{\mathtt{n}})$**.**

Classical $\Rightarrow$ **linear!**

gaussian $:\Leftrightarrow$ **linear, Borel probability gaussian.**

Deterministic system

$(\mathbb{W}, \mathscr{E}, P)$ **is said to be** ***deterministic*** **if**

$$\mathscr{E} = \{\emptyset, \mathbb{B}, \mathbb{B}^{complement}, \mathbb{W}\} \text{ and } P(\mathbb{B}) = 1.$$

Deterministic system

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Atoms of deterministic system: $\mathbb{B}$ **and** $\mathbb{B}^{complement}$**.**

Linear deterministic $\Leftrightarrow$ $\mathbb{B}$ **linear subspace of** $\mathbb{R}^{\mathtt{n}}$**.**

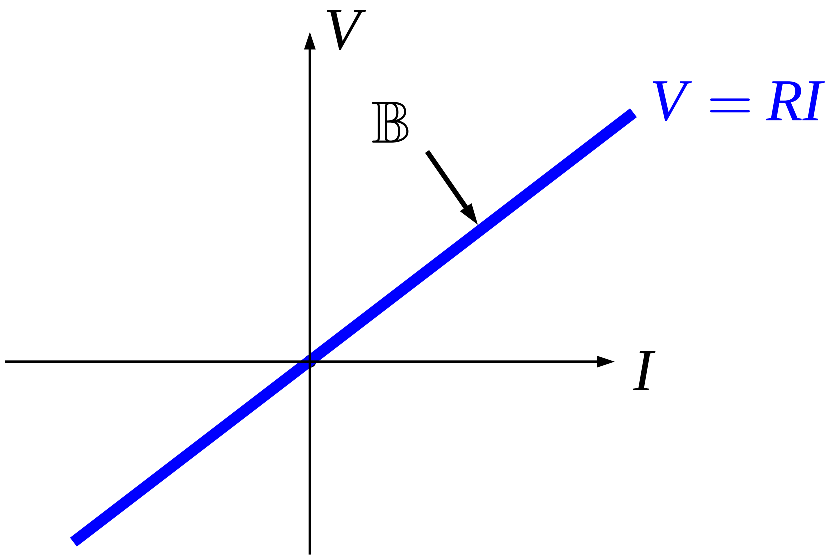
$\mathbb{B}$ **is called the behavior.**

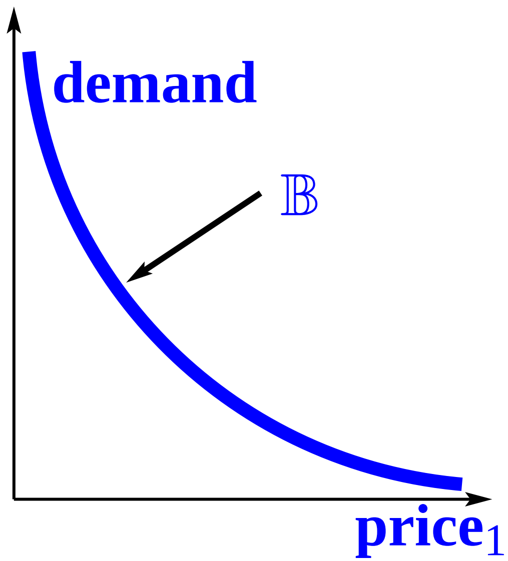
Deterministic examples

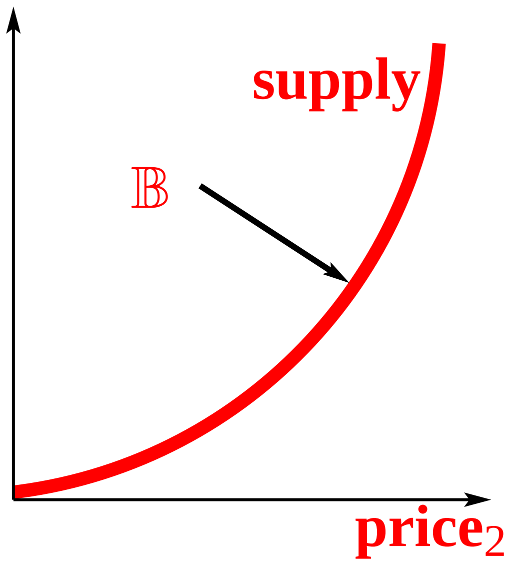
An Ohmic resistor,

$\mathbb{W} = \mathbb{R}^2$,

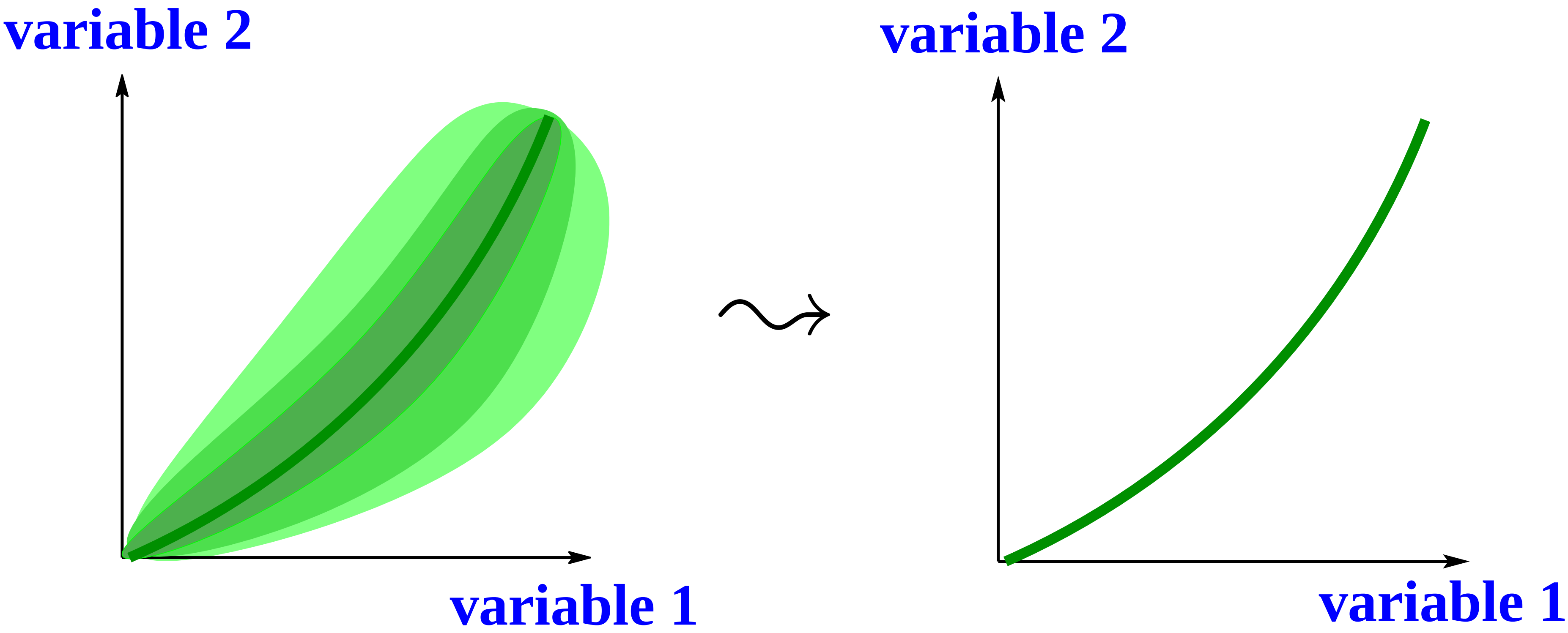
$\mathbb{B} = \{ \begin{bmatrix} V \\ I \end{bmatrix} \in \mathbb{R}^2 \mid V = RI \}$.







The need for ‘coarse’ σ-algebras



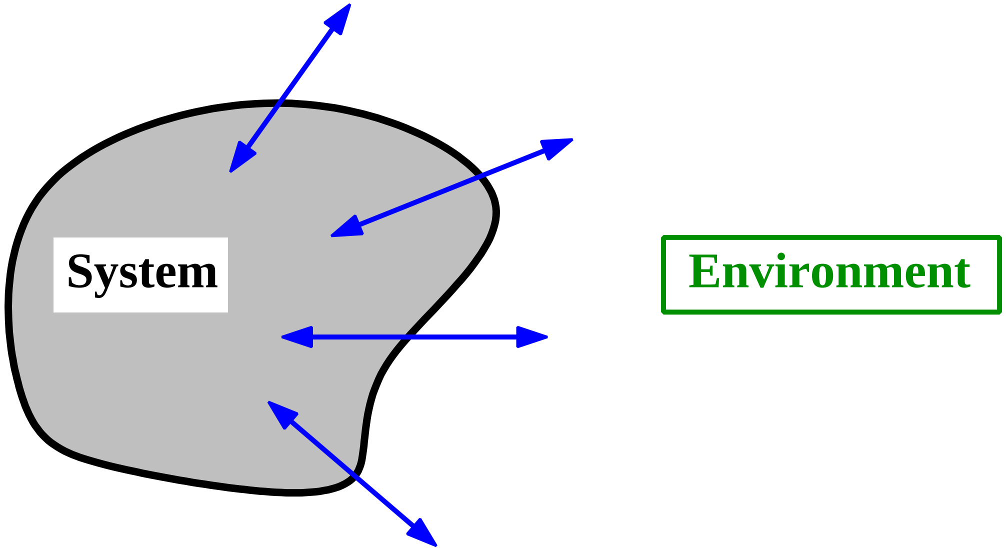
For a classical random vector, the deterministic limit

$\simeq$ a (singular) probability distribution.

Awkward from the modeling point of view.

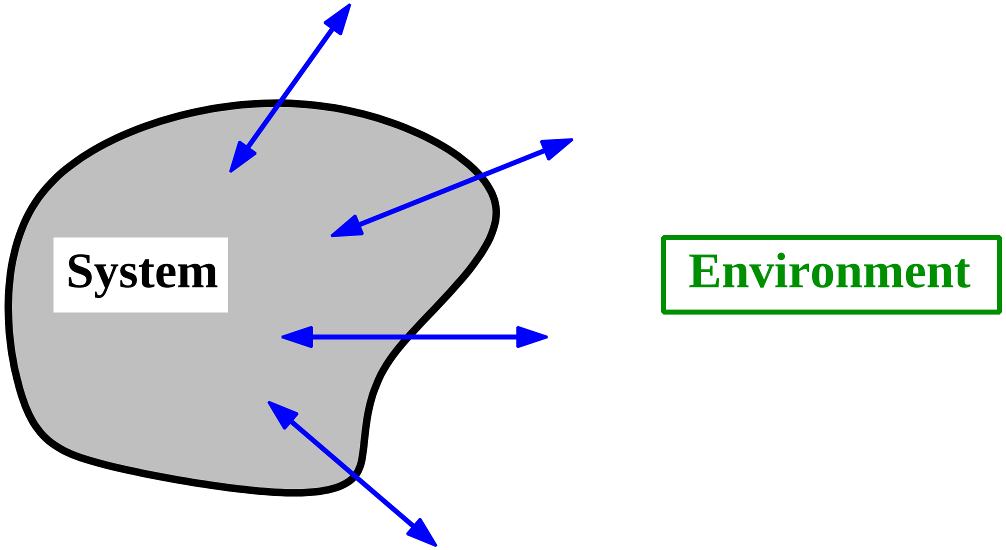
Interconnection

System theoretic musts

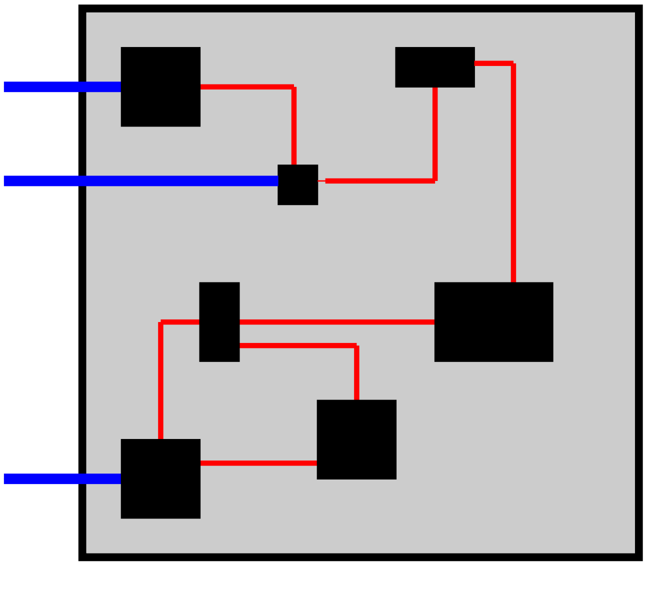


Open

System theoretic musts

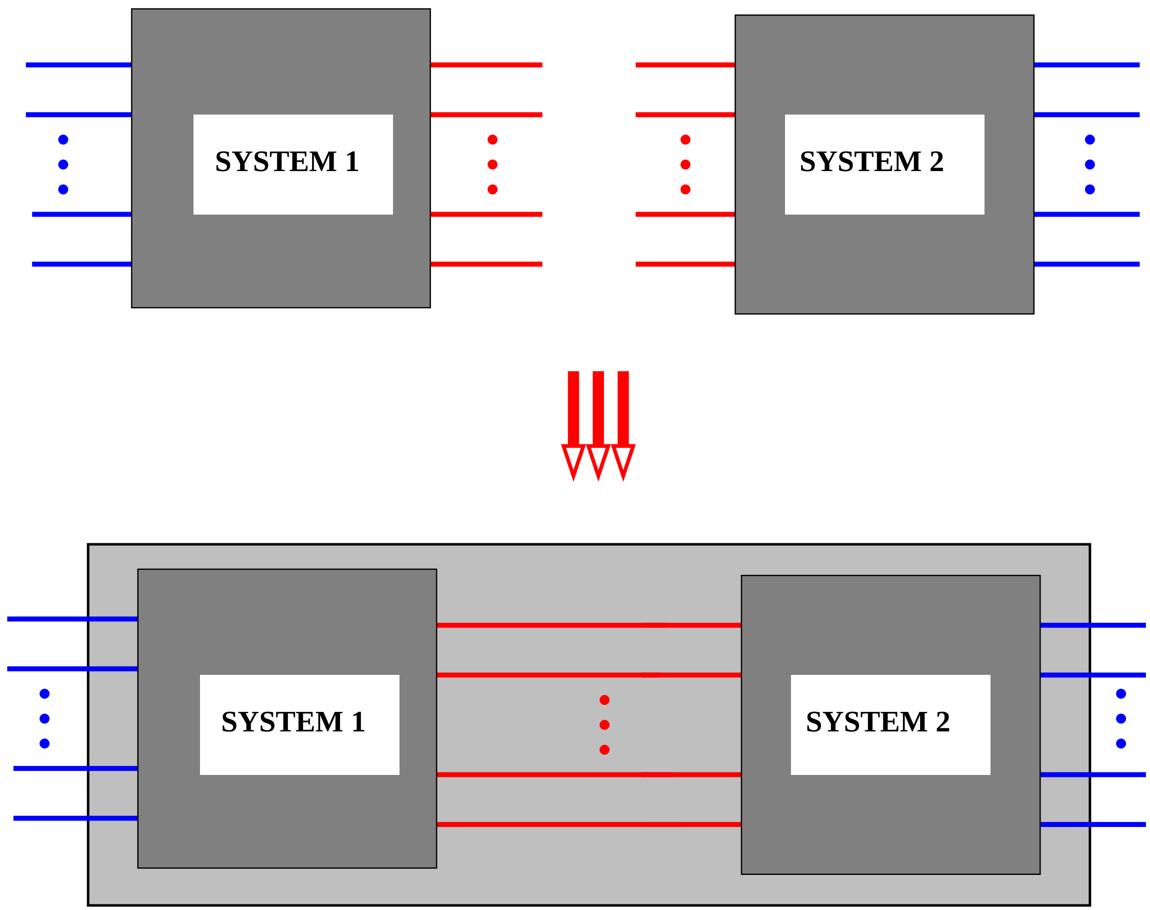


Open

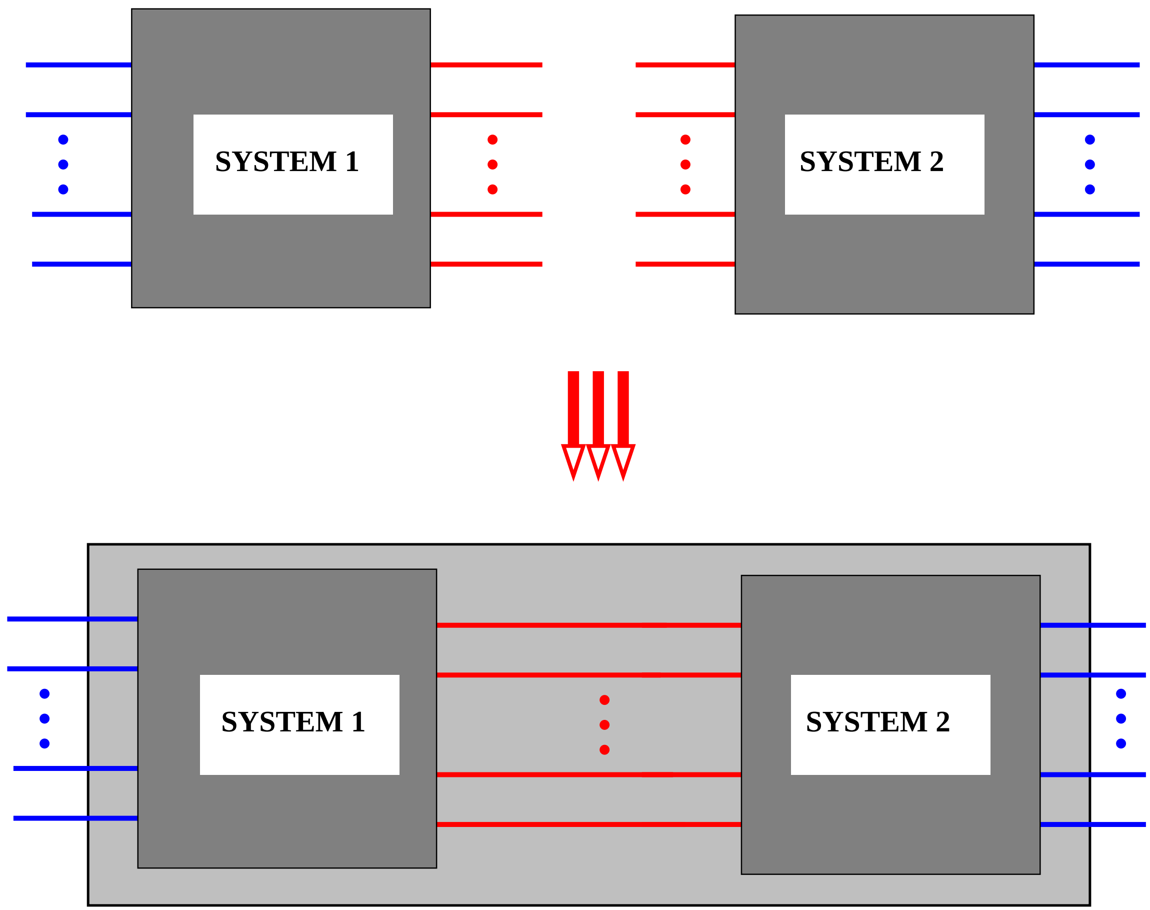


Connectable

Interconnection

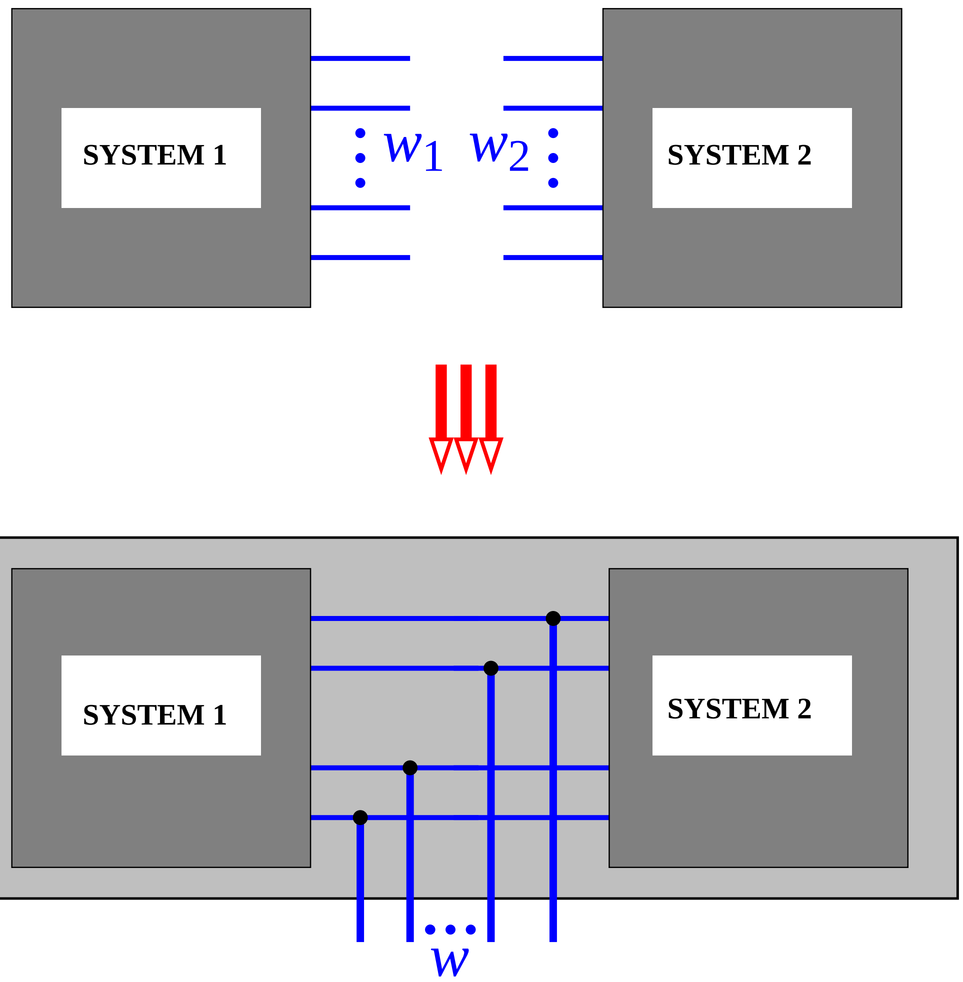


Interconnection



Consider the following (seemingly) special case.

Interconnection



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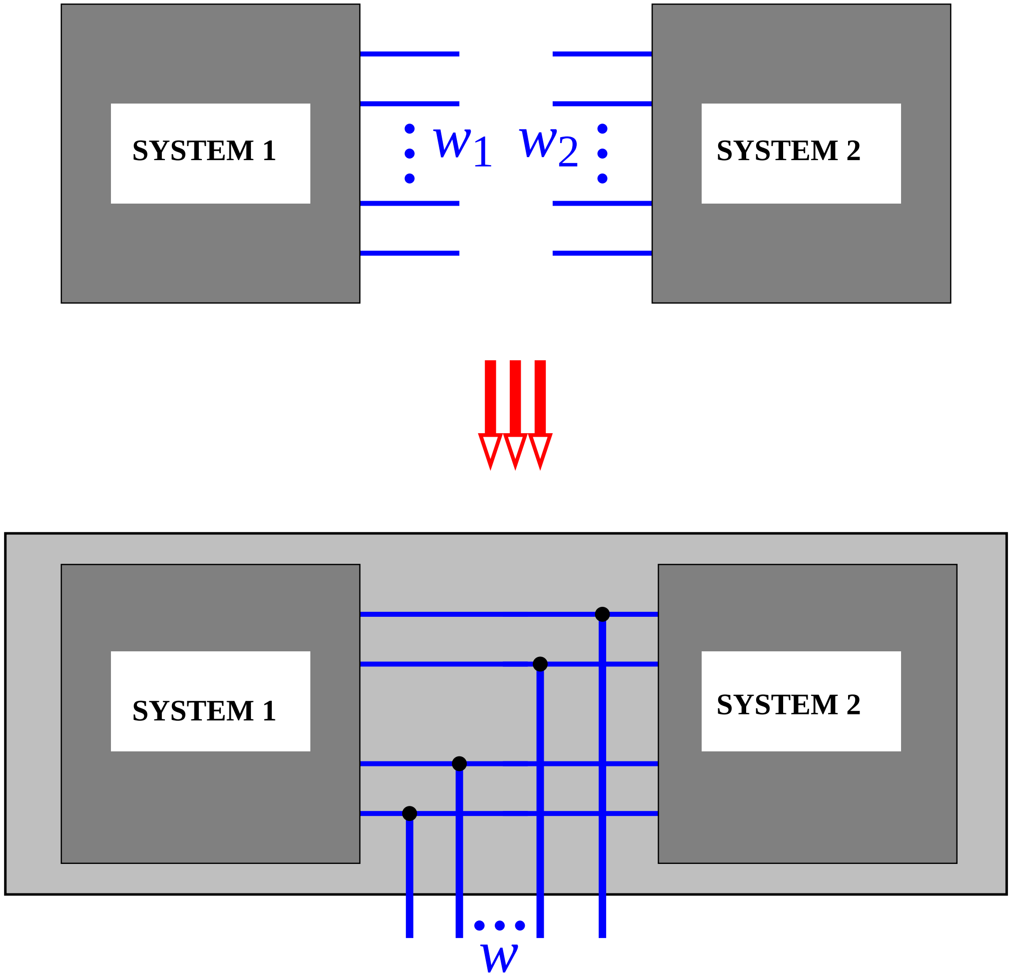
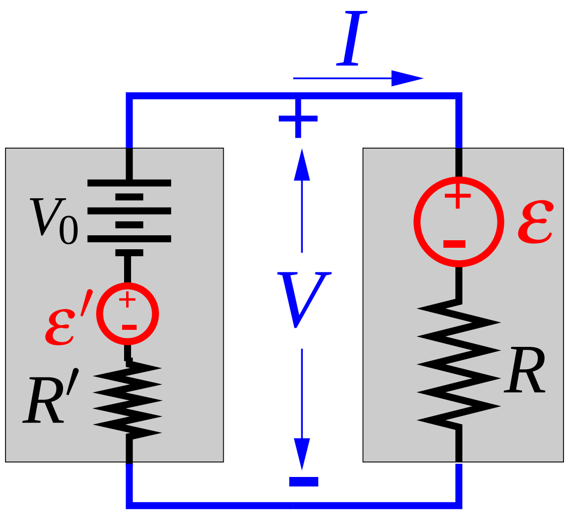
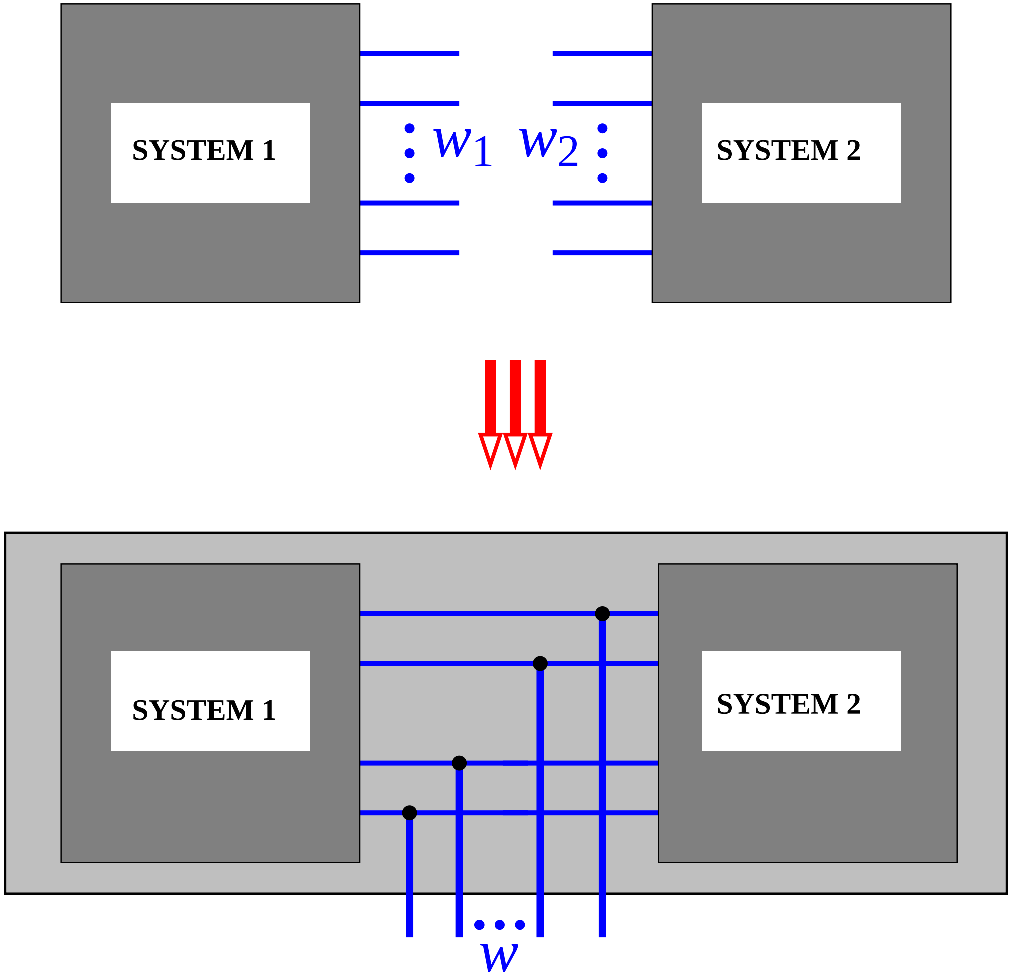


Example:



Interconnection

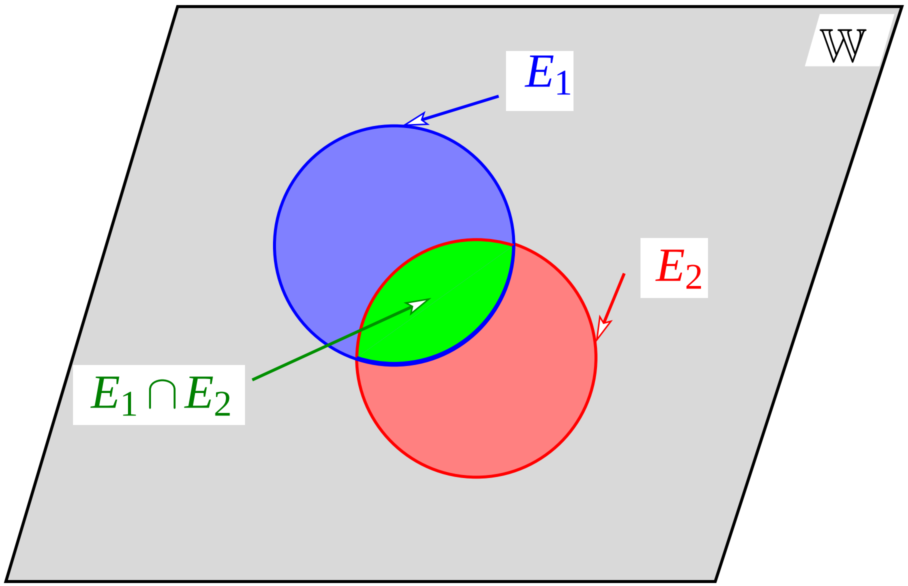


Can we impose two distinct probabilistic laws on the same set of variables?

Complementarity of σ-algebras

$\mathscr{E}_1$ **and** $\mathscr{E}_2$ **are** ***complementary σ-algebras*** **:**$\Leftrightarrow$
for all nonempty sets $E_1, E_1' \in \mathscr{E}_1, E_2, E_2' \in \mathscr{E}_2$

$$[\![E_1 \cap E_2 = E_1' \cap E_2']\!] \Rightarrow [\![E_1 = E_1' \textbf{ and } E_2 = E_2']\!].$$



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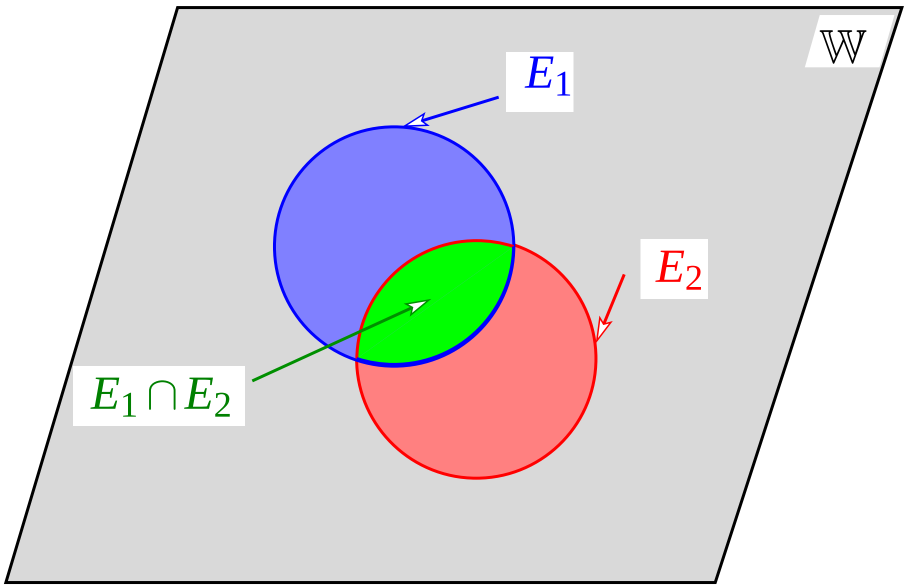
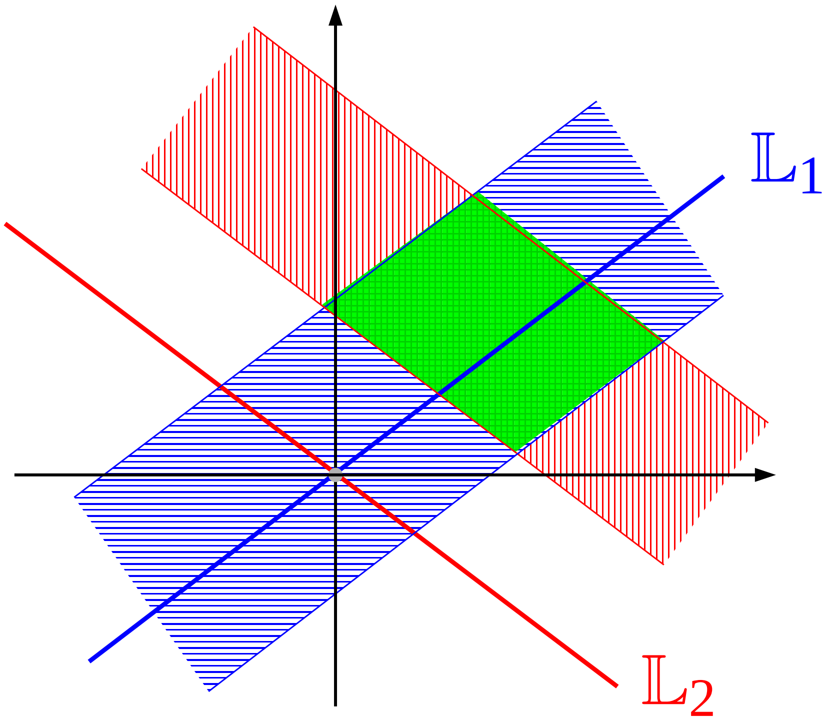


The intersection determines the intersectants.

Linear example

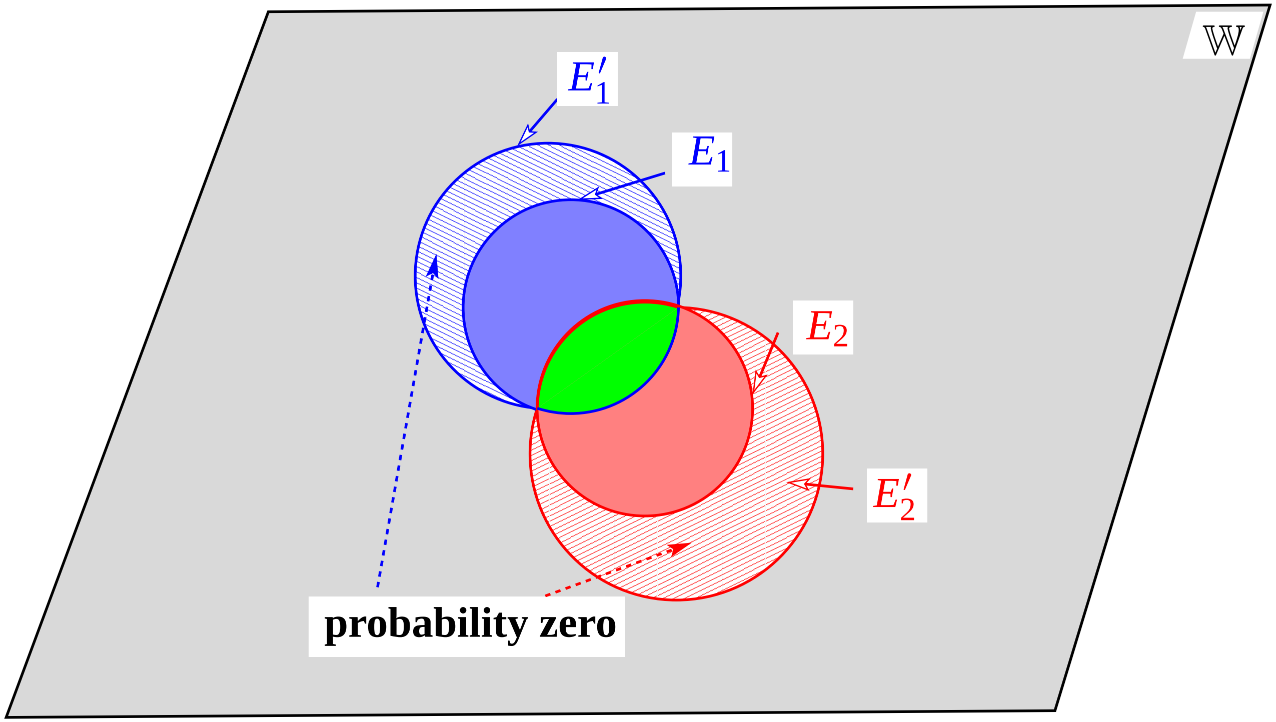


$$\mathbb{L}_1 + \mathbb{L}_2 = \mathbb{R}^n$$

Complementarity of systems

$(\mathbb{W}, \mathscr{E}_1, P_1)$ **and** $(\mathbb{W}, \mathscr{E}_2, P_2)$ **are said to be** ***complementary*** $:\Leftrightarrow$ **for** $E_1, E_1' \in \mathscr{E}_1$ **and** $E_2, E_2' \in \mathscr{E}_2$**:**

$$[\![E_1 \cap E_2 = E_1' \cap E_2']\!] \Rightarrow [\![P_1(E_1)P_2(E_2) = P_1(E_1')P_2(E_2')]\!].$$



Intersection $\Rightarrow$ product of probabilities of intersectants.

Interconnection of complementary systems

Let $(\mathbb{W}, \mathscr{E}_1, P_1)$ **and** $(\mathbb{W}, \mathscr{E}_2, P_2)$ **be stochastic systems (stochastically independent). Assume complementarity. Their** ***interconnection*** **is defined as**

$$(\mathbb{W}, \mathscr{E}, P)$$

with $\mathscr{E} :=$ **the** σ**-algebra generated by 'rectangles'**

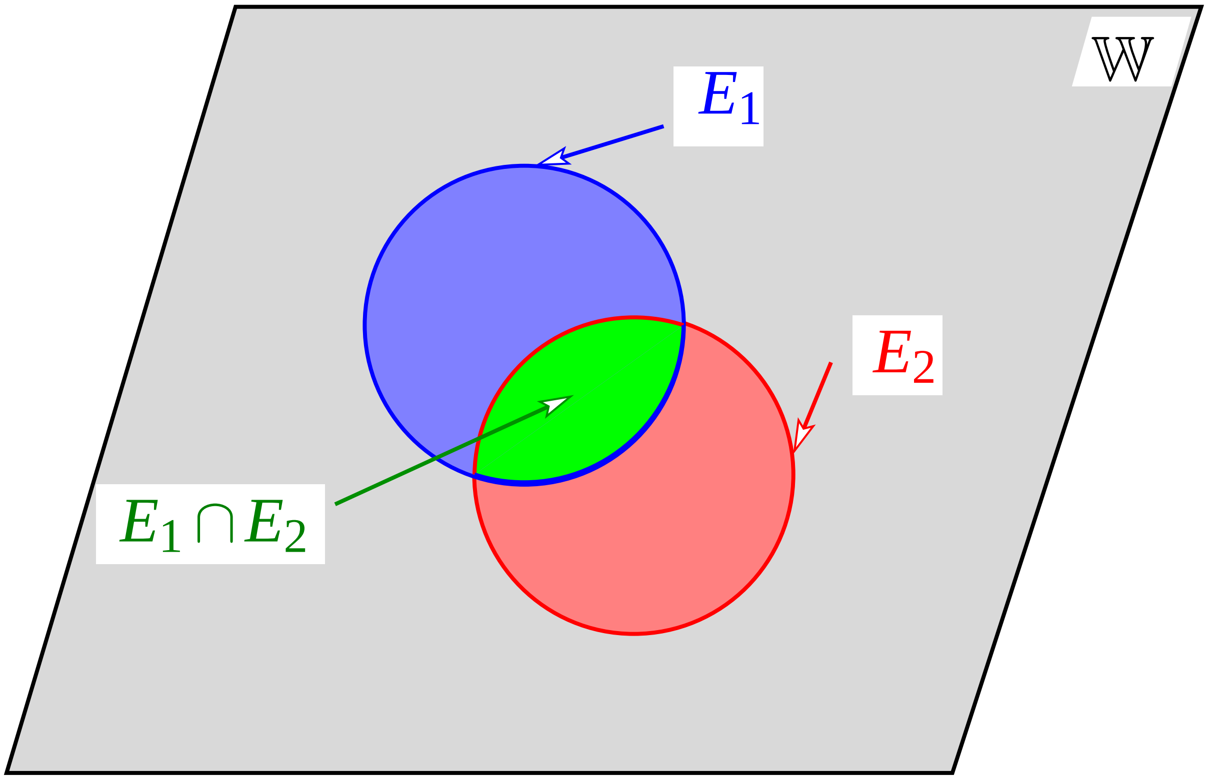
$$\{E_1 \cap E_2 \mid E_1 \in \mathscr{E}_1, E_2 \in \mathscr{E}_2\},$$

and P **defined through the rectangles by**

$$P(E_1 \cap E_2) := P_1(E_1) P_2(E_2).$$

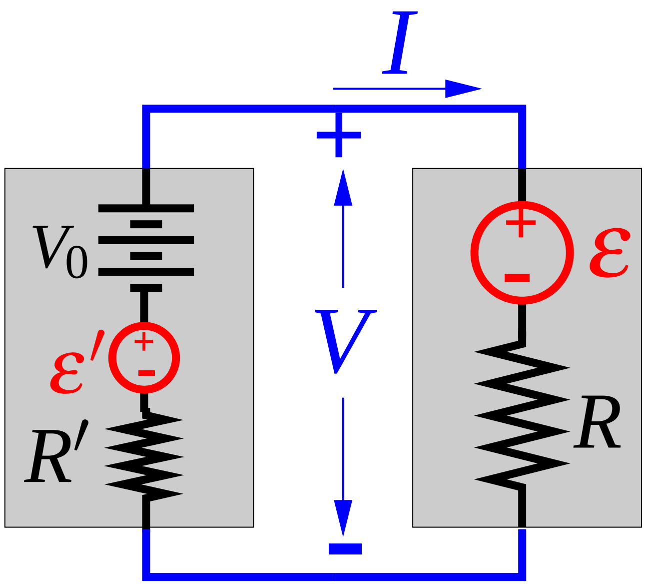
for $E_1 \in \mathscr{E}_1, E_2 \in \mathscr{E}_2$**.**

Interconnection of complementary systems

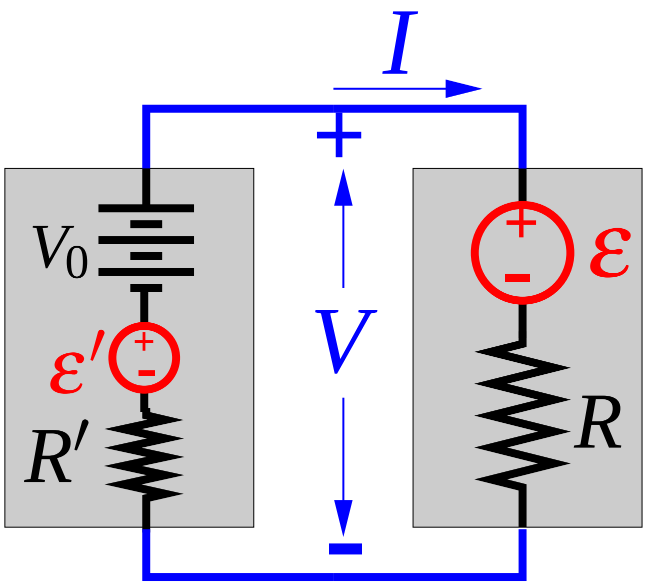


$$P(E_1 \cap E_2) := P_1(E_1)P_2(E_2).$$

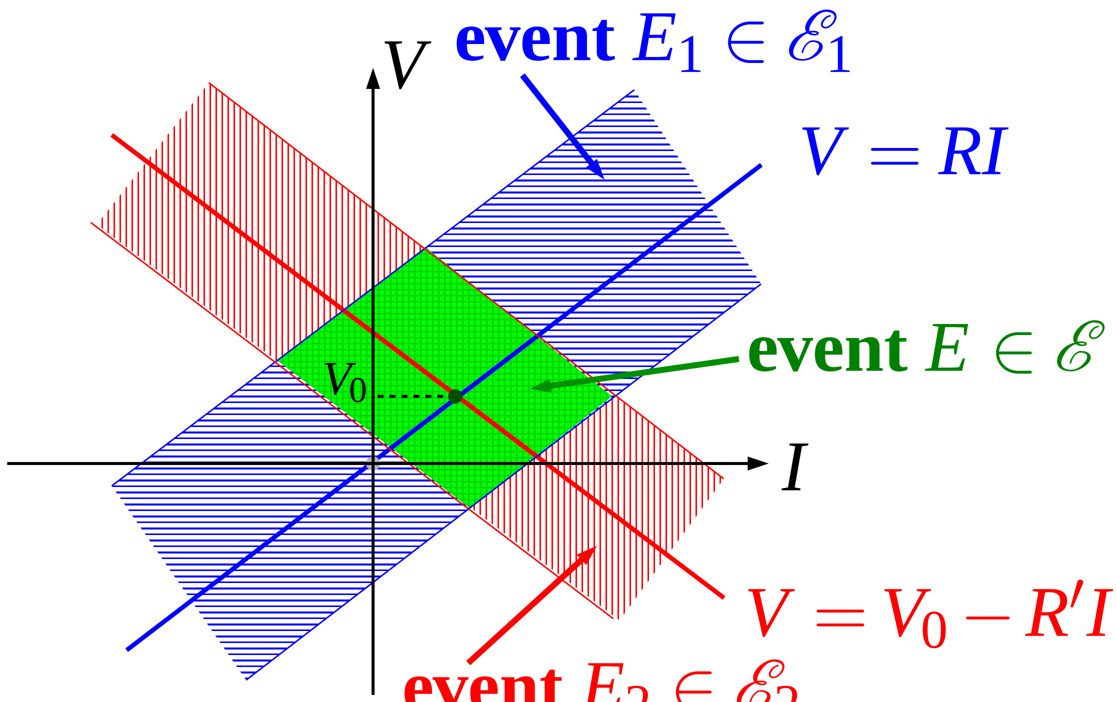
Noisy resistor terminated by voltage source



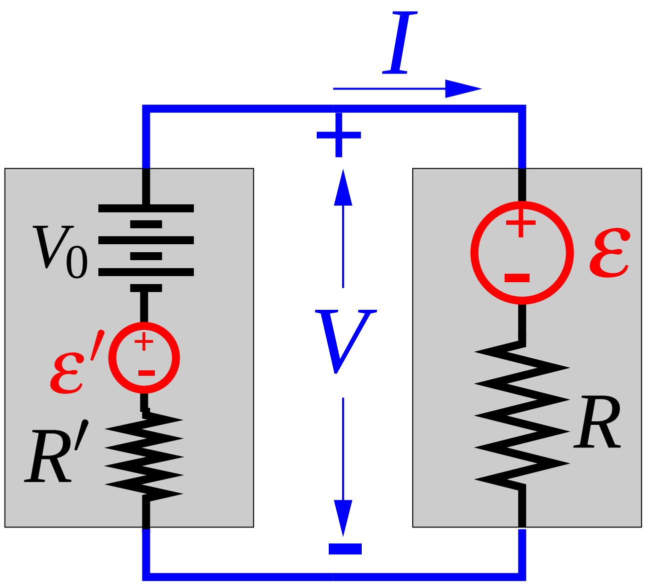
Noisy resistor terminated by voltage source



$$P(E) = P_1(E_1)P_2(E_2)$$



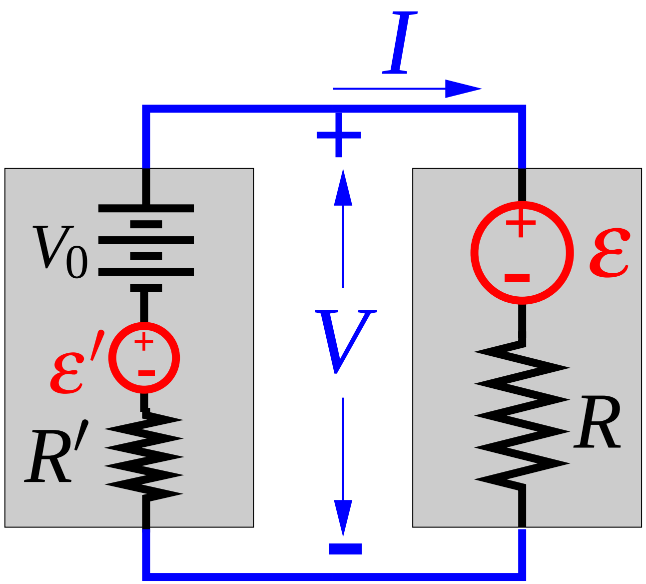
In terms of equations



$$V = RI + \varepsilon$$

$$V = V_0 - R'I + \varepsilon'$$

In terms of equations



$$V = RI + \varepsilon$$

$$V = V_0 - R'I + \varepsilon'$$

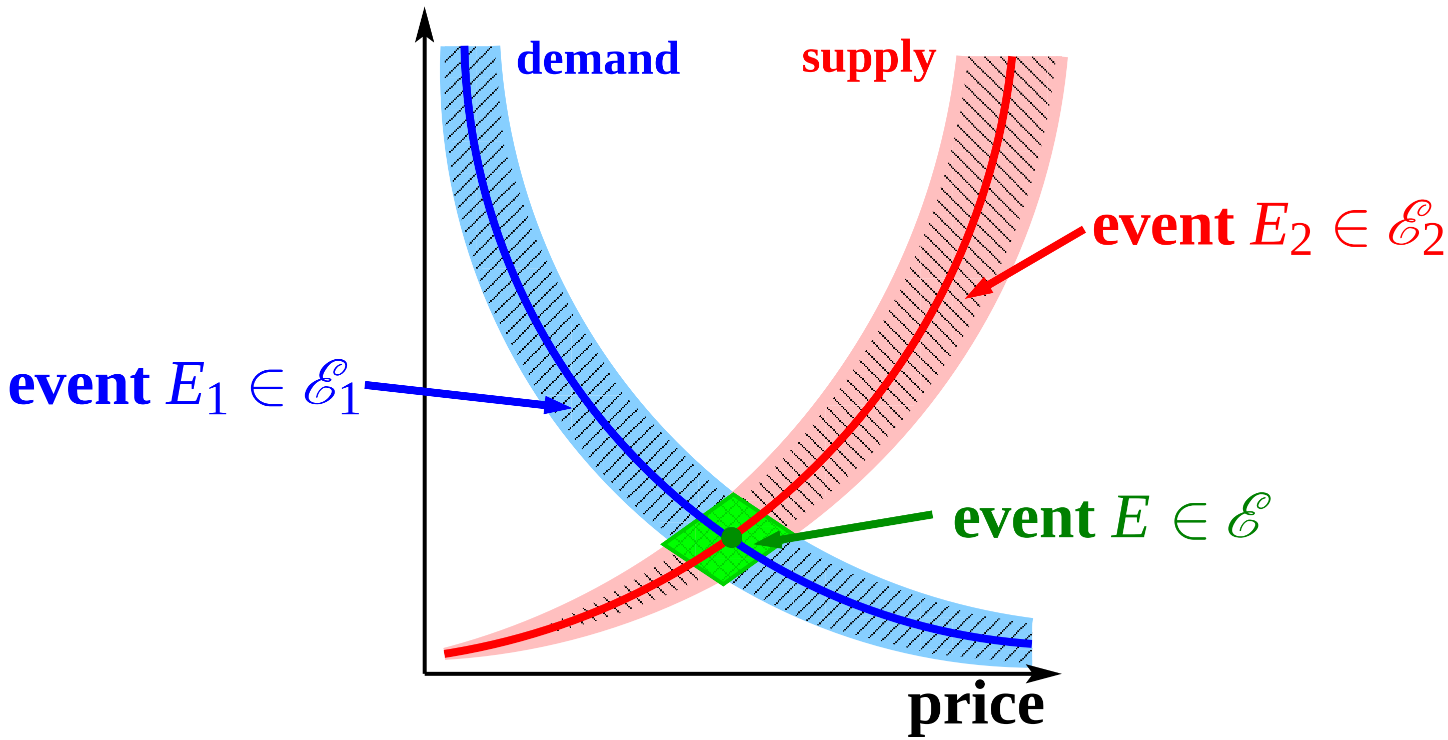
$$V = \frac{1}{R+R'}\left(R'\varepsilon + R(V_0 + \varepsilon')\right)$$

$$I = \frac{1}{R+R'}(-\varepsilon + V_0 + \varepsilon').$$

Shows that $\begin{bmatrix} V \\ I \end{bmatrix}$ is indeed a classical random vector.

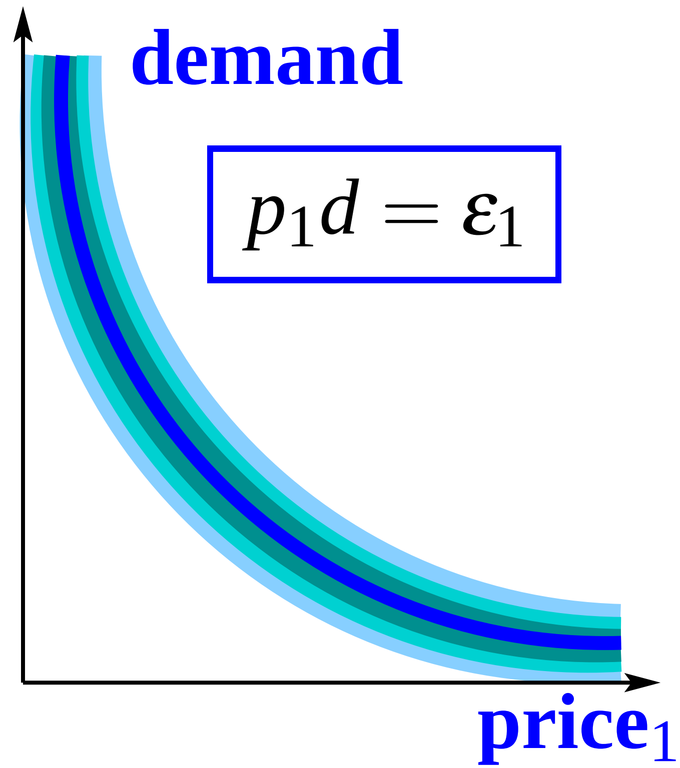
Complementarity $\Rightarrow$ this construction geometrically.

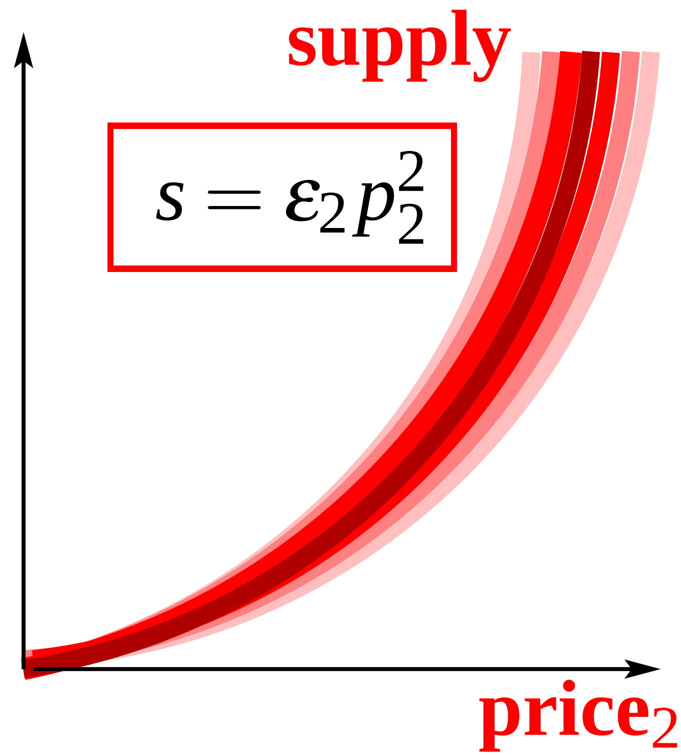
Equilibrium price/demand/supply



$$P(E) = P_1(E_1)P_2(E_2).$$

In terms of equations





$$p_1 = p_2 = \sqrt[3]{\frac{\varepsilon_1}{\varepsilon_2}}, \qquad d = s = \sqrt[3]{\varepsilon_1^2 \varepsilon_2}.$$

Complementarity $\Rightarrow$ this construction geometrically.

Open stochastic systems

Open versus closed

Consider $\Sigma_1 = (\mathbb{R}^{\mathtt{n}}, \mathscr{E}_1, P_1)$**.**

If $\mathscr{E}_1$ **= the Borel** σ**-algebra, and** $\mathtt{support}(P_1) = \mathbb{R}^{\mathtt{n}}$**, then** Σ_1 **is interconnectable only with the free system** $(\mathbb{R}^{\mathtt{n}}, \mathscr{E}_2, P_2)$**,** $\mathscr{E}_2 = \{\emptyset, \mathbb{R}^{\mathtt{n}}\}$**.**

$\Rightarrow$ **classical** Σ_1 **= ‘closed’ system.**

Open versus closed

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$\Rightarrow$ **classical** Σ_1 **= ‘closed’ system.**

Coarse $\mathscr{E}_1$

$\Rightarrow$ Σ_1 **is interconnectable.**

$\Rightarrow$ **‘open’ system.**

Open versus closed

In the Kolmogorov philosophy, random variables, random vectors, and random processes are (measurable) functions defined on the probability space $(\Omega, \mathscr{A}, P)$.

We view the randomness as 'internal' to the system.

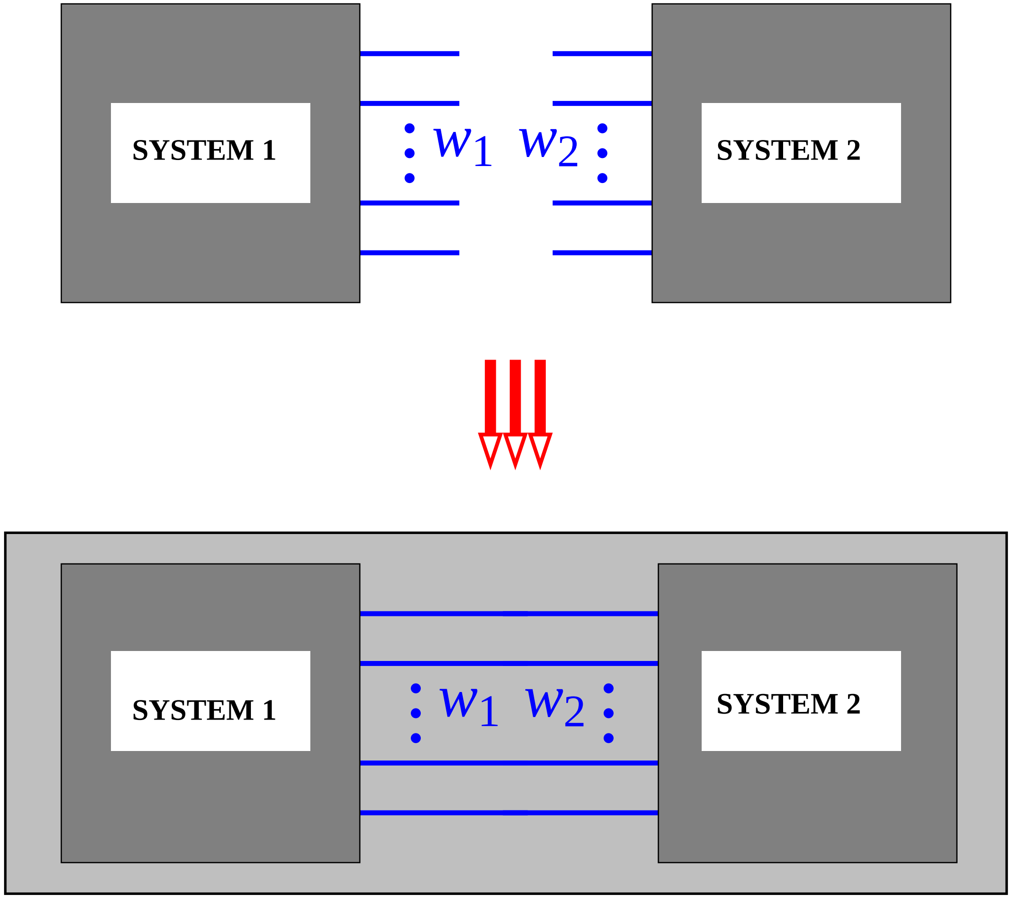
So, once the Gods choose $\omega \in \Omega$, all the random variables are determined.

The environment has no influence on the outcomes.

$\Rightarrow$ **'closed' systems.**

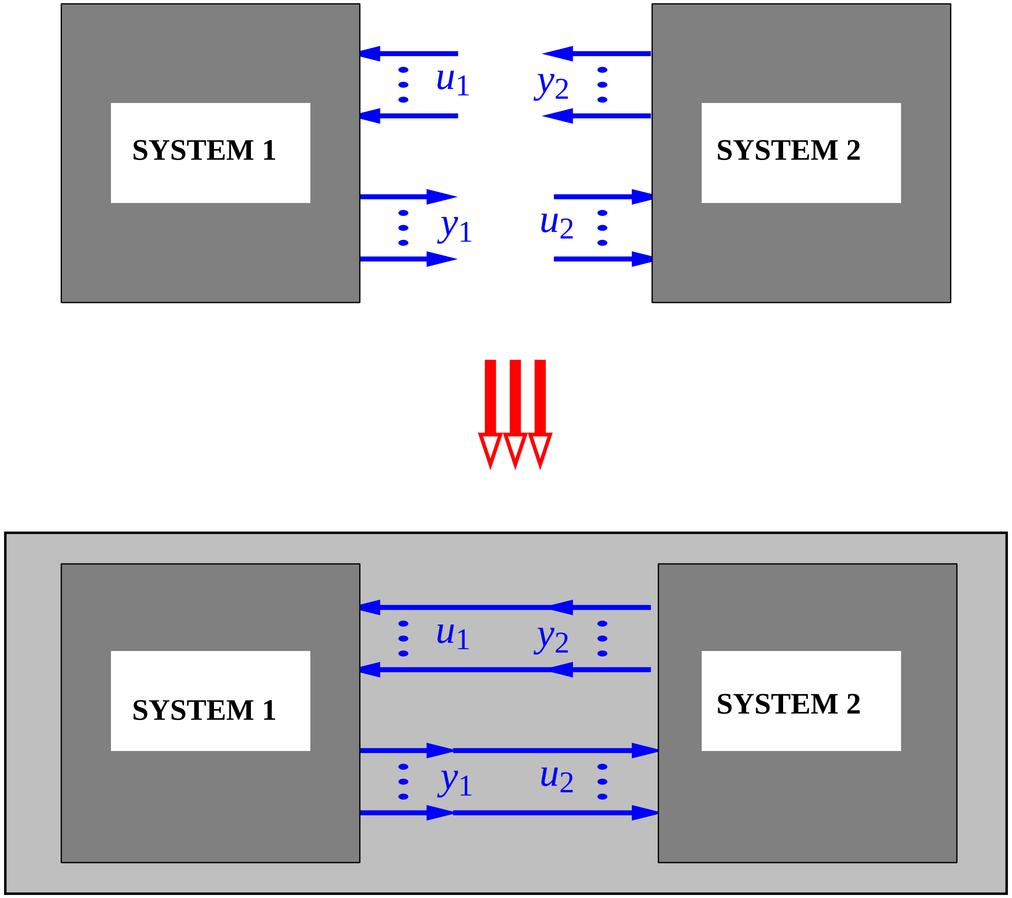
Interconnection $\Leftrightarrow$ variable sharing

Variable sharing



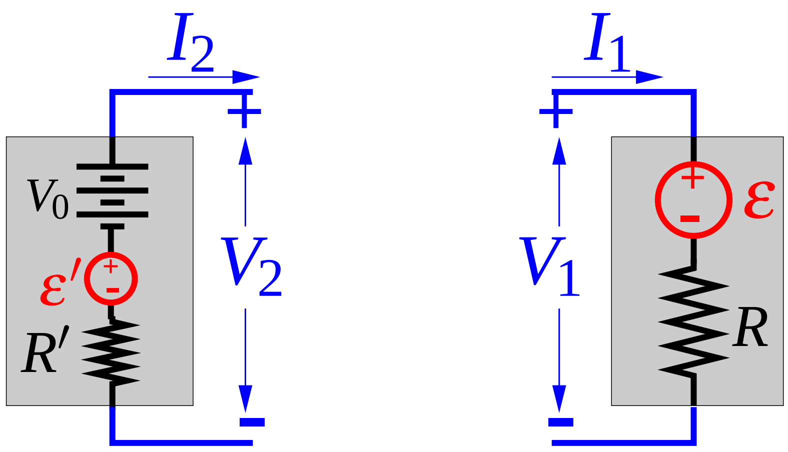
$$w_1 = w_2$$

Output-to-input assignment



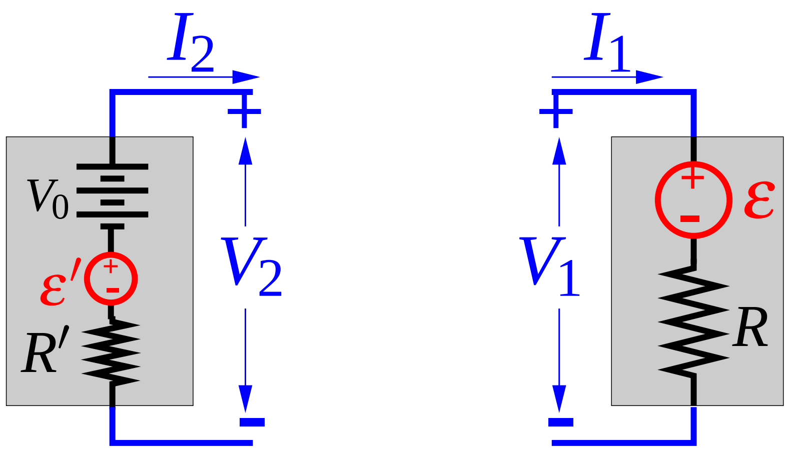
$$u_1 = y_2, \quad u_2 = y_1$$

Resistor interconnection

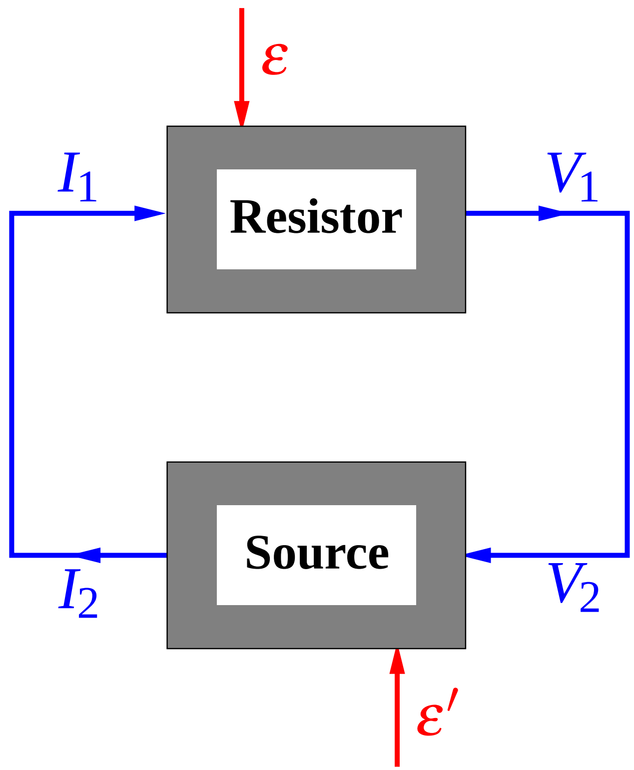


$$V_1 = V_2, \quad I_1 = I_2$$

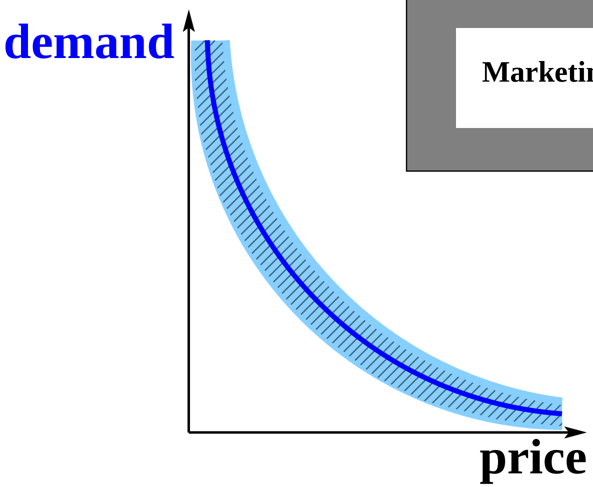
Resistor interconnection

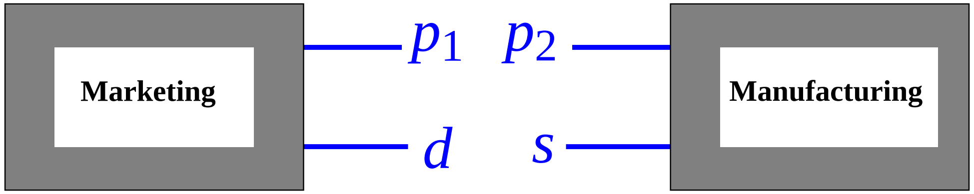


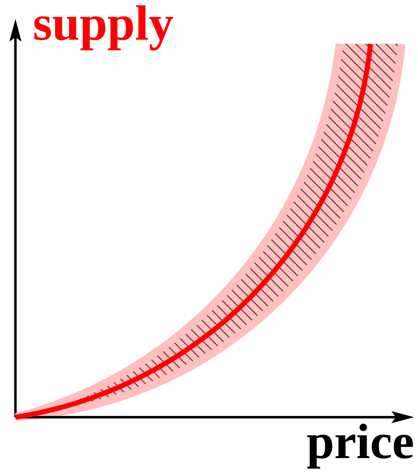
$$V_1 = V_2, \quad I_1 = I_2$$



Price/demand/supply interconnection

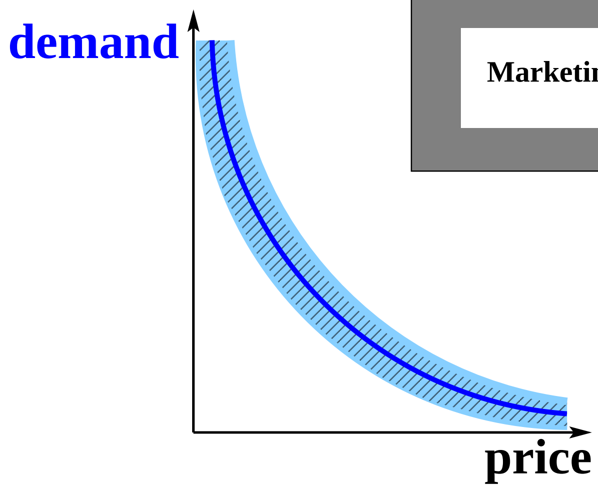


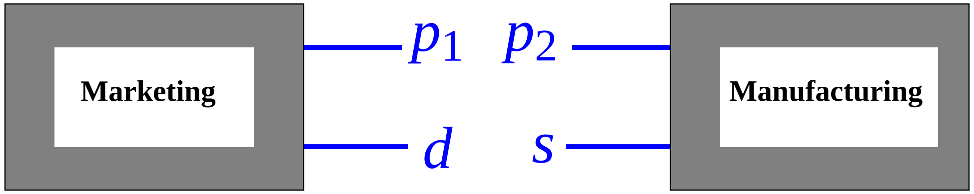


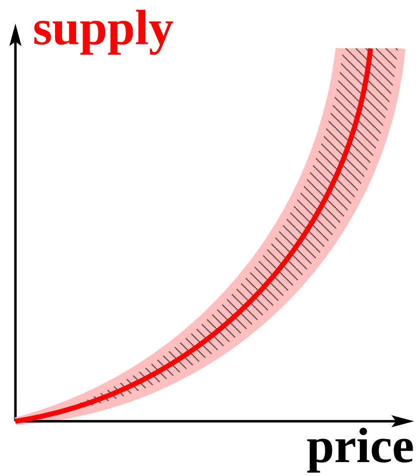


$$p_1 = p_2, \quad d = s$$

Price/demand/supply interconnection

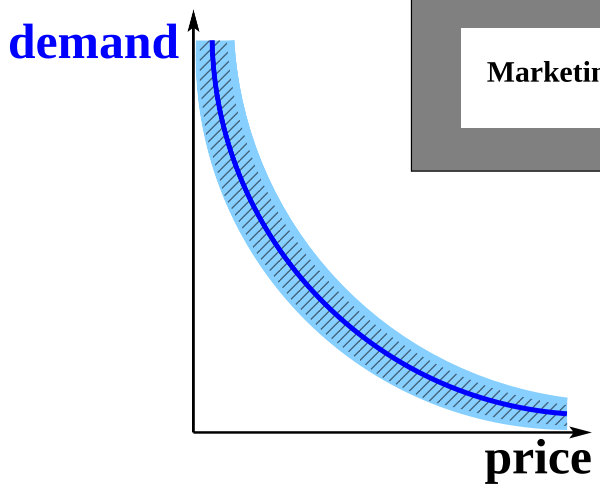


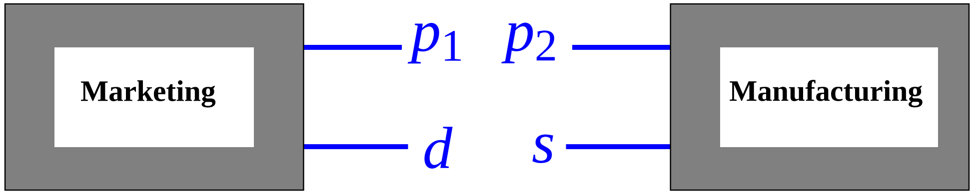


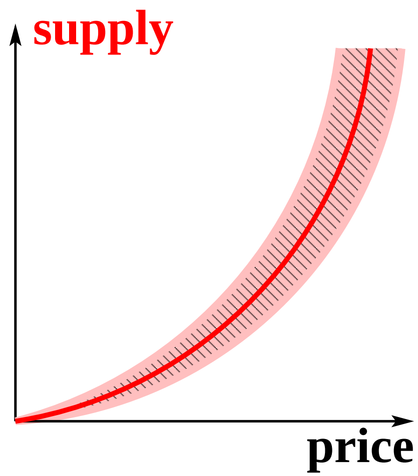


$$p_1 = p_2, \quad d = s$$

Price/demand/supply interconnection







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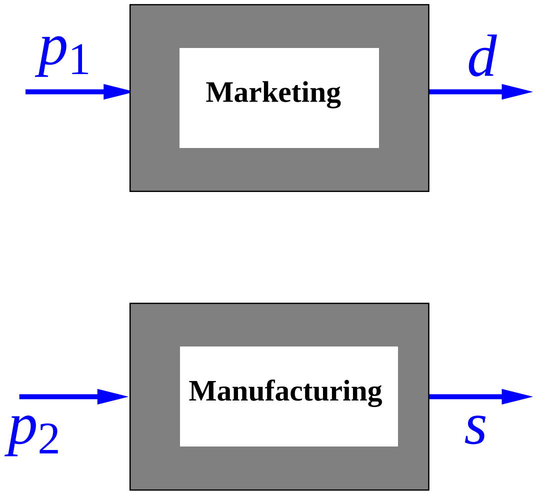
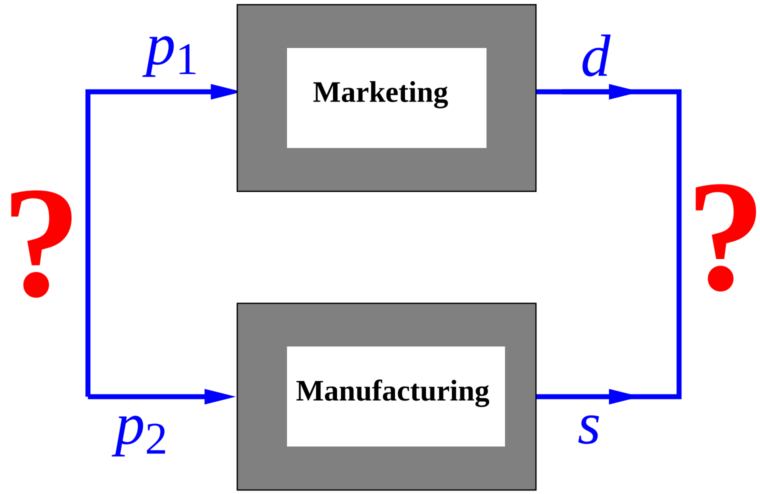
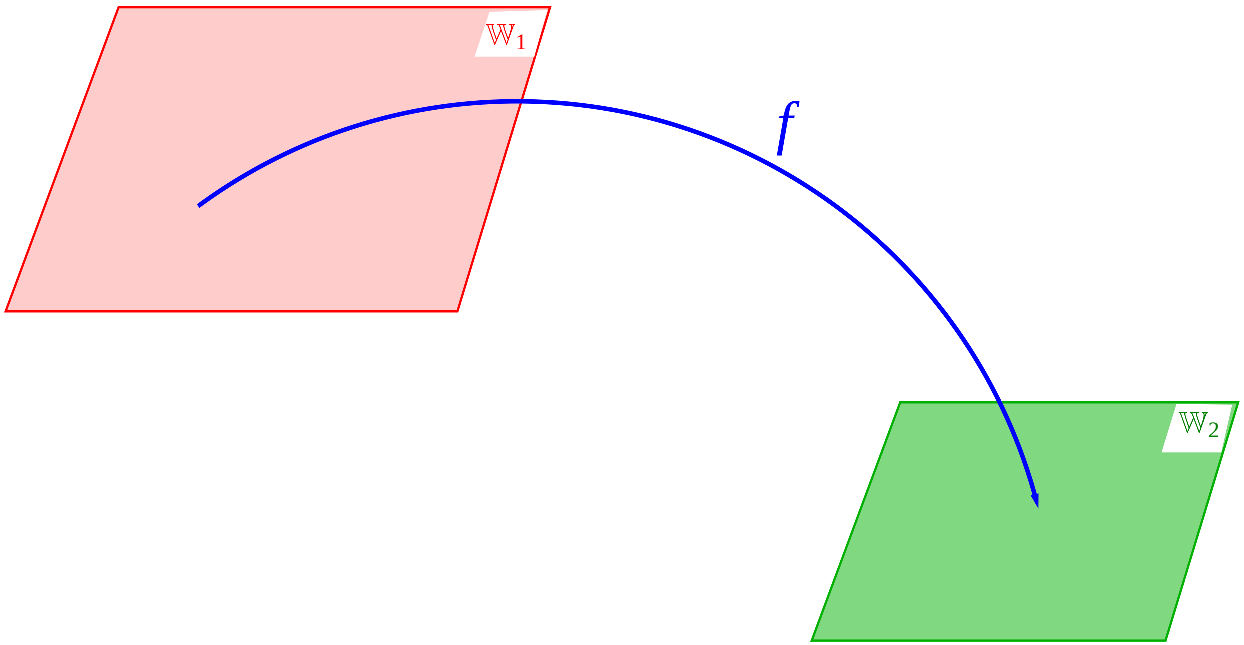




Functions

Problem

Consider the stochastic system $(\mathbb{W}_1, \mathscr{E}_1, P_1)$**,**
and the map $\mathbb{W}_1 \xrightarrow{f} \mathbb{W}_2$**.**

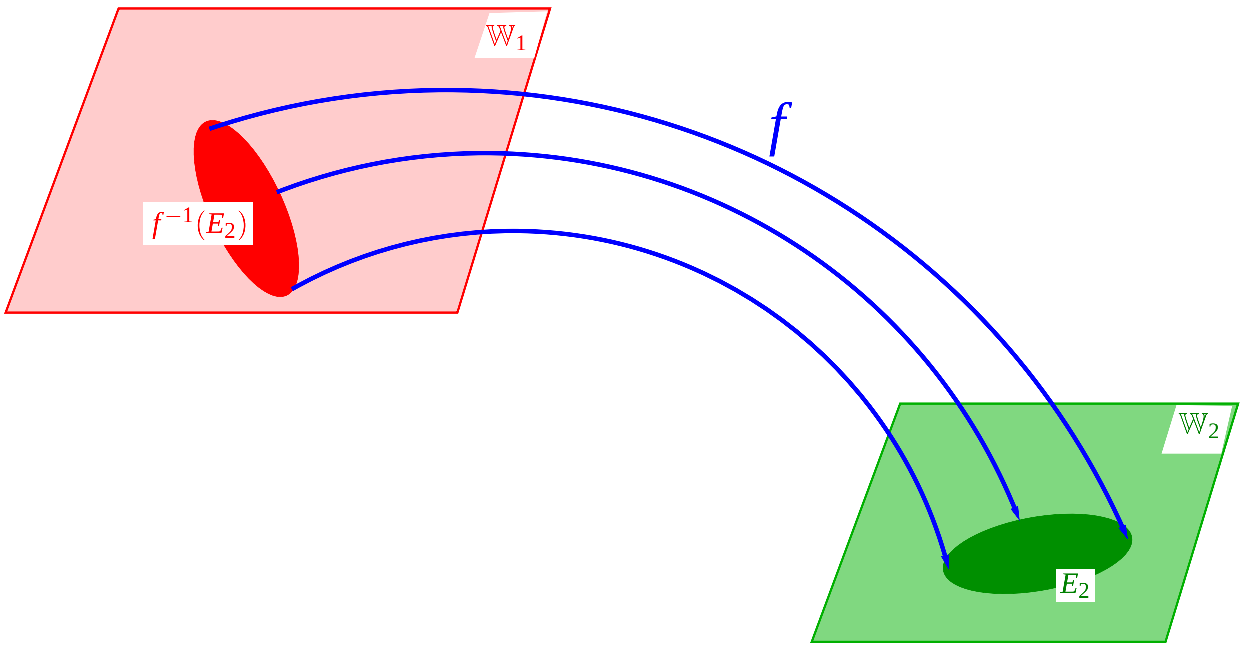


Which stochastic system on $\mathbb{W}_2$ ***does*** f ***generate?***

Pullback construction

If $(\mathbb{W}_1, \mathscr{E}_1) \xrightarrow{f} (\mathbb{W}_2, \mathscr{E}_2)$ **is a measurable map:** $\Leftrightarrow$

$$[\![E_2 \in \mathscr{E}_2]\!] \Rightarrow [\![f^{-1}(E_2) \in \mathscr{E}_1]\!],$$



then $P_2(E_2) := P_1(f^{-1}(E_2)) \rightsquigarrow (\mathbb{W}_2, \mathscr{E}_2, P_2)$.

Pullback construction

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$$[\![E_2 \in \mathscr{E}_2]\!] \Rightarrow [\![f^{-1}(E_2) \in \mathscr{E}_1]\!],$$

then $P_2(E_2) := P_1(f^{-1}(E_2)) \rightsquigarrow (\mathbb{W}_2, \mathscr{E}_2, P_2)$**.**

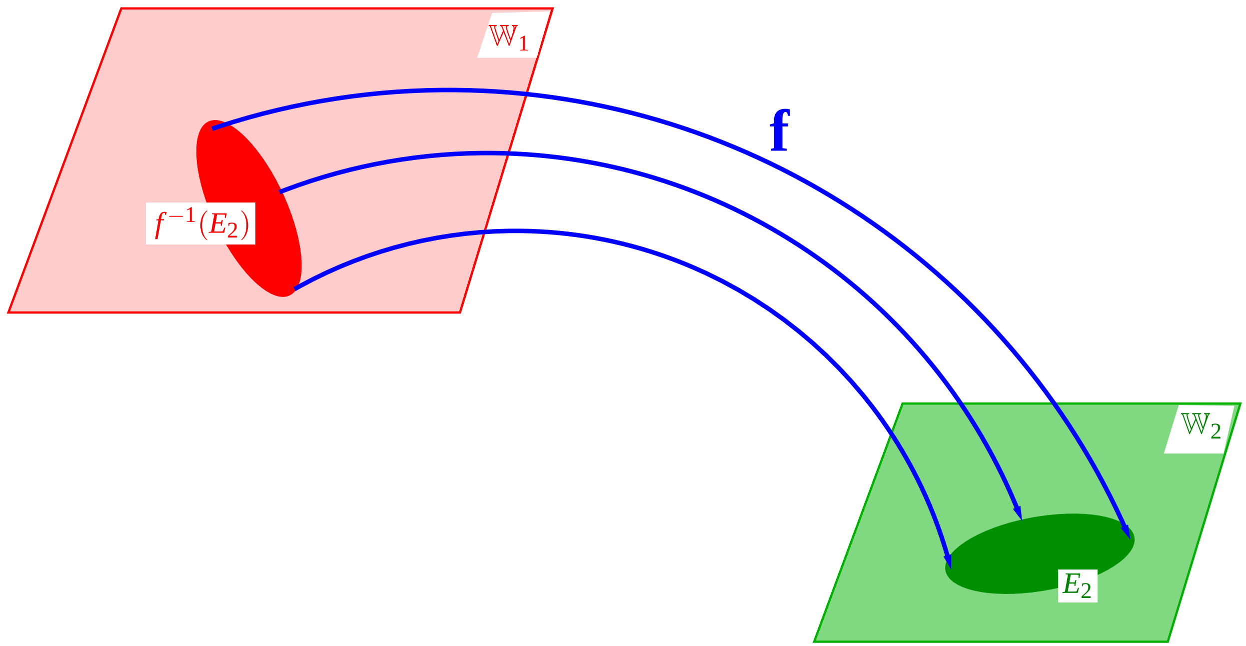
Usually, it is assumed that $\mathscr{E}_2$ **is given, say as** $\mathscr{B}(\mathbb{R}^{\mathtt{n}})$**, and that** f **is measurable.**

However, since the events are part of the stochastic phenomenon, $\mathscr{E}_2$ **ought to be constructed.**

Construction of $\mathscr{E}_2$

$$\mathscr{E}_2 := \{E_2 \subseteq \mathbb{W}_2 \mid f^{-1}(E_2) \in \mathscr{E}_1\}.$$

$\mathscr{E}_2$ **is a σ-algebra,** $\rightsquigarrow$ **stochastic system** $(\mathbb{W}_2, \mathscr{E}_2, P_2)$
with $P_2(E_2) = P_1\left(f^{-1}(E_2)\right)$.



$\mathscr{E}_2$ **= those subsets to which a probability can be assigned.**
$\mathscr{E}_2$ **is modeled, not obtained from the topology on** $\mathbb{W}_2$**.**

Example

Noisy resistor, $V = RI + \varepsilon$**,** $\quad R \neq 0$**.**

- $$f : \begin{bmatrix} V \\ I \end{bmatrix} \mapsto V.$$

  $\rightsquigarrow$ **stochastic system** $(\mathbb{R}, \{\emptyset, \mathbb{R}\}, P_2)$**.**

Example

Noisy resistor, $V = RI + \varepsilon$**,** $\quad R \neq 0$**.**

- $$f : \begin{bmatrix} V \\ I \end{bmatrix} \mapsto V.$$

  $\rightsquigarrow$ **stochastic system** $(\mathbb{R}, \{\emptyset, \mathbb{R}\}, P_2)$**.**

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  $\rightsquigarrow$ **stochastic system** $(\mathbb{R}, \{\emptyset, \mathbb{R}\}, P_2)$**.**

Example

Noisy resistor, $V = RI + \varepsilon$**,** $\quad R \neq 0$**.**

- $$f : \begin{bmatrix} V \\ I \end{bmatrix} \mapsto V.$$
  $\rightsquigarrow$ **stochastic system** $(\mathbb{R}, \{\emptyset, \mathbb{R}\}, P_2)$**.**
- $$f : \begin{bmatrix} V \\ I \end{bmatrix} \mapsto I.$$
  $\rightsquigarrow$ **stochastic system** $(\mathbb{R}, \{\emptyset, \mathbb{R}\}, P_2)$**.**
- $$f : \begin{bmatrix} V \\ I \end{bmatrix} \mapsto V - RI.$$
  $\rightsquigarrow$ **stochastic system** $(\mathbb{R}, \mathscr{B}(\mathbb{R}), P_2)$
  with $P_2 = \mathscr{N}(0, \sim \sqrt{RT})$**.**

Independence

Independence of stochastic variables

Independence of events is a measure theoretic concept. Does not need adjustment.

Let $(\mathbb{W}, \mathscr{E}, P)$ **be a stochastic system.**

Consider $\mathbb{W} \xrightarrow{f_1} \mathbb{W}_1$**,** $\mathbb{W} \xrightarrow{f_2} \mathbb{W}_2$

$\rightsquigarrow$ **stochastic systems** $(\mathbb{W}_1, \mathscr{E}_1, P_1)$**, and** $(\mathbb{W}_2, \mathscr{E}_2, P_2)$**.**

Are the outcomes w_1 ***and*** w_2 ***stochastically independent?***

Independence of stochastic variables

$$\mathbb{W} \xrightarrow{f_1} \mathbb{W}_1, \qquad \mathbb{W} \xrightarrow{f_2} \mathbb{W}_2.$$

Consider also $\mathbb{W} \xrightarrow{(f_1,f_2)} \mathbb{W}_1 \times \mathbb{W}_2$**.**

$$\rightsquigarrow \quad \Sigma_{12} = (\mathbb{W}_1 \times \mathbb{W}_2, \mathscr{E}_{12}, P_{12}).$$

Independence $:\Leftrightarrow (\mathbb{W}_1 \times \mathbb{W}_2, \mathscr{E}_{12}, P_{12})$
is the product of $(\mathbb{W}_1, \mathscr{E}_1, P_1)$ **and** $(\mathbb{W}_2, \mathscr{E}_2, P_2)$**.**

Noisy resistor $R \neq 0$**:** V **and** I **are not independent.**
$R = 0$**:** V **and** I **are independent.**

Conditioning & Constraining

Conditional probability

Let $\Sigma = (\mathbb{W}, \mathscr{E}, P)$. **Look at the outcomes** $\boxed{w \in \mathbb{S}.}$

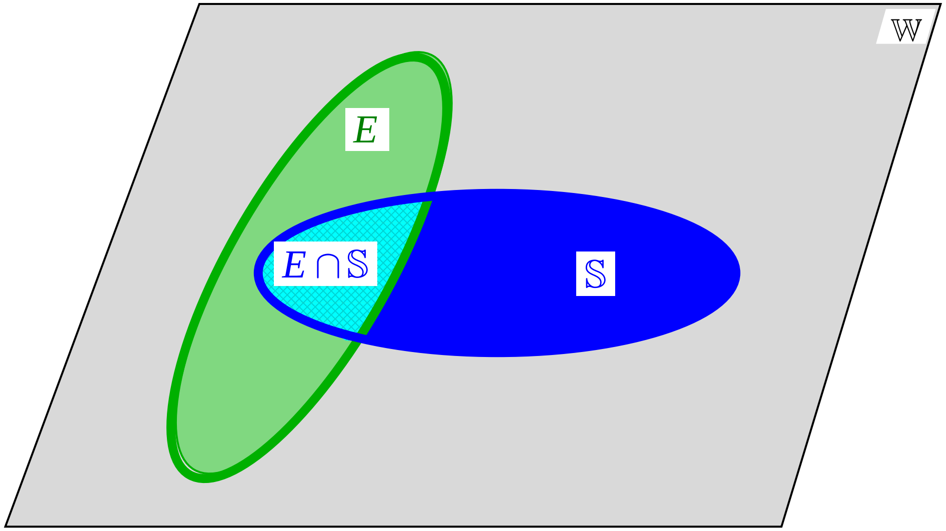
When $\mathbb{S}$ **is an event, that is** $\mathbb{S} \in \mathscr{E}$,

$\rightsquigarrow$ **conditional probability. Assume** $P(S) > 0$. **Then**

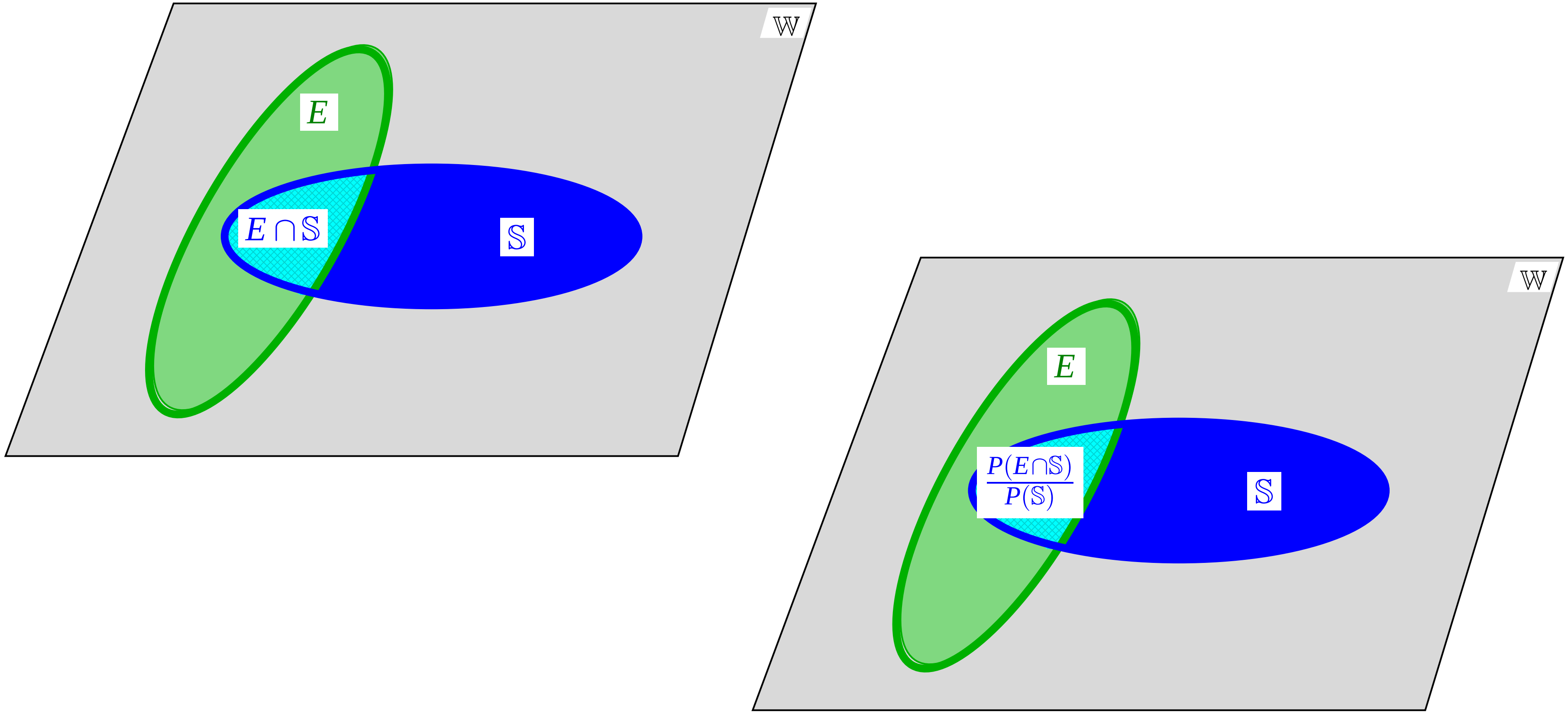
$$\Sigma|_{\mathbb{S}} := (\mathbb{S}, \mathscr{E} \cap \mathbb{S}, P(\cdot|\mathbb{S})), \text{ with } P(E \cap \mathbb{S}|\mathbb{S}) := \frac{P(E \cap \mathbb{S})}{P(\mathbb{S})}.$$

The construction of $P(\cdot|\mathbb{S})$ **is more complicated when** $P(\mathbb{S}) = 0$, **but well-known.**

Conditional probability



Conditional probability



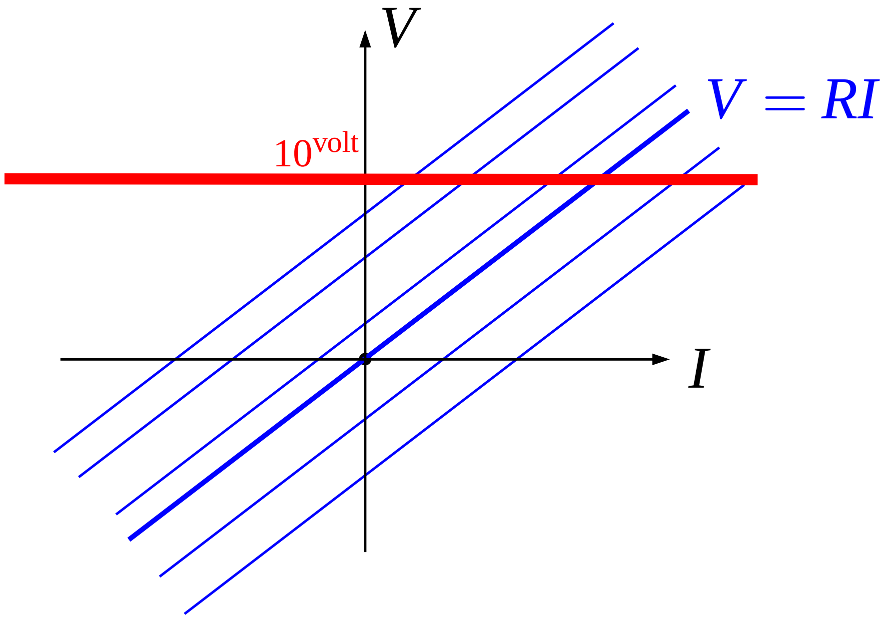
Constrained probability

Let $\Sigma = (\mathbb{W}, \mathscr{E}, P)$. Impose the constraint $\boxed{\text{“ } w \in \mathbb{S} \text{ ”}}$ **with** $\mathbb{S} \subset \mathbb{W}$, $\boxed{\mathbb{S} \notin \mathscr{E}.}$

What is the stochastic nature of the outcomes in $\mathbb{S}$ **?**

Is this a meanigful question?

Noisy resistor

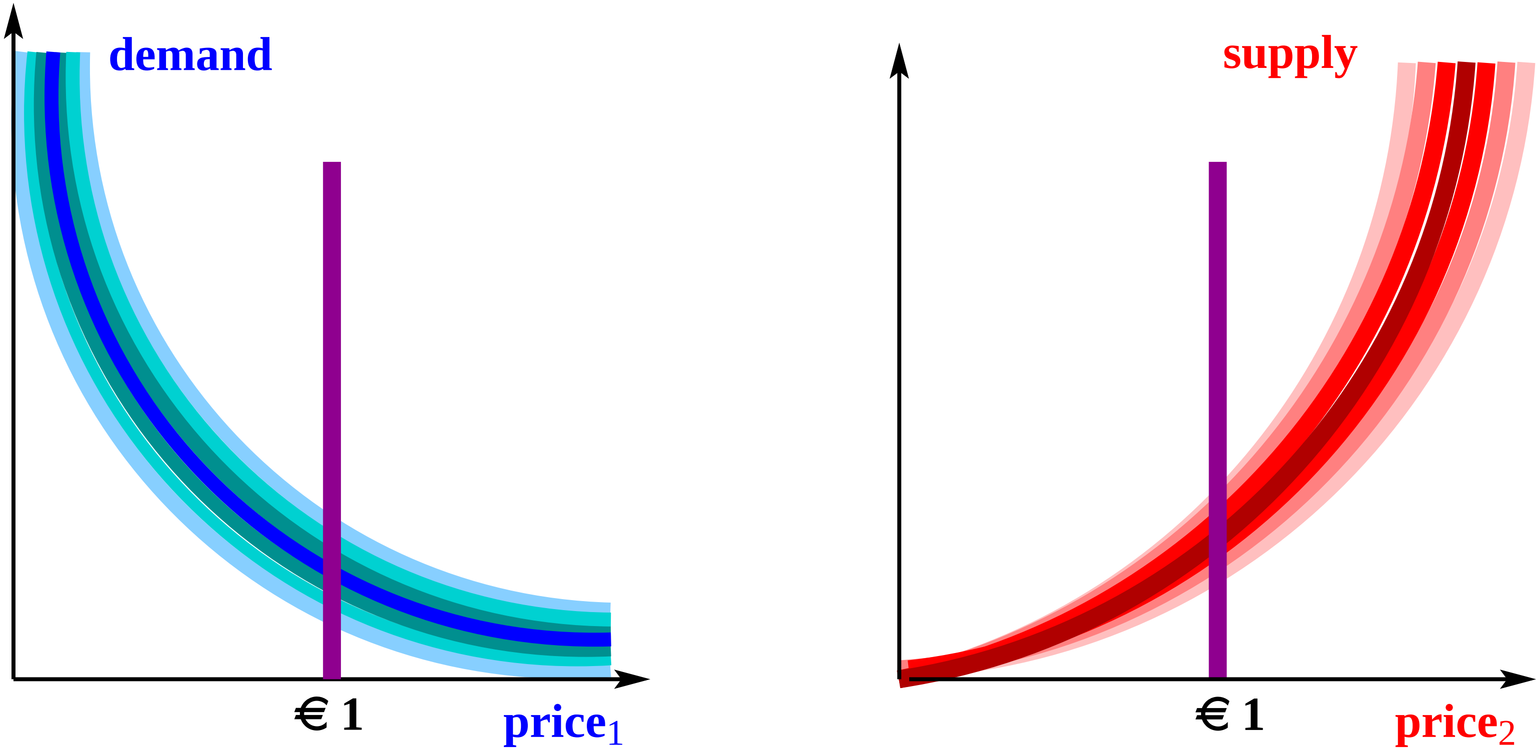


Impose $V = 10^{\text{volt}}$. What is the distribution of I?

$$V = RI + \varepsilon, V = 10^{\text{volt}} \;\Rightarrow\; I = \frac{V_0}{10} - \frac{\varepsilon}{10}.$$

I is a well-defined random variable!

Price/demand/supply example



Impose price = € 1. Distribution of demand, supply?

$p_1 d = \varepsilon_1, p_1 = e\ 1 \Rightarrow d = \varepsilon_1; s = \varepsilon_2 p_2^2, p_2 = e\ 1 \Rightarrow 2 = \varepsilon_2$**.**

d, s **become well-defined random variables.**

Constrained probability

Let $\Sigma = (\mathbb{W}, \mathscr{E}, P)$**. Impose the constraint** $\boxed{\text{“ } w \in \mathbb{S} \text{ ”}}$ **with** $\mathbb{S} \subset \mathbb{W}$, $\boxed{\mathbb{S} \notin \mathscr{E}.}$

What is the stochastic nature of the outcomes in $\mathbb{S}$ **?**

Is this a meanigful question? Yes!

Constrained probability

Conditioning $\simeq$ **interconnection of** $\Sigma = (\mathbb{W}, \mathscr{E}, P)$ **and the deterministic system with behavior** $\mathbb{S}$**.**

Assume complementarity:

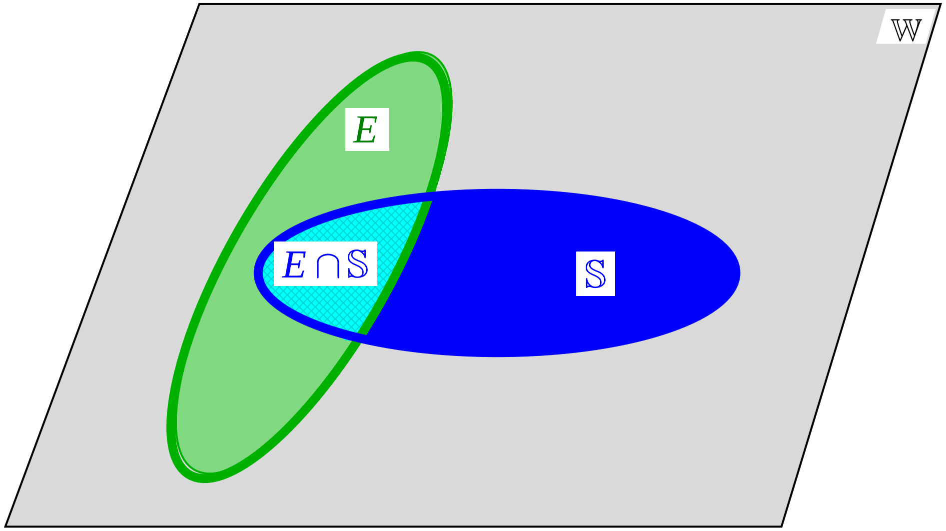
$[\![E_1, E_2 \in \mathscr{E}$ **and** $E_1 \cap \mathbb{S} = E_2 \cap \mathbb{S}]\!] \Rightarrow [\![P(E_1) = P(E_2)]\!]$

Interconnection $\rightsquigarrow$

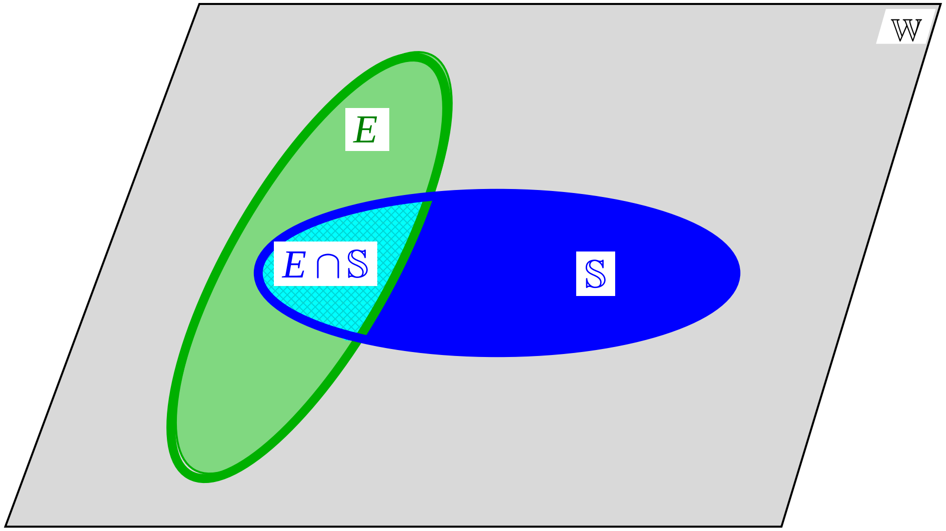
$\Sigma|_{\mathbb{S}} = (\mathbb{S}, \mathscr{E} \cap \mathbb{S}, P(\cdot|\mathbb{S}))$ **with** $P(E \cap \mathbb{S}|\mathbb{S}) := P(E)$.

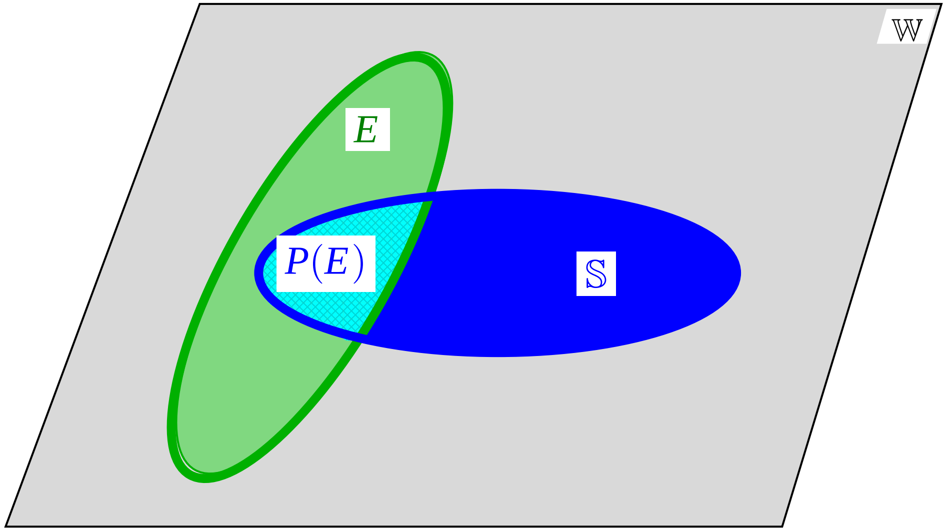
$P(\cdot|\mathbb{S})$ **= "probability of** w **constrained by** $w \in \mathbb{S}$**".**

Constrained probability

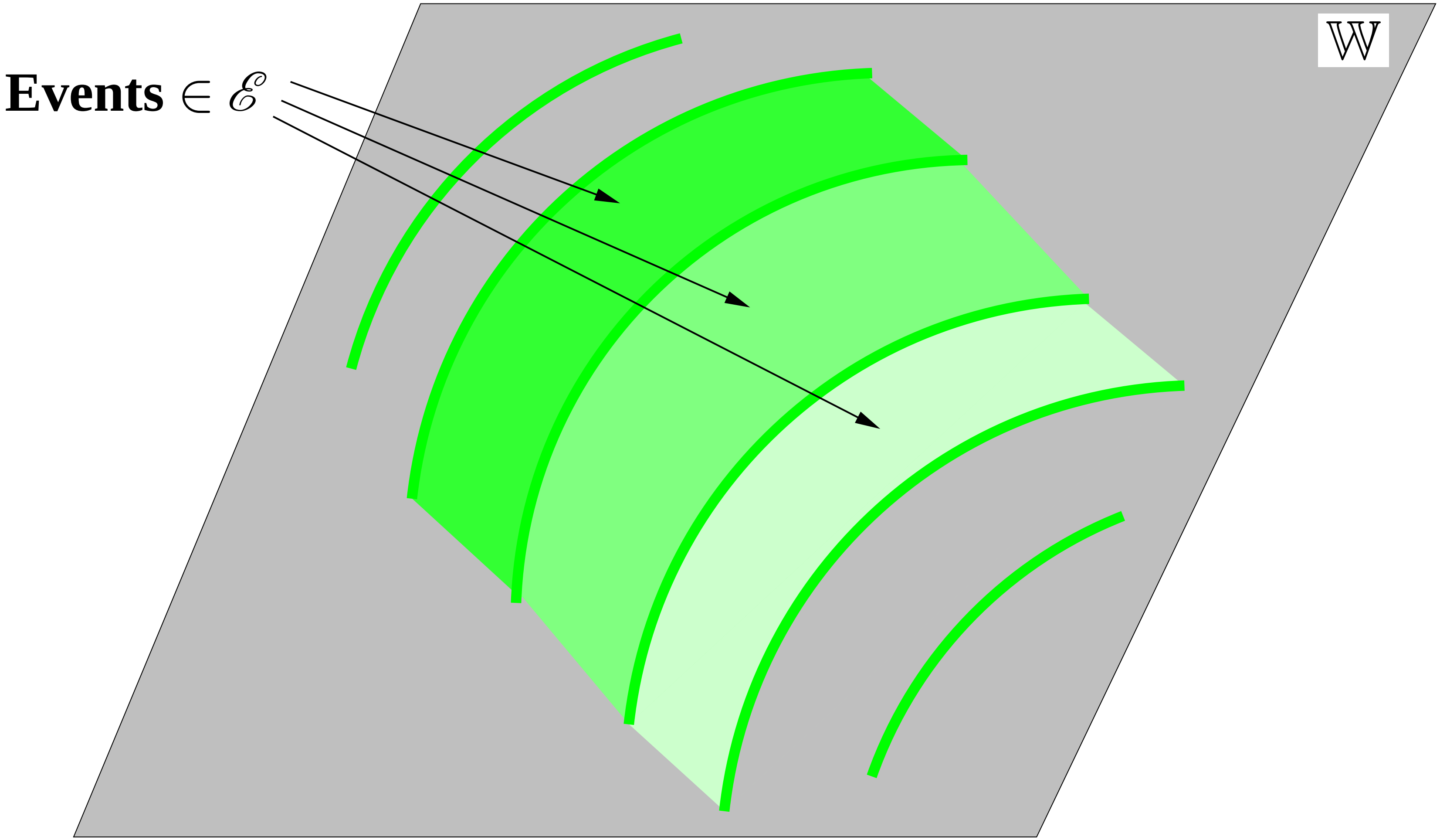


Constrained probability

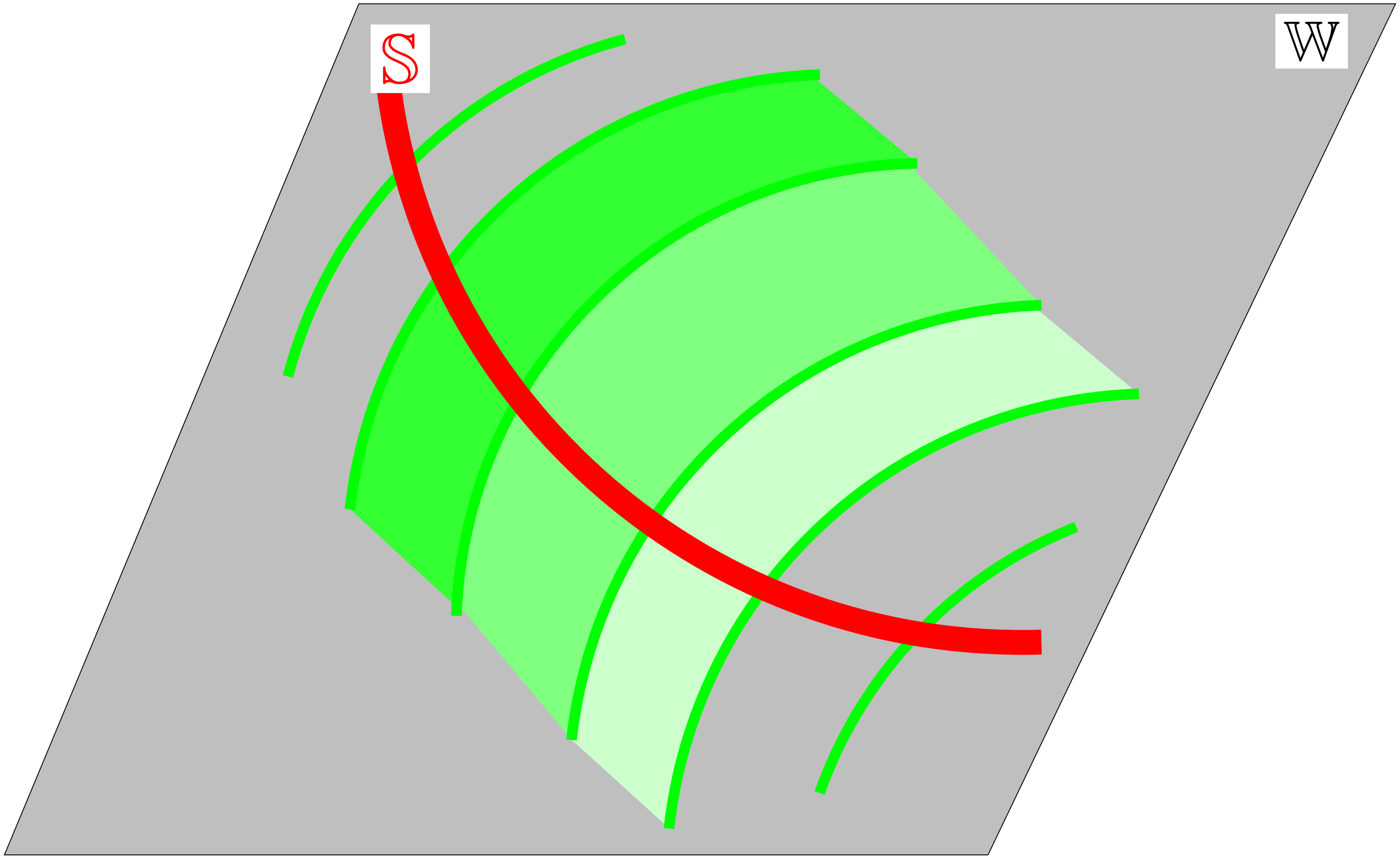




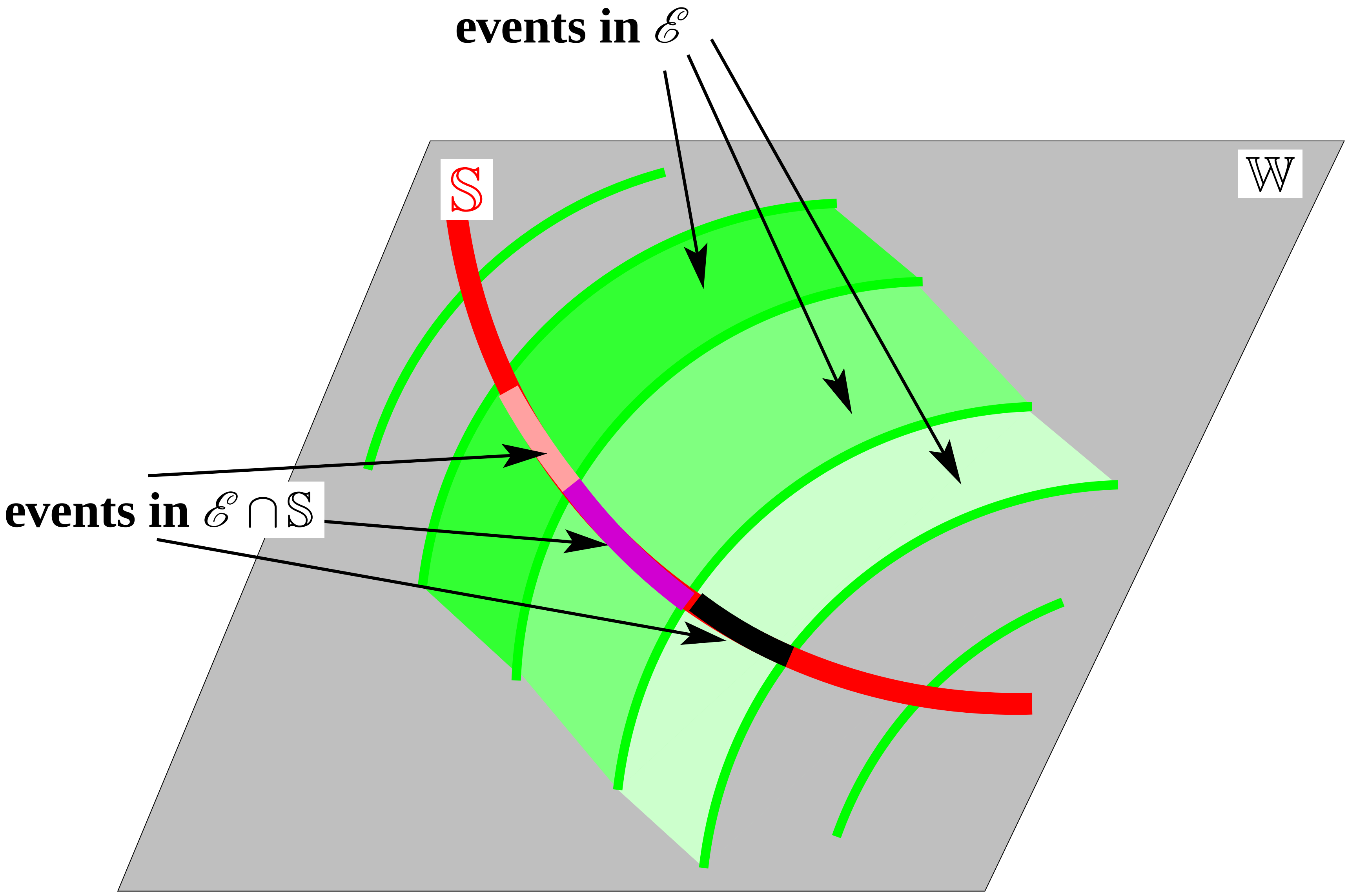
In pictures



In pictures

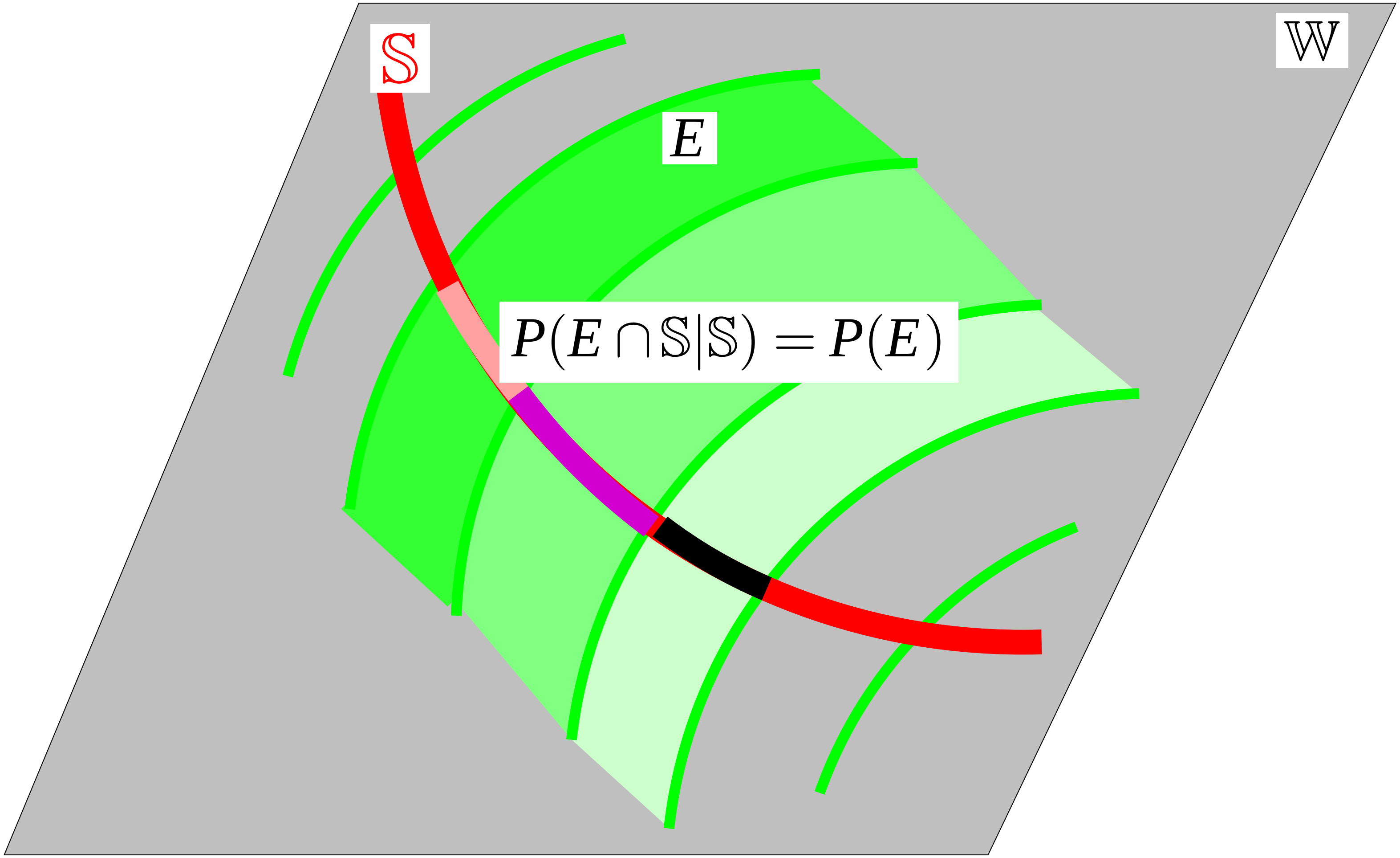


In pictures



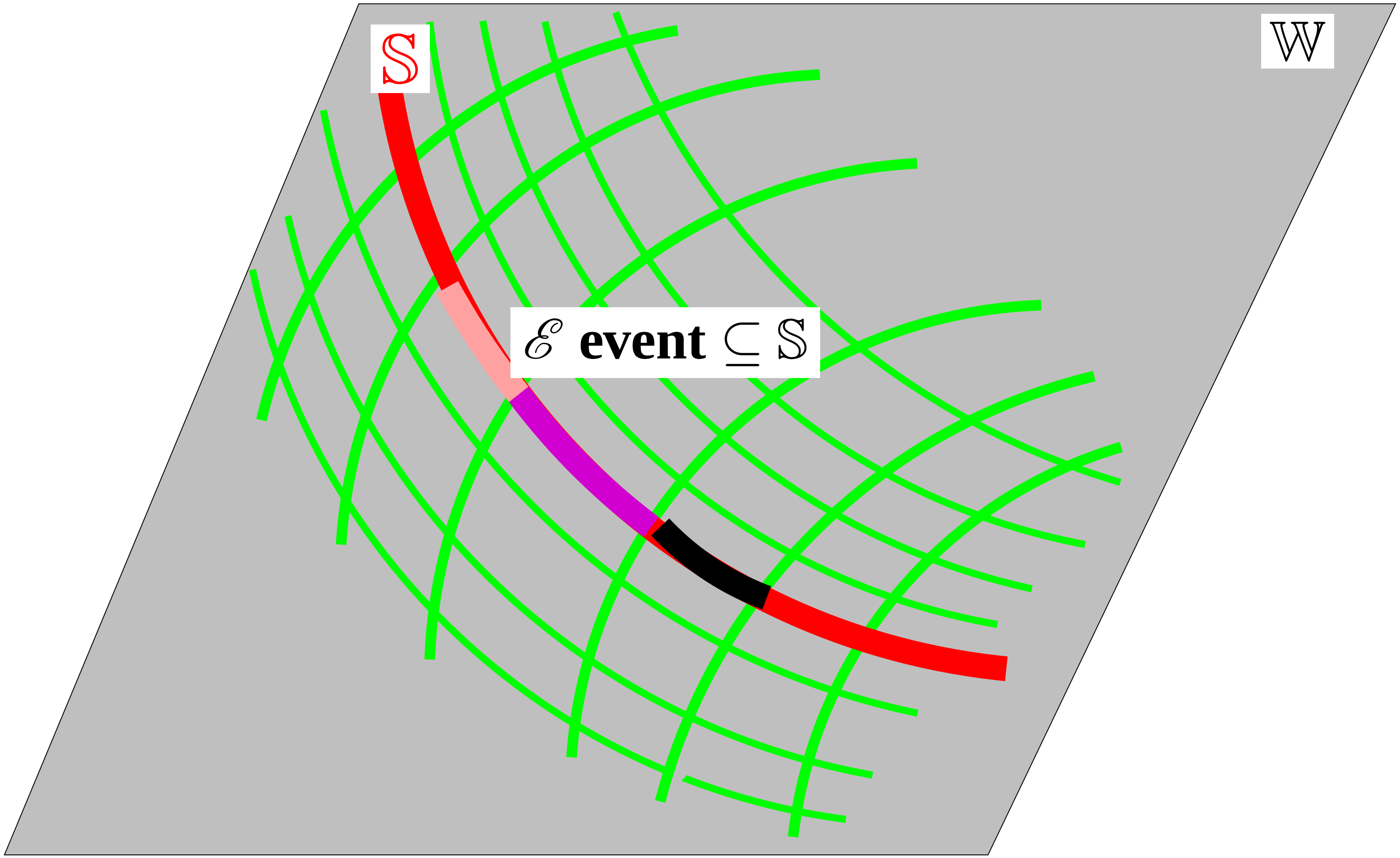
events in $\mathscr{E}$
$\mathbb{S}$
$\mathbb{W}$
events in $\mathscr{E} \cap \mathbb{S}$

In pictures

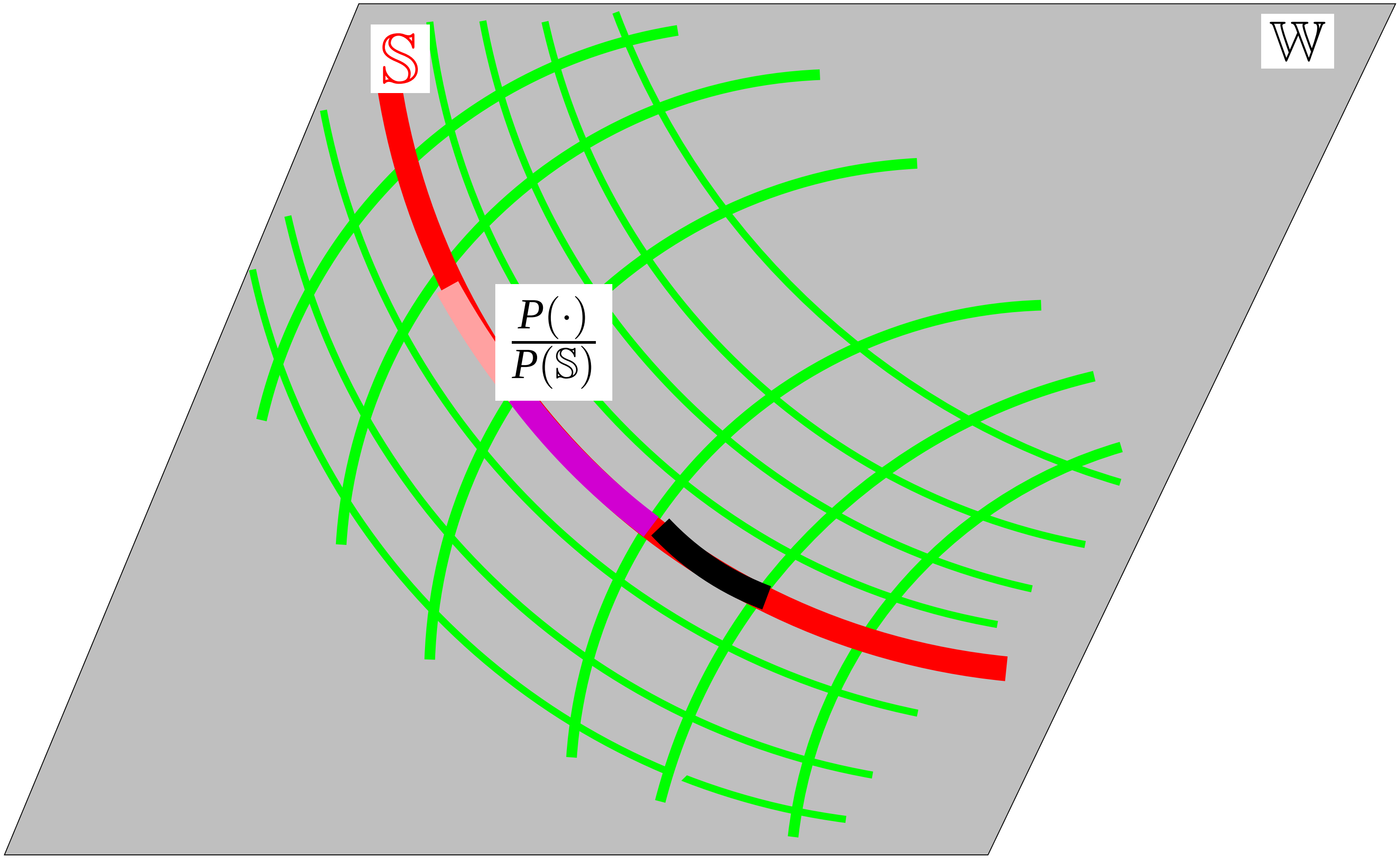


Probability of E drawn globally into $\mathbb{S}$.

Contrast with conditional probability



Contrast with conditional probability



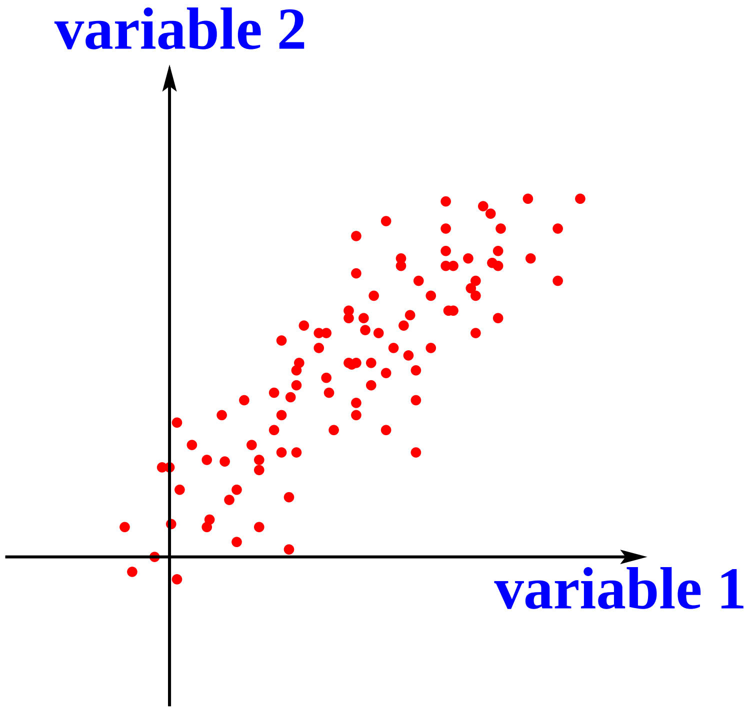
Probability locally computed, with renormalization.

Constraining and conditioning

Is there a point of view from which yields both concepts are special cases of one unifying idea?

Identification

Sampling

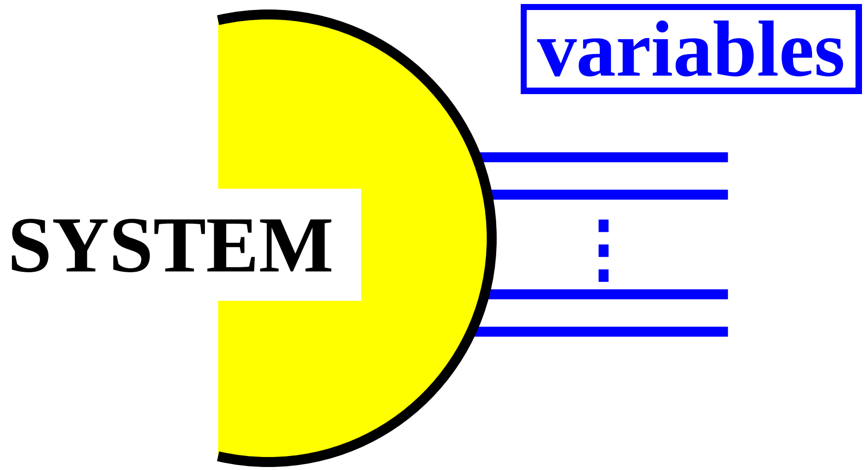


System identification: deduce the stochastic model

$\mathscr{E}$ and P

from the samples.

Measurements

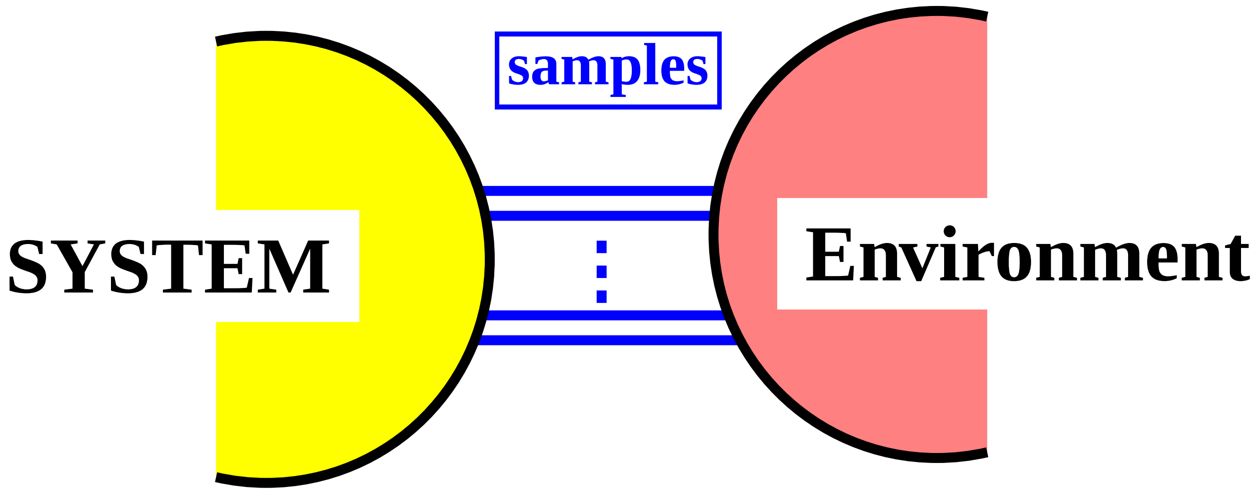


Data collection implies observing a stochastic system
in interaction with an environment.

Measurements

Data collection implies observing a stochastic system

in interaction with an environment.



Measurements

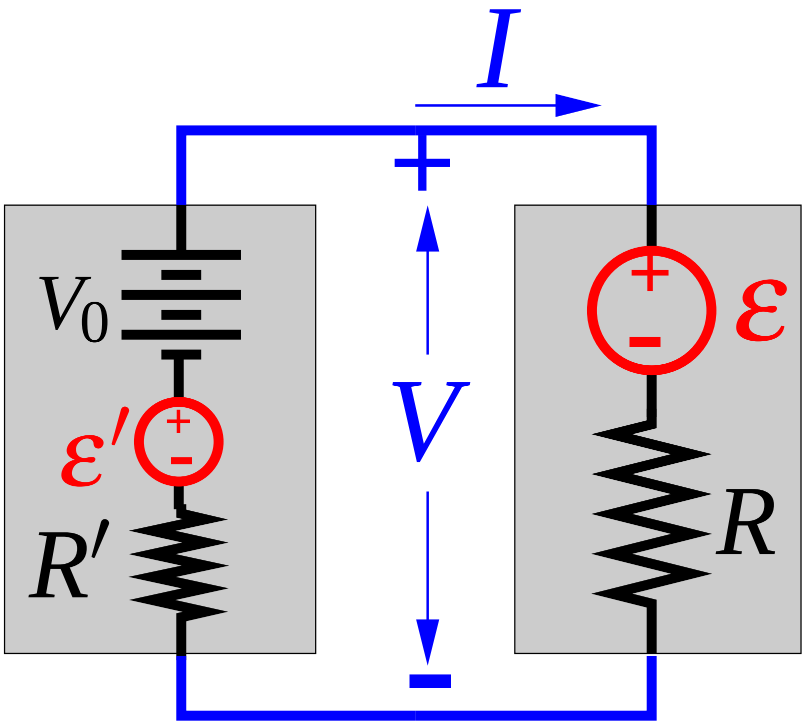
Data collection implies observing a stochastic system ***in interaction with an environment*****.**

Is it possible to disentangle the laws of a system from the laws of the environment?

In engineering, it may be possible to set the experimental conditions.

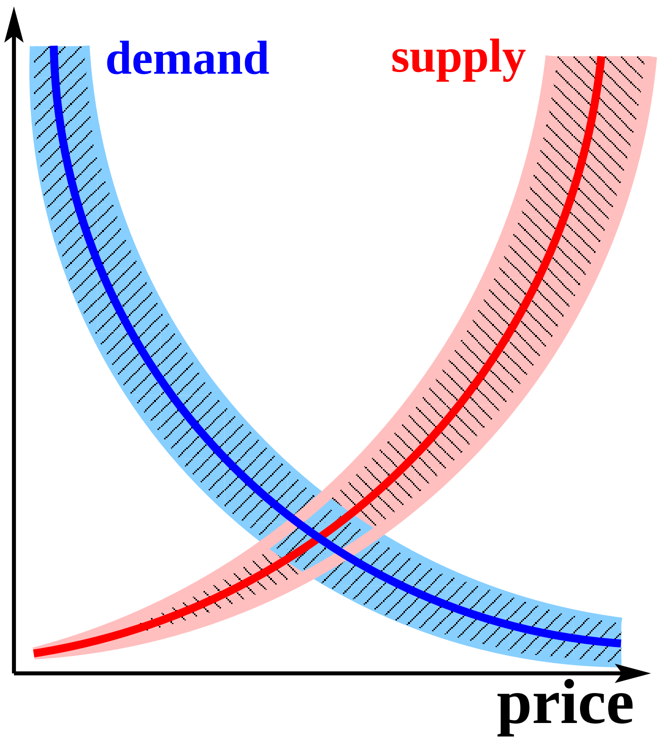
In economics and the social sciences (and biology?), data often gathered passively ‘in vivo’.

Disentangling



Can R and σ be deduced by sampling (V,I)?

Disentangling



Can the price/demand characteristic be deduced

by sampling (p, d) in equilibrium?

SYSID for gaussian systems

Let Σ_1 and Σ_2 be complementary gaussian systems and assume that the interconnection $\Sigma_1 \wedge \Sigma_2$ is a classical random system.

Sampling $\rightsquigarrow$ the mean and covariance of $\Sigma_1 \wedge \Sigma_2$. Assume that the covariance is nonsingular.

SYSID for gaussian systems

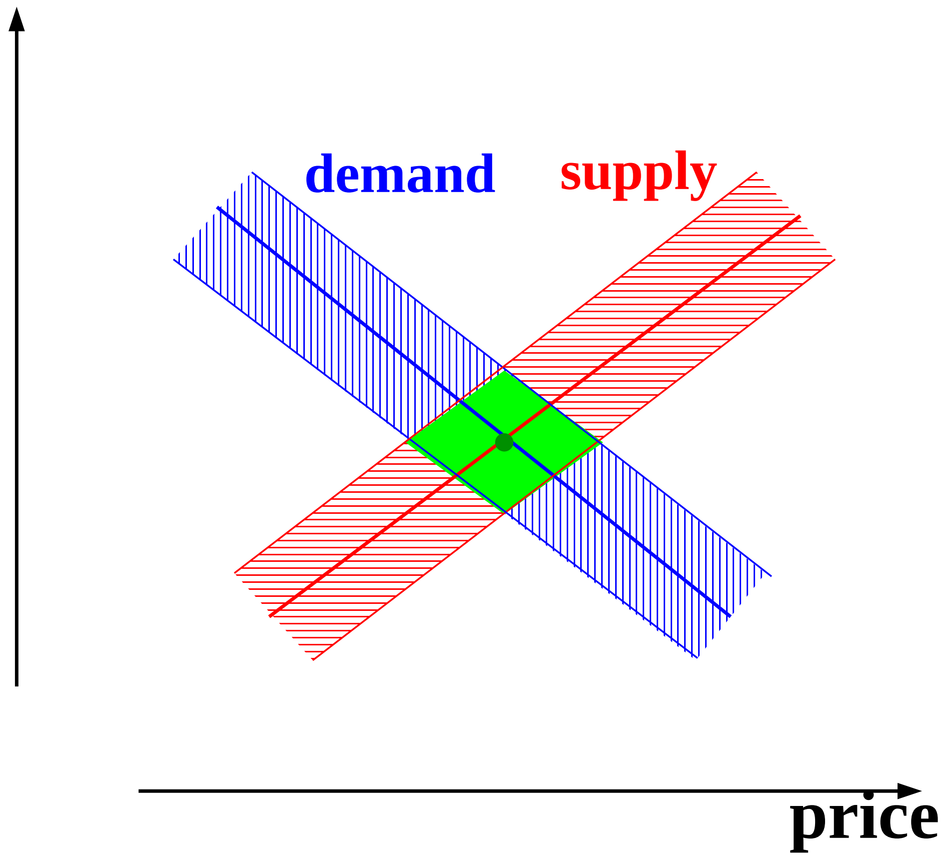
Let Σ_1 and Σ_2 be complementary gaussian systems and assume that the interconnection $\Sigma_1 \wedge \Sigma_2$ is a classical random system.

Sampling $\leadsto$ the mean and covariance of $\Sigma_1 \wedge \Sigma_2$. Assume that the covariance is nonsingular.

Given the fiber of either Σ_1 or Σ_2, then all the other parameters of Σ_1 and Σ_2 can be deduced from $\Sigma_1 \wedge \Sigma_2$.

The fiber of Σ_1 can be chosen freely.

Linearized gaussian price/demand/supply



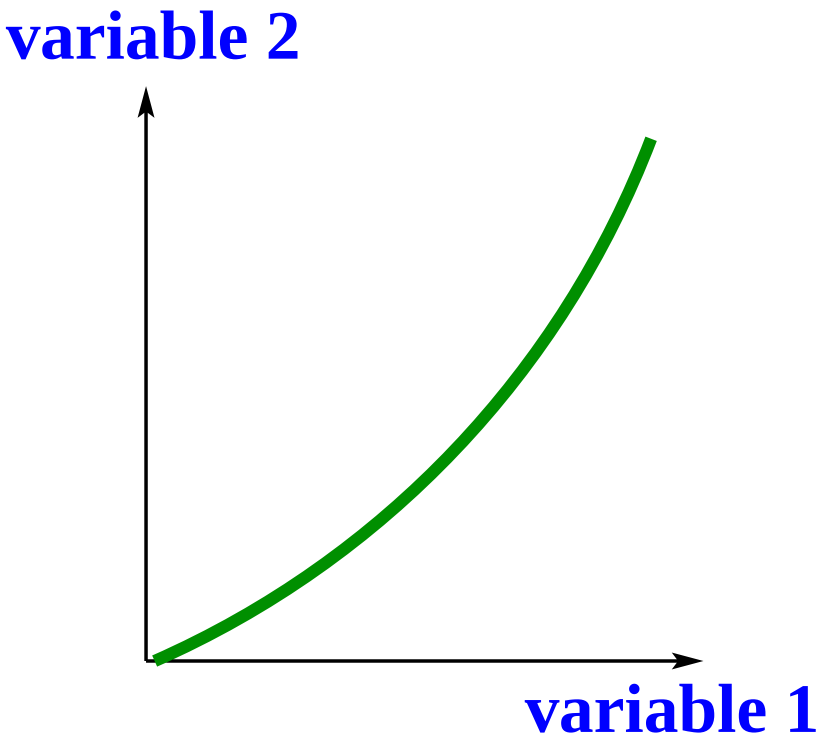
Identifiability provided one of the fibers is known.

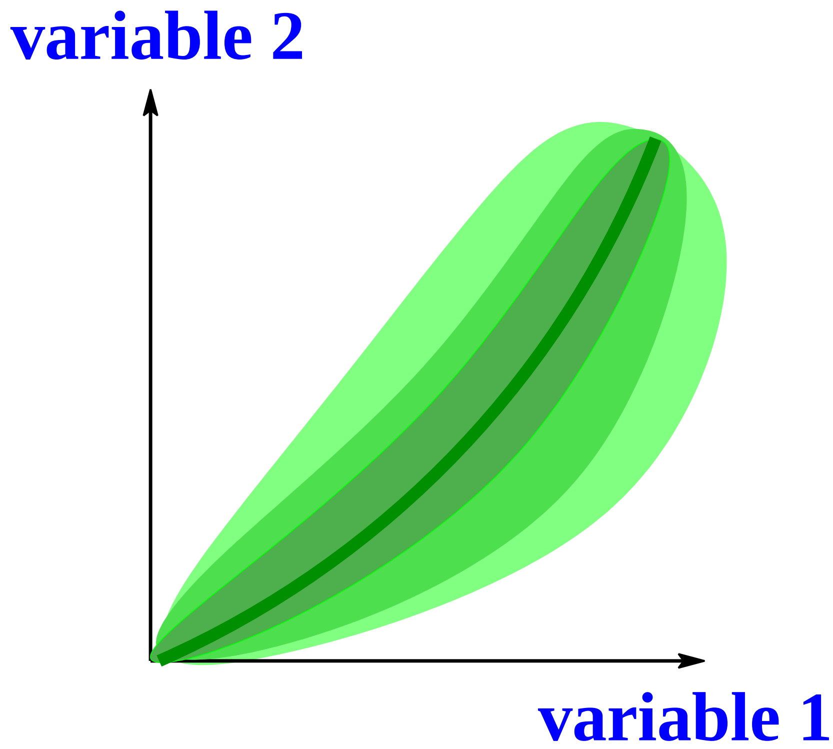
Sampling alone does not give these elasticities.

Conclusions

Stochastic systems

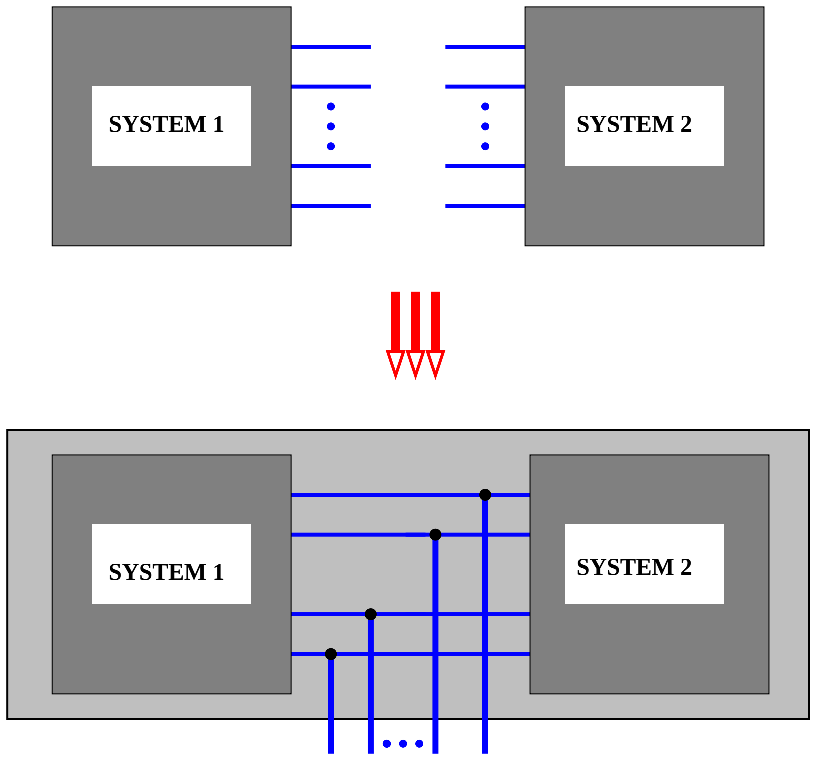
- **The Borel σ-algebra is inadequate even for elementary applications.**





Stochastic systems

- Complementary stochastic systems can be interconnected: two distinct laws imposed on one set of variables.

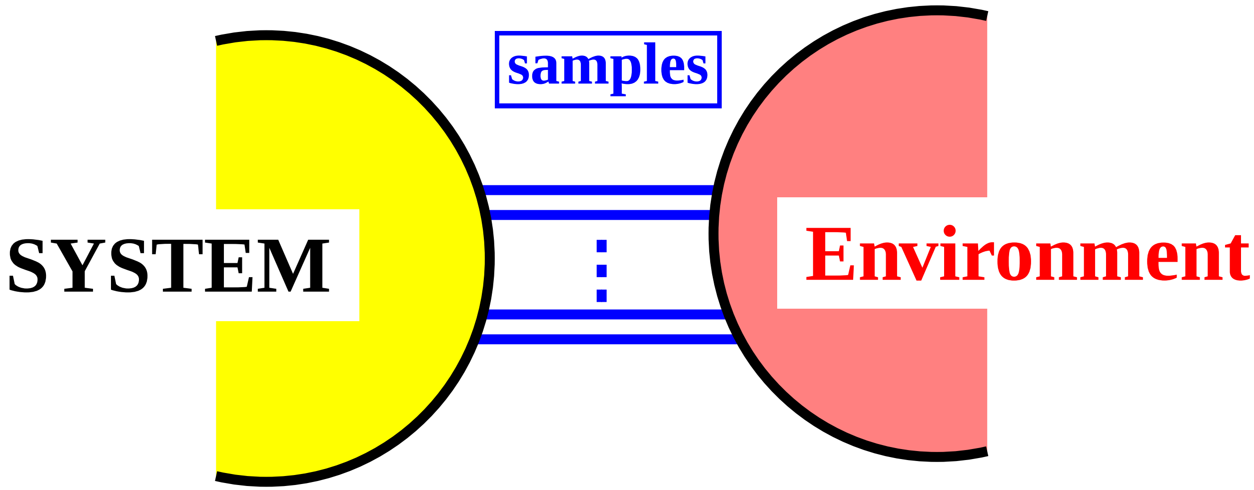


Stochastic systems

- **Open stochastic systems require a coarse σ-algebra.**

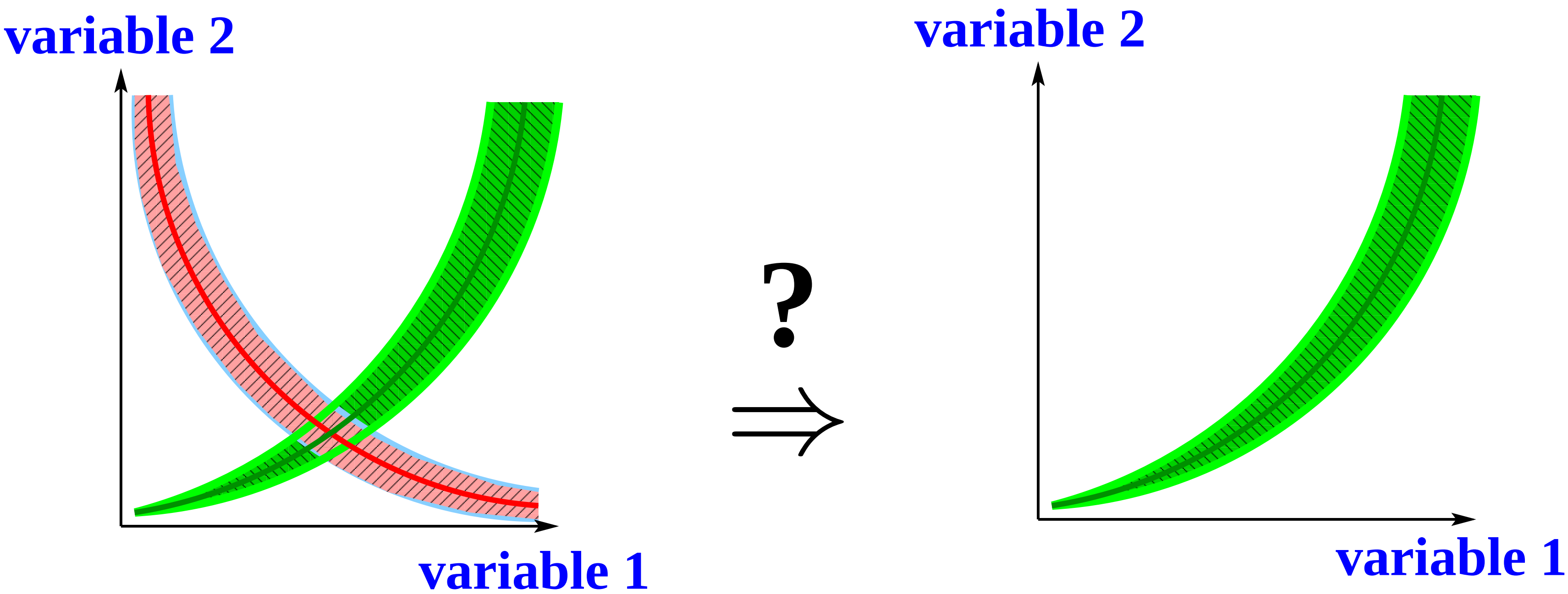
  Classical random vectors imply closed systems.

- Measurements are the result of interaction with an environment.



Modeling from data requires disentanglement.

- **Modeling from data requires disentanglement.**
  The data alone are insufficient for identifiability.



Future work

Urgent:

Generalization to stochastic processes.

<u>Reference</u>: *Open stochastic systems*, IEEE Tr. AC, submitted.

Copies of the lecture frames available from/at

`http://www.esat.kuleuven.be/∼jwillems`

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Thank you

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