OPEN STOCHASTIC SYSTEMS

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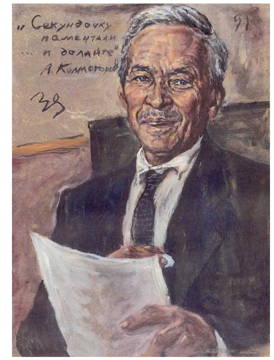
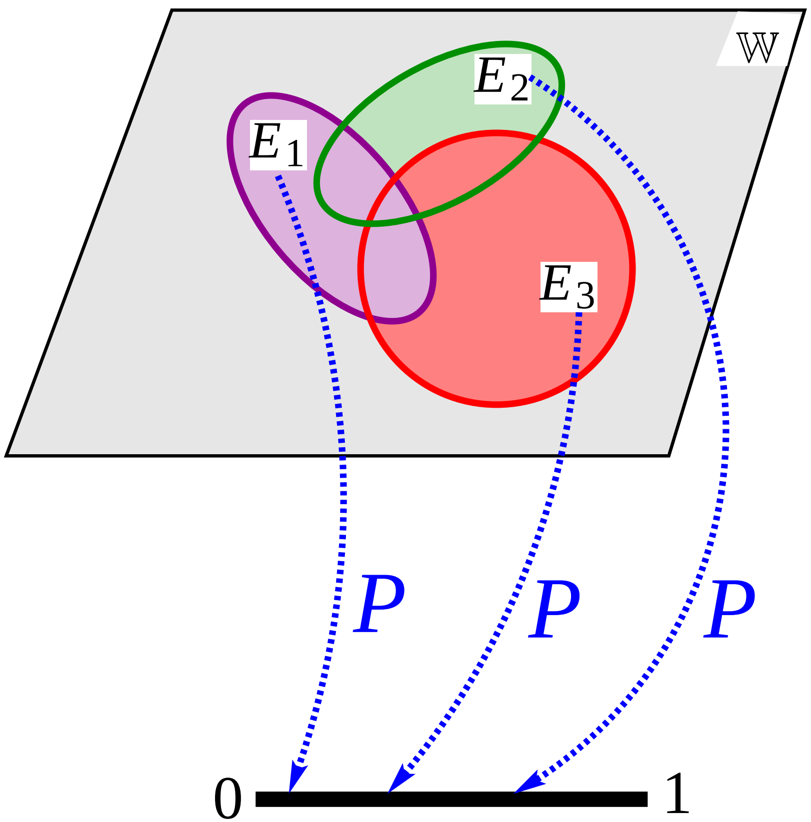
Basic probability



A.N. Kolmogorov
1903 – 1987



A **probability** $P(E) \in [0, 1]$ **is assigned to certain subsets** E **(*‘events’*) of the *outcome space*** $\mathbb{W}$**.**

$\mathscr{E}$ **:= the class of ‘measurable’ subsets of** $\mathbb{W}$**,**

= the sets that are assigned a probability.

Main (not all) axioms

$\mathscr{E}$ **is a σ-algebra** $:\Rightarrow$

- $[\![E \in \mathscr{E}]\!] \Rightarrow [\![E^{\text{complement}} \in \mathscr{E}]\!]$
- $[\![E_1, E_2 \in \mathscr{E}]\!] \Rightarrow [\![E_1 \cap E_2 \in \mathscr{E},\ \ E_1 \cup E_2 \in \mathscr{E}]\!]$

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- $P(\mathbb{W}) = 1$

P **is** **additive** $:\Rightarrow$

- $[\![E_1 \cap E_2 = \emptyset]\!] \Rightarrow [\![P(E_1 \cup E_2) = P(E_1) + P(E_2)]\!]$

Borel

Émile Borel
1871 – 1956

In most applications it is assumed that the σ-algebra of measurable sets are the *Borel sets*.

$\mathscr{B}(\mathbb{R}^n)$ **= the Borel σ-algebra on $\mathbb{R}^n$;**

random variable: $\mathbb{W} = \mathbb{R}$ **(or $\mathbb{C}$), and** $\mathscr{E} = \mathscr{B}(\mathbb{R})$

random vector: $\mathbb{W} = \mathbb{R}^n$, **and** $\mathscr{E} = \mathscr{B}(\mathbb{R}^n)$

random process: a family of random vectors, etc.

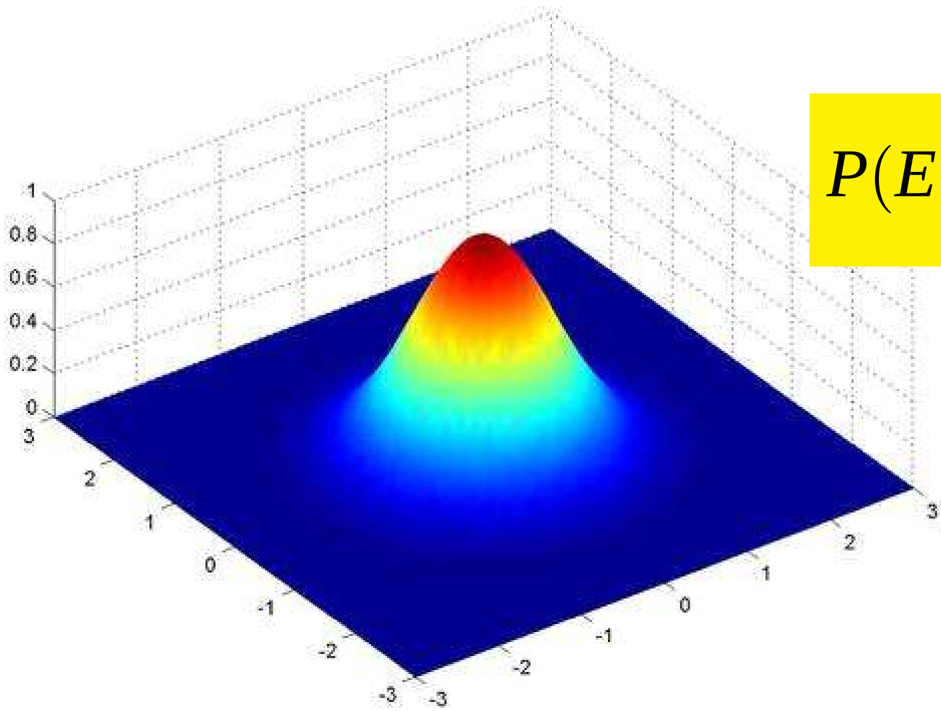
$\mathscr{B}(\mathbb{R}^n)$ **contains 'basically every' subset of $\mathbb{R}^n$.**

Theme

We consider stochastic models with outcomes in $\mathbb{R}^n$.

Usual framework:

- **The event σ-algebra $\mathscr{E}$ consists of the Borel sets.**
  $\rightsquigarrow$ **'Every' subset of $\mathbb{R}^n$ is assigned a probability.**
- $\rightsquigarrow$ **probability distributions, probability densities, marginal distributions, conditional densities, ...**



$$P(E) = \int_E p(x)\,dx, \quad E \subseteq \mathbb{R}^n \textbf{ Borel}$$

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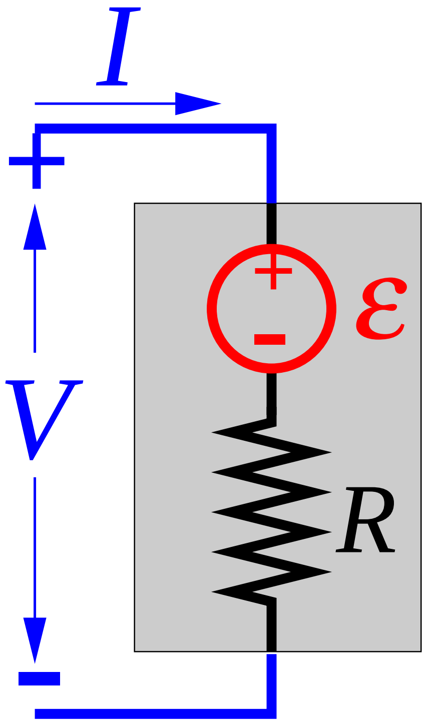
<u>Thesis</u>:

**This is unduly restrictive,
even for elementary applications.**

The event set is crucial, and often less than $\mathscr{B}(\mathbb{R}^{\mathtt{n}})$.

Motivating examples

Noisy resistor



$$V = RI + \varepsilon$$

ε gaussian

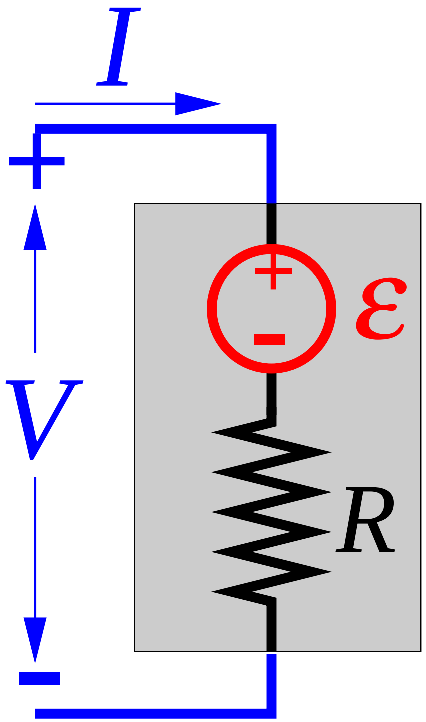
mean $= 0$

standard deviation

$\sim \sqrt{RT}$, T **temp.**

'Johnson-Nyquist resistor'

Noisy resistor



$$\boxed{V = RI + \varepsilon}$$

ε gaussian

mean $= 0$

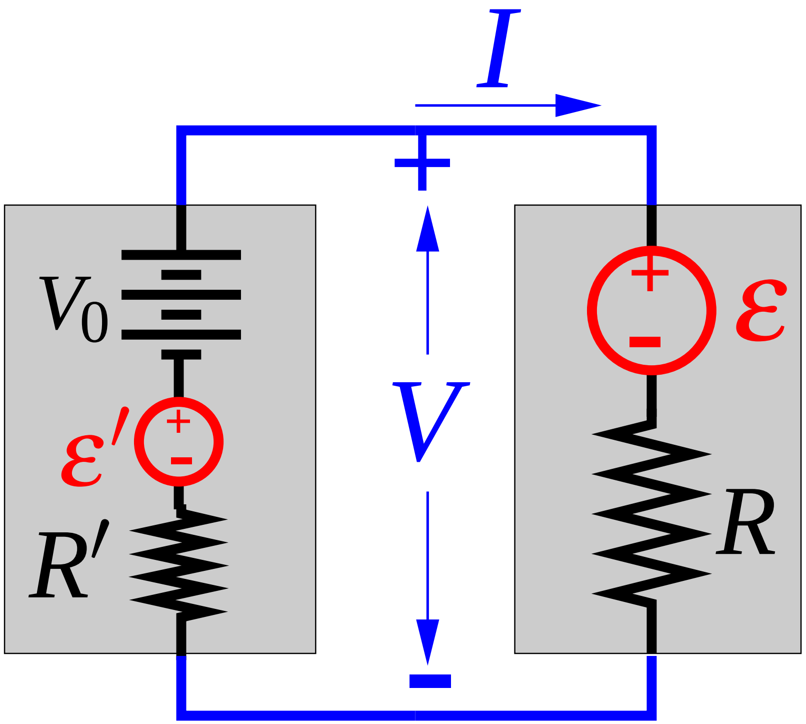
standard deviation

$\sim \sqrt{RT}, \quad T$ **temp.**

‘Johnson-Nyquist resistor’

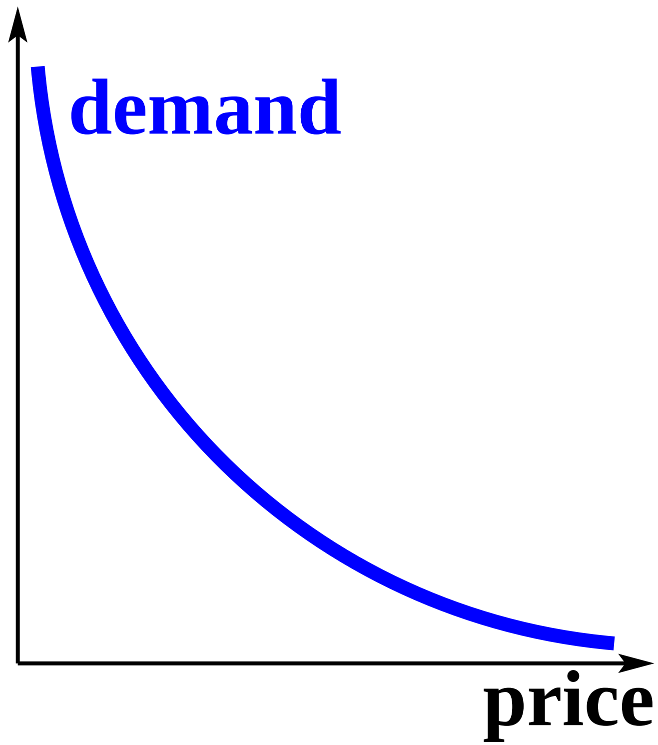
What is $\begin{bmatrix} V \\ I \end{bmatrix}$ ***as a stochastic object?***

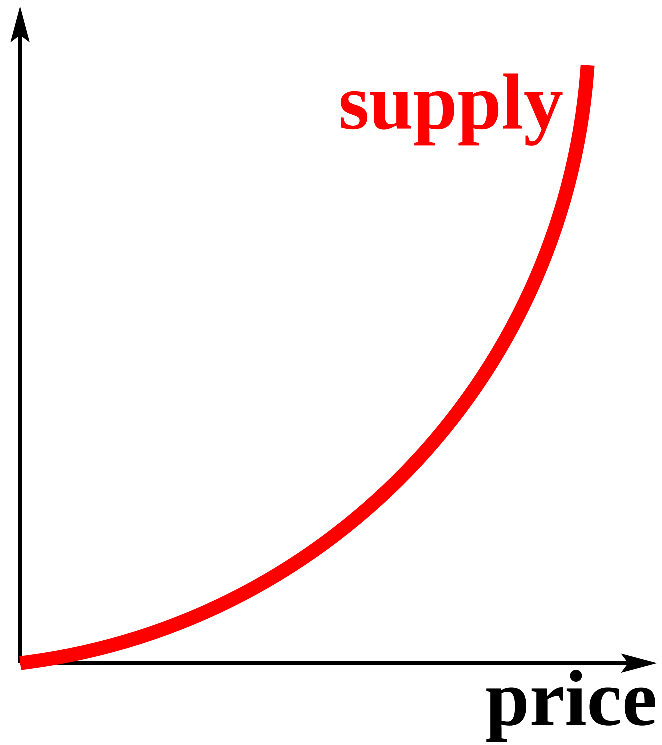
Noisy resistor



How do we deal with interconnection?

Deterministic price/demand/supply





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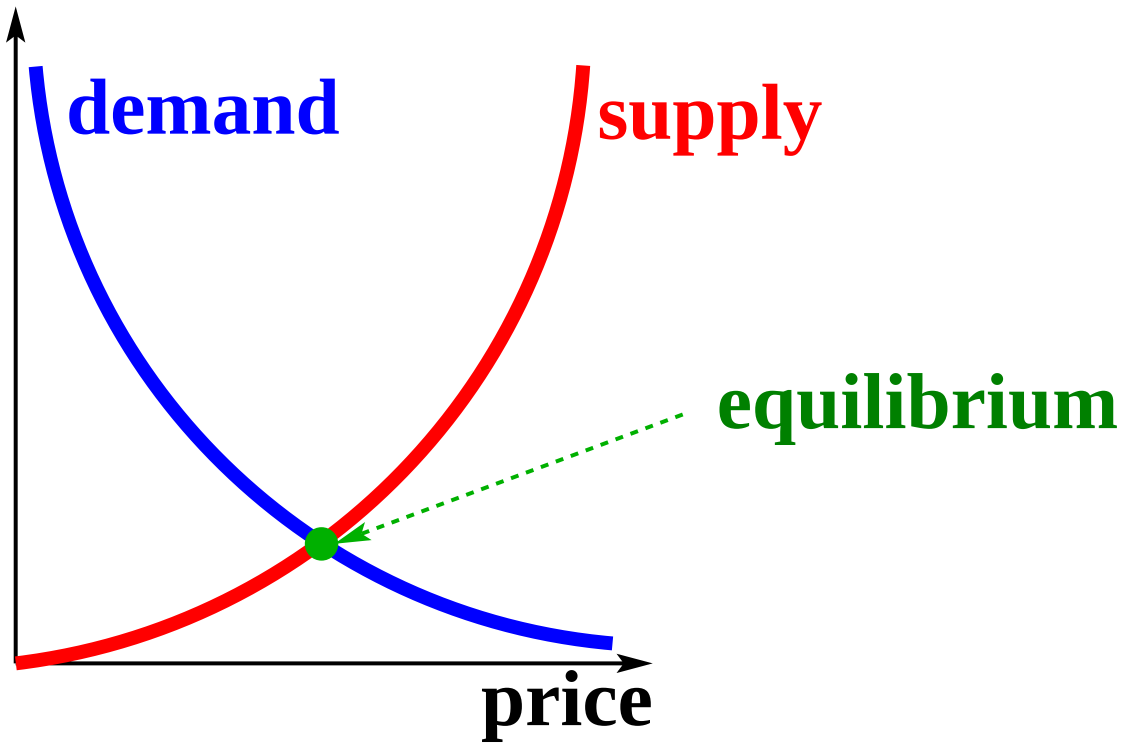
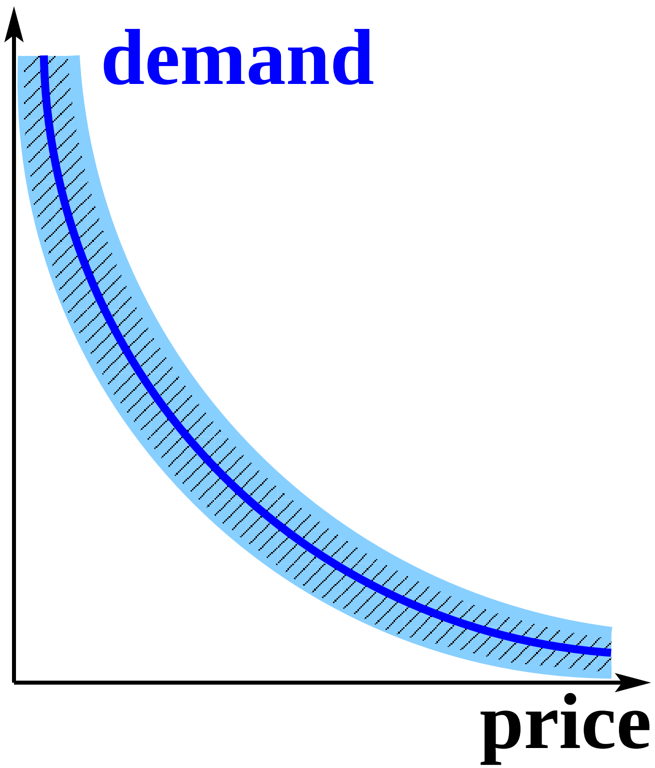


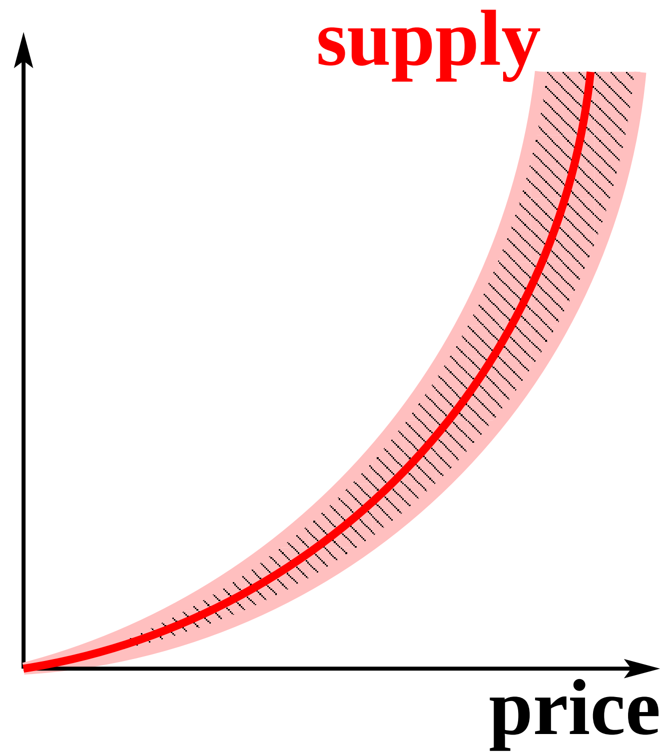
Equilibrium:

- **prices pertain to same product,**
- **supply = demand.**

$\leadsto$ **‘Interconnection’**

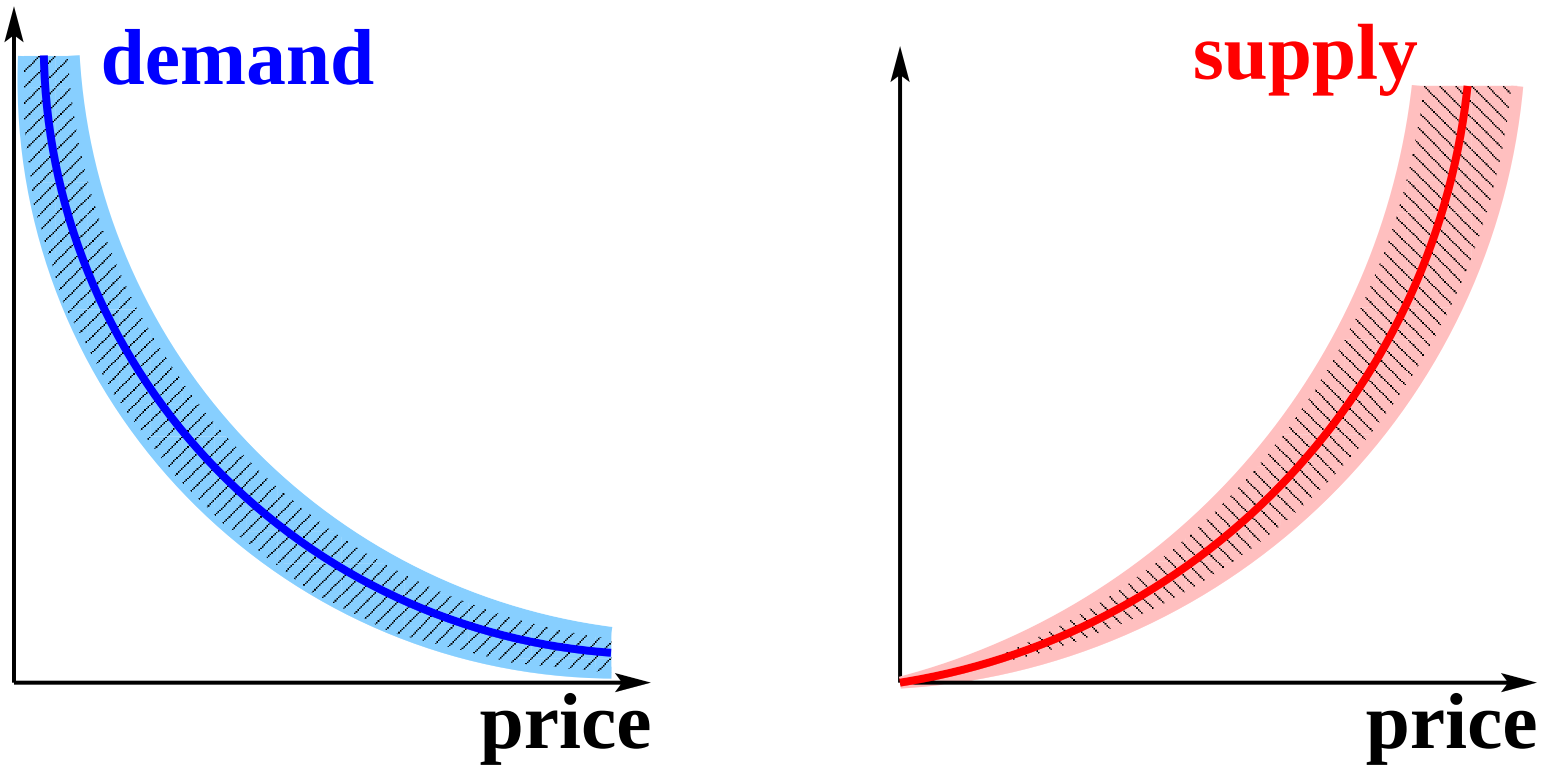
Stochastic price/demand/supply





Only certain regions of the [price demand] plane and of the [price supply] plane are assigned a probability.

Stochastic price/demand/supply



Only certain regions of the $\begin{bmatrix} \text{price} \\ \text{demand} \end{bmatrix}$ plane and of the $\begin{bmatrix} \text{price} \\ \text{supply} \end{bmatrix}$ plane are assigned a probability.

How do we deal with equilibrium supply = demand?

Formal definitions

Definition

A *stochastic system* is a probability triple $(\mathbb{W}, \mathscr{E}, P)$

- $\mathbb{W}$ **a non-empty set, the *outcome space*,**
- $\mathscr{E}$ **a σ-algebra of subsets of** $\mathbb{W}$**, the *events*,**
- $P : \mathscr{E} \to [0,1]$ **a *probability measure*.**

$\mathscr{E}$**: the subsets that are assigned a probability.**

Probability that outcome $\in E$, $E \in \mathscr{E}$**, is** $P(E)$**.**

Model $\cong \mathscr{E}$ **and** P**;** $\mathscr{E}$ **is an essential part.**

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'Classical' random vector on $\mathbb{R}^{\mathrm{n}}$**:**

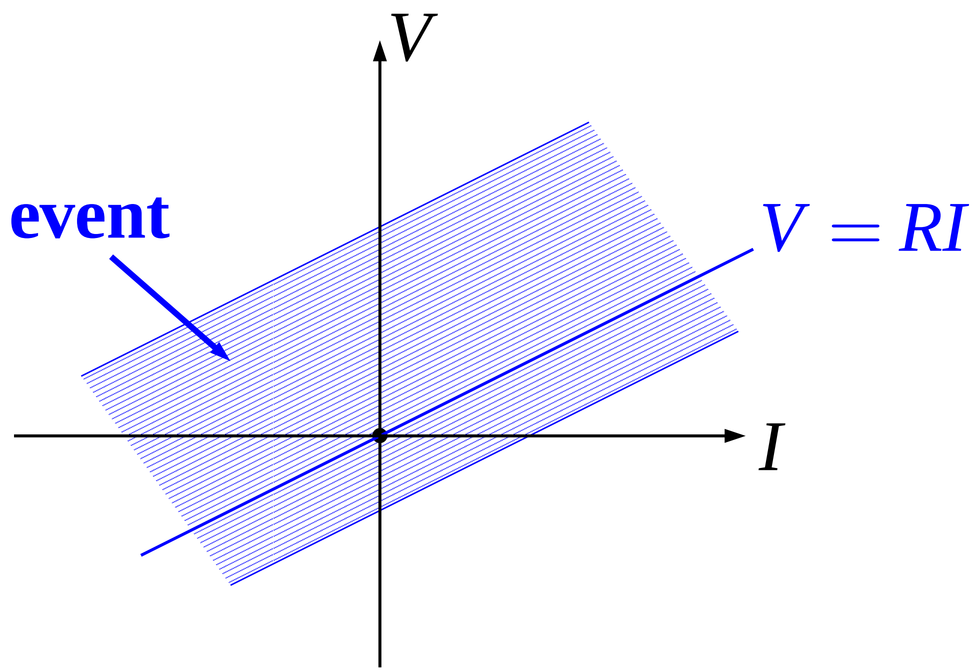
$\mathbb{W} = \mathbb{R}^{\mathrm{n}}$ **and** $\mathscr{E}$ **= the Borel subsets of** $\mathbb{R}^{\mathrm{n}}$**.**

$\mathscr{E}$ **is inherited from the topology on** $\mathbb{R}^{\mathrm{n}}$**,**

it does not involve the random phenomenon.

P **can then be specified by a probability distribution.**

Noisy resistor



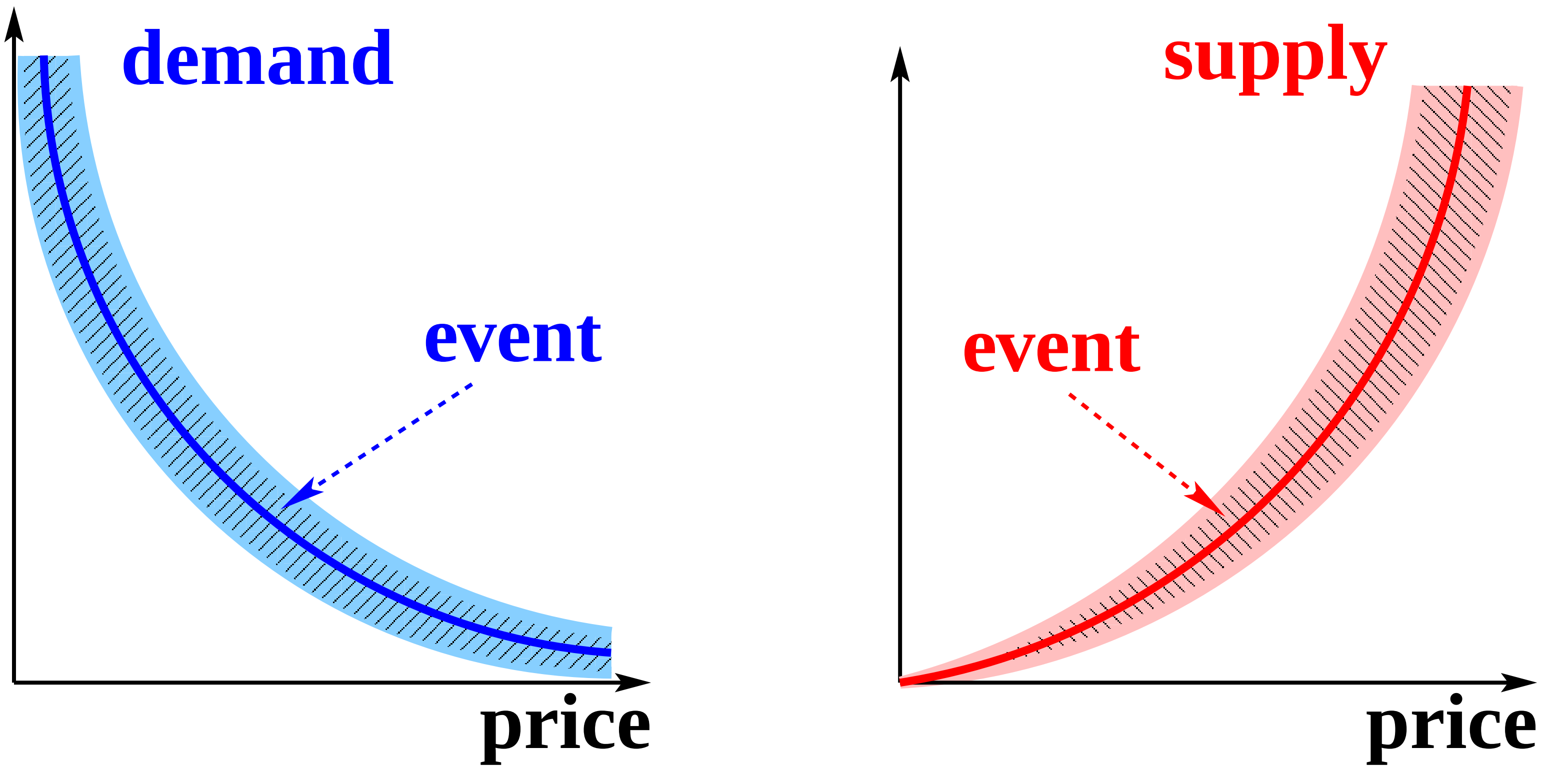
$V = RI + \varepsilon$: **stoch. system,** $\mathbb{W} = \mathbb{R}^2$**, outcomes** $\begin{bmatrix} V \\ I \end{bmatrix}$**.**

Events: $\left\{ \begin{bmatrix} V \\ I \end{bmatrix} \in \mathbb{R}^2 \mid V - RI \in A \text{ with } A \text{ a Borel subset of } \mathbb{R} \right\}$**.**

$P(\textbf{event}) =$ **gaussian measure of** A**.**

V **and** I **are** ***not*** **classical real random variables.**

Stochastic price/demand/supply



$\mathscr{E}$ = **the regions that are assigned a probability.**

p**,** d**, and** s **are** ***not*** **classical real random variables.**

linear stochastic system

:$\Leftrightarrow$ Borel probability on $\mathbb{R}^{\mathtt{n}}/\mathbb{L}$, $\mathbb{L}$ linear, 'fiber'.

***Events*:** **cylinders with sides parallel to $\mathbb{L}$,**

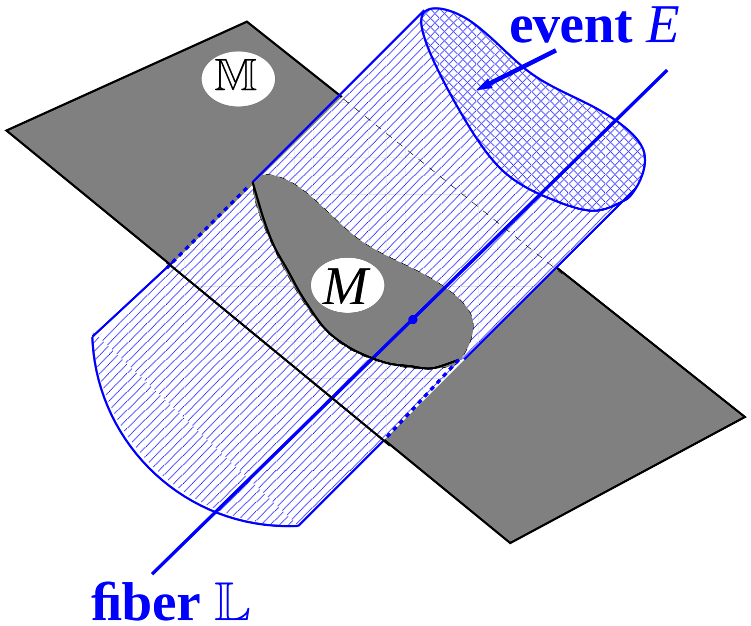
that is, subsets of the form $A + \mathbb{L}$,

$\mathbb{L}$ a linear subspace of $\mathbb{R}^{\mathtt{n}}$,

$A \subseteq \mathbb{R}^{\mathtt{n}}/\mathbb{L}$ a Borel set.

$\boxed{\textit{linear stochastic system}}$

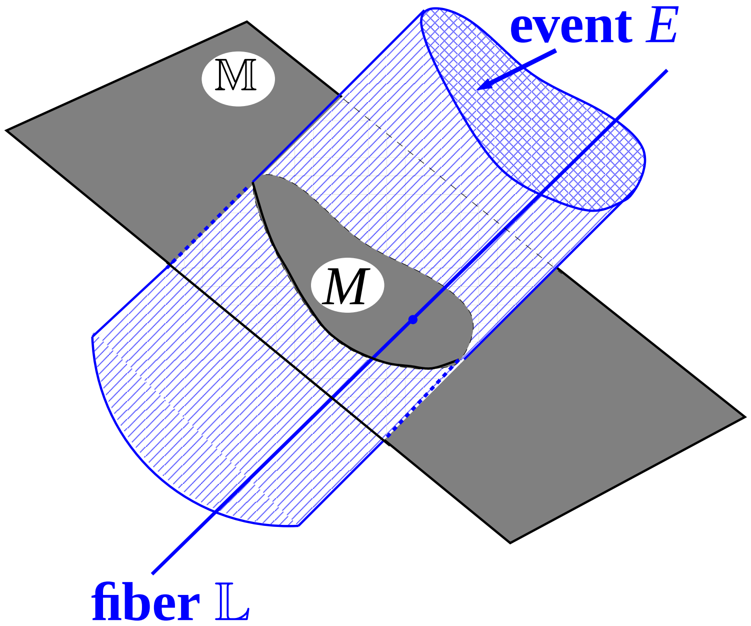
:⇔ Borel probability on $\mathbb{R}^n/\mathbb{L}$, $\mathbb{L}$ linear, 'fiber'.



Borel probability on $\mathbb{M} \cong \mathbb{R}^n/\mathbb{L}$. Classical $\Rightarrow$ linear.

linear stochastic system

:⇔ **Borel probability on $\mathbb{R}^{\mathtt{n}}/\mathbb{L}$, $\mathbb{L}$ linear, 'fiber'.**



Borel probability on $\mathbb{M} \cong \mathbb{R}^{\mathtt{n}}/\mathbb{L}$. Classical $\Rightarrow$ linear.

gaussian :⇔ **linear & Borel probability gaussian.**

Deterministic

$(\mathbb{W}, \mathscr{E}, P)$ **is said to be *deterministic* if**

$$\mathscr{E} = \{\emptyset, \mathbb{B}, \mathbb{B}^{complement}, \mathbb{W}\} \quad \textbf{with} \quad P(\mathbb{B}) = 1.$$

$\mathbb{B}$ **is called the *behavior* of the deterministic system.**

Only valid probabilistic statements:

$$P(\mathbb{B}) = 1, P(\mathbb{W}) = 1,$$

$$P(\mathbb{B}^{complement}) = 0, P(\emptyset) = 0.$$

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If $\mathbb{B} = \mathbb{W}$**, the variables are said to be *free*.**

Noisy resistor

noisy resistor: linear, gaussian, fiber $V = RI$**.**

Let $\alpha \in \mathbb{R}$**. Map** $\begin{bmatrix} V \\ I \end{bmatrix} \mapsto V - \alpha I = w$**.**

w **is classical random variable iff** $\alpha = R$**.**

If $\alpha \neq R$**, then** w **is free.**

In particular, for $R \neq 0$**,** V **and** I **are both free.**

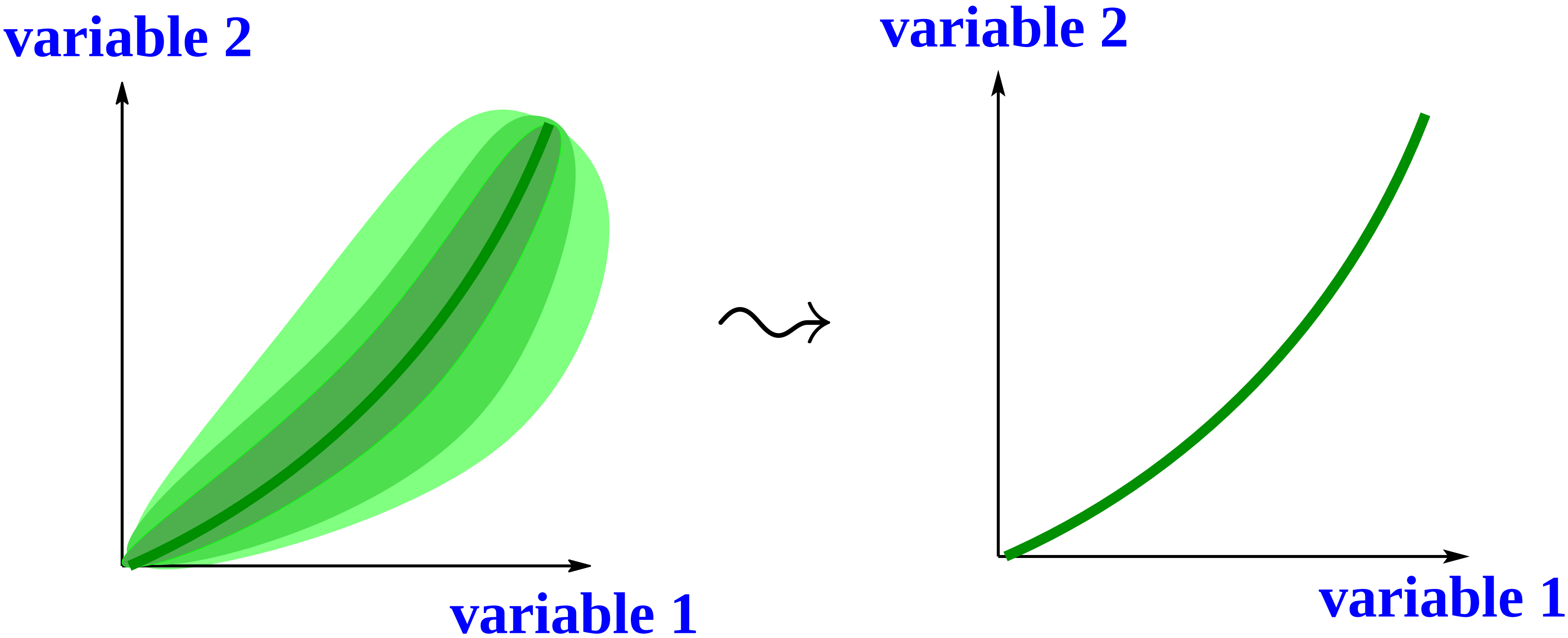
Only statements: $P(\{V \in \mathbb{R}\}) = 1$**,** $P(\{I \in \mathbb{R}\}) = 1$**.**

$\begin{bmatrix} V \\ I \end{bmatrix}$ **no distribution, no pdf,**

no marginal distributions,

no conditional distributions.

The need for coarse σ-algebras



If $\begin{bmatrix}\text{variable 1}\\ \text{variable 2}\end{bmatrix}$ were a classical random vector,

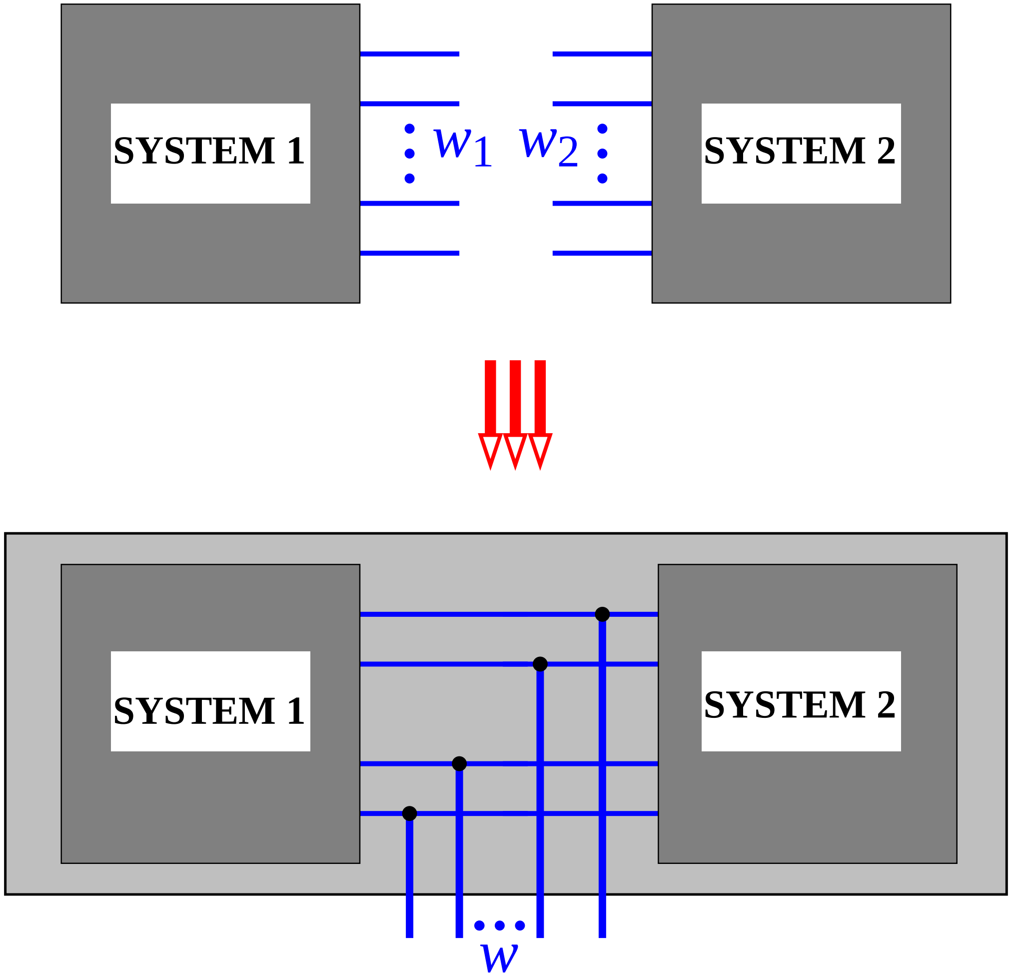
then the deterministic limit becomes a (singular) pdf.

Awkward from the modeling point of view:

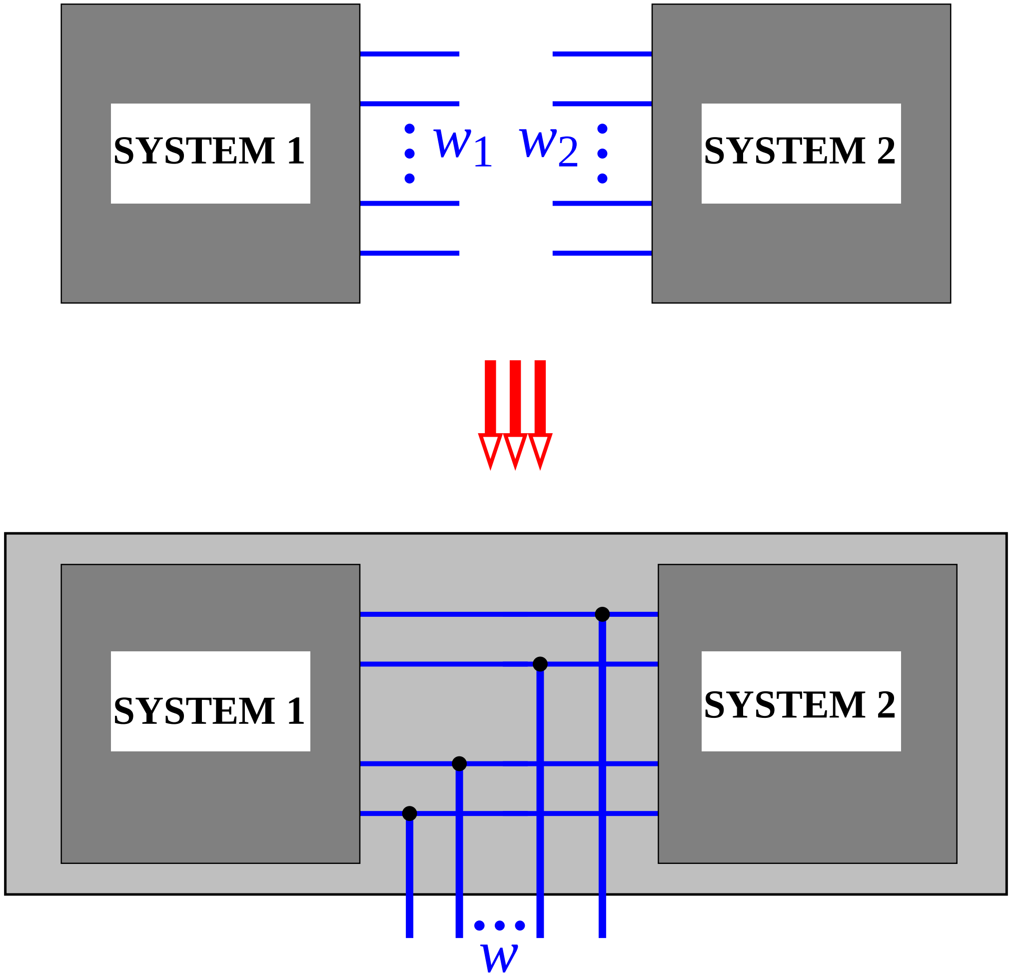
determinism $\not\Rightarrow$ stochastic laws.

Interconnection

Interconnection



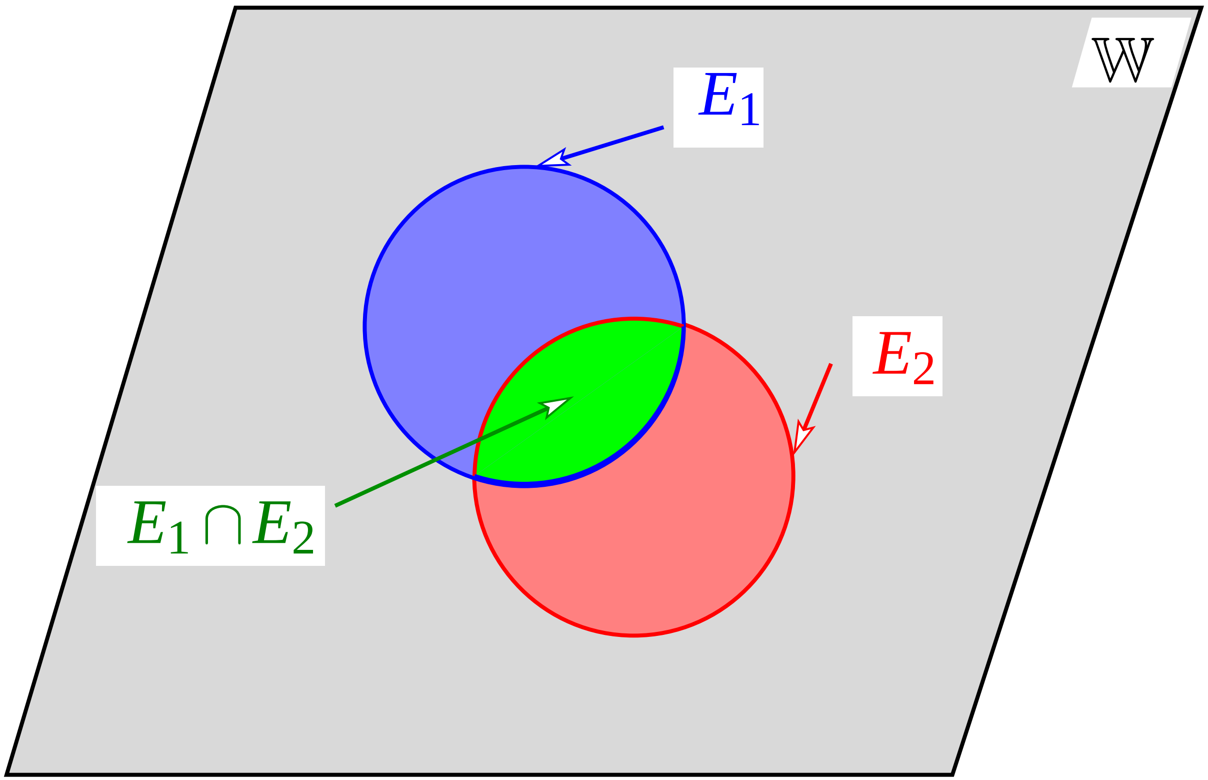
Interconnection



Is it possible to impose two distinct probabilistic laws on the same set of variables?

Complementarity

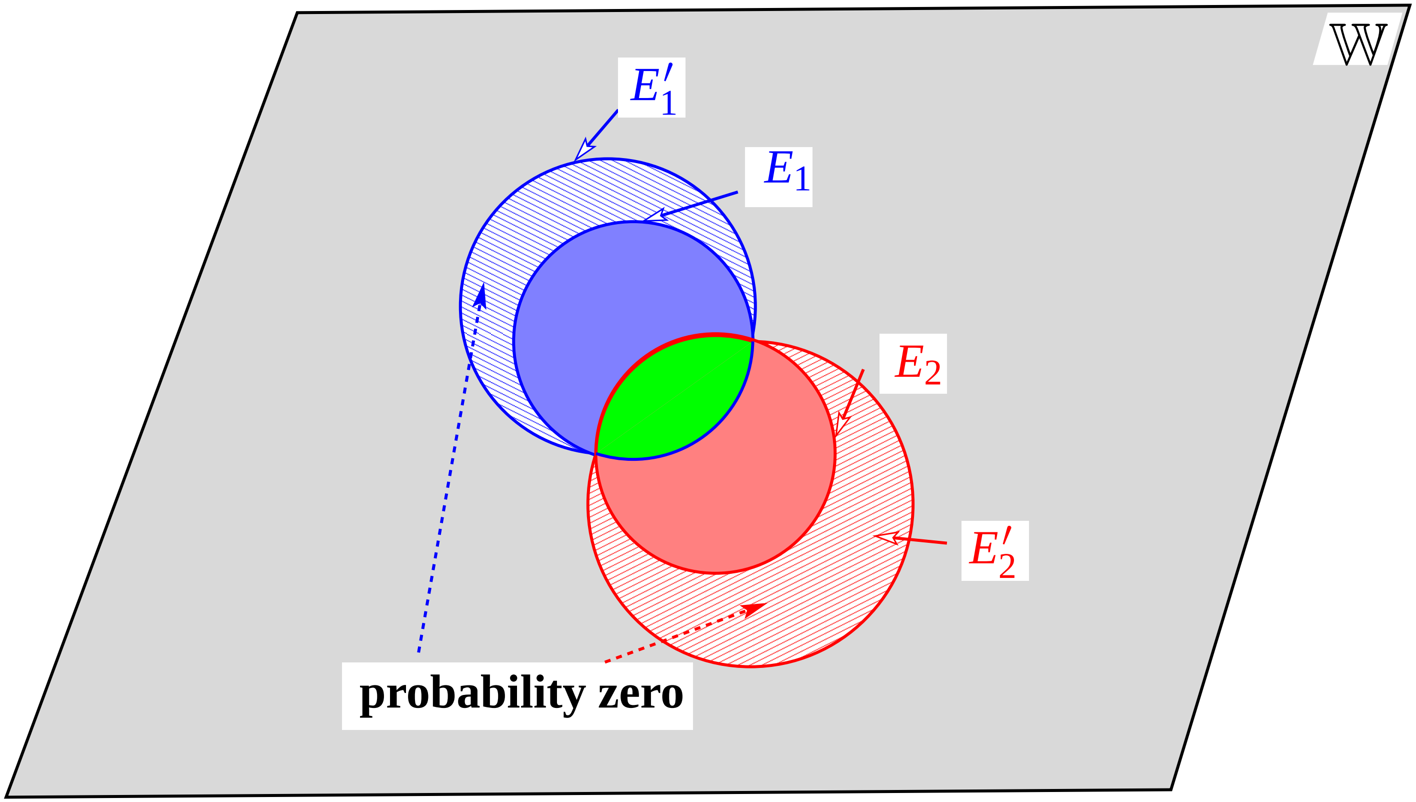
The σ-algebras $\mathscr{E}_1$ and $\mathscr{E}_2$ on $\mathbb{W}$ are said to be complementary :⇔ for non-empty $E_1 \in \mathscr{E}_1, E_2 \in \mathscr{E}_2$, the intersection $E_1 \cap E_2$ determines E_1 and E_2.



Complementarity

$\Sigma_1 = (\mathbb{W}, \mathscr{E}_1, P_1)$ **and** $\Sigma_2 = (\mathbb{W}, \mathscr{E}_2, P_2)$ **are said to be** ***complementary*** $:\Leftrightarrow$ **for** $E_1, E_1' \in \mathscr{E}_1$ **and** $E_2, E_2' \in \mathscr{E}_2$**:**

$$[\![E_1 \cap E_2 = E_1' \cap E_2']\!] \Rightarrow [\![P_1(E_1)P_2(E_2) = P_1(E_1')P_2(E_2')]\!].$$



Interconnection of complementary systems

Let $\Sigma_1 = (\mathbb{W}, \mathscr{E}_1, P_1)$ **and** $\Sigma_2 = (\mathbb{W}, \mathscr{E}_2, P_2)$ **be complementary stochastic systems (assumed stochastically independent). Their *interconnection* is**

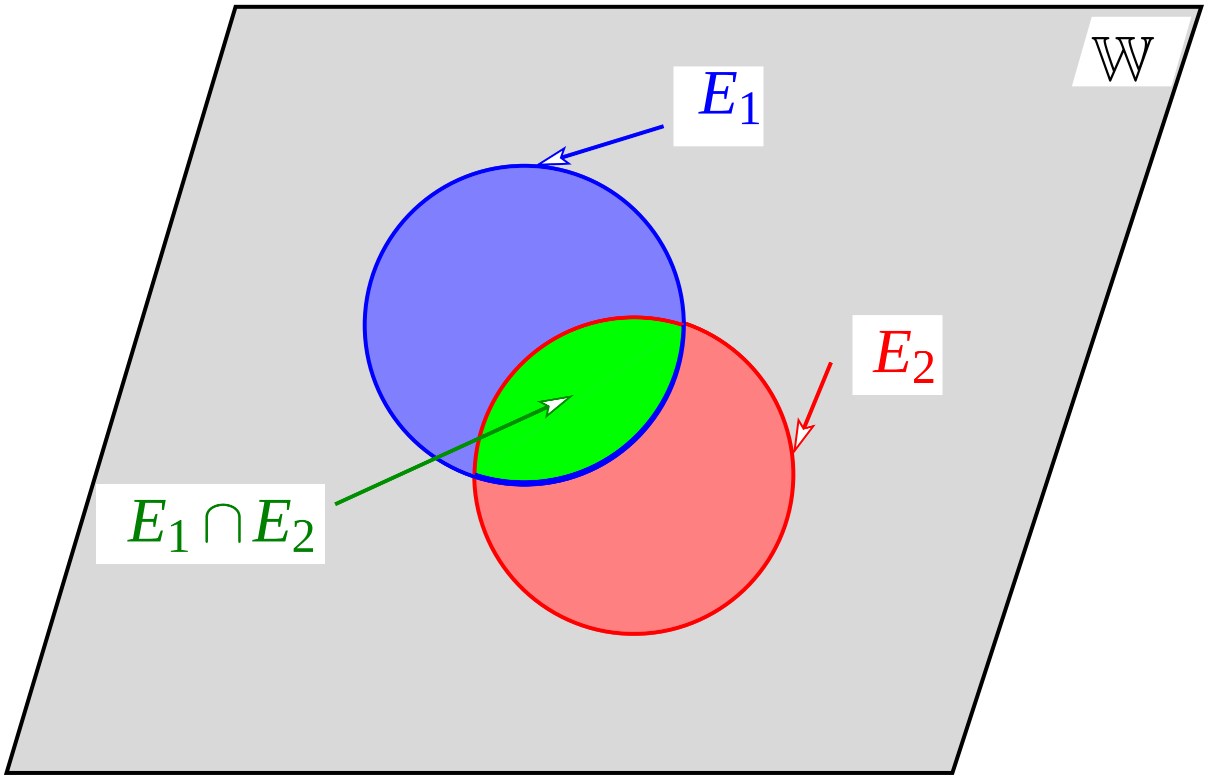
$$(\mathbb{W}, \mathscr{E}, P)$$

with $\mathscr{E} :=$ **the** σ**-algebra generated by** $\mathscr{E}_1 \cup \mathscr{E}_2$**,**
and P **defined through the 'rectangles' by**

$$\boxed{P(E_1 \cap E_2) := P_1(E_1)P_2(E_2).}$$

for $E_1 \in \mathscr{E}_1, E_2 \in \mathscr{E}_2$**.**

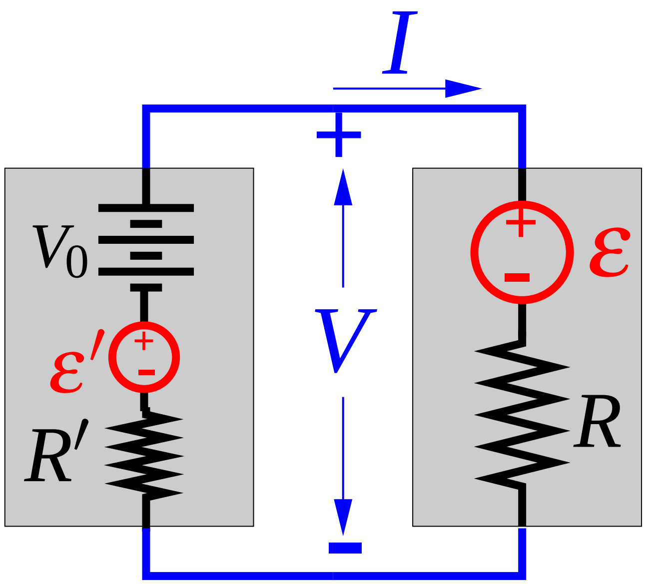
Interconnection of complementary systems



$$P(E_1 \cap E_2) := P_1(E_1)P_2(E_2).$$

Needs complementarity.

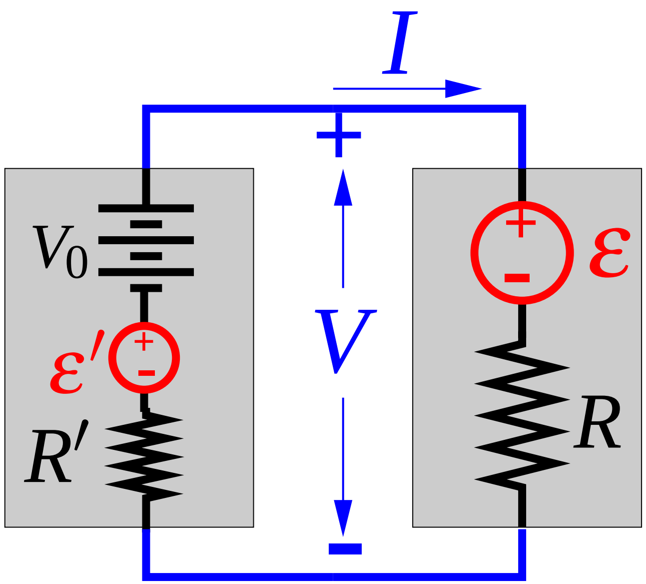
Noisy resistor terminated by voltage source



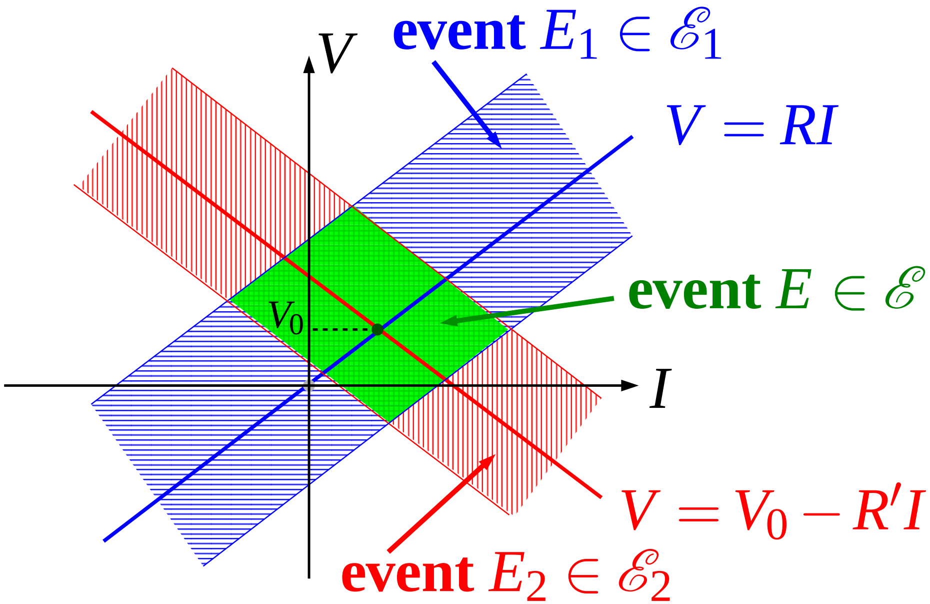
Complementary σ-algebras

if $R + R' \neq 0.$

Noisy resistor terminated by voltage source



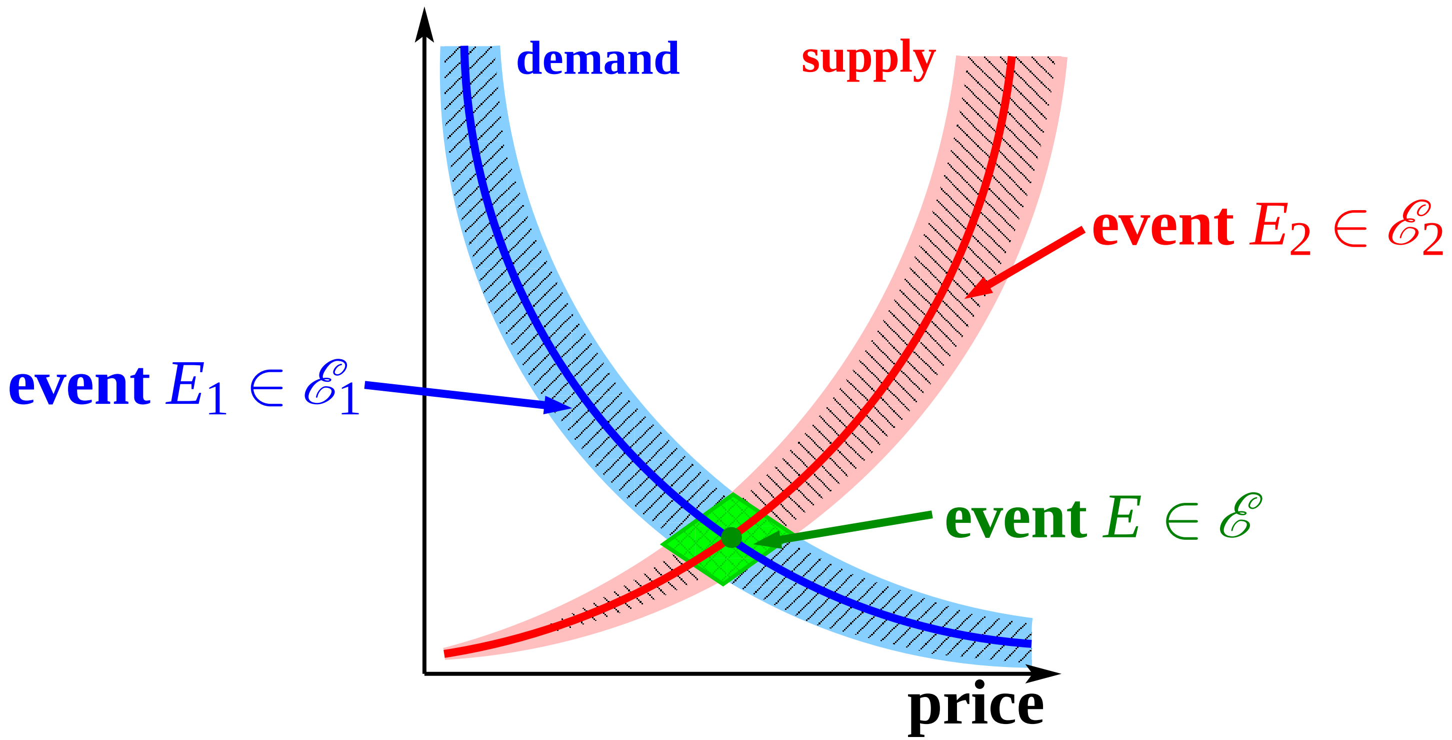
Complementary σ-algebras if $R + R' \neq 0$.



$$P(E) = P_1(E_1)P_2(E_2)$$

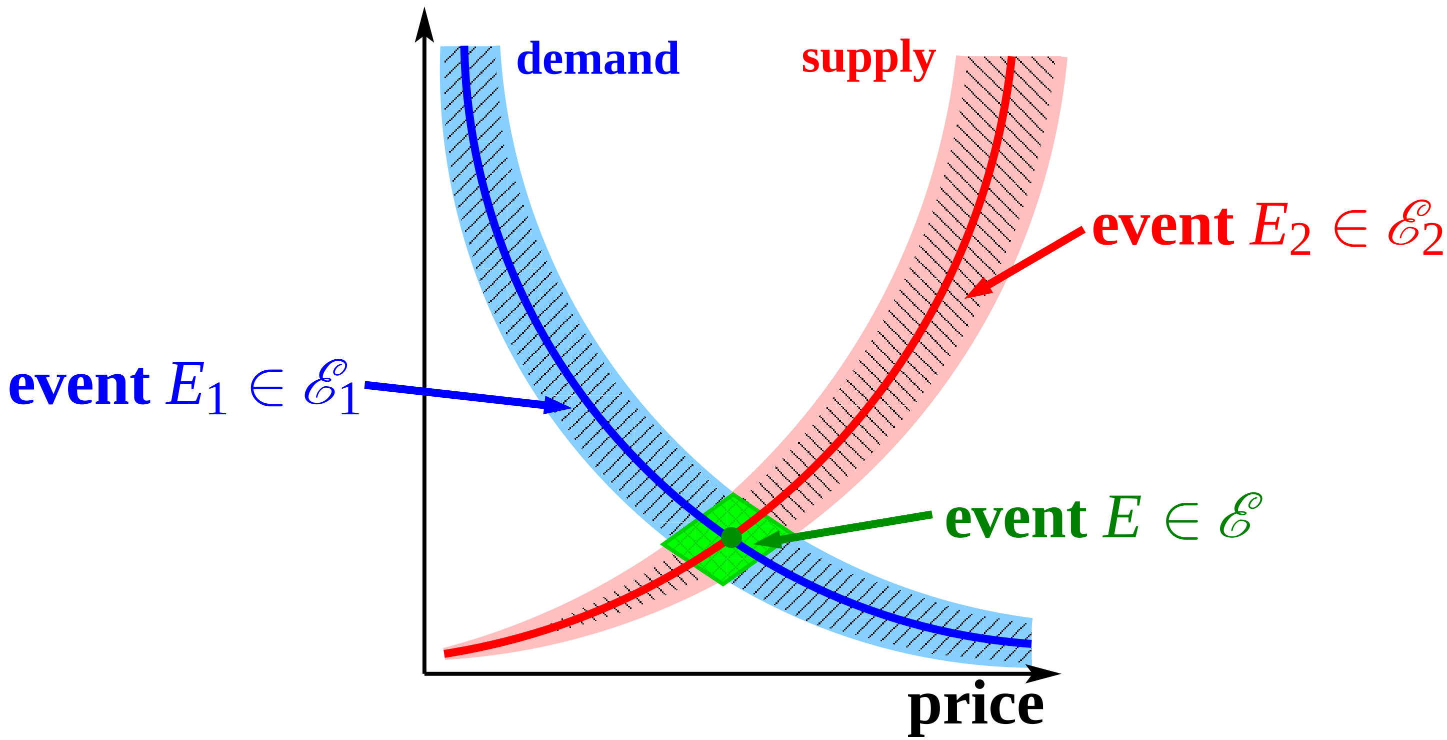
$\mathscr{E}$ = Borel σ-algebra on $\mathbb{R}^2$.

Equilibrium price/demand/supply



$\mathscr{E}_1$ and $\mathscr{E}_2$ typically complementary.

Equilibrium price/demand/supply



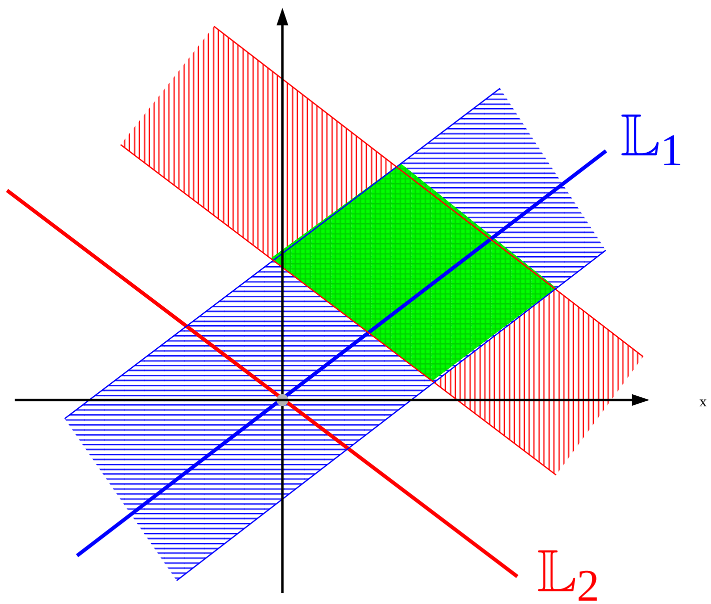
$$P(E) = P_1(E_1)P_2(E_2).$$

$\mathscr{E}$ **typically = Borel** σ**-algebra on** $[0,\infty) \times [0,\infty)$**.**

Interconnection of linear systems

Assume $\Sigma_1 = (\mathbb{R}^{\mathtt{n}}, \mathscr{E}_1, P_1), \Sigma_2 = (\mathbb{R}^{\mathtt{n}}, \mathscr{E}_2, P_2)$ **linear.**

$\mathscr{E}_1$ **and** $\mathscr{E}_2$ **are complementary iff** $\mathbb{L}_1 + \mathbb{L}_2 = \mathbb{R}^{\mathtt{n}}$.



The interconnection of Σ_1 **and** Σ_2 **is a classical random vector if** $\mathbb{L}_1 \oplus \mathbb{L}_2 = \mathbb{R}^{\mathtt{n}}$.

Open stochastic systems

Open versus closed

$\Sigma_1 = (\mathbb{R}^n, \mathscr{E}_1, P_1)$, $\mathscr{E}_1$ = **Borel σ-algebra** $\mathscr{B}(\mathbb{R}^n)$.

$\mathscr{E}_2 \subseteq \mathscr{B}(\mathbb{R}^n)$ **sub-σ-algebra.**

$[\![\mathscr{E}_1$ **and** $\mathscr{E}_2$ **complementary**$]\!] \Rightarrow [\![\mathscr{E}_2 = \{\emptyset, \mathbb{R}^n\}]\!]$,

that is, $\Sigma_2 = (\mathbb{R}^n, \mathscr{E}_2, P_2)$ **is free.**

$\Rightarrow$ **classical** Σ_1 = **'closed' system.**

Open versus closed

$\Sigma_1 = (\mathbb{R}^{\mathtt{n}}, \mathscr{E}_1, P_1)$, $\mathscr{E}_1$ **= Borel** σ**-algebra** $\mathscr{B}(\mathbb{R}^{\mathtt{n}})$**.**

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that is, $\Sigma_2 = (\mathbb{R}^{\mathtt{n}}, \mathscr{E}_2, P_2)$ **is free.**

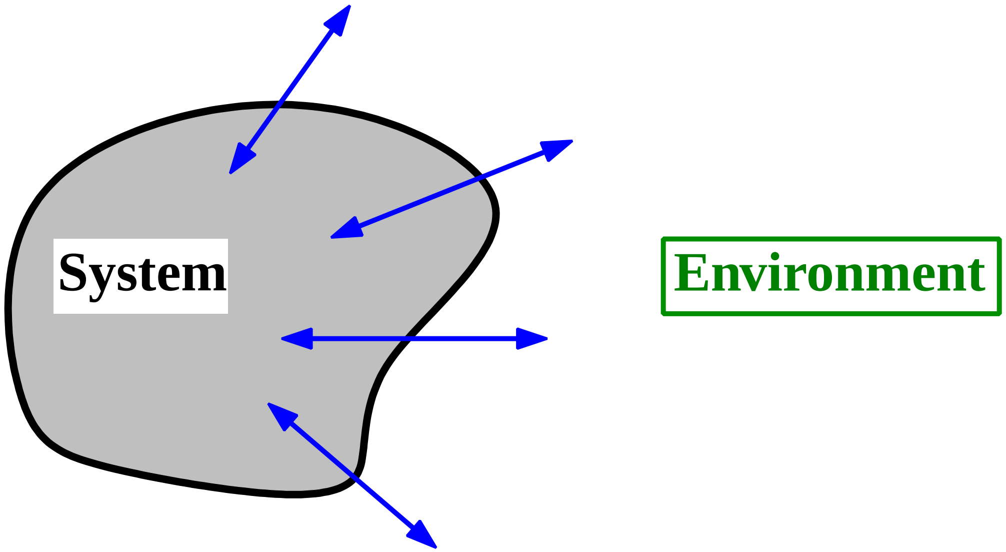
$\Rightarrow$ **classical** Σ_1 **= 'closed' system.**

Coarse $\mathscr{E}_1$

$\Rightarrow \Sigma_1$ **is interconnectable.**

$\Rightarrow$ **'open' system.**

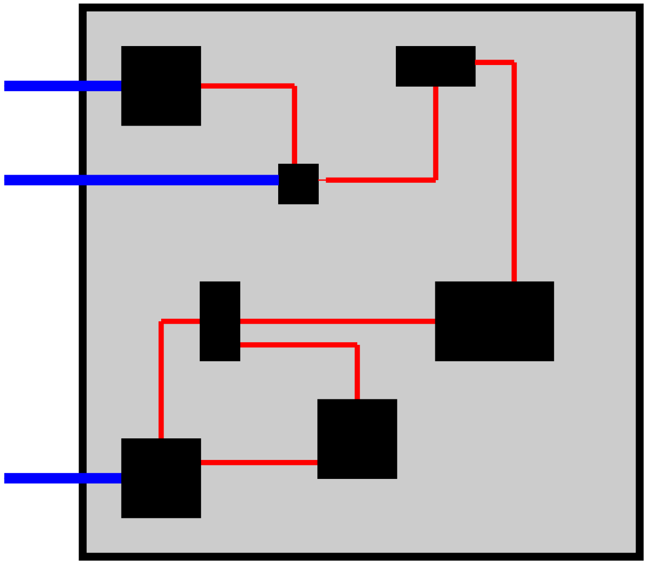
The system theorist's requirements for a good notion



Open

Models should incorporate influence of environment.

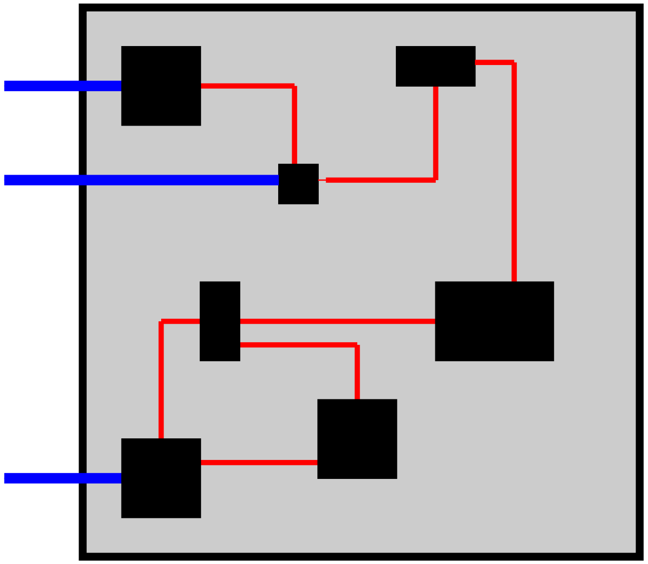
The system theorist's requirements for a good notion



Connectable

Models should allow interconnection.

The system theorist's requirements for a good notion



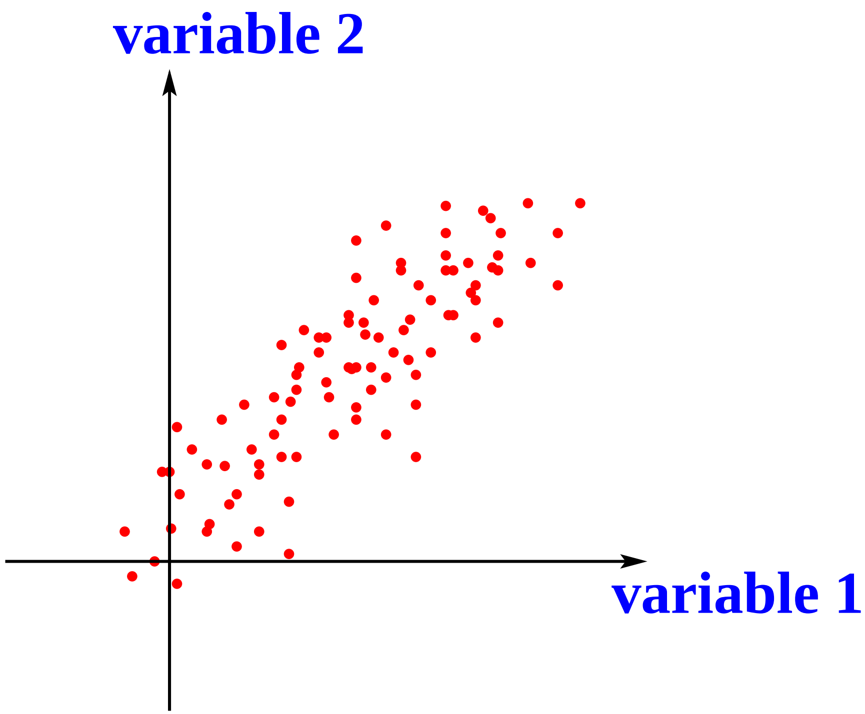
Connectable

Models should allow interconnection.

Classical random vectors fail these requirements.

Identification

Sampling

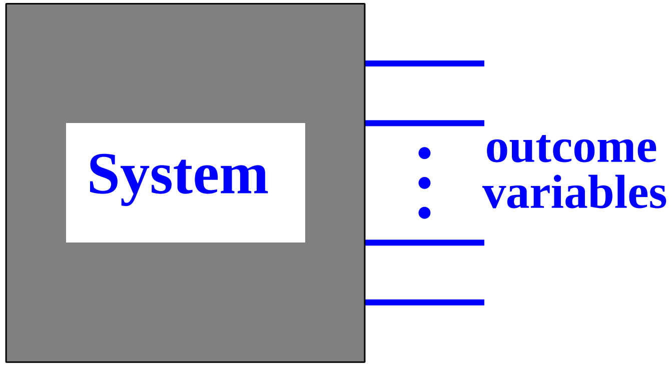


System identification: deduce the stochastic model

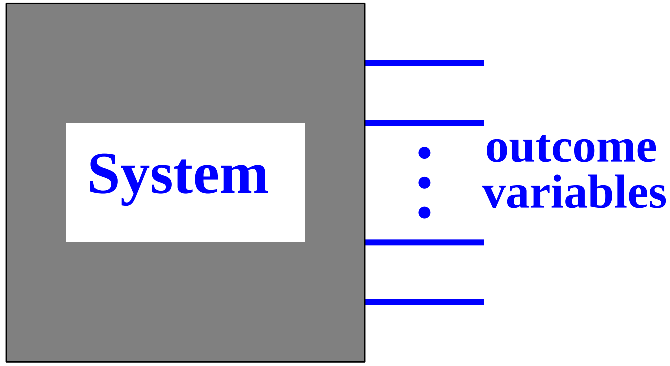
$\mathscr{E}$ and P

from the samples.

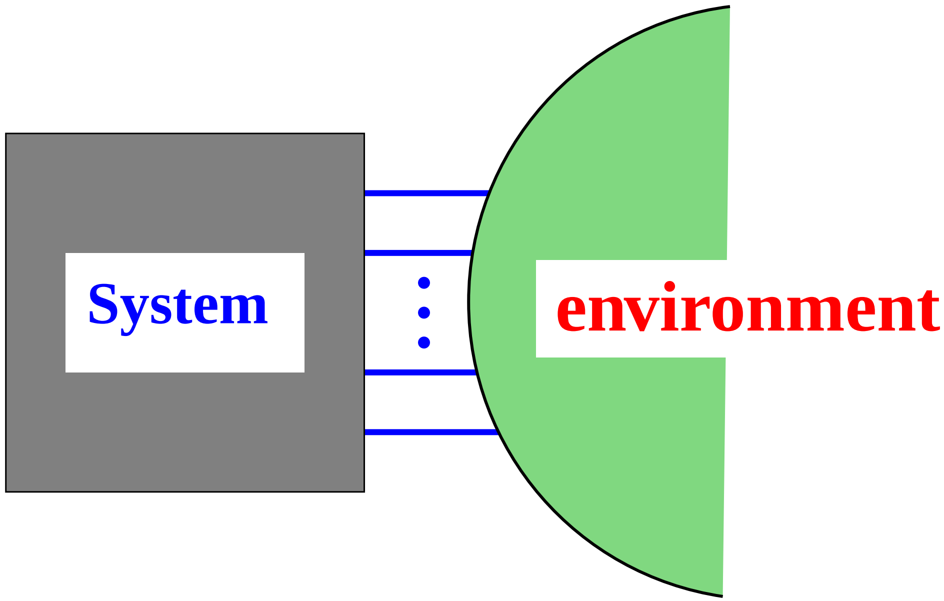
Measurements



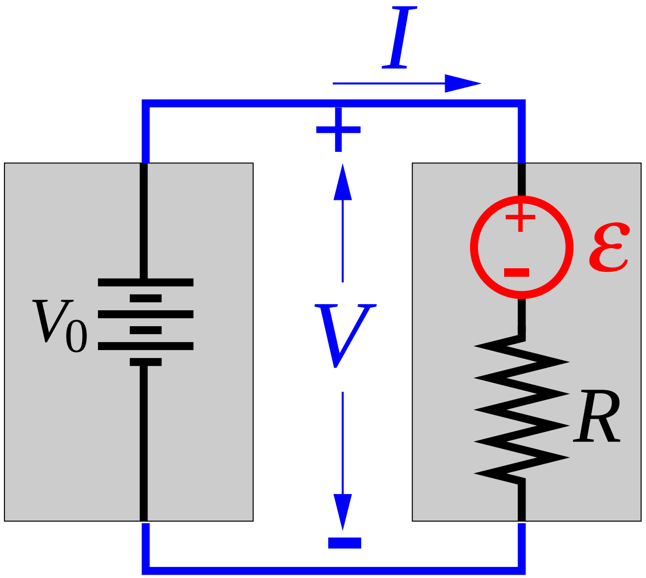
Measurements

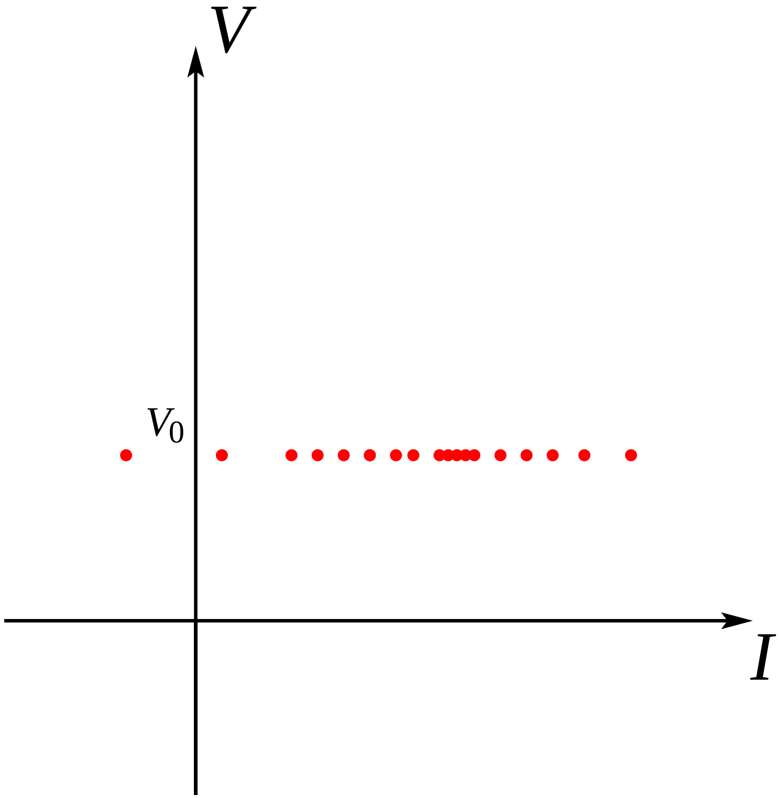


Measurements occur under experimental conditions.

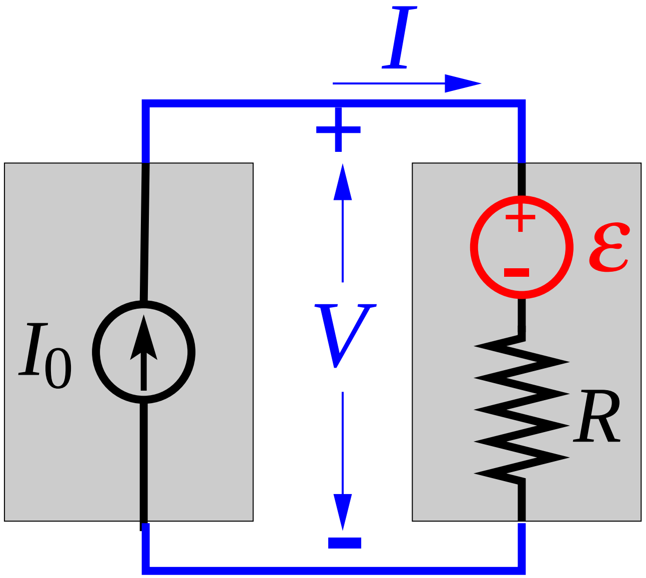


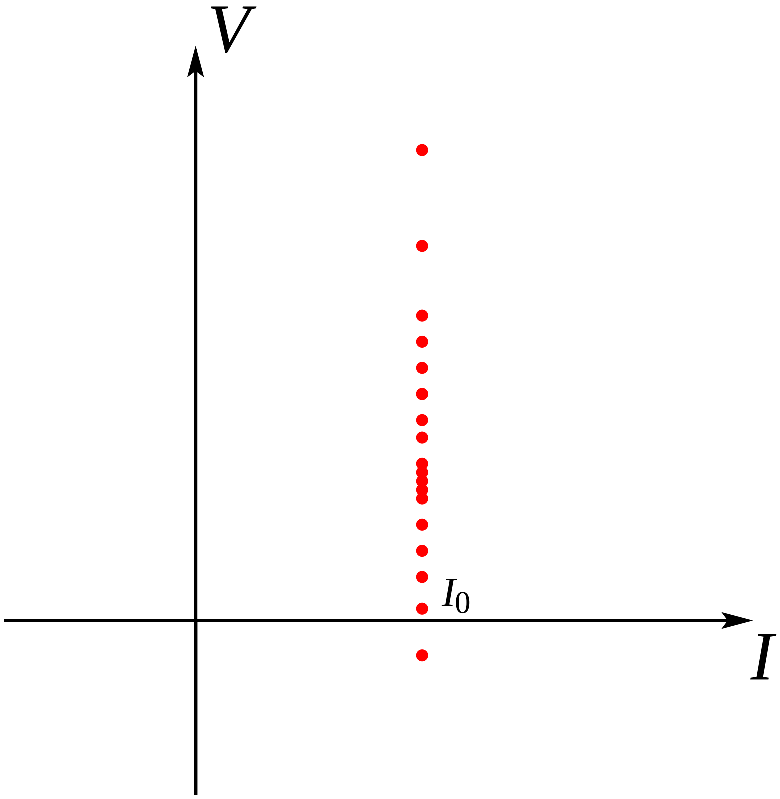
Noisy resistor with a voltage source



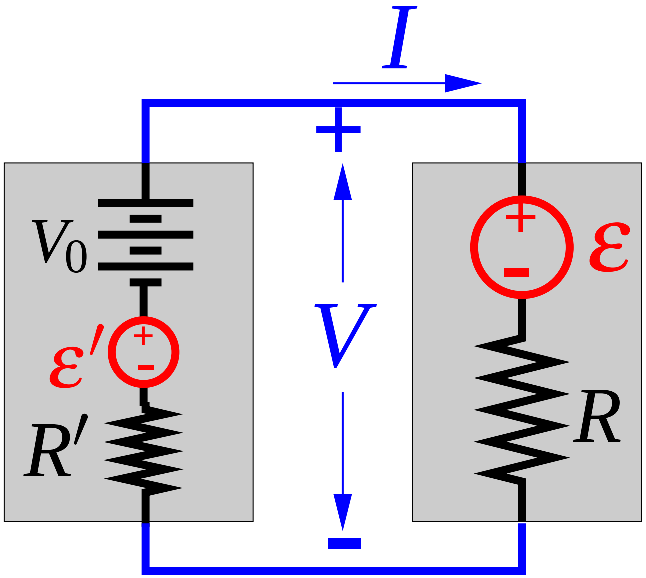


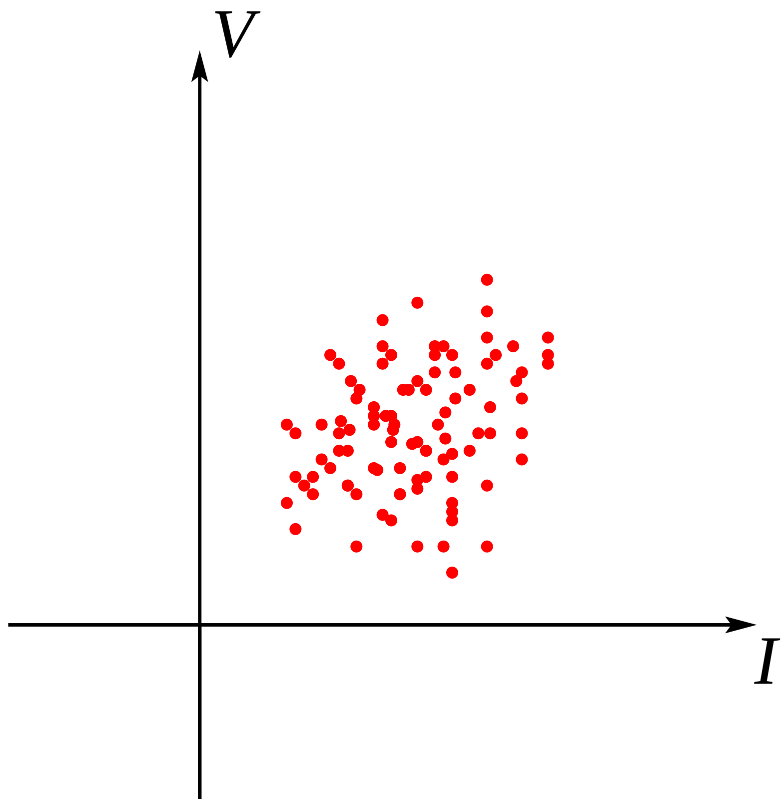
Noisy resistor with a current source





Noisy resistor with a noisy voltage source





Measurements

Data collection requires observing a stochastic system *in interaction with an environment*. The samples depend on both the system & the environment.

Is it possible to disentangle the laws of a system from the laws of the environment?

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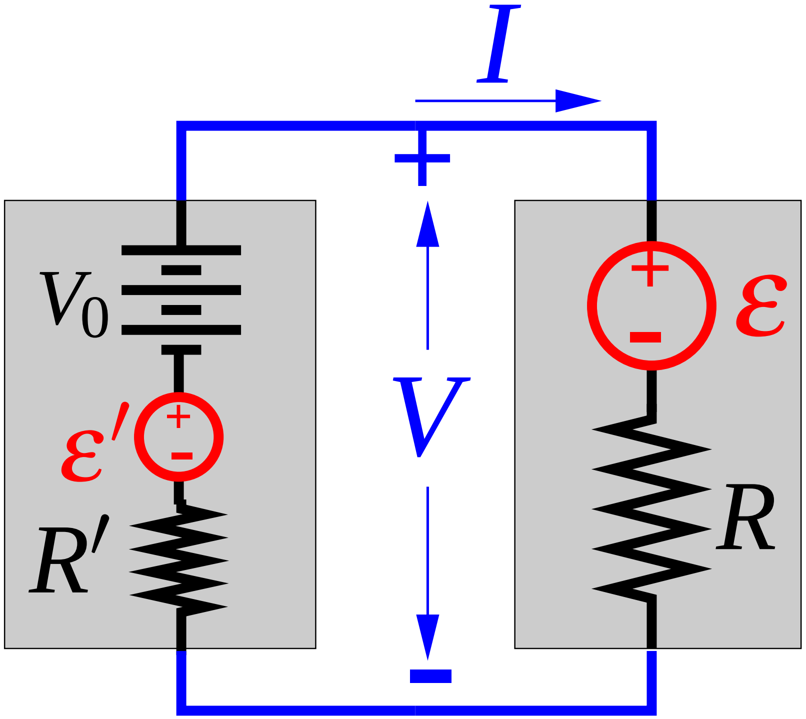
Is it possible to disentangle the laws of a system from the laws of the environment?

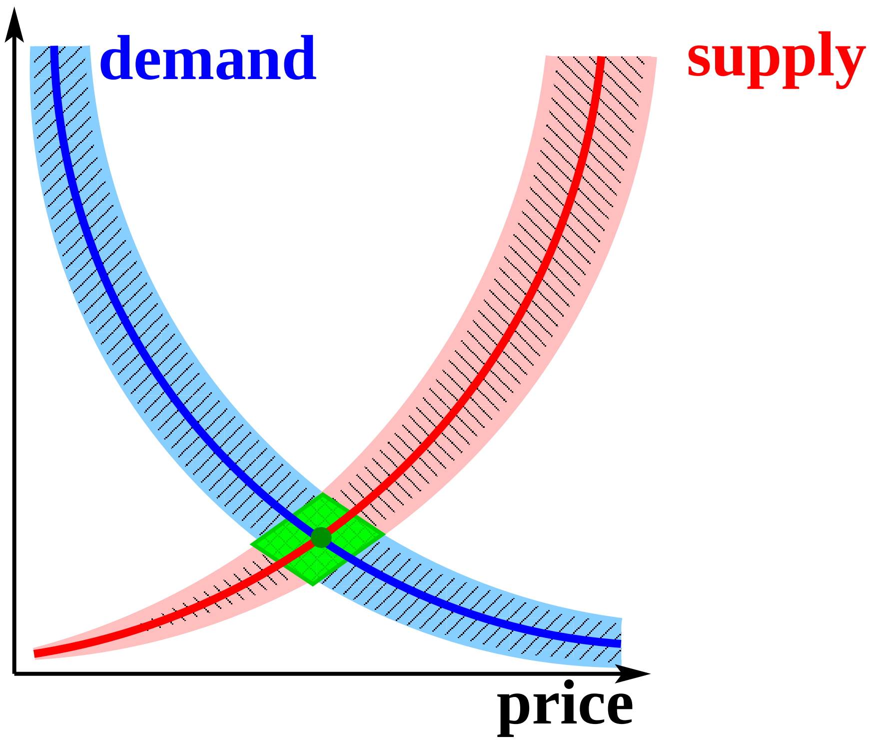
In engineering, it may be possible to set the experimental conditions. In economics and the social sciences (and biology?), data are often gathered passively, 'in vivo'.

Disentangling



Can R and σ be deduced by sampling (V, I)?

Disentangling



Can the price/demand characteristic be deduced by sampling (p, d) in equilibrium?

SYSID for gaussian stochastic systems

Let $\Sigma_1 = (\mathbb{R}^{\mathtt{n}}, \mathscr{E}_1, P_1)$ **and** $\Sigma_2 = (\mathbb{R}^{\mathtt{n}}, \mathscr{E}_2, P_2)$ **be complementary gaussian systems. Assume that the interconnection** $\Sigma_1 \wedge \Sigma_2$ **is a classical random vector.**

Sampling $\rightsquigarrow$ **the mean and covariance of** $\Sigma_1 \wedge \Sigma_2$**. Assume that the covariance is nonsingular.**

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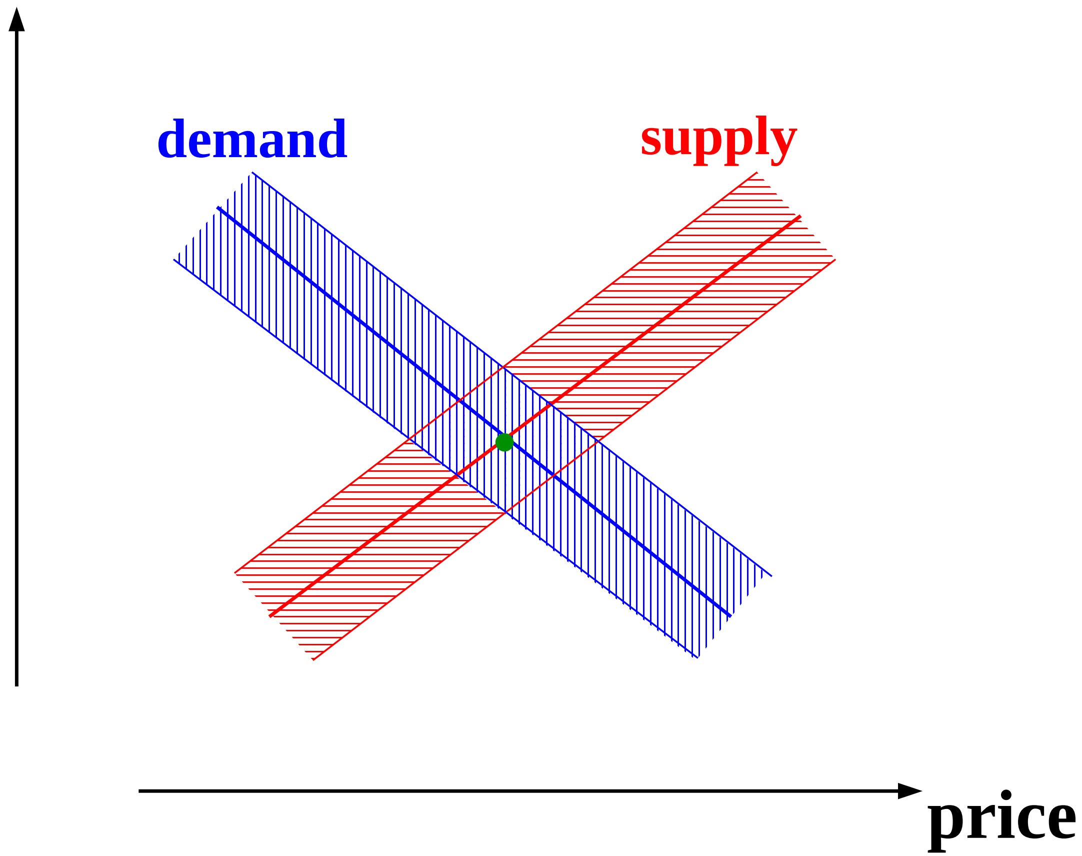
Given fiber of either Σ_1 **or** Σ_2**, then the parameters of** Σ_1 **and** Σ_2 **can be deduced from** $\Sigma_1 \wedge \Sigma_2$**.**

The fiber of Σ_1 **can be chosen freely.**

Linearized gaussian price/demand/supply

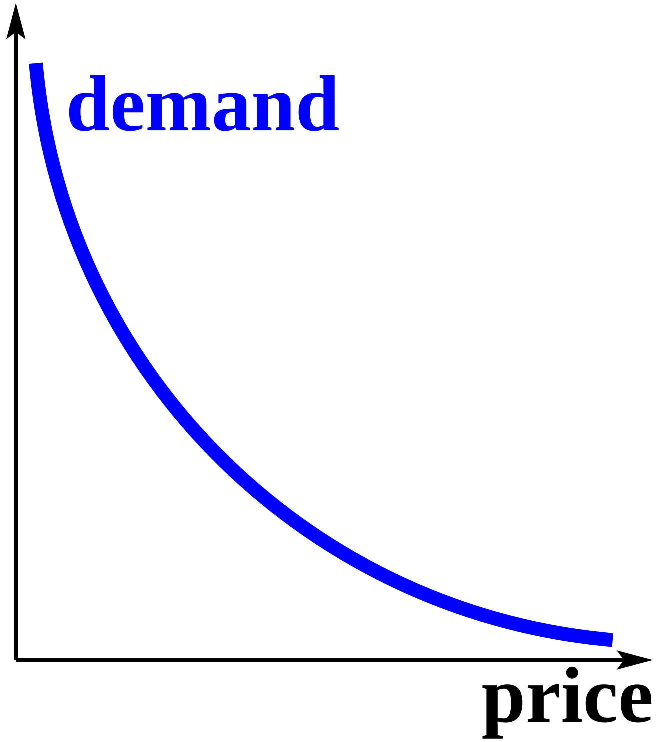


Identifiability provided one of the fibers is known.

Sampling alone does not give the elasticities.

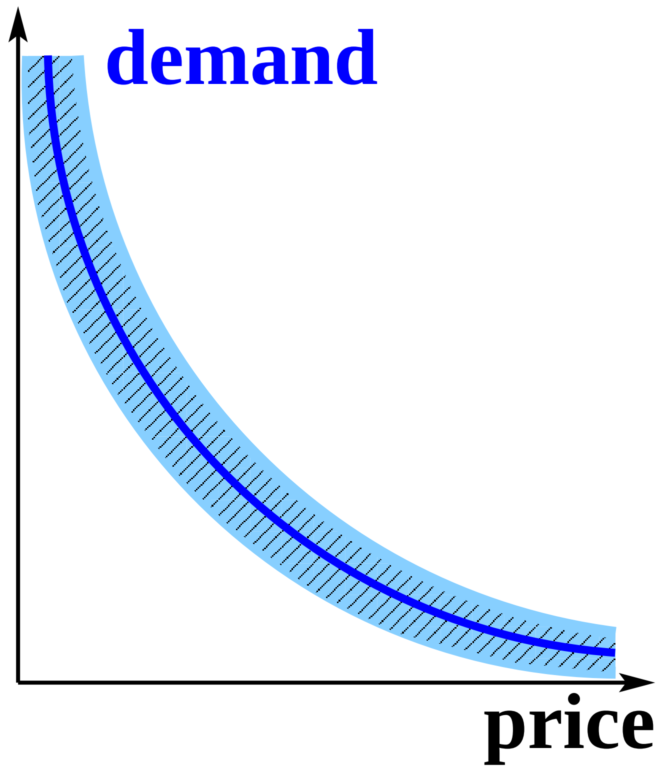
Summary: an example

Deterministic price/demand



Price ⇒ demand.

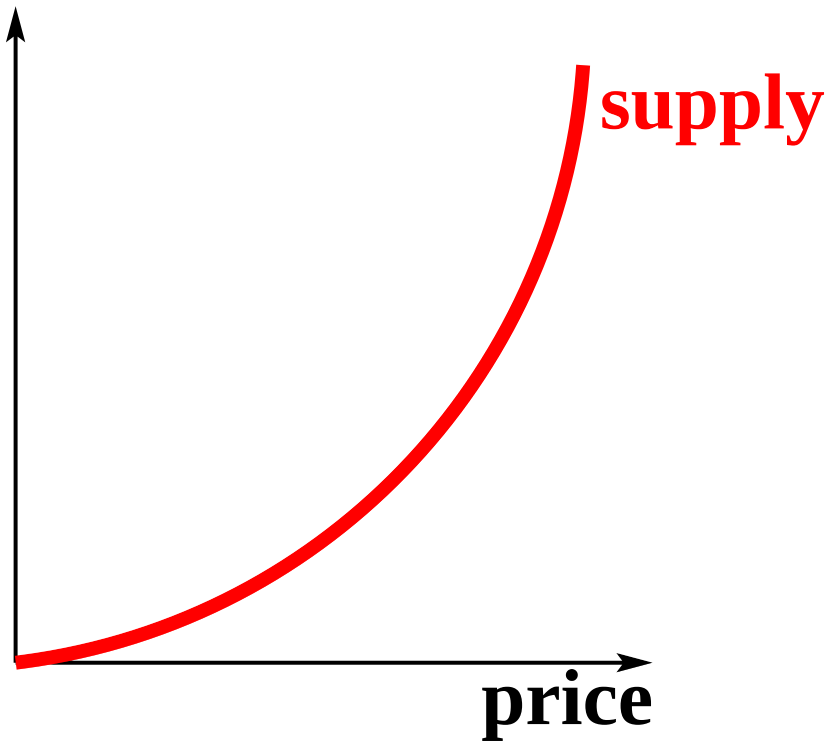
Stochastic price/demand



Certain $\left[\begin{array}{c}\text{price}\\ \text{demand}\end{array}\right]$ regions are assigned a probability.

Borel σ-algebra $\Rightarrow$ trouble in the deterministic limit.

Deterministic price/supply



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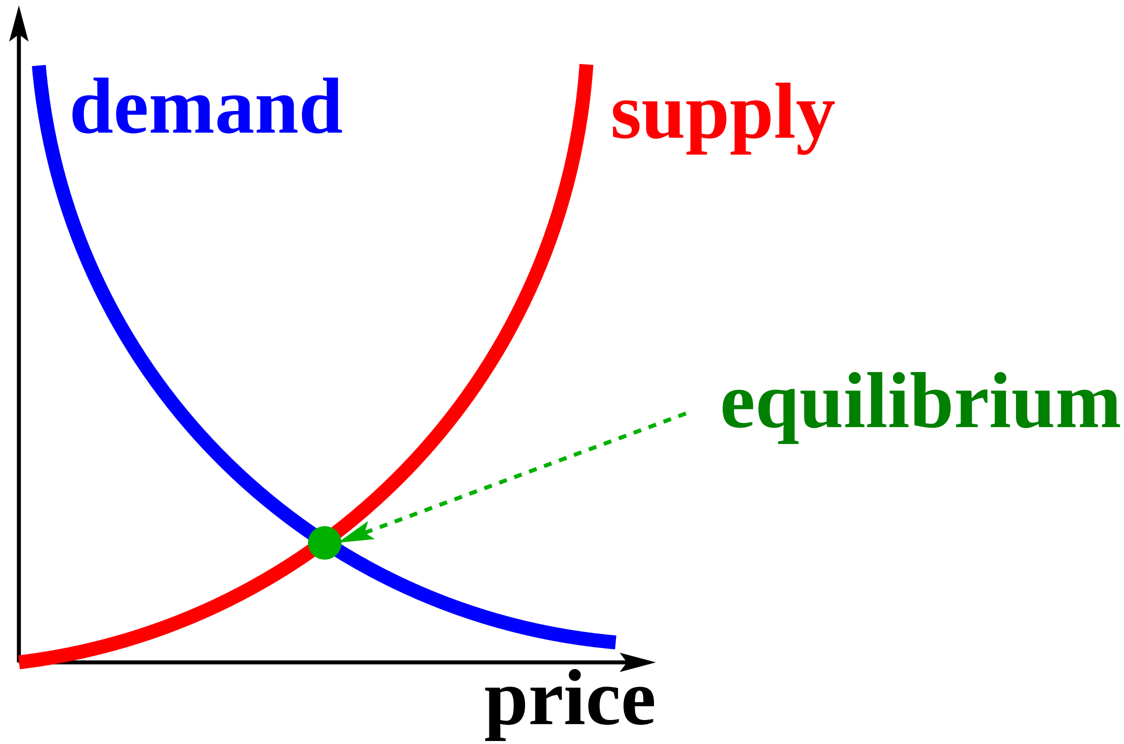
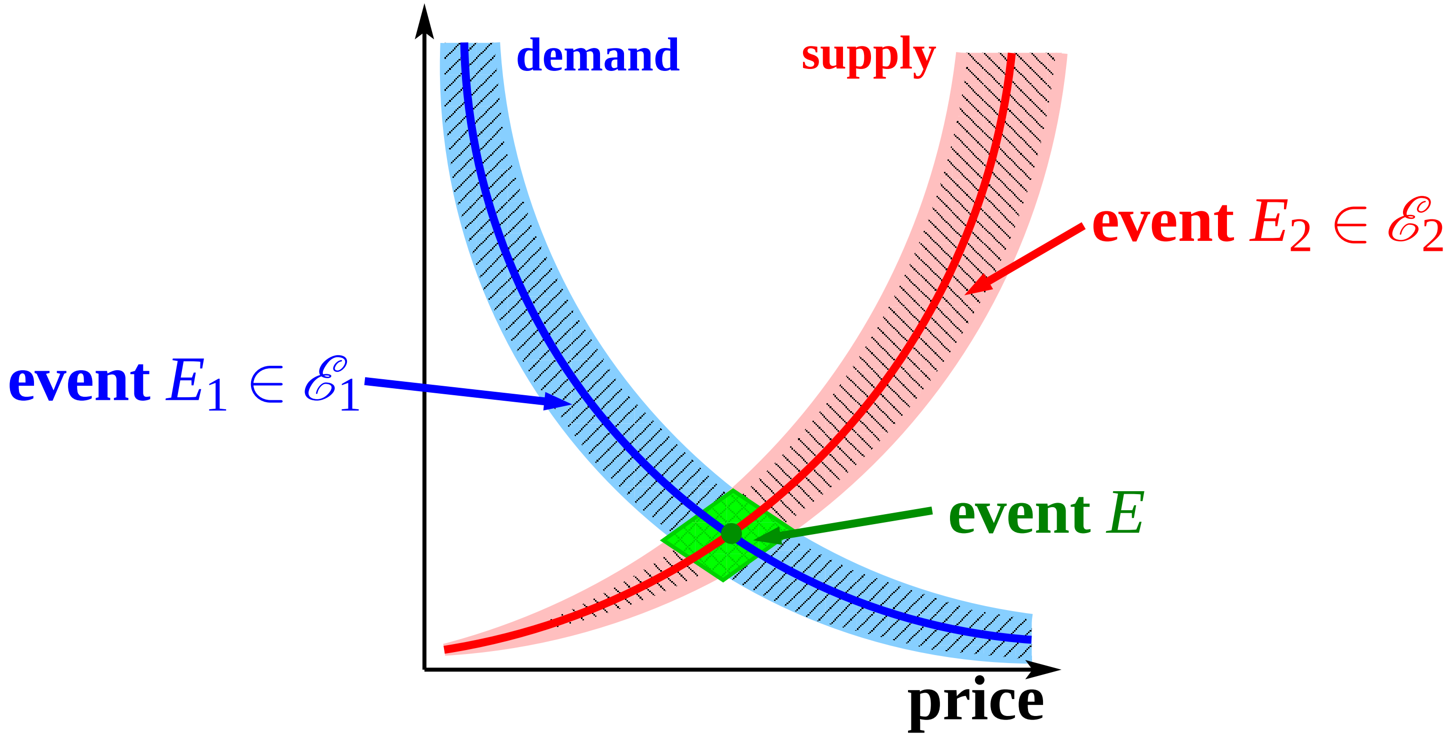
Deterministic equilibrium

Prices pertain to same product & demand = supply.



Equilibrium ⇒ price, demand, and supply.

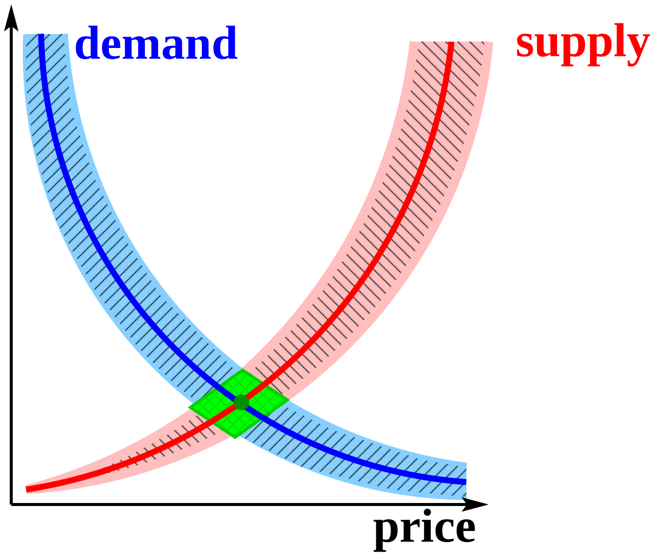
Prices pertain to same product & demand = supply.

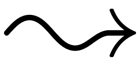


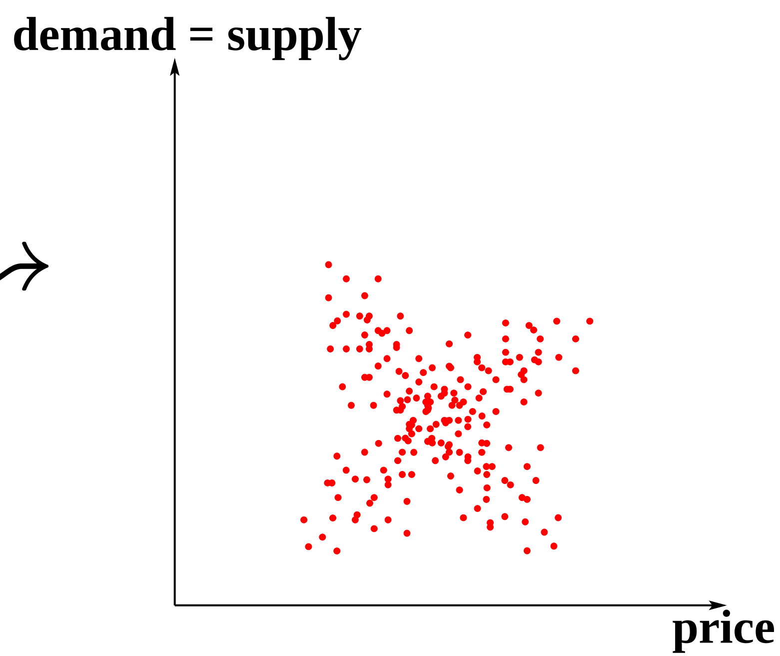
Complementarity $\Rightarrow \begin{bmatrix} \text{price} \\ \text{demand = supply} \end{bmatrix}$ **well-defined stochastic system. Typically the equilibrium yields a classical random vector** $\begin{bmatrix} \text{price} \\ \text{demand = supply} \end{bmatrix}$.

Identification

Sample $\left[\begin{array}{c}\textbf{price}\\ \textbf{demand = supply}\end{array}\right]$.

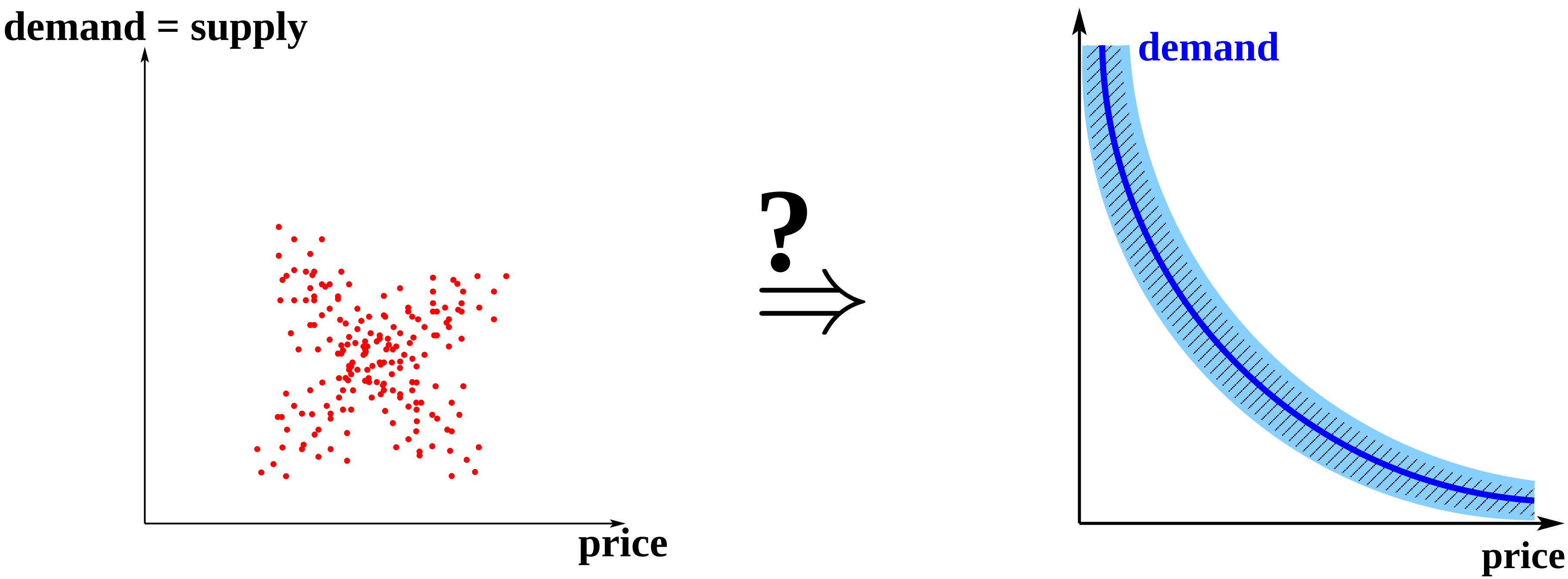






Disentanglement

Does passive sampling $\left[\begin{array}{c}\text{price}\\ \text{demand = supply}\end{array}\right]$ imply the price/demand elasticity?



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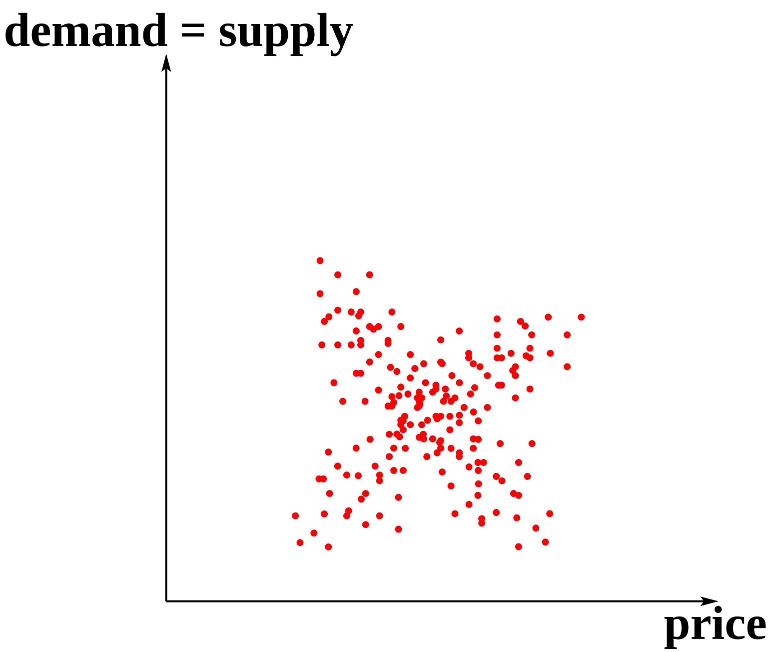
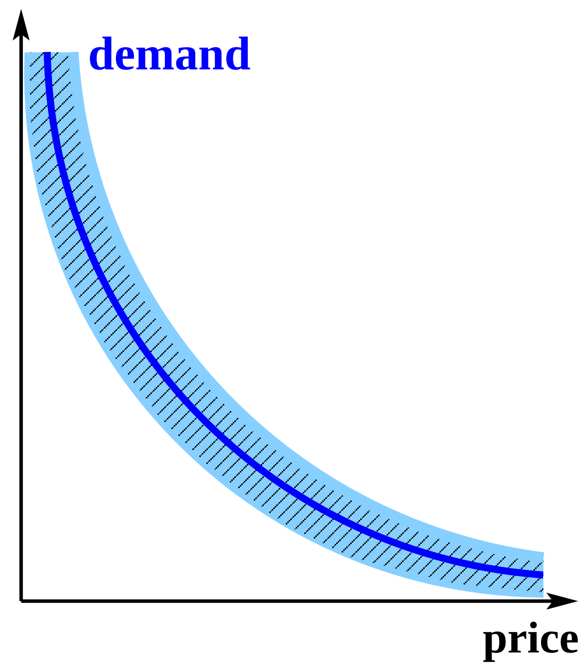


$\overset{?}{\Rightarrow}$



Sampling alone $\not\Rightarrow$ identification.

Requires more a priori knowledge.

Conclusions

Stochastic systems

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Stochastic systems

- The Borel σ-algebra is inadequate even for elementary applications.
- Complementary stochastic systems can be interconnected:
  two distinct laws imposed on one set of variables.
- Classical random vectors are closed systems.
  Open stochastic systems require a coarse σ-algebra of events.
- Measurements are the result of interaction with an environment.
  Modeling from data requires disentanglement.
  Sampling alone is insufficient for identifiability.

Reference: ***Open stochastic systems*, IEEE TAC, submitted.**

Copies of the lecture frames available from/at

`http://www.esat.kuleuven.be/~jwillems`

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