SYNTHESIS of RECIPROCAL CIRCUITS

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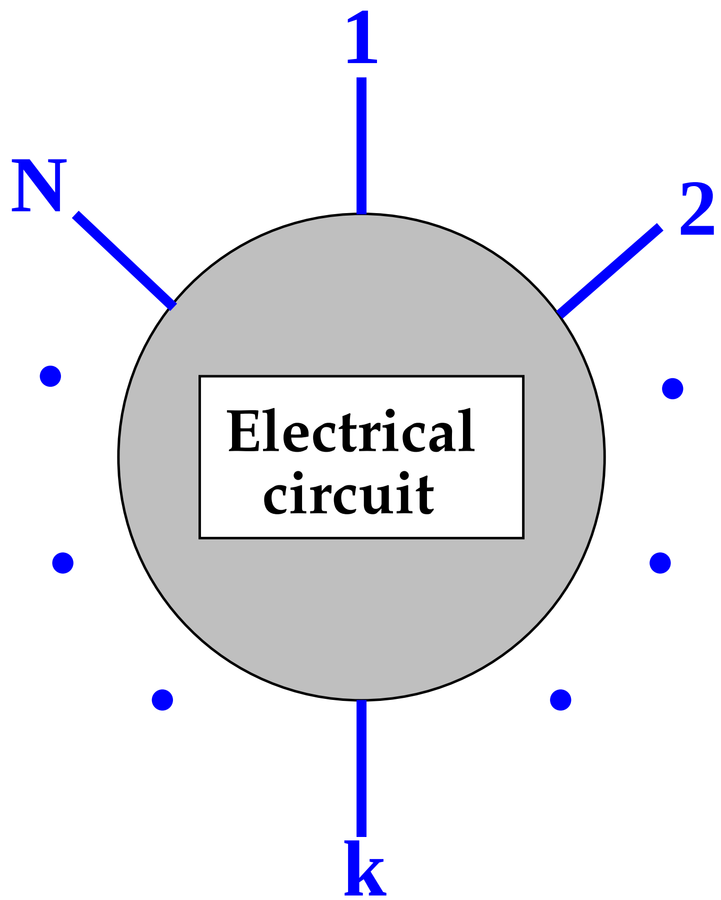
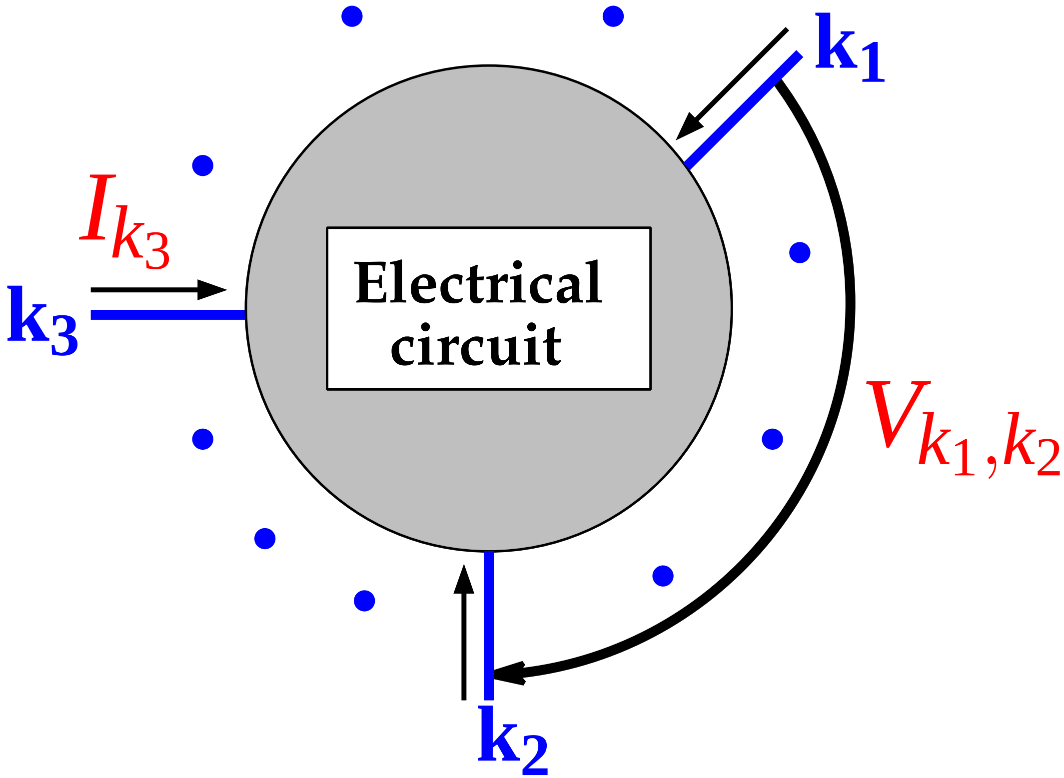
Circuits

Electrical circuits



An N-terminal circuit

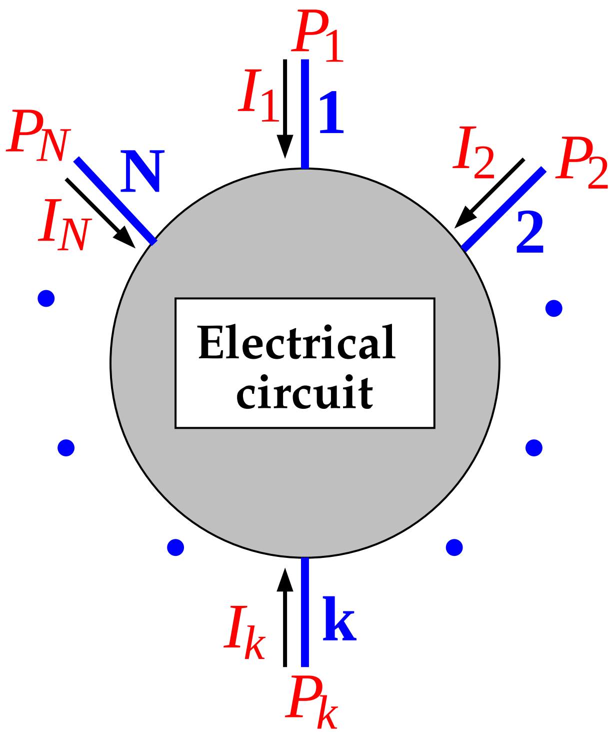
Electrical circuits



Variables: terminal currents and voltages

Electrical circuits

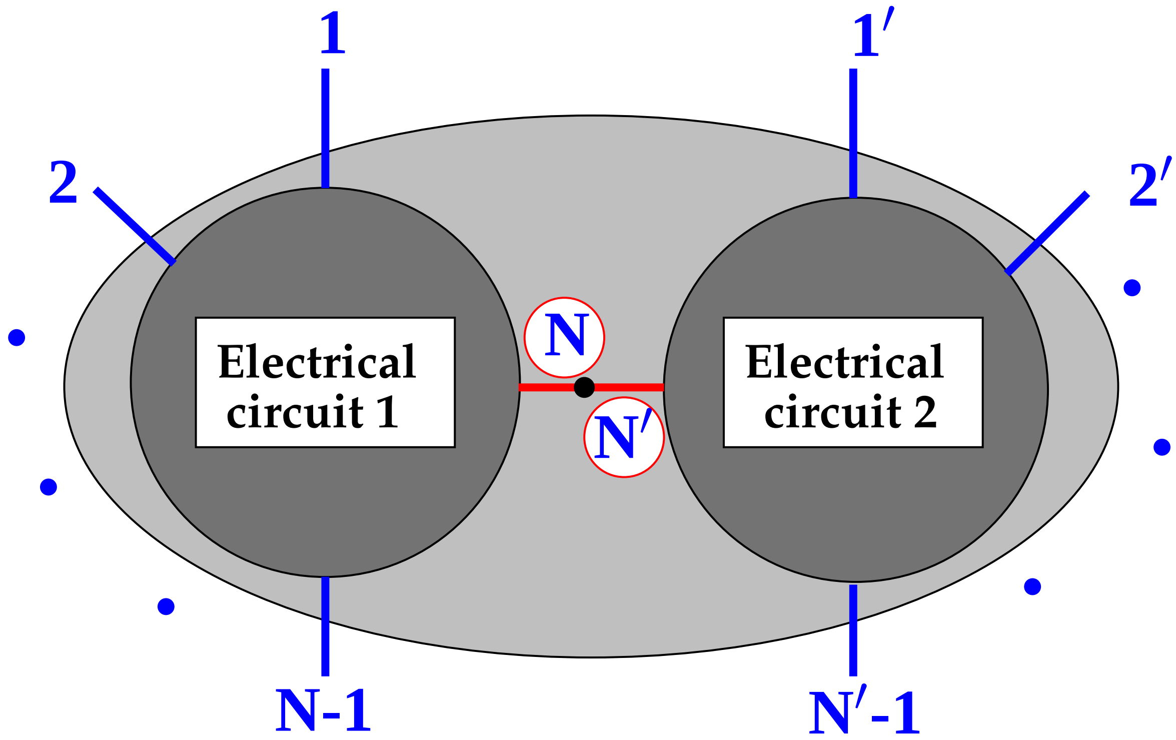
Assume KVL: $V_{k_1,k_2} + V_{k_2,k_3} + V_{k_3,k_1} = 0.$



<u>Variables</u>: terminal currents and potentials

$$V_{k_1,k_2} = P_{k_1} - P_{k_2}.$$

Terminal connection

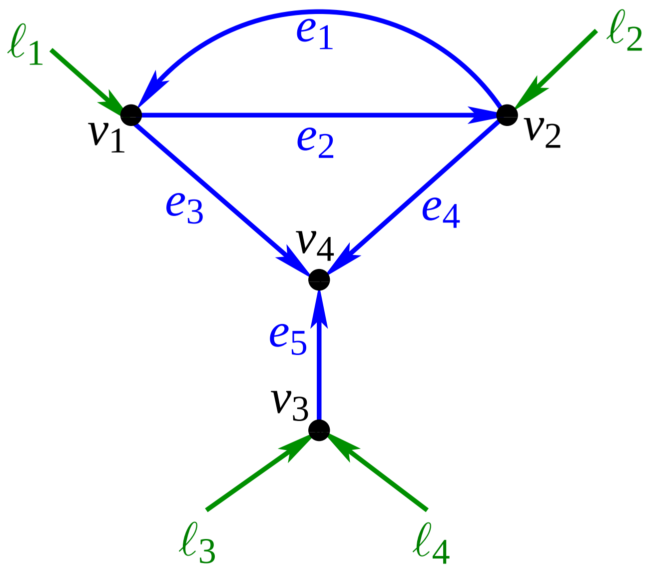


Imposes : $$I_N + I_{N'} = 0, \quad P_N = P_{N'}.$$

Extendable to more than 2 terminals, than 2 circuits.

Circuit architecture

Digraph with leaves



vertices: subcircuits, 'building blocks',

edges: connections,

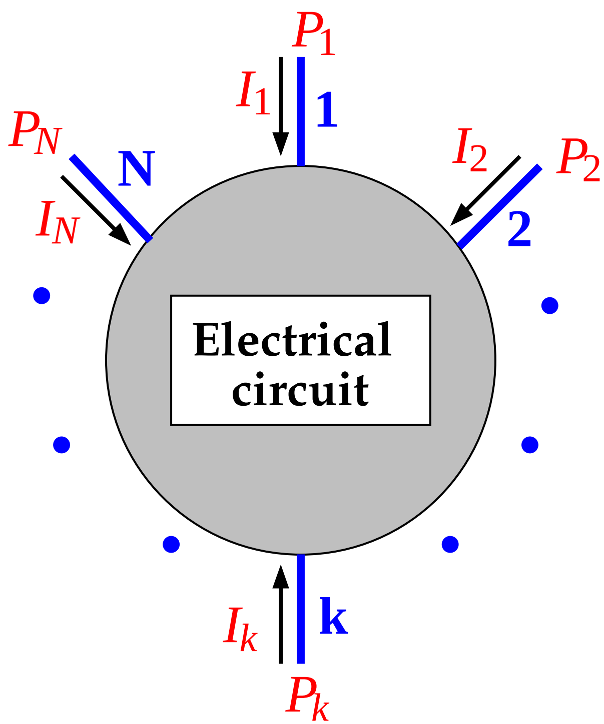
leaves: external terminals.

The terminal behavior

The current/potential behavior

$$\mathscr{B}_{IP} \subseteq (\mathbb{R}^N \times \mathbb{R}^N)^{\mathbb{R}}.$$

$\mathscr{B}_{IP}$ **= all possible** $(I_1, P_1, I_2, P_2, \ldots, I_N, P_N) : \mathbb{R} \to \mathbb{R}^N \times \mathbb{R}^N$.



KCL: $I_1 + I_2 + \cdots + I_N = 0.$

Analysis & synthesis

Analysis:

Given circuit architecture and element laws,

find $\mathscr{B}_{IP}$.

Synthesis:

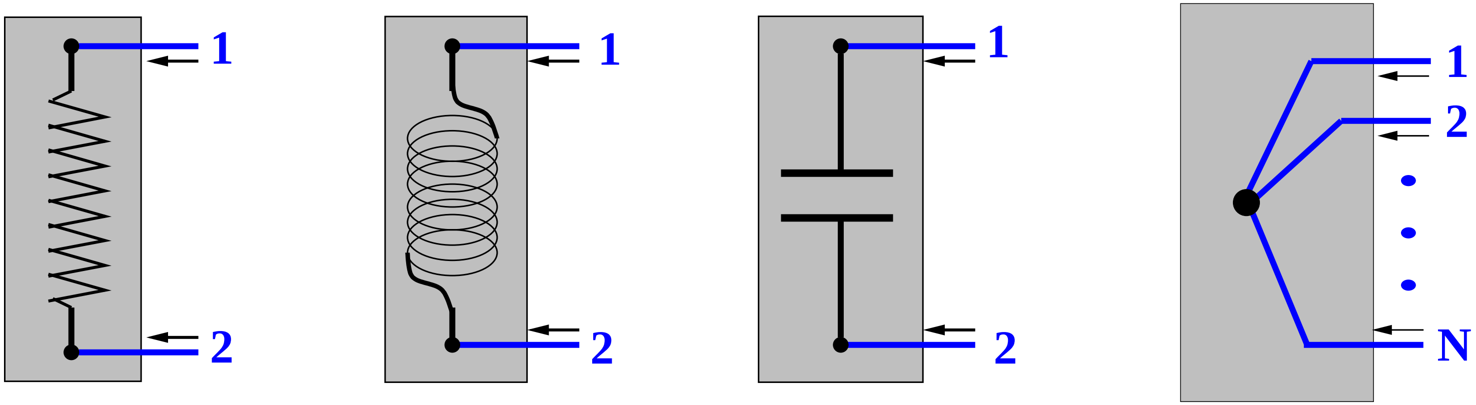
Given $\mathscr{B}_{IP}$ and building blocks,

find circuit architecture.

Which $\mathscr{B}_{IP}$'s can be synthesized?

Building blocks

Resistor, inductor, capacitor, connector

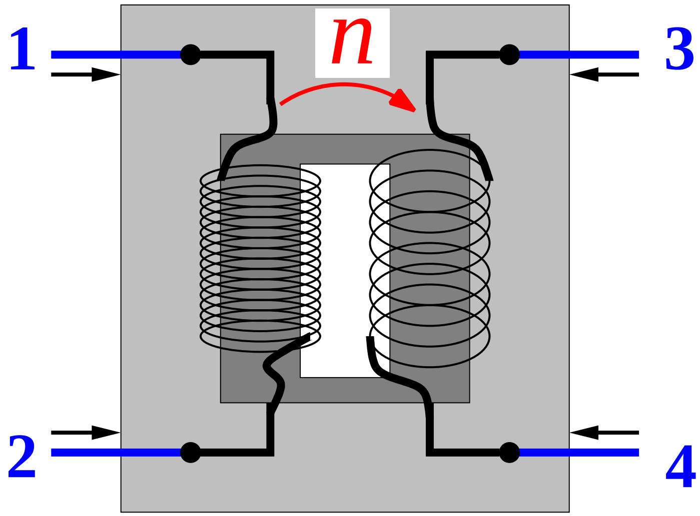


Element laws, e.g. inductor

$$I_1 + I_2 = 0, \qquad \boxed{P_1 - P_2 = L\,\frac{d}{dt}I_1.}$$

L **= inductance.**

Transformer

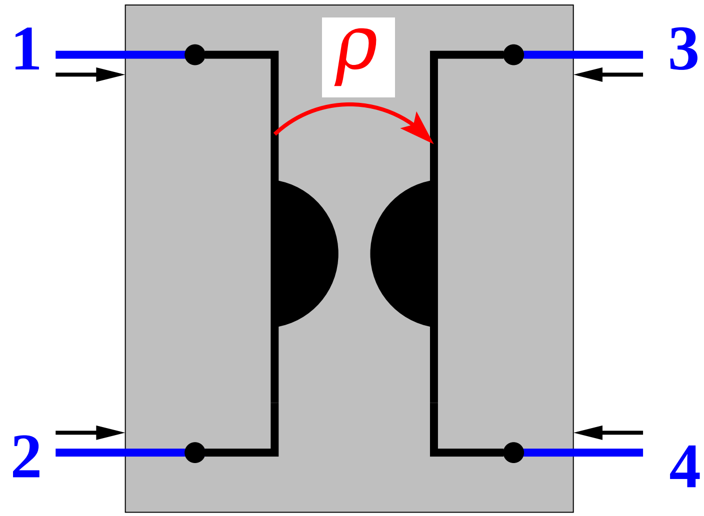


$$I_1 + I_2 = 0, I_3 + I_4 = 0,$$

$$I_3 = -n\, I_1, \qquad (P_1 - P_2) = n\,(P_3 - P_4).$$

n = **'turns ratio'.**

Gyrator



$$I_1 + I_2 = 0, I_3 + I_4 = 0,$$

$$(P_1 - P_2) = \rho\, I_3, \quad (P_3 - P_4) = -\rho\, I_1.$$

ρ = 'gyrator resistance'.

Tellegen B.D.H. : 'The gyrator, a new electric network element', *Philips Research Reports*, 3, pp. 81-101, 1948.

RLCTG Synthesis

Terminal synthesis

Determine the terminal behaviors $\mathscr{B}_{IP}$ that are achievable by interconnecting a finite number of

positive resistors,

positive capacitors,

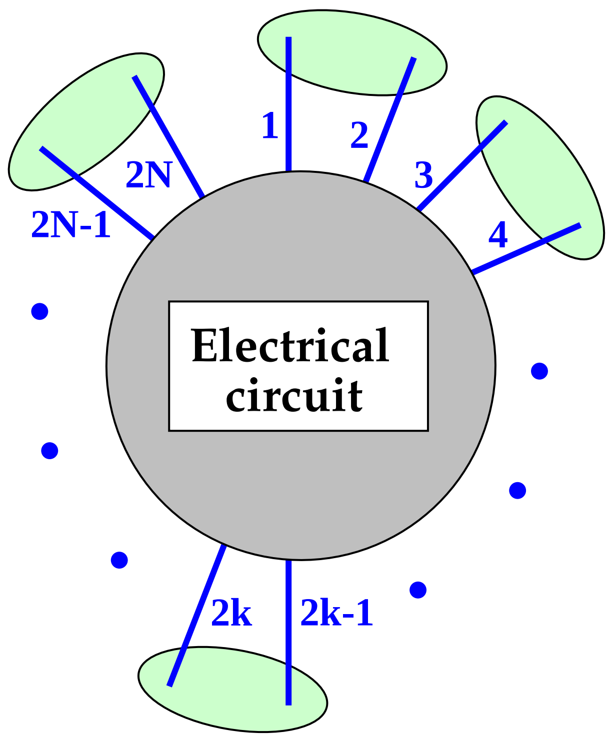
positive inductors,

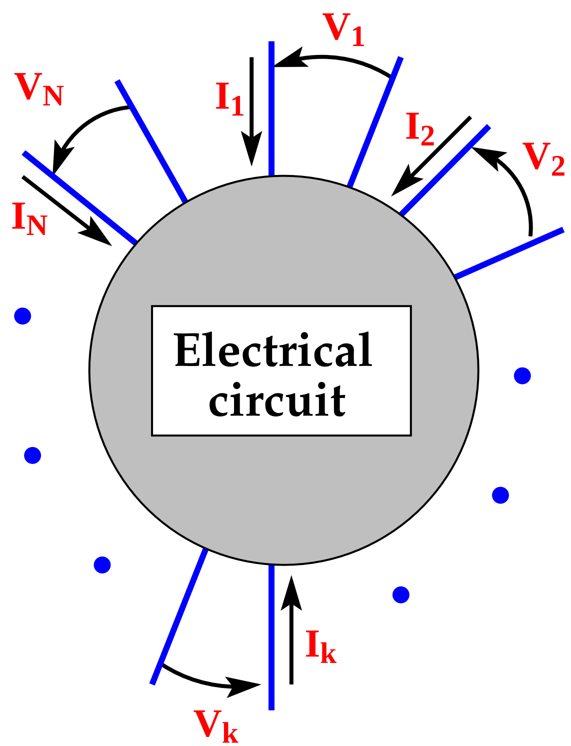
transformers,

and gyrators.

Ports

The port behavior

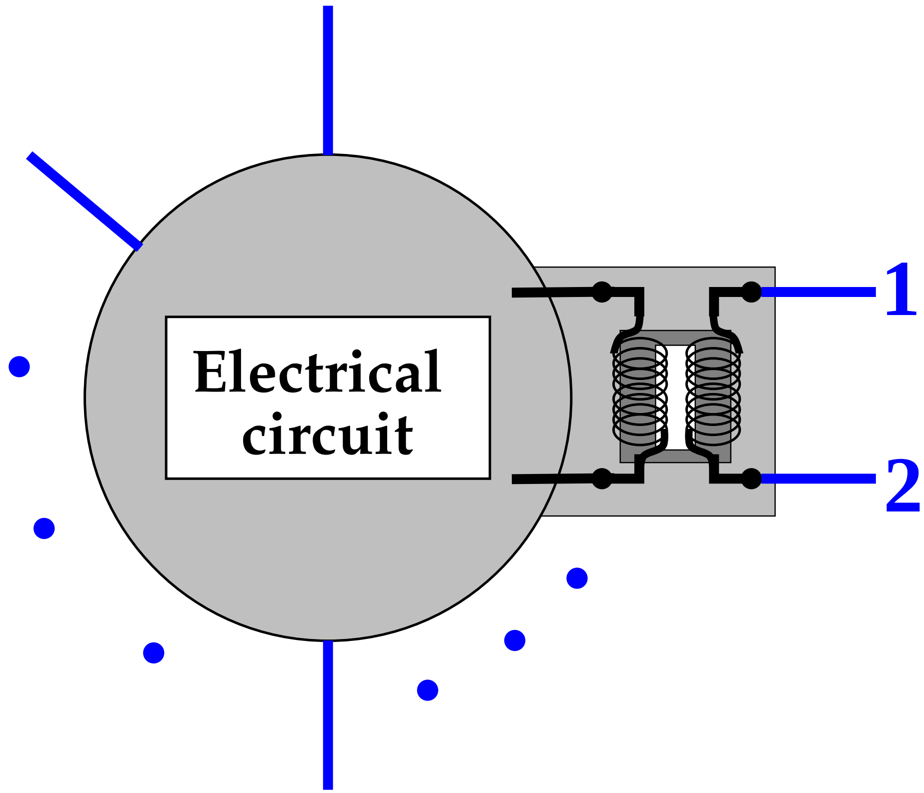




Port behavior:

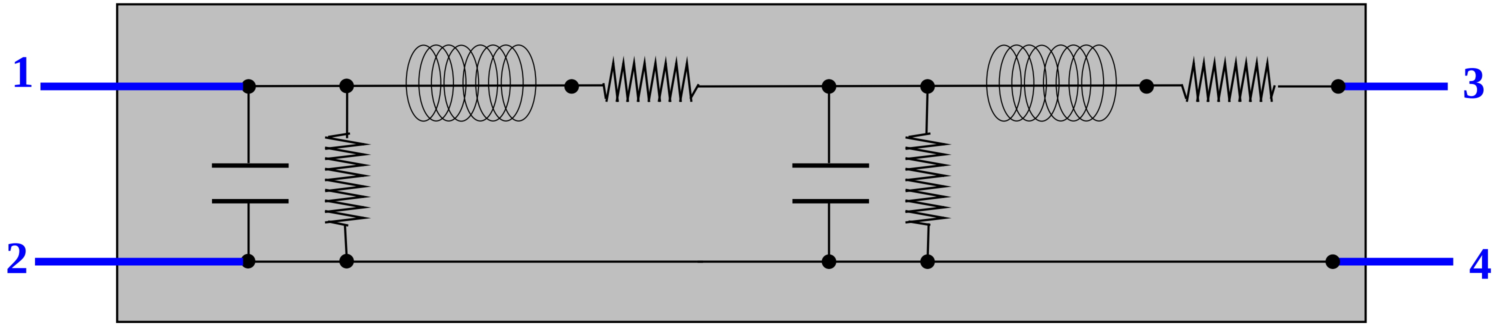
$$\mathscr{B}_{IV} = \{(I_1, V_1, I_2, V_2, \ldots, V_N, I_N) \mid \exists\, P_1, P_3, \ldots, P_{2N-1} :$$
$$(I_1, P_1, -I_1, P_1 - V_1, \ldots, I_N, P_{2N-1}, -I_N, P_{2N-1} - V_N) \in \mathscr{B}_{IP}\}$$

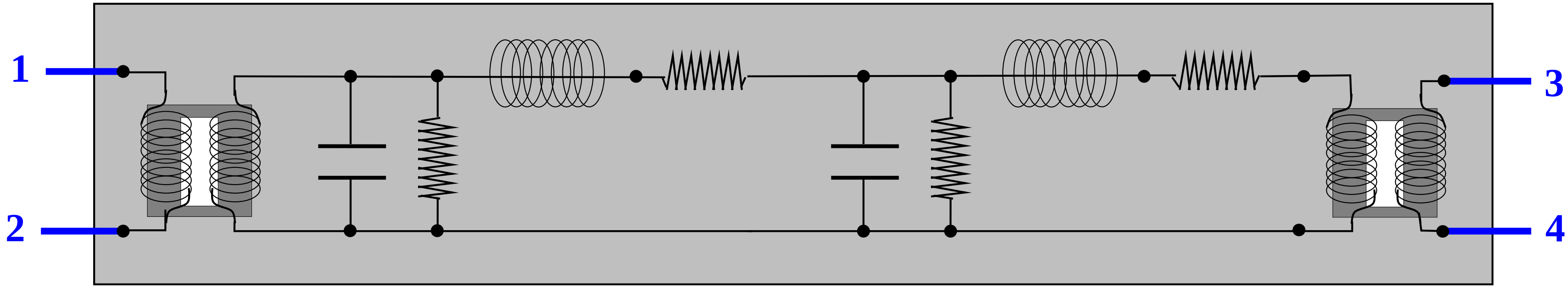
Unit transformer termination

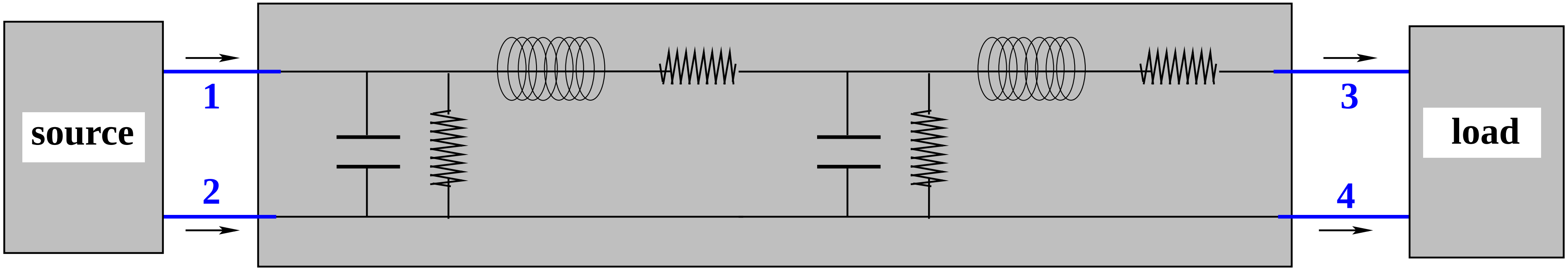


Port behavior $\Leftrightarrow$ unit transformer termination.

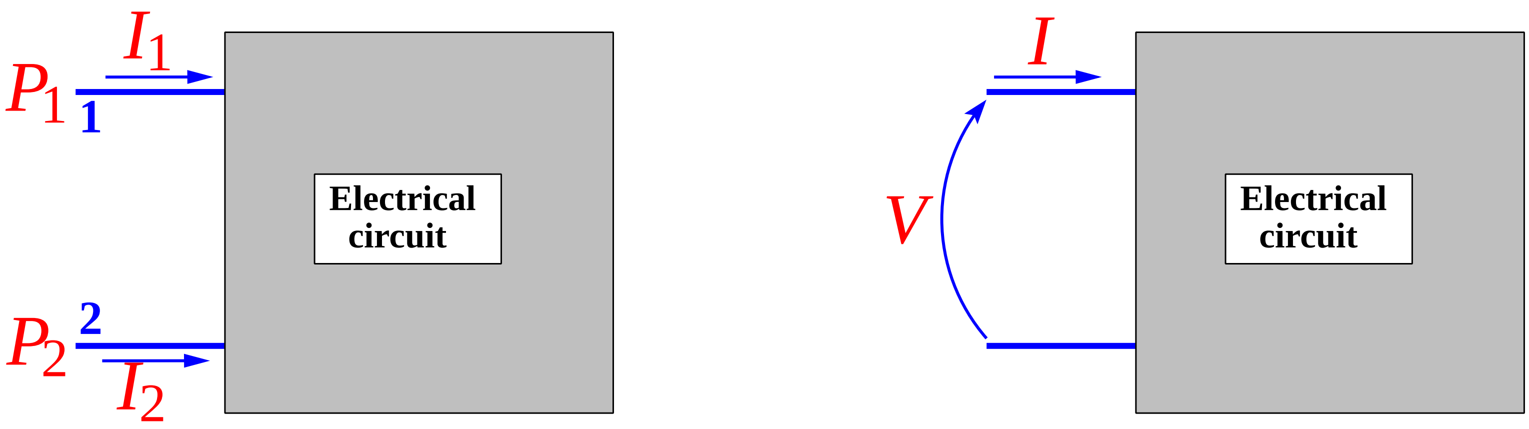
4-terminal circuit or 2-port circuit?







2-terminal case



A 2-terminal circuit is a 1-port,

(assuming KCL and KVL).

Port = WLOG in 2-terminal case.

Port synthesis

Determine the port behaviors $\mathscr{B}_{IV}$ that are achievable by interconnecting a finite number of

positive resistors,

positive capacitors,

positive inductors,

transformers,

and gyrators.

Synthesis conditions

Properties of the port behavior

The port behavior $B_{IV} \subseteq (\mathbb{R}^N \times \mathbb{R}^N)^{\mathbb{R}}$ **of an interconnection of RLCTG's is**

- **linear,**
- **time-invariant,**
- **finite-dimensional,**
- **hybrid,**
- **passive.**

Properties of the port behavior

'Hybrid' :$\Leftrightarrow$ **Each port is either current driven or voltage driven, meaning $\exists$ an input/output partition**

$$\textbf{input} \quad \begin{bmatrix} I_1 \\ \vdots \\ I_{N'} \\ V_{N'+1} \\ \vdots \\ V_N \end{bmatrix} \mapsto \begin{bmatrix} V_1 \\ \vdots \\ V_{N'} \\ I_{N'+1} \\ \vdots \\ I_N \end{bmatrix} \quad \textbf{output}$$

LTIFD $\rightsquigarrow$ rational transfer f'n $G \in \mathbb{R}(\xi)^{N\times N}$.

E.g. 'impedance' $V = ZI$, **'admittance'** $I = YV$.

Properties of the port behavior

'Passive' :$\Leftrightarrow$

$$[\![(I,V) \in \mathscr{B}_{IV}]\!] \;\Rightarrow\; [\![\int_{-\infty}^{0} I^\top(t)V(t)\,dt \geq 0]\!].$$

$\Leftrightarrow$ **G is 'positive real'**

i.e. $[\![s \in \mathbb{C}, \textbf{real part}(s) > 0]\!] \Rightarrow$

$$[\![G(s) + G(\bar{s})^{\text{Hermitian conjugate}} \succeq 0]\!].$$

Basic synthesis result

A hybrid transfer function

$$G \in \mathbb{R}(\xi)^{N\times N}$$

is synthesizable as the

N-port behavior of an RLCTG circuit

if and only if G is

positive real.

V. Belevitch, *Classical Network Theory*, Holden-Day, 1968.

B.D.O. Anderson, S. Vongpanitlerd, *Network Analysis and Synthesis. A Modern Systems Approach*, Prentice Hall, 1972.

Other synthesis questions

Of interest:

- **RLCTG,**
- **RLCT,**
- **RLC,**
- **LC, RL, RC,**
- **RTG, RT, GT, T,**
- **R,**
- **etc.**

Memoryless synthesis

$$G \in \mathbb{R}^{N \times N}$$

Transformer circuit

$$\begin{bmatrix} V_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} 0 & N \\ -N^\top & 0 \end{bmatrix} \begin{bmatrix} I_1 \\ V_2 \end{bmatrix}$$

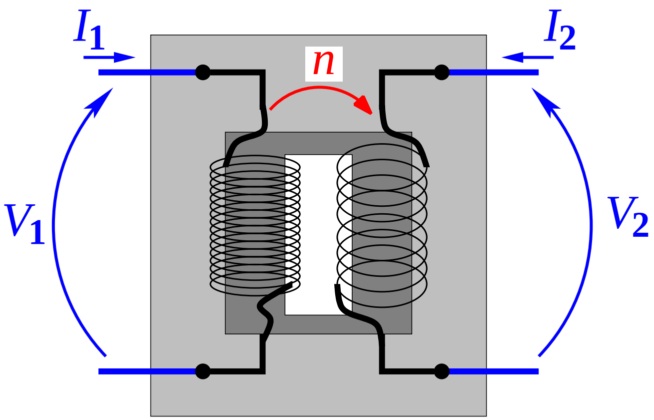
is realizable using transformers.

Transformer circuit

$$\begin{bmatrix} V_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} 0 & N \\ -N^\top & 0 \end{bmatrix} \begin{bmatrix} I_1 \\ V_2 \end{bmatrix}$$

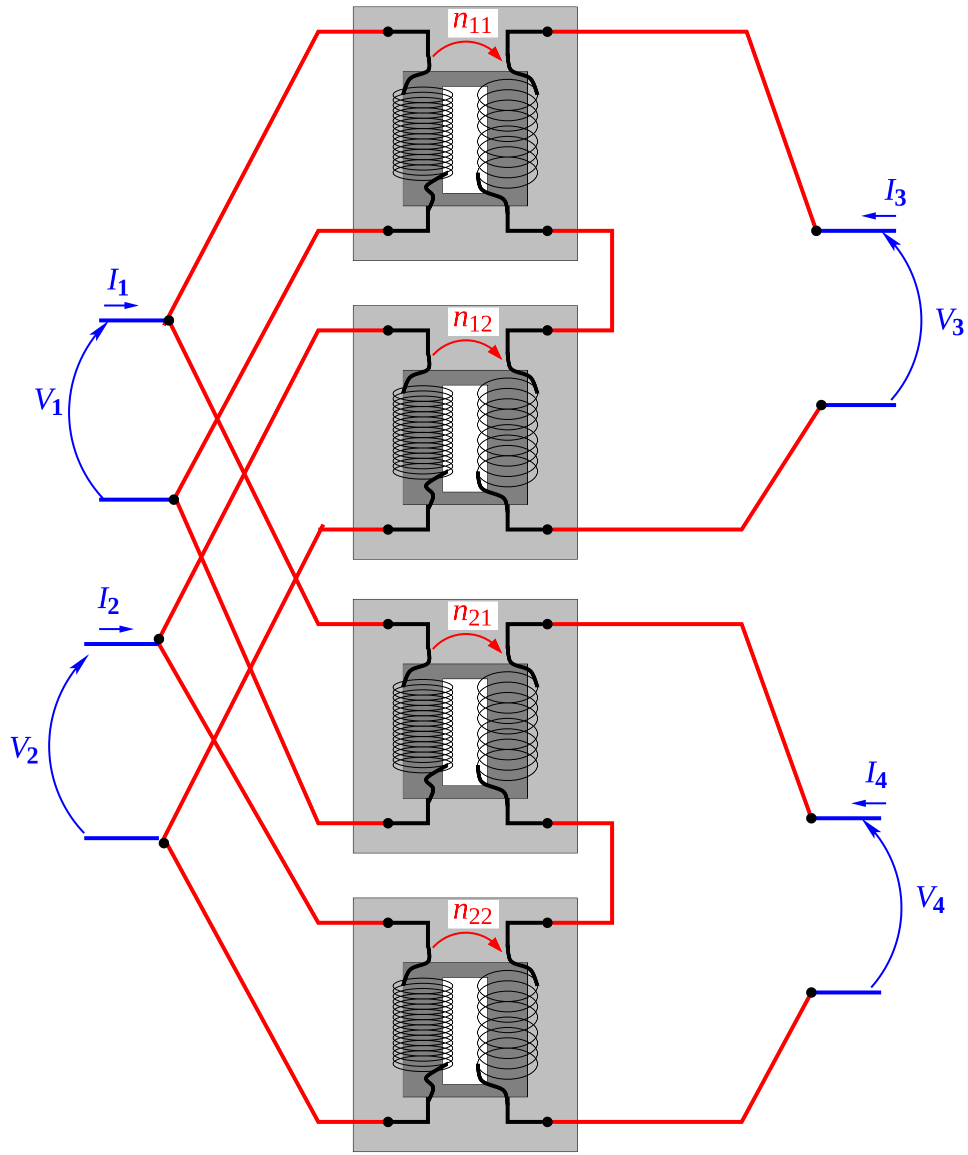
is realizable using transformers.

Multivariable generalization of

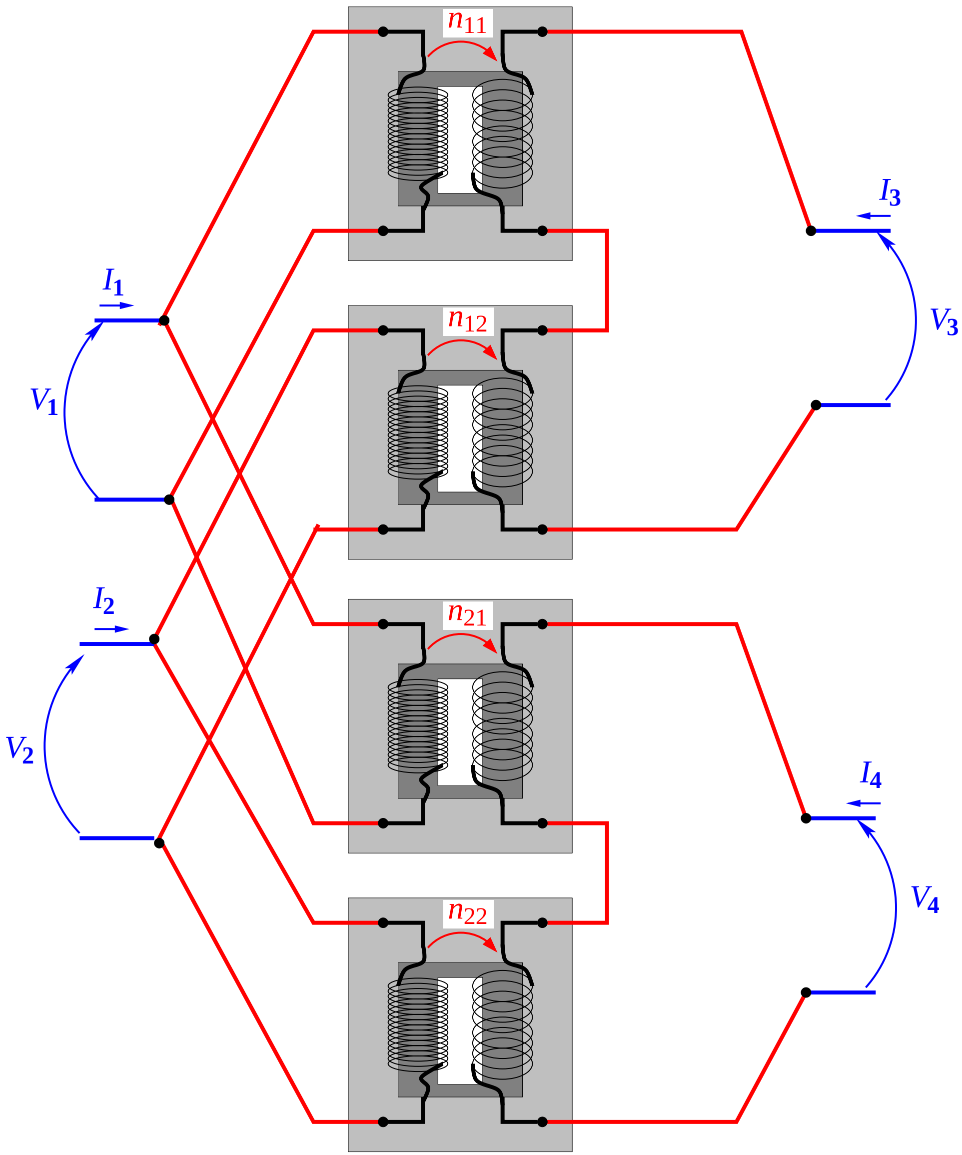


$$\begin{bmatrix} V_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} 0 & n \\ -n & 0 \end{bmatrix} \begin{bmatrix} I_1 \\ V_2 \end{bmatrix}$$

Transformer synthesis



Transformer synthesis



$$\begin{bmatrix} V_1 \\ V_2 \\ -- \\ I_3 \\ I_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & n_{11} & n_{12} \\ 0 & 0 & n_{21} & n_{22} \\ -n_{11} & -n_{21} & 0 & 0 \\ -n_{12} & -n_{22} & 0 & 0 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ -- \\ V_3 \\ V_4 \end{bmatrix}$$

Transformers & Gyrators

$$\begin{bmatrix} V_1 \\ I_2 \end{bmatrix} = G \begin{bmatrix} I_1 \\ V_2 \end{bmatrix}$$

with

$$G + G^\top = 0$$

is realizable using gyrators and transformers.

Transformers & Gyrators

$$\begin{bmatrix} V_1 \\ I_2 \end{bmatrix} = G \begin{bmatrix} I_1 \\ V_2 \end{bmatrix}$$

with

$$G + G^\top = 0$$

is realizable using gyrators and transformers.

Assume (WLOG) $V = GI$ **(impedance)**

Factor $G = NJN^\top$ **with** $J = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}$

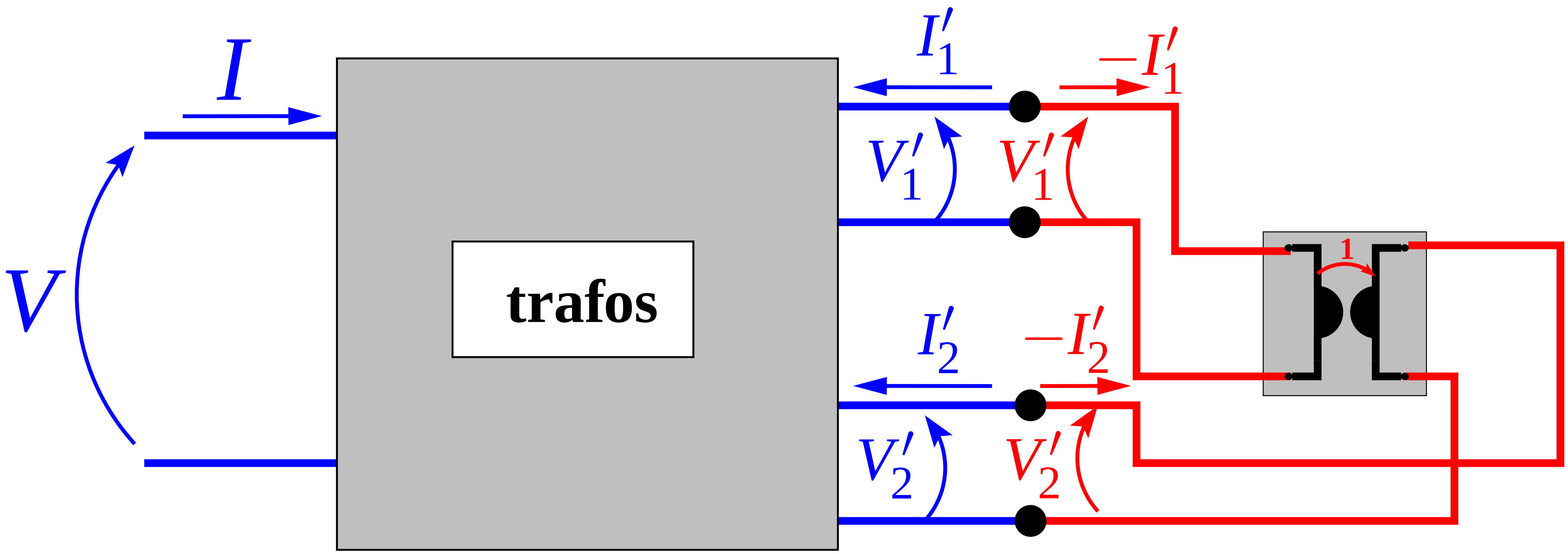
Gyrator extraction

Synthesize using transformers $$\begin{bmatrix} V \\ I' \end{bmatrix} = \begin{bmatrix} 0 & N \\ -N^\top & 0 \end{bmatrix} \begin{bmatrix} I \\ V' \end{bmatrix}.$$

Synthesize using gyrators $$V' = -JI'.$$

$$V = NV' = -NJI' = NJN^\top I = GI.$$

Gyrator extraction



Transformers, gyrators, & resistors

$$\begin{bmatrix} V_1 \\ I_2 \end{bmatrix} = G \begin{bmatrix} I_1 \\ V_2 \end{bmatrix}$$

with

$$G + G^\top \succeq 0$$

is realizable using resistors, gyrators, and trafos.

Resistor extraction

Assume (WLOG) $V = GI$ **(impedance)**

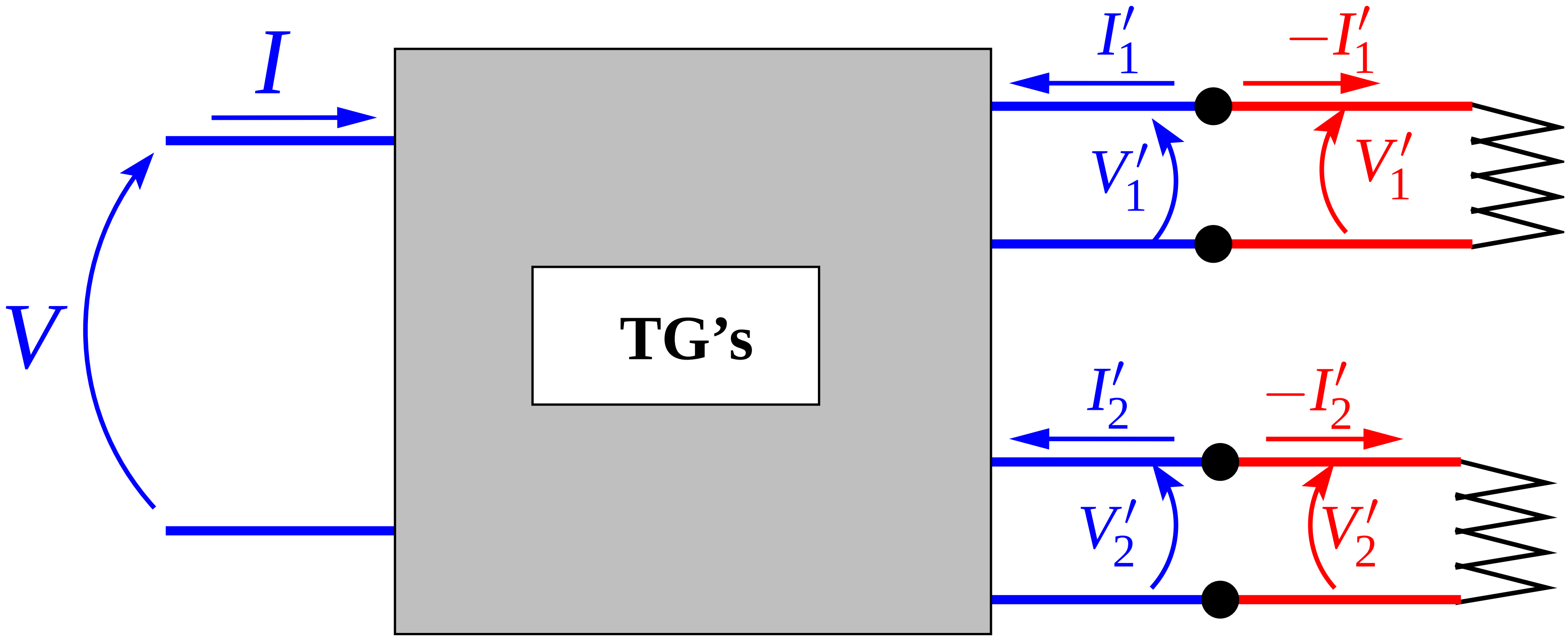
Write $\frac{1}{2}(G + G^\top) = S^\top S.$
Synthesize using gyrators and transformers:

$$\begin{bmatrix} V \\ V' \end{bmatrix} = \begin{bmatrix} G - S^\top S & S^\top \\ -S & 0 \end{bmatrix} \begin{bmatrix} I \\ I' \end{bmatrix}.$$

Synthesize using resistors: $V' = -I'.$

$$V = (G - S^\top S)I + S^\top I' = (G - S^\top S)I - S^\top V' = (G - S^\top S)I + S^\top S I = GI.$$

Resistor extraction



Memoryless synthesis

We have hence proven that

$$\begin{bmatrix} V_1 \\ I_2 \end{bmatrix} = G \begin{bmatrix} I_1 \\ V_2 \end{bmatrix}$$

with

$$G + G^\top \succeq 0$$

realizable using resistors, gyrators, transformers.

Dynamic synthesis

$$G \in \mathbb{R}(\xi)^{N \times N}$$

Realization theory

$G \in \mathbb{R}(\xi)^{N \times N}$. **(WLOG) proper, impedance.**

to be synthesized: $V = GI$

Realization theory

$G \in \mathbb{R}(\xi)^{N\times N}$. **(WLOG) proper, impedance.**

to be synthesized: $V = GI$

Well-known <u>theorem</u>: $\exists \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ **such that**

$$\frac{d}{dt}x = Ax + BI, V = Cx + DI \rightsquigarrow G(s) = D + C(Is - A)^{-1}B.$$

$\exists \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ **such that the system is state-controllable and state-observable ('minimality').**

KYP lemma

$G(s) = D + C(Is - A)^{-1}B$ **is positive real**

if and only if **there exists** $P = P^\top \succ 0$ **such that**

$\frac{d}{dt}x = Ax + BI, V = Cx + DI$ **implies**

$$\frac{d}{dt}\frac{1}{2}x^\top Px \leq I^\top V.$$

'dissipation inequality'.

KYP lemma

$G(s) = D + C(Is - A)^{-1}B$ **is positive real**

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$\frac{d}{dt}x = Ax + BI, V = Cx + DI$ **implies**

$$\frac{d}{dt}\frac{1}{2}x^\top Px \leq I^\top V.$$

'dissipation inequality'.

$$\Leftrightarrow \quad \begin{bmatrix} A^\top P + PA & PB - C^\top \\ B^\top P - C & -D - D^\top \end{bmatrix} \preceq 0.$$

KYP lemma

$$P = S^\top S, \qquad \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \mapsto \left[\begin{array}{c|c} S^{-1}AS & S^{-1}B \\ \hline CS & D \end{array}\right]$$

G **positive real** **if and only if** $\exists \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ **such that**

$$G(s) = D + C(Is - A)^{-1}B,$$

$$\begin{bmatrix} -A & -B \\ C & D \end{bmatrix} + \begin{bmatrix} -A & -B \\ C & D \end{bmatrix}^\top \succeq 0.$$

i.e., $\dfrac{d}{dt}\dfrac{1}{2}||x||^2 \leq I^\top V$. **'Passive realization'.**

Reactance extraction

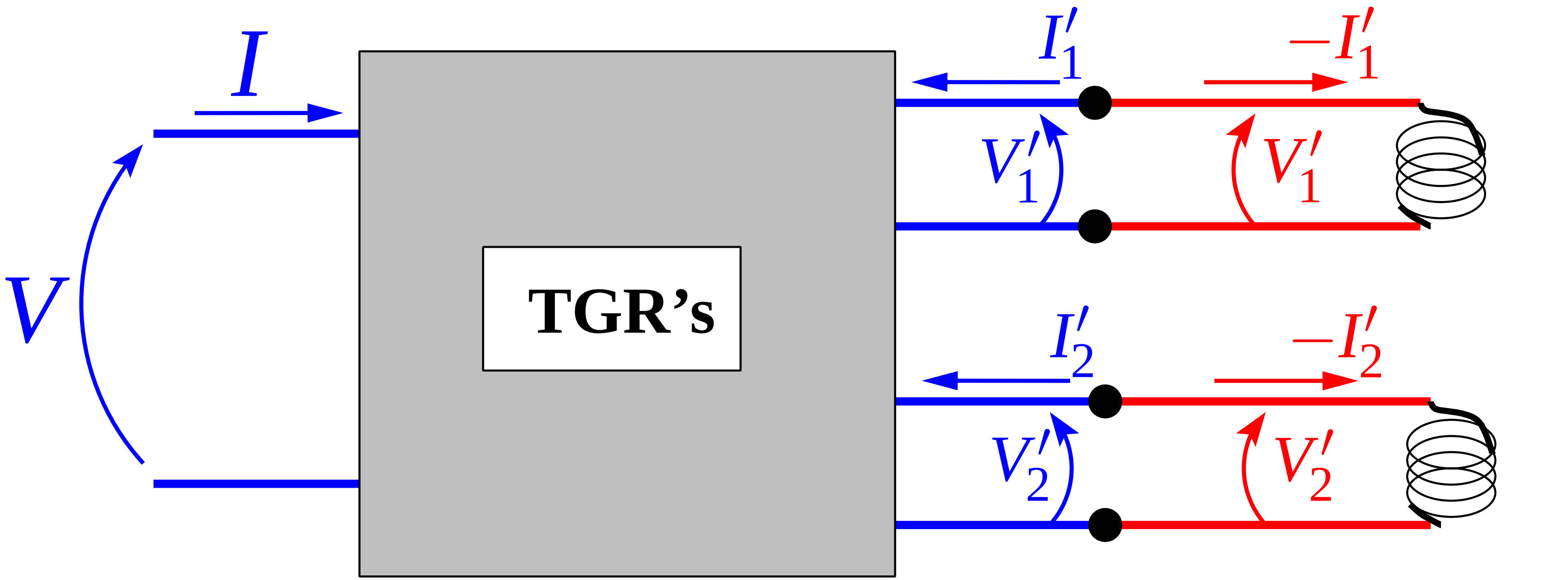
Synthesize using trafos, gyrators, and resistors:

$$\begin{bmatrix} V' \\ V \end{bmatrix} = \begin{bmatrix} -A & -B \\ C & D \end{bmatrix} \begin{bmatrix} I' \\ I \end{bmatrix}.$$

Synthesize using inductors: $$-\frac{d}{dt} I' = V'.$$

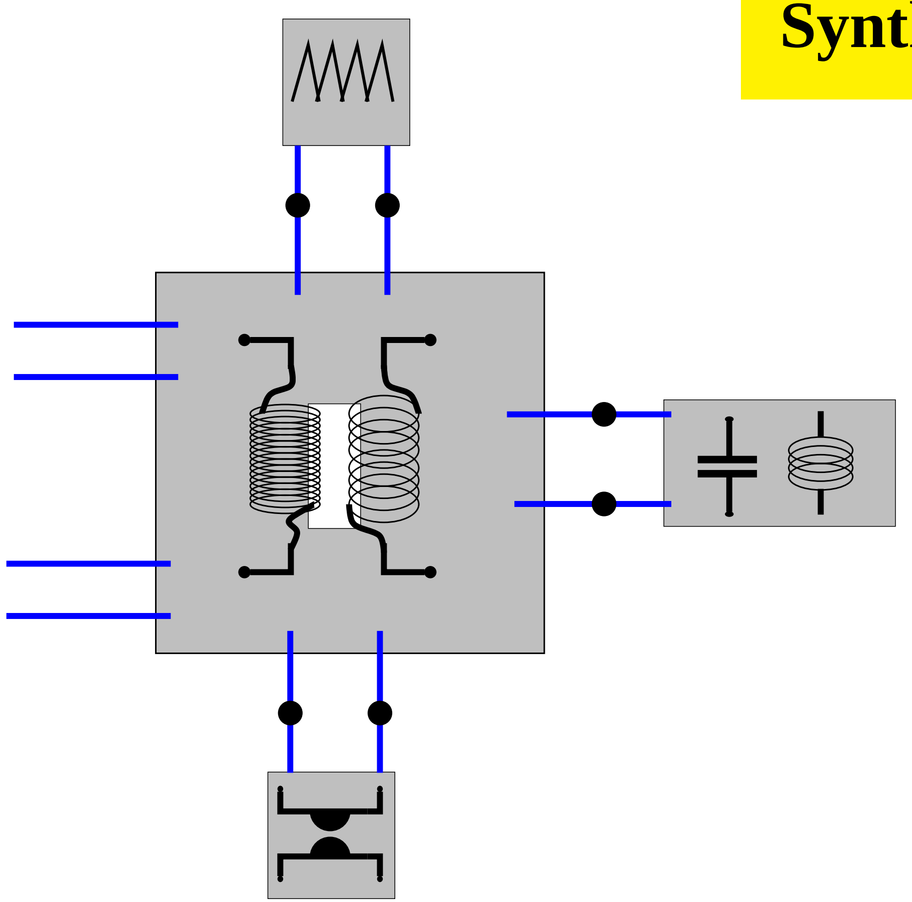
$$\begin{bmatrix} V' = -AI' - BI \\ I = CI' + DI \end{bmatrix} \Leftrightarrow \begin{bmatrix} \frac{d}{dt} I' = AI' + BI \\ I = CI' + DI \end{bmatrix} \Leftrightarrow \textbf{ tf fn equals } G.$$

Reactance extraction



Summary

Synthesis



Positive realness $\Rightarrow$ KYP lemma $\Rightarrow$ $\exists$ passive realization.

Passive realization + reactance extraction $\Rightarrow$ memoryless case.

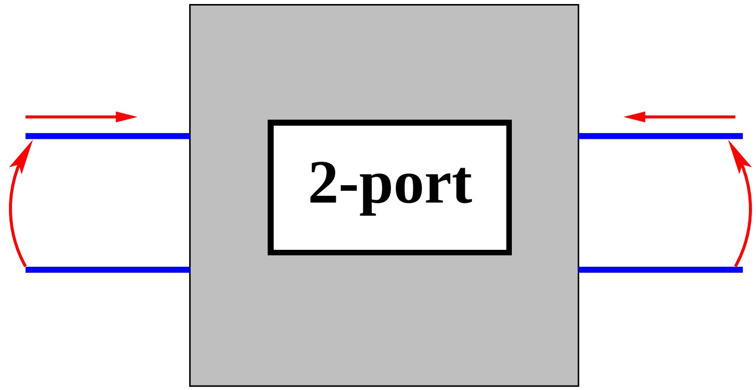
$G+G^\top \succeq 0$ **+ resistor extraction $\Rightarrow$** $G+G^\top = 0$.

$G+G^\top = 0$ **+ gyrator extraction $\Rightarrow$ symmetric** $G+G^\top = 0$.

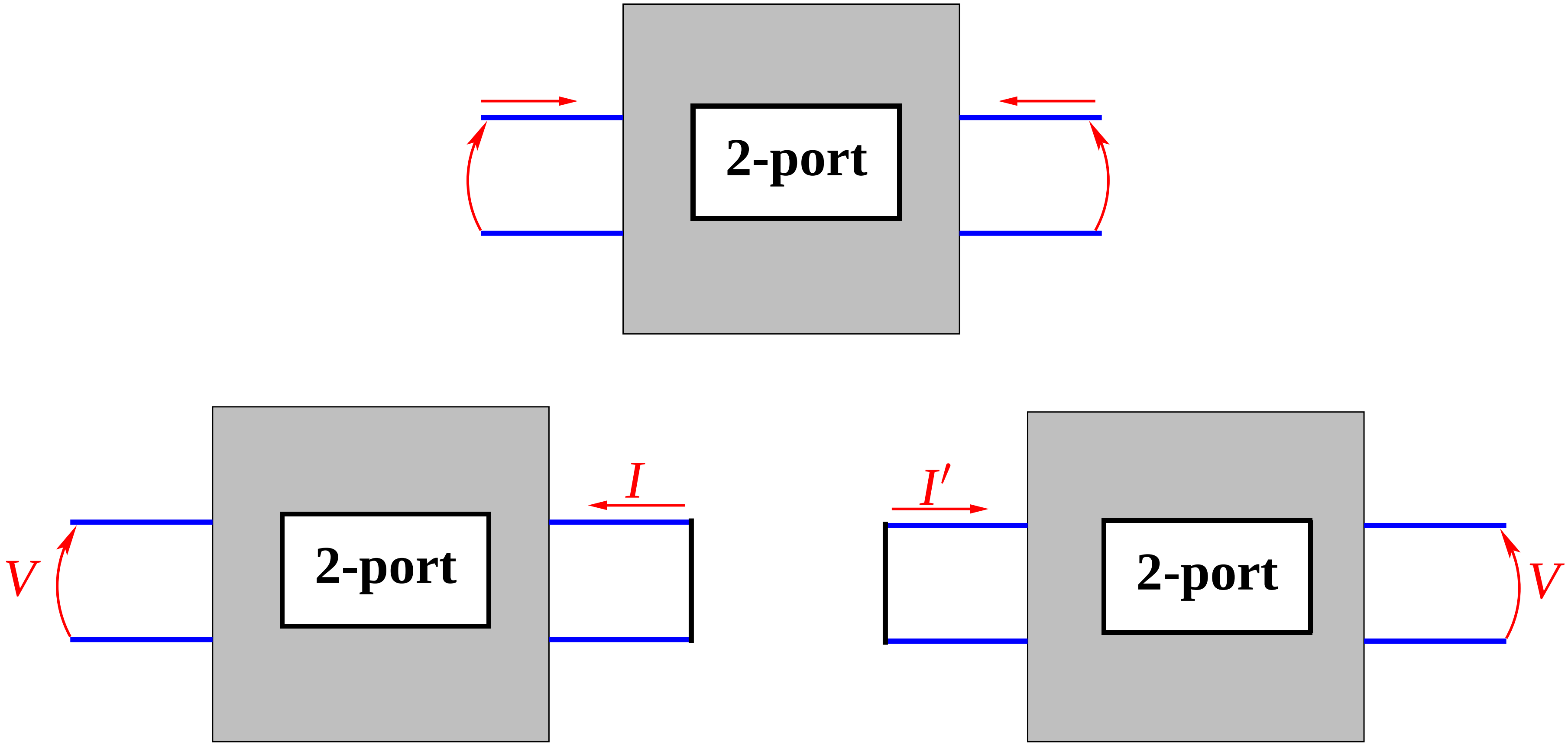
symmetric $G+G^\top = 0 \Rightarrow$ **bank of transformers.**

Reciprocity

Symmetry



Symmetry



$$[\![\textbf{reciprocity}]\!] :\Leftrightarrow [\![I = I']\!].$$

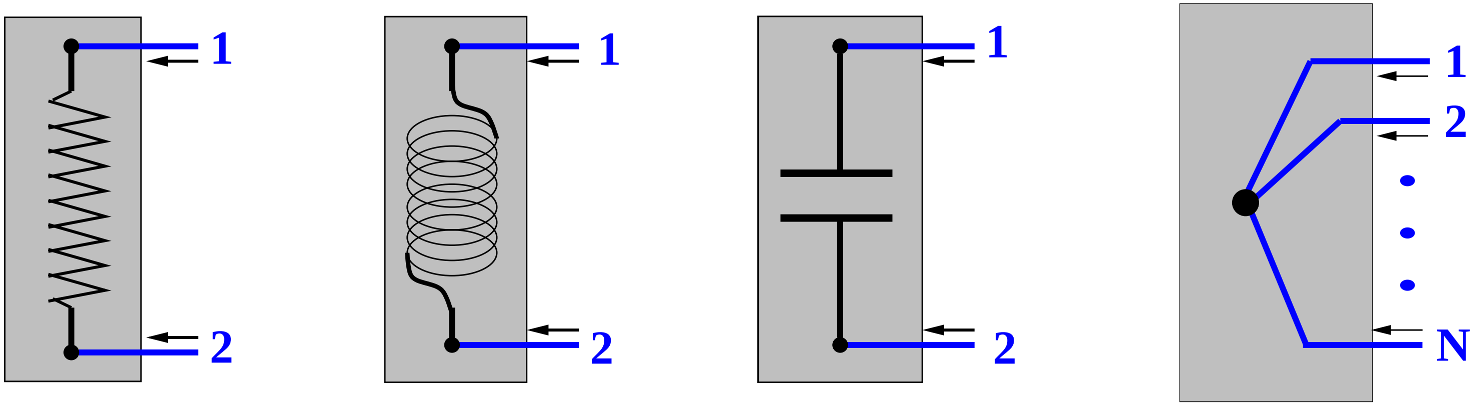
Reciprocity

$V = ZI$ **is reciprocal** $:\Leftrightarrow$ $Z = Z^\top.$

$$\begin{bmatrix} V_1 \\ \vdots \\ V_{N'} \\ --- \\ I_{N'+1} \\ \vdots \\ I_N \end{bmatrix} = G \begin{bmatrix} I_1 \\ \vdots \\ I_{N'} \\ --- \\ V_{N'+1} \\ \vdots \\ V_N \end{bmatrix}$$ **is reciprocal**

$:\Leftrightarrow \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}$ **satisfies** $G_{11} = G_{11}^\top, G_{22} = G_{22}^\top, G_{12} = -G_{21}^\top.$

Reciprocal building blocks

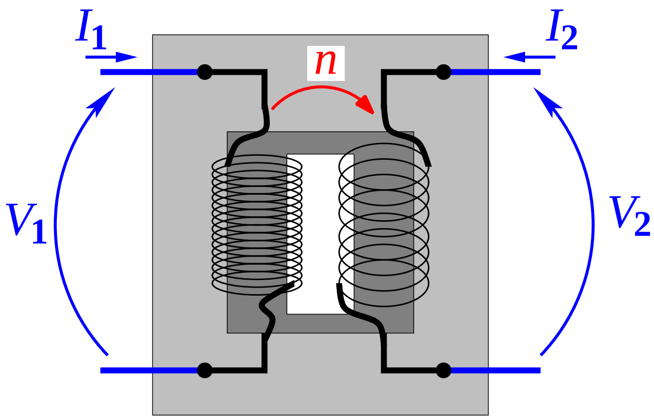


$V = RI, \frac{d}{dt}I = LV, I = C\frac{d}{dt}V$, **trivially reciprocal.**

Connector:

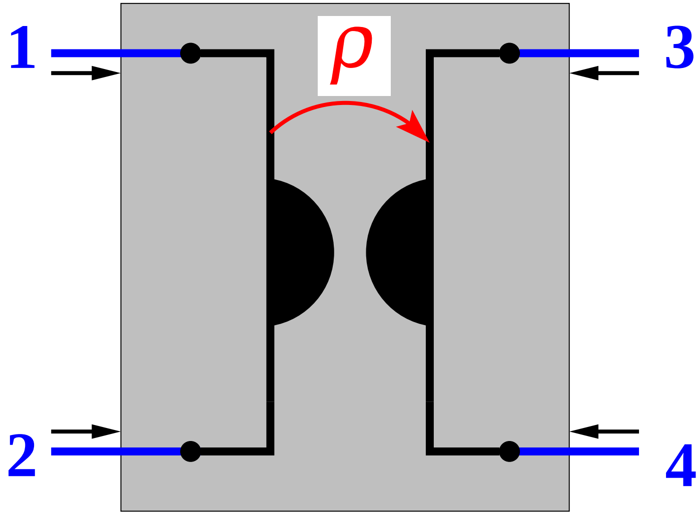
$$\begin{bmatrix} P_1 \\ P_2 \\ \vdots \\ P_{n-1} \\ --- \\ I_n \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 1 \\ & & \vdots & & \\ 0 & 0 & \cdots & 0 & 1 \\ -1 & -1 & \cdots & -1 & 0 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ \vdots \\ I_{n-1} \\ --- \\ P_n \end{bmatrix} \Rightarrow$$ **reciprocal.**

Reciprocal building blocks



$$\begin{bmatrix} V_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} 0 & n \\ -n & 0 \end{bmatrix} \begin{bmatrix} I_1 \\ V_2 \end{bmatrix} \Rightarrow$$ **trafo is reciprocal**.

Non-reciprocal building block



$$V_1 = \rho\, I_2, \quad V_2 = -\rho\, I_1;$$

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} 0 & \rho \\ -\rho & 0 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$ $\Rightarrow$ **gyrator is NOT reciprocal**.

Interconnection

The interconnection of reciprocal is reciprocal.

RLCT circuits are therefore

- **linear, time-invariant, finite-dimensional,**
- **hybrid,**
- **passive,**
- **reciprocal.**

Reciprocal synthesis

Reciprocal synthesis conditions

$$\begin{bmatrix} V_1 \\ \vdots \\ V_{N'} \\ --- \\ I_{N'+1} \\ \vdots \\ I_N \end{bmatrix} = G \begin{bmatrix} I_1 \\ \vdots \\ I_{N'} \\ --- \\ V_{N'+1} \\ \vdots \\ V_N \end{bmatrix}$$

is synthesizable as a RLCT circuit if and only if

$G \in \mathbb{R}(\xi)^{N \times N}$ **is positive real &**

$G = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}$ **satisfies** $G_{11} = G_{11}^\top, G_{22} = G_{22}^\top, G_{12} = -G_{21}^\top$.

Memoryless reciprocal synthesis procedure

Memoryless case. First impedance:

$$V = ZI, \qquad Z \in \mathbb{R}^{N \times N}, \quad Z = Z^\top \succeq 0.$$

Memoryless reciprocal synthesis procedure

Memoryless case. First impedance:

$$V = ZI, \qquad Z \in \mathbb{R}^{N \times N}, \; Z = Z^\top \succeq 0.$$

Write $Z = NN^\top$. **Synthesize using transformers:**

$$\begin{bmatrix} V \\ V' \end{bmatrix} = \begin{bmatrix} 0 & N \\ -N^\top & 0 \end{bmatrix} \begin{bmatrix} I \\ I' \end{bmatrix}.$$

Synthesize using resistors: $V' = -I'.$

$$V = NI' = -NV' = NN^\top I = ZI.$$

Memoryless case. General case:

$$\begin{bmatrix} V_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} G_{11} & G_{12} \\ -G_{12}^\top & G_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ V_2 \end{bmatrix}$$

Maximally current driven

$$\Rightarrow G_{22} = 0, G_{11} = G_{11}^\top \succeq 0.$$

Memoryless reciprocal synthesis procedure

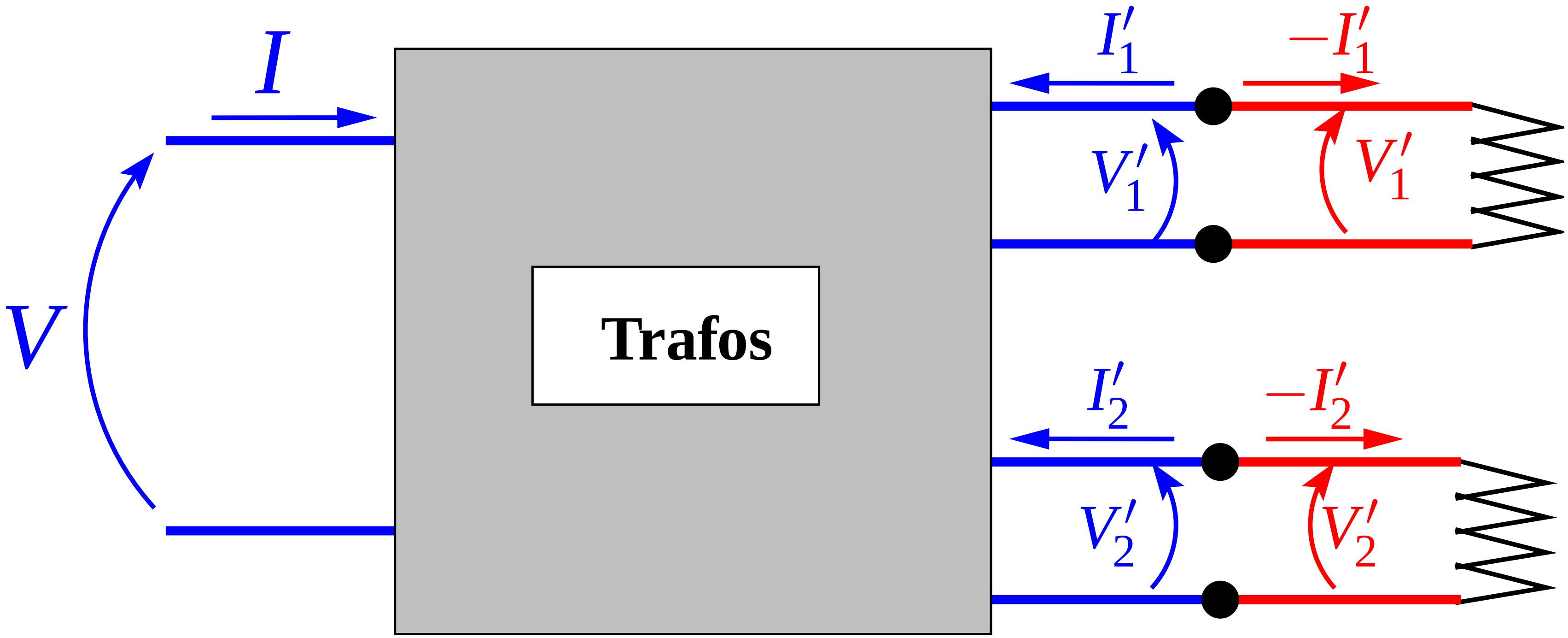
Write $G_{11} = NN^\top$. **Synthesize using transformers:**

$$\begin{bmatrix} V_1 \\ V' \\ I_2 \end{bmatrix} = \begin{bmatrix} 0 & N & G_{12} \\ -N^\top & 0 & 0 \\ -G_{12}^\top & 0 & 0 \end{bmatrix} \begin{bmatrix} I_1 \\ I' \\ V_2 \end{bmatrix}.$$

Synthesize using resistors: $V' = -I'$.

$$\begin{bmatrix} V_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} NI' + G_{12}V_2 \\ -G_{12}^\top I_1 \end{bmatrix} = \begin{bmatrix} -NV' + G_{12}V_2 \\ -G_{12}^\top I_1 \end{bmatrix} = \begin{bmatrix} NN^\top I_1 + G_{12}V_2 \\ -G_{12}^\top I_1 \end{bmatrix} = G \begin{bmatrix} I_1 \\ V_2 \end{bmatrix}.$$

Resistor extraction



Dynamic reciprocal synthesis procedure

Reciprocity and passivity:

$G \in \mathbb{R}(\xi)^{N\times N}$ **is positive real &**

$$G = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}$$ **satisfies** $G_{11} = G_{11}^{\top}, G_{22} = G_{22}^{\top}, G_{12} = -G_{21}^{\top}$.

i.e.

$$G \textbf{ p.r.} \quad \textbf{and} \quad G\Sigma_e = \Sigma_e G^{\top}, \Sigma_e = \begin{bmatrix} I_{n_1} & 0 \\ 0 & -I_{n_2} \end{bmatrix}.$$

Dynamic reciprocal synthesis procedure

$\exists$ **realization** $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ **such that simultaneously**

$$\begin{bmatrix} -A & -B \\ C & D \end{bmatrix} + \begin{bmatrix} -A & -B \\ C & D \end{bmatrix}^{\top} \succeq 0.$$

$$\begin{bmatrix} -A & -B \\ C & D \end{bmatrix} \begin{bmatrix} \Sigma_i & 0 \\ 0 & \Sigma_e \end{bmatrix} = \begin{bmatrix} -A & -B \\ C & D \end{bmatrix}^{\top} \begin{bmatrix} \Sigma_i & 0 \\ 0 & \Sigma_e \end{bmatrix}.$$

$\Sigma_e = \begin{bmatrix} I_{n_3} & 0 \\ 0 & -I_{n_4} \end{bmatrix}$ **: 'internal passivity' and 'internal reciprocity'.**

Various proofs

- **Several constructions in**
  B.D.O. Anderson, S. Vongpanitlerd, *Network Analysis and Synthesis. A Modern Systems Approach*, Prentice Hall, 1972.
- **JCW, Dissipative dynamical systems, Part II,**
  ***Archive for Rational Mechanics and Analysis*, 45, pp. 352-393, 1972.**
  Sol. set to KYP is convex compact, and Brouwer's fixpoint thm.
- **Reciprocal synthesis is also in the classical literature.**
  V. Belevitch, *Classical Network Theory*, Holden-Day, 1968.
- **T. Reis and JCW, A balancing approach to the realization of systems with internal passivity and reciprocity, *SCLetters*, 60, pp. 69-74, 2011.**
  Positive real balancing $\Rightarrow$ internal passive and reciprocal.

Reactance extraction

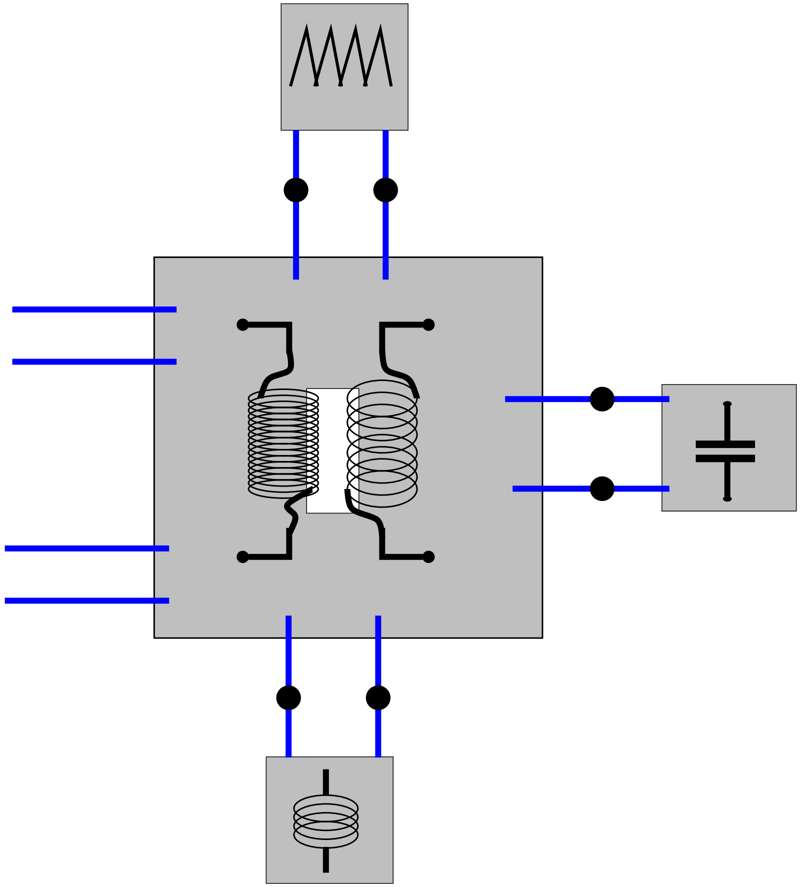
Trafos and resistors:

$$\begin{bmatrix} V_1' \\ I_2' \\ V_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} -A & -B \\ C & D \end{bmatrix} \begin{bmatrix} I_1' \\ V_2' \\ I_1 \\ V_2 \end{bmatrix}.$$

Inductors: $-\frac{d}{dt} I_1' = V_1'.$ **Capacitors:** $-\frac{d}{dt} V_2' = I_2'.$

$$\begin{bmatrix} \frac{d}{dt} I_1' \\ \frac{d}{dt} V_2' \\ V_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} -V_1' \\ -I_2' \\ V_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} I_1' \\ V_2' \\ I_1 \\ V_2 \end{bmatrix} \Leftrightarrow$$ **tf fn equals** G.

Summary



Conclusions

Conclusions

- **Circuit synthesis shows the power and effectiveness of state space thinking.**
- **Why ports instead of terminals in case $N > 1$, surely $N > 2$?**
- **Open problem: Do Bott-Duffin from state point of view. Requires non-minimal, non-controllable realizations.**
- **Open problem: Resistive N-ports. Transformerless N-terminal resistive case solved.**

Copies of the lecture frames available from/at

http://www.esat.kuleuven.be/∼jwillems

Thank you

Thank you

Thank you

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Thank you