OPEN STOCHASTIC SYSTEMS

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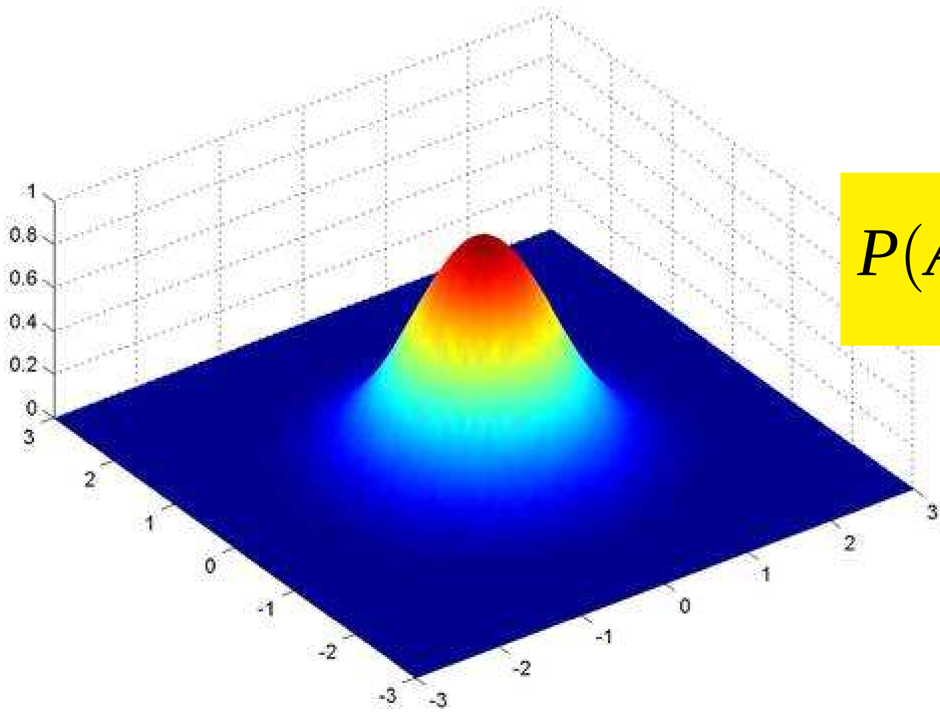
The idea

Theme

Model a phenomenon stochastically; outcomes in $\mathbb{R}^n$.

Usual framework:

- **probability distributions, probability density functions;**
- **means that the event σ-algebra consists of the Borel sets.**
  $\rightsquigarrow$ **'Every' subset of $\mathbb{R}^n$ is assigned a probability.**



$$P(A) = \int_A p(x)\,dx,\ \ A \subseteq \mathbb{R}^n \textbf{ Borel}$$

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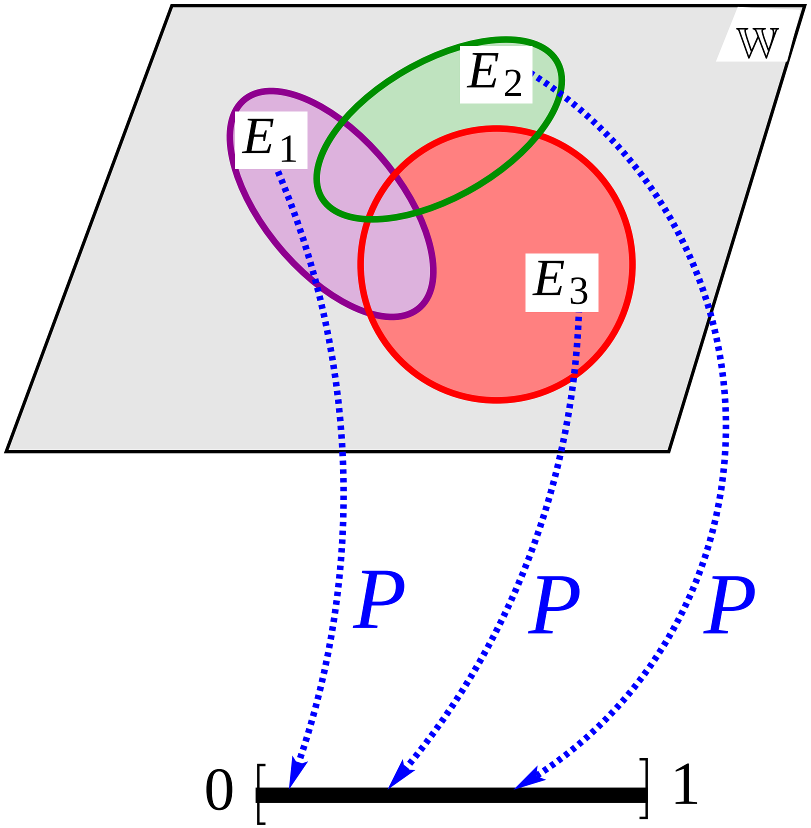
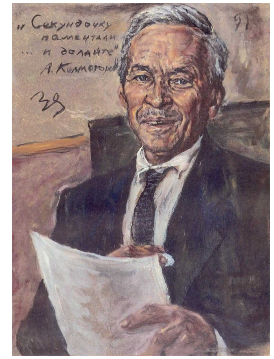
Thesis:

This is unduly restrictive,
even for elementary applications.

Basic probability

Events





A.N. Kolmogorov
1903 – 1987

A probability $P(E) \in [0,1]$ is assigned to certain subsets E (*'events'*) of the *outcome space* $\mathbb{W}$.

$\mathscr{E}$:= the class of 'measurable' subsets of $\mathbb{W}$,

= the sets that are assigned a probability.

Main (not all) axioms

The set of events $\mathscr{E}$ is a σ-algebra :⇒

- $[\![E \in \mathscr{E}]\!] \Rightarrow [\![E^{\text{complement}} \in \mathscr{E}]\!]$
- $[\![E_1, E_2 \in \mathscr{E}]\!] \Rightarrow [\![E_1 \cap E_2 \in \mathscr{E},\ \ E_1 \cup E_2 \in \mathscr{E}]\!]$

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- $[\![E_1, E_2 \in \mathscr{E}]\!] \Rightarrow [\![E_1 \cap E_2 \in \mathscr{E},\ \ E_1 \cup E_2 \in \mathscr{E}]\!]$

The probability $P : \mathscr{E} \to [0,1]$ satisfies

- $P(\mathbb{W}) = 1$**,**
- $[\![E_1, E_2 \in \mathscr{E}$ **and** $E_1 \cap E_2 = \emptyset]\!]$

  $$\Rightarrow [\![P(E_1 \cup E_2) = P(E_1) + P(E_2)]\!]$$

  (*P* is additive).

Borel

Émile Borel
1871 – 1956

In most applications it is assumed that the σ-algebra of measurable sets are the *Borel sets.*

$\mathscr{B}(\mathbb{R}^n)$ **= the Borel σ-algebra on $\mathbb{R}^n$;**

random variable: $\mathbb{W} = \mathbb{R}$ **(or $\mathbb{C}$), and** $\mathscr{E} = \mathscr{B}(\mathbb{R})$

random vector: $\mathbb{W} = \mathbb{R}^n$**, and** $\mathscr{E} = \mathscr{B}(\mathbb{R}^n)$

random process: **a family of random vectors, etc.**

$\mathscr{B}(\mathbb{R}^n)$ **contains 'basically every' subset of $\mathbb{R}^n$.**

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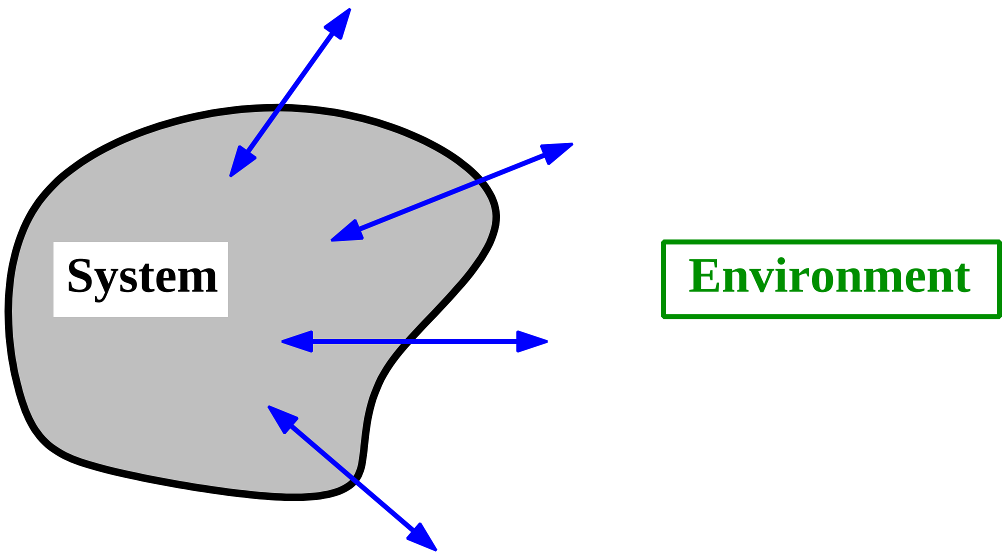
Thesis:

Borel is unduly restrictive for system theory.

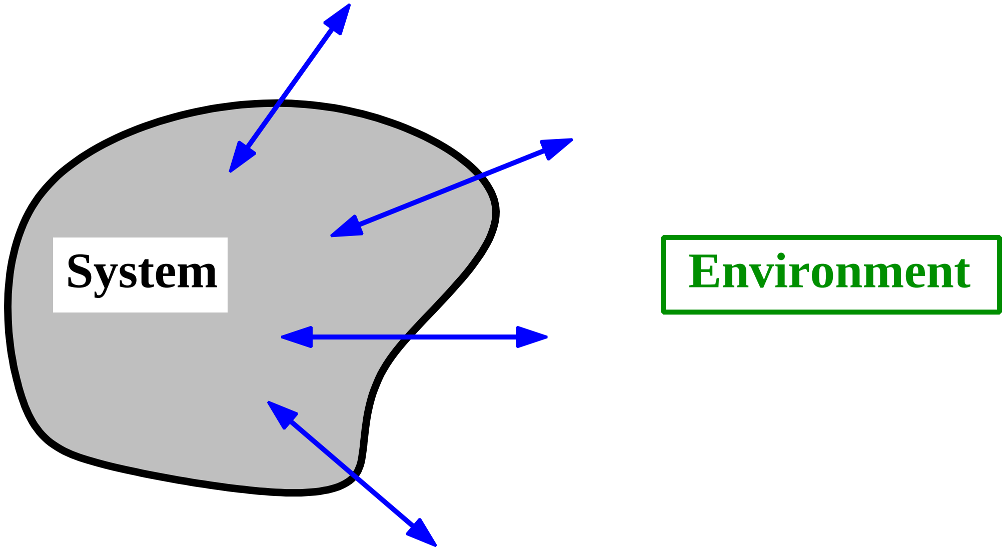
Stochastic systems

Requirements for a good concept

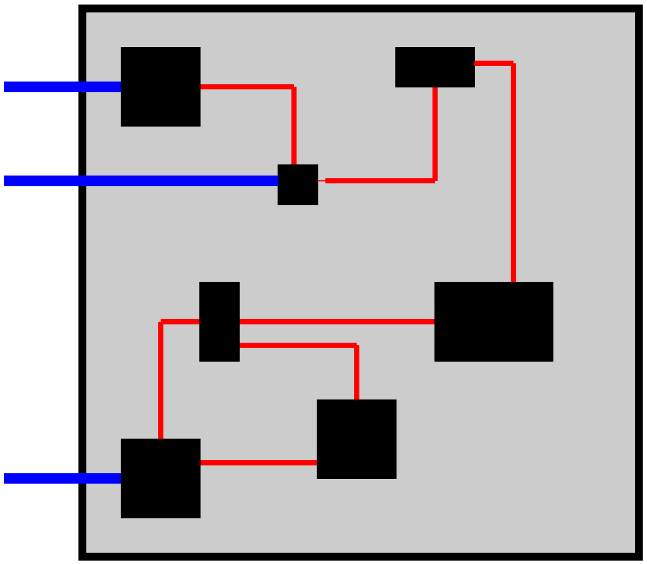


Open

Requirements for a good concept



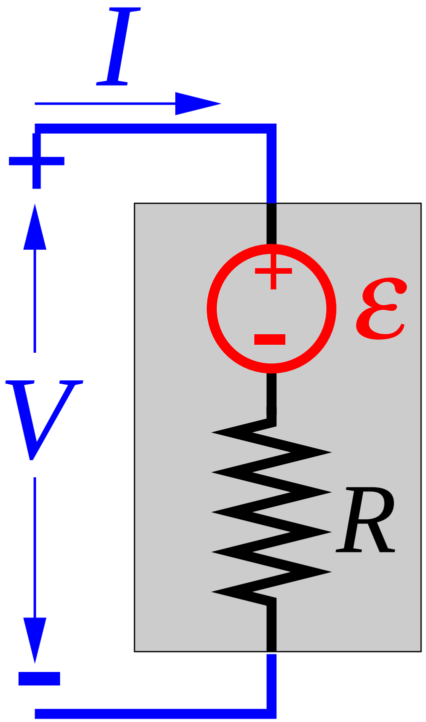
Open



Connectable

Motivating examples

Noisy resistor



$$\boxed{V = RI + \varepsilon}$$

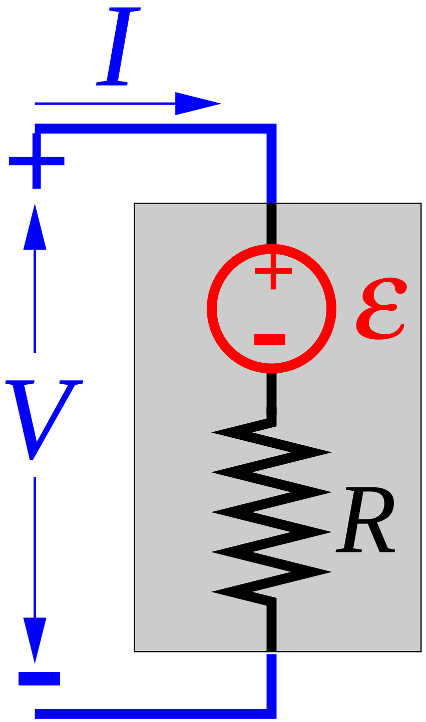
ε gaussian

zero mean

variance $\sigma \sim \sqrt{RT}$

'Johnson-Nyquist resistor'

Noisy resistor



$$\boxed{V = RI + \varepsilon}$$

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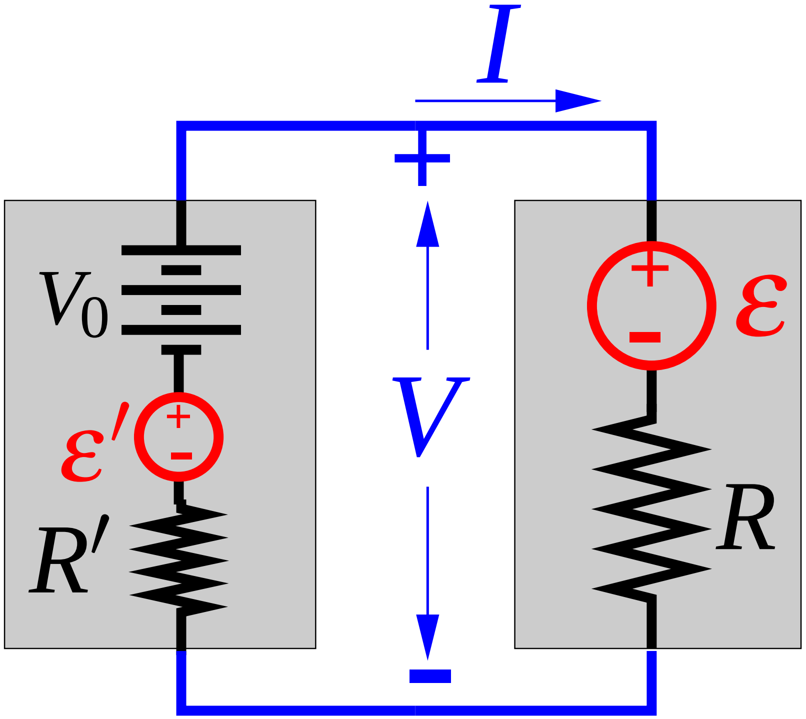
'Johnson-Nyquist resistor'

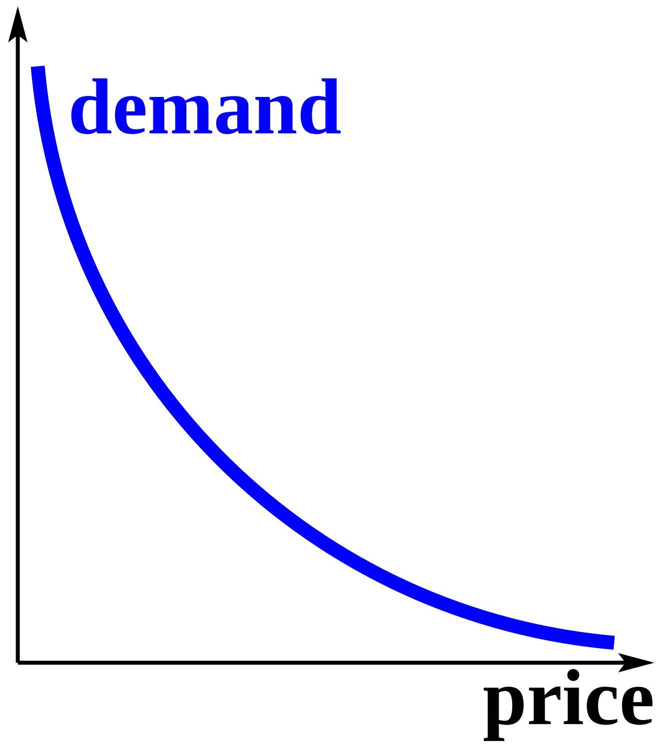
What is $\begin{bmatrix} V \\ I \end{bmatrix}$ as a mathematical entity?

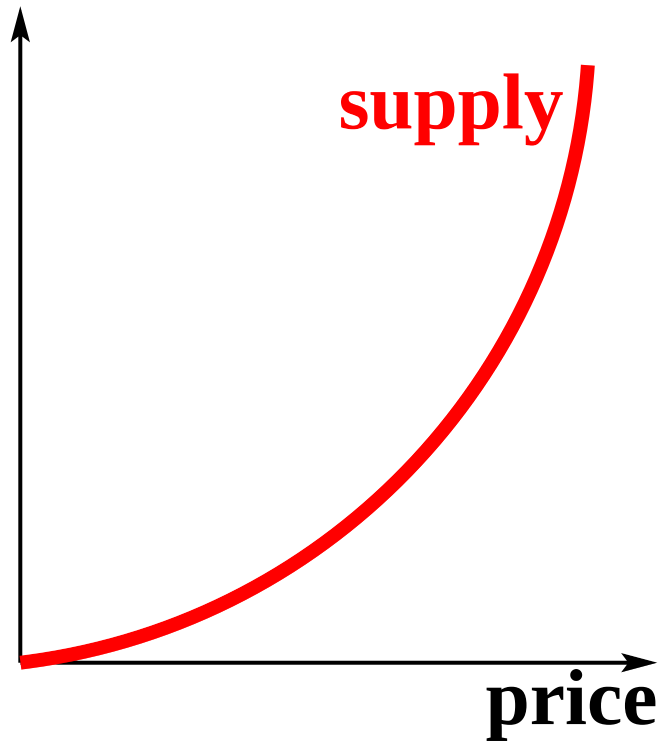
Noisy resistor



How do we deal with interconnection?

Deterministic price/demand/supply





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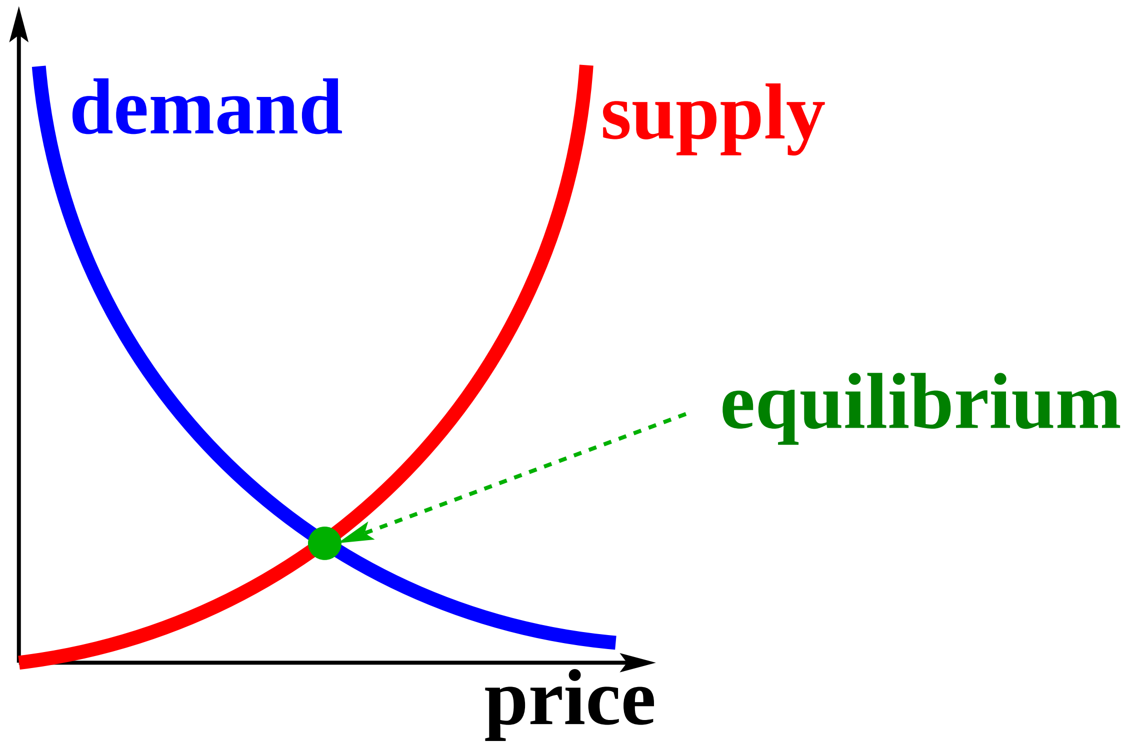
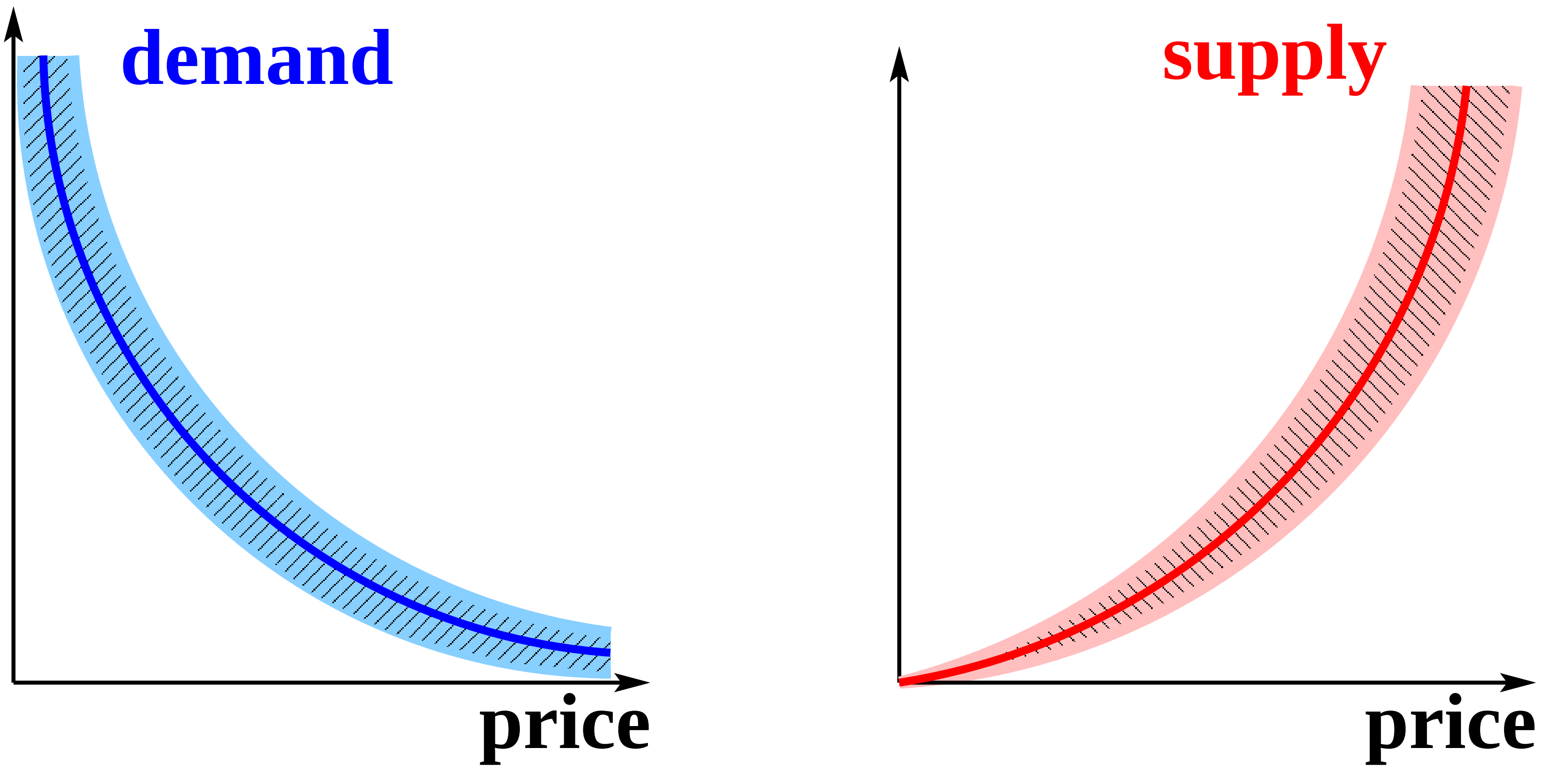


‘Interconnection’

Stochastic price/demand/supply



(Only) certain regions of the $\begin{bmatrix} \text{price} \\ \text{demand} \end{bmatrix}$ and $\begin{bmatrix} \text{price} \\ \text{supply} \end{bmatrix}$ planes are assigned a probability.

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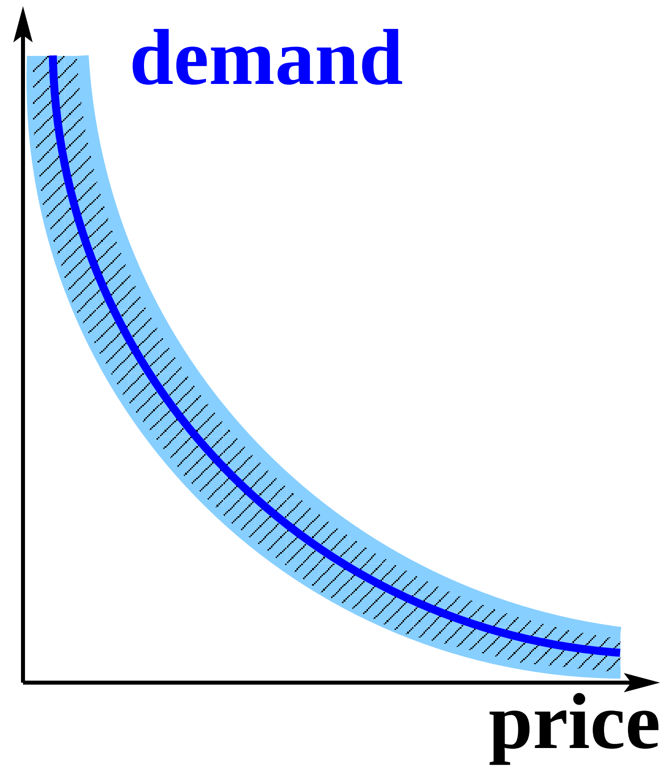

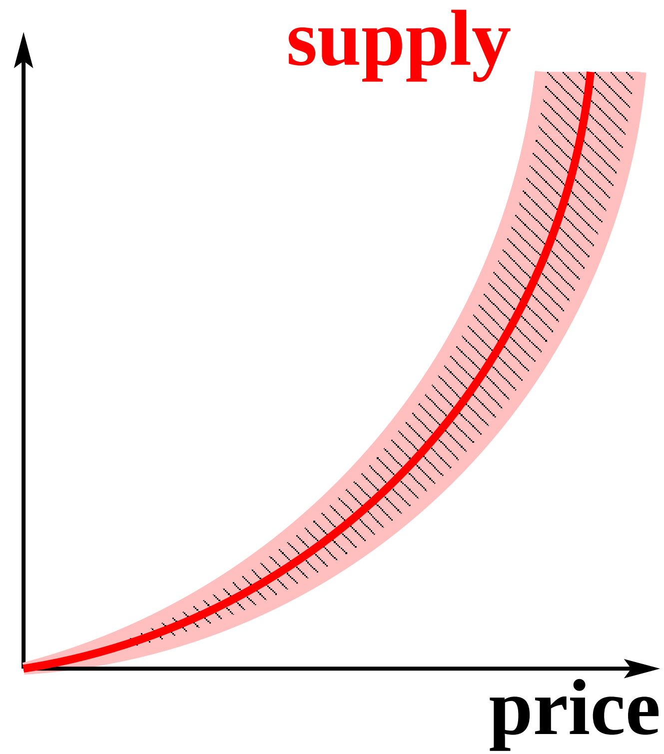

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How do we deal with equilibrium: supply = demand?

Formal definitions

Definition

A *stochastic system* is a probability triple $(\mathbb{W}, \mathscr{E}, P)$

- $\mathbb{W}$ **a non-empty set, the *outcome space*,**
- $\mathscr{E}$ **a σ-algebra of subsets of $\mathbb{W}$: the *events*,**
- $P : \mathscr{E} \to [0,1]$ **a *probability measure*.**

$\mathscr{E}$**: the subsets that are assigned a probability.**

Probability that outcomes $\in E$, $E \in \mathscr{E}$, **is** $P(E)$**.**

Model $\cong$ $\mathscr{E}$ **and** P**;**

$\mathscr{E}$ **is an essential part!**

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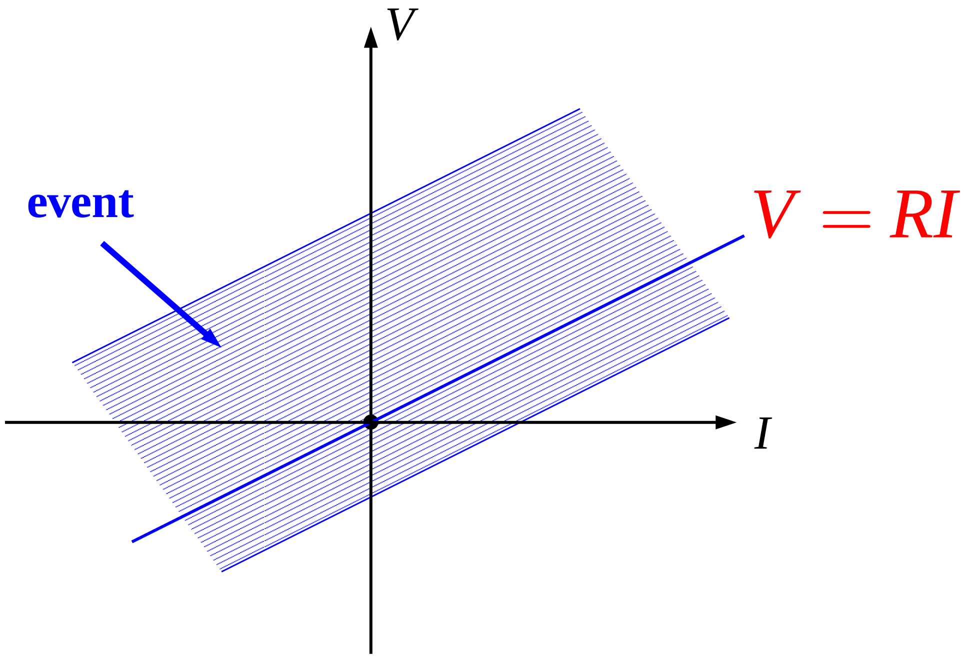
'Classical' stochastic system:

$\mathbb{W} = \mathbb{R}^n$ **and** $\mathscr{E}$ **= the Borel subsets of** $\mathbb{R}^n$**.**

P **specified by a probability distribution or a pdf.**

$\mathscr{E}$ **is inherited from the topology, it does not involve the random phenomenon, only the outcome space.**

Noisy resistor



$V = RI + \varepsilon$: **stoch. system, outcomes** $\begin{bmatrix} V \\ I \end{bmatrix}$, $\mathbb{W} = \mathbb{R}^2$.

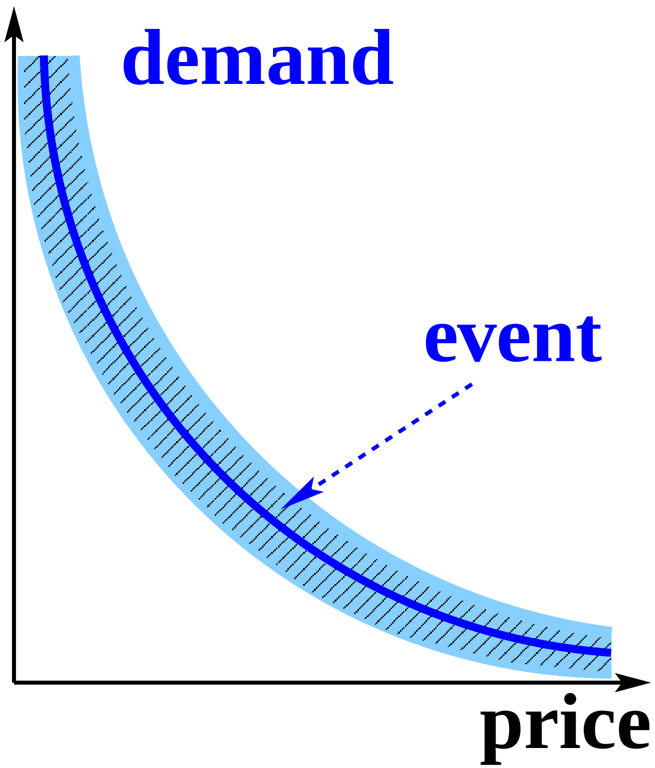
Events: $\left\{ \begin{bmatrix} V \\ I \end{bmatrix} \in \mathbb{R}^2 \mid V - RI \in A \right.$ **with** A **a Borel subset of** $\left. \mathbb{R} \right\}$.

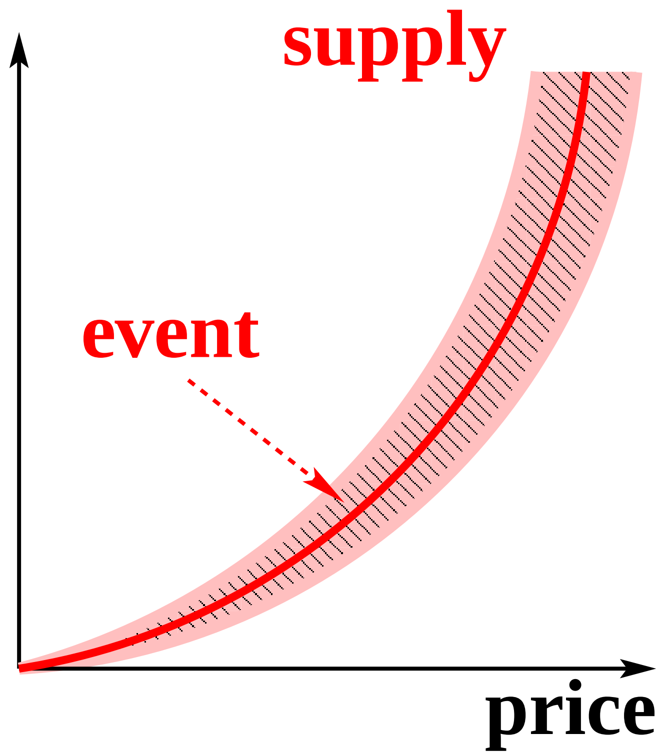
$P($**event**$) =$ **gaussian measure of** A.

V **and** I **are** ***not*** **classical real random variables.**

Neither $\begin{bmatrix} V \\ I \end{bmatrix}$ **nor** I **nor** V **possess a pdf.**

Stochastic price/demand/supply





$\mathscr{E}$ = **the regions that are assigned a probability.**

p, d, **and** s **are not classical real random variables.**

Linearity

linear stochastic system

$:\Leftrightarrow$ **Borel probability on** $\mathbb{R}^{\mathtt{n}}/\mathbb{L}$**,**

$\mathbb{L} \subseteq \mathbb{R}^{\mathtt{n}}$ **a linear subspace, called the 'fiber'.**

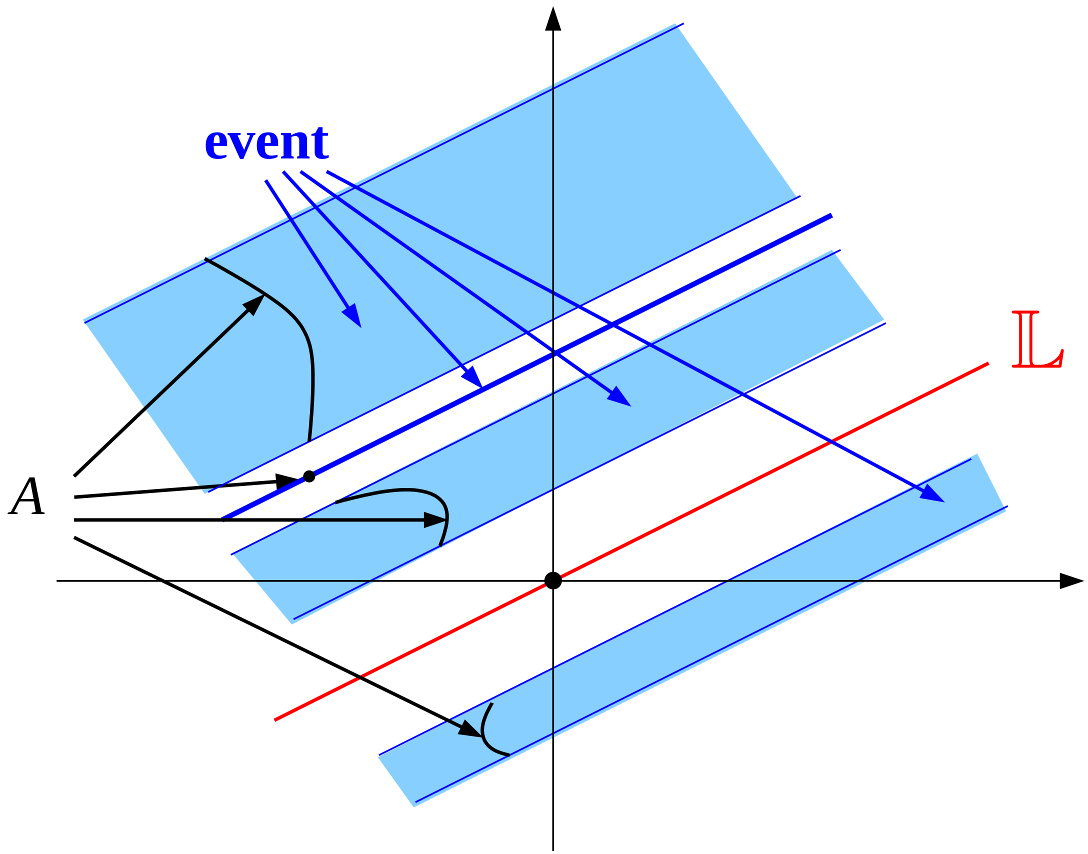
Events: cylinders with sides parallel to $\mathbb{L}$**.**

Subsets of $\mathbb{R}^{\mathtt{n}}$ **as** $A + \mathbb{L}$, $\mathbb{L}$ **linear subspace,** A **Borel.**

Linearity

Events: cylinders with sides parallel to $\mathbb{L}$.

Subsets of $\mathbb{R}^n$ as $A + \mathbb{L}$, $\mathbb{L}$ linear subspace, A Borel.



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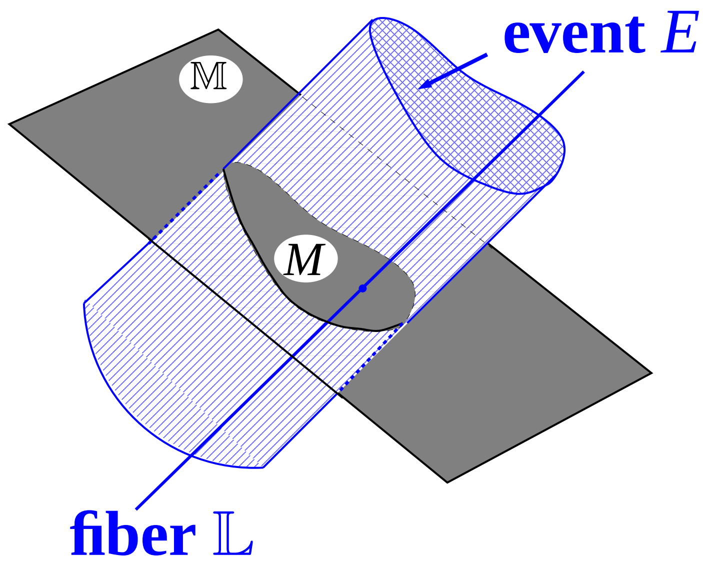

Borel probability on $\mathbb{M} \cong \mathbb{R}^n/\mathbb{L}, \quad (\mathbb{M} \oplus \mathbb{L} = \mathbb{R}^n)$**.**

Classical $\Rightarrow$ **linear!**

gaussian $:\Leftrightarrow$ **linear, Borel probability gaussian.**

Deterministic

$(\mathbb{W}, \mathscr{E}, P)$ **is said to be *deterministic* if**

$$\mathscr{E} = \{\emptyset, \mathbb{B}, \mathbb{B}^{complement}, \mathbb{W}\} \text{ and } P(\mathbb{B}) = 1.$$

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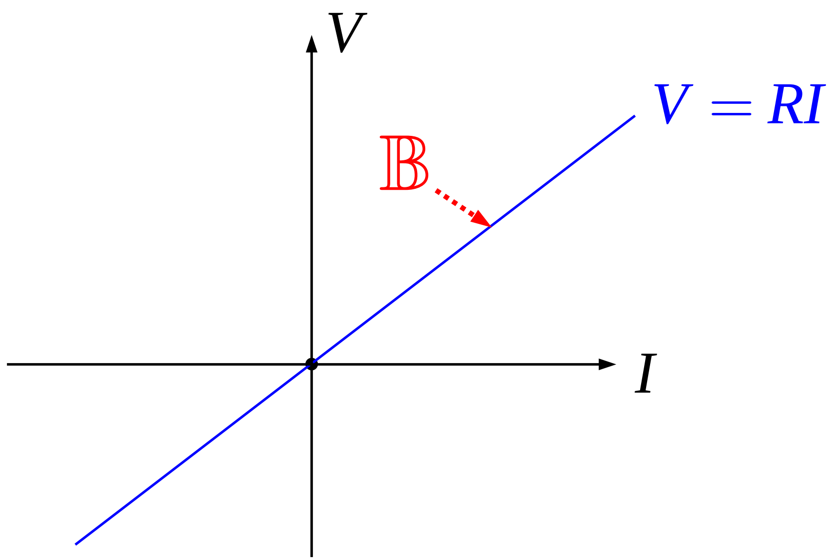
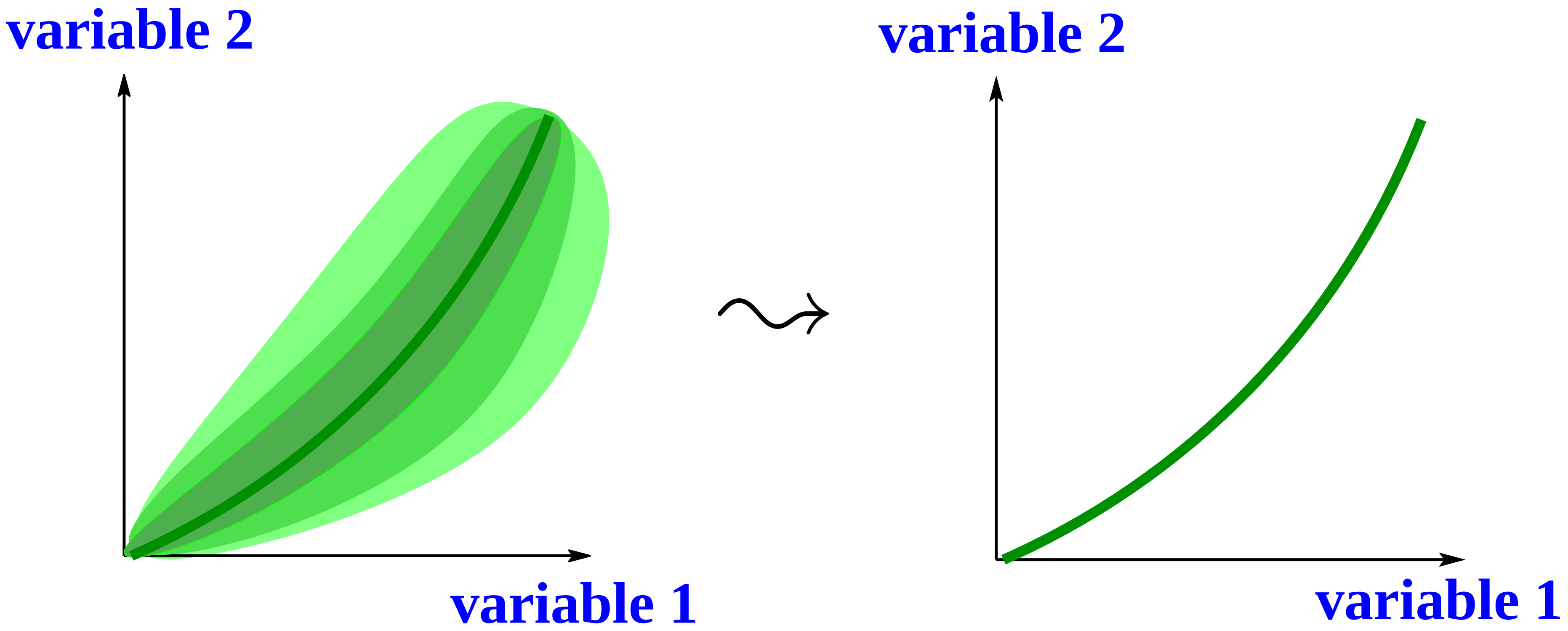
Example: An Ohmic resistor,

$$\mathbb{W} = \mathbb{R}^2, \quad \mathbb{B} = \{\begin{bmatrix} V \\ I \end{bmatrix} \in \mathbb{R}^2 \mid V = RI\}.$$



The need for 'parsimonious' ('coarse') σ-algebras



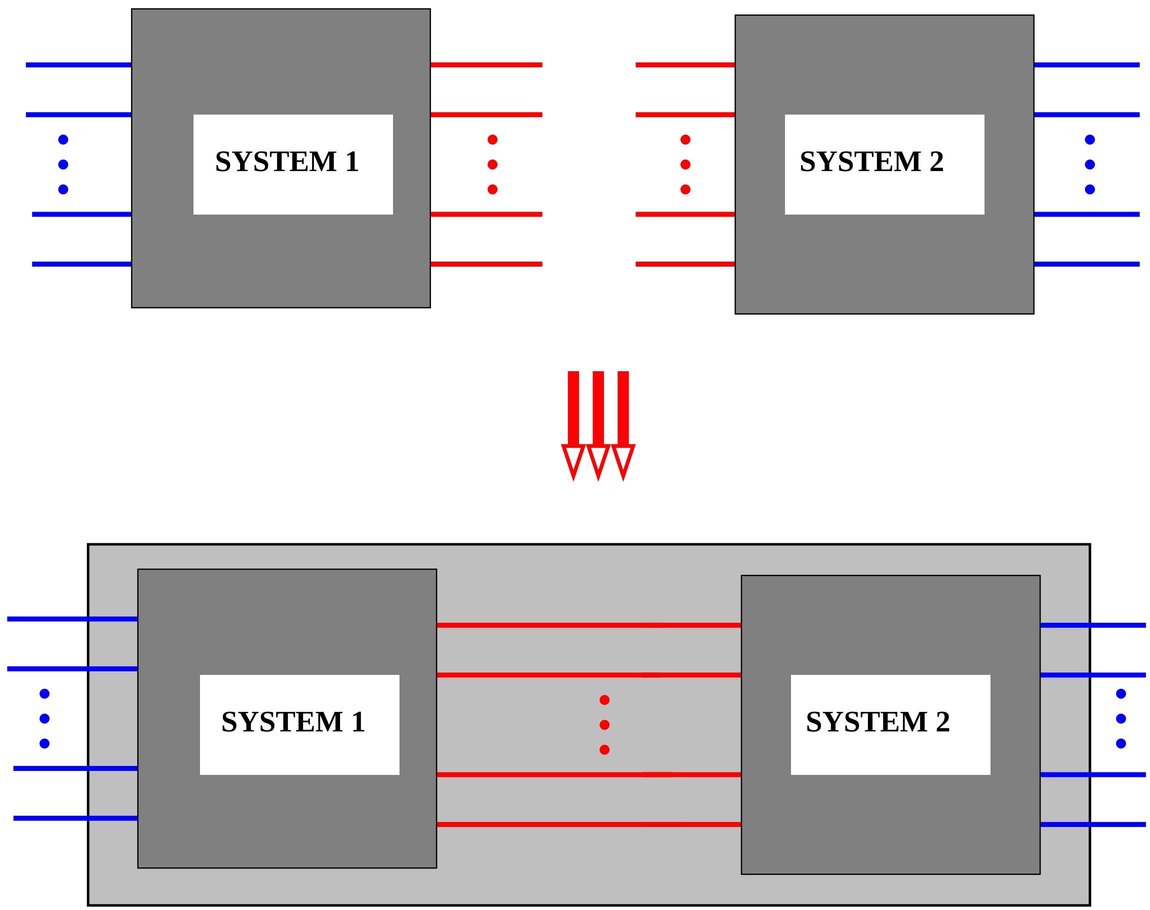
For a classical random vector,

the deterministic limit becomes a (singular) pdf.

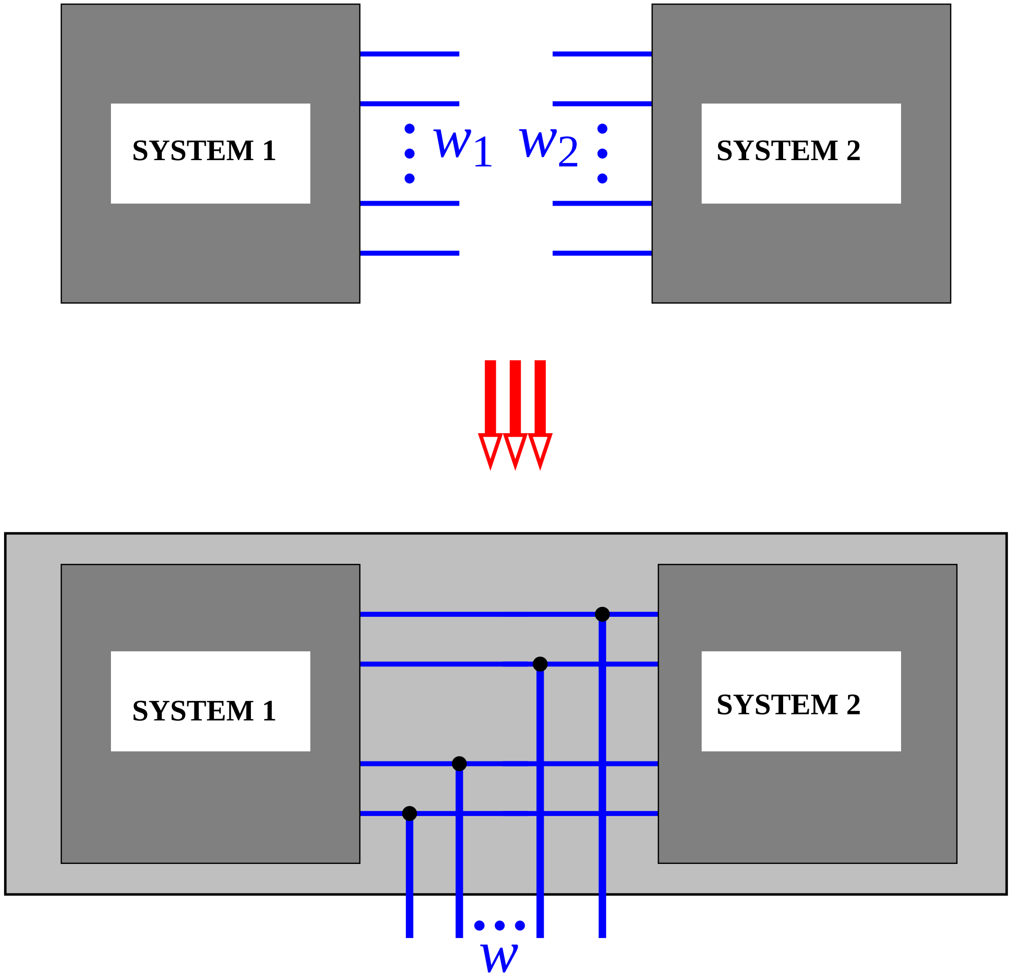
Awkward from the modeling point of view.

Interconnection

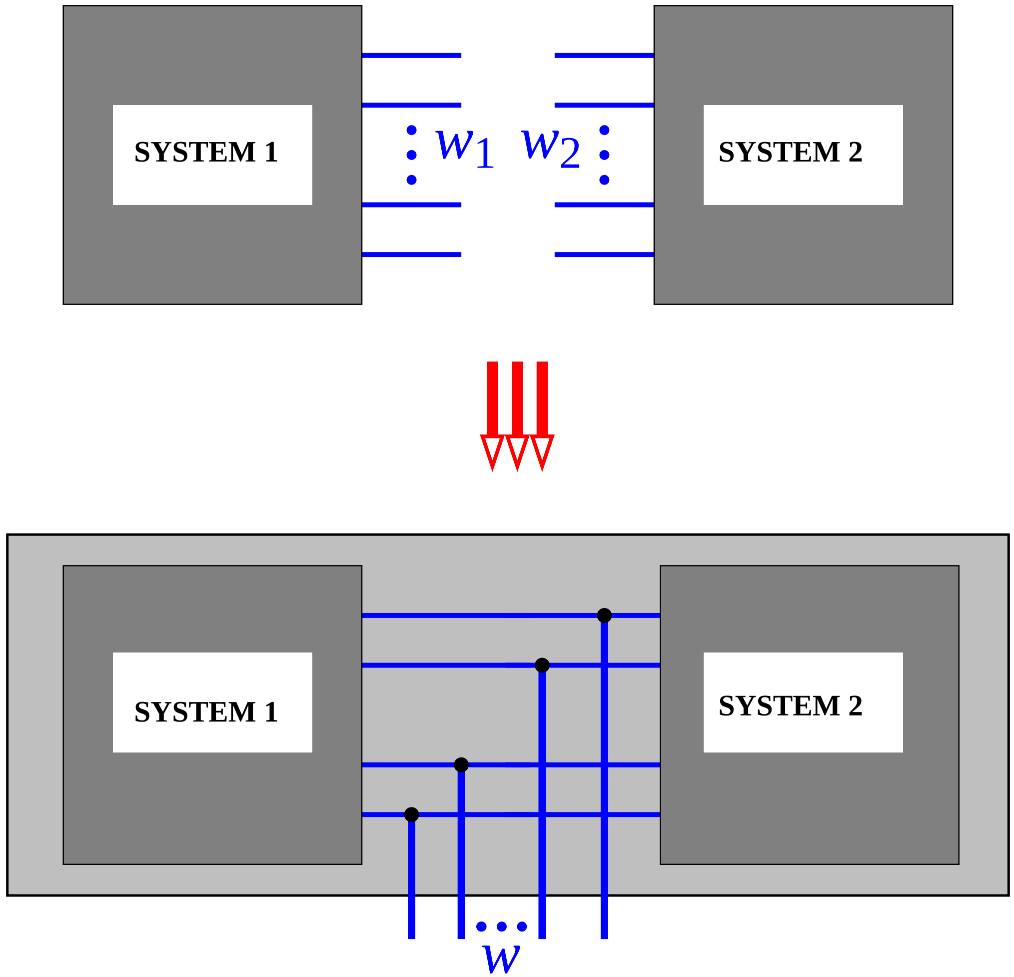
Interconnection



Interconnection



Interconnection

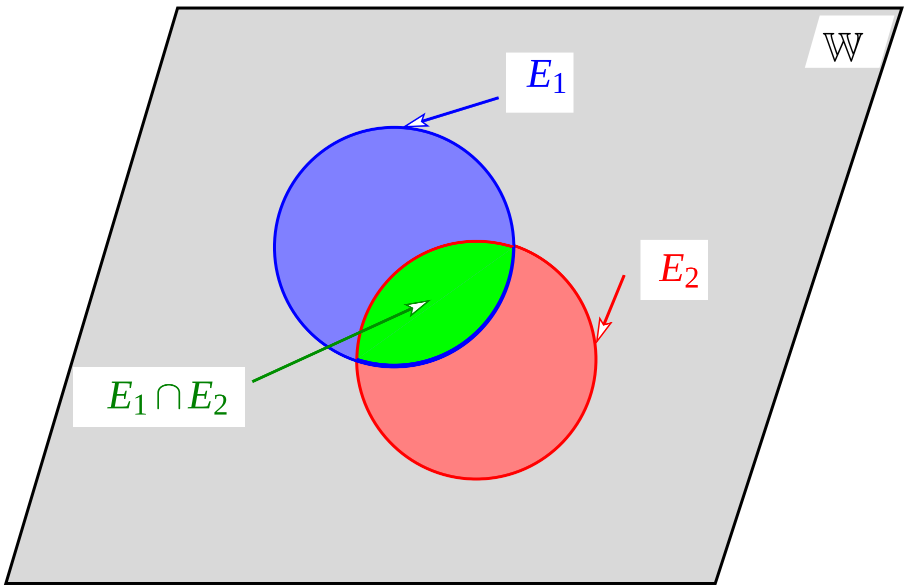


Can we impose two distinct probabilistic laws on the same set of variables?

Complementarity of σ-algebras

$\mathscr{E}_1$ **and** $\mathscr{E}_2$ **are** ***complementary σ-algebras*** **:**$\Leftrightarrow$
for all nonempty sets $E_1, E_1' \in \mathscr{E}_1, E_2, E_2' \in \mathscr{E}_2$

$$[\![E_1 \cap E_2 = E_1' \cap E_2']\!] \Rightarrow [\![E_1 = E_1' \text{ and } E_2 = E_2']\!].$$



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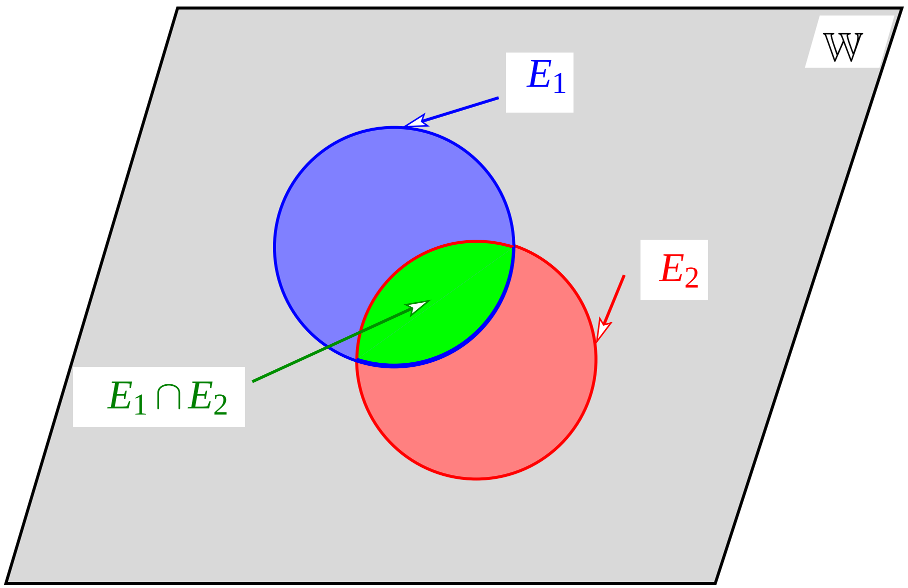
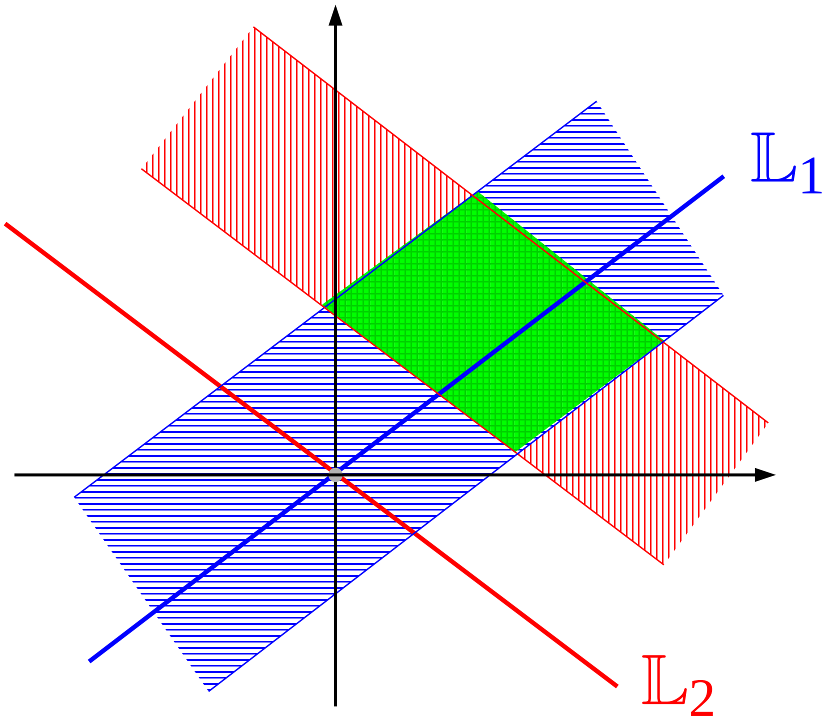


The intersection determines the intersectants.

Linear example

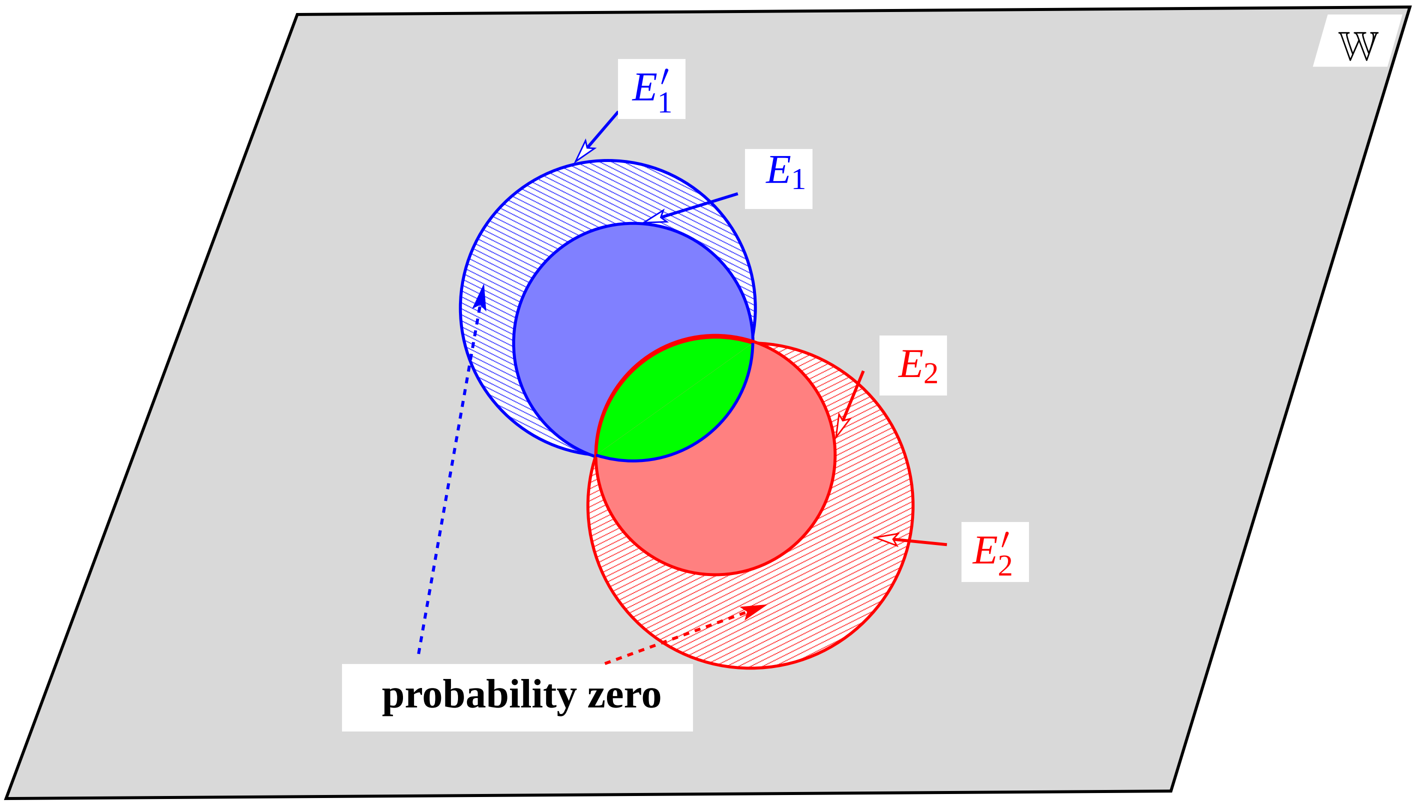


$$\mathbb{L}_1 + \mathbb{L}_2 = \mathbb{R}^n$$

Complementarity of systems

$\Sigma_1 = (\mathbb{W}, \mathscr{E}_1, P_1)$ **and** $\Sigma_2 = (\mathbb{W}, \mathscr{E}_2, P_2)$ **are said to be** ***complementary*** $:\Leftrightarrow$ **for** $E_1, E_1' \in \mathscr{E}_1$ **and** $E_2, E_2' \in \mathscr{E}_2$**:**

$$[\![E_1 \cap E_2 = E_1' \cap E_2']\!] \Rightarrow [\![P_1(E_1)P_2(E_2) = P_1(E_1')P_2(E_2')]\!].$$



Interconnection of complementary systems

Let $\Sigma_1 = (\mathbb{W}, \mathscr{E}_1, P_1)$ and $\Sigma_2 = (\mathbb{W}, \mathscr{E}_2, P_2)$ be complementary stochastic systems (assumed stochastically independent). Their *interconnection* is

$$(\mathbb{W}, \mathscr{E}, P)$$

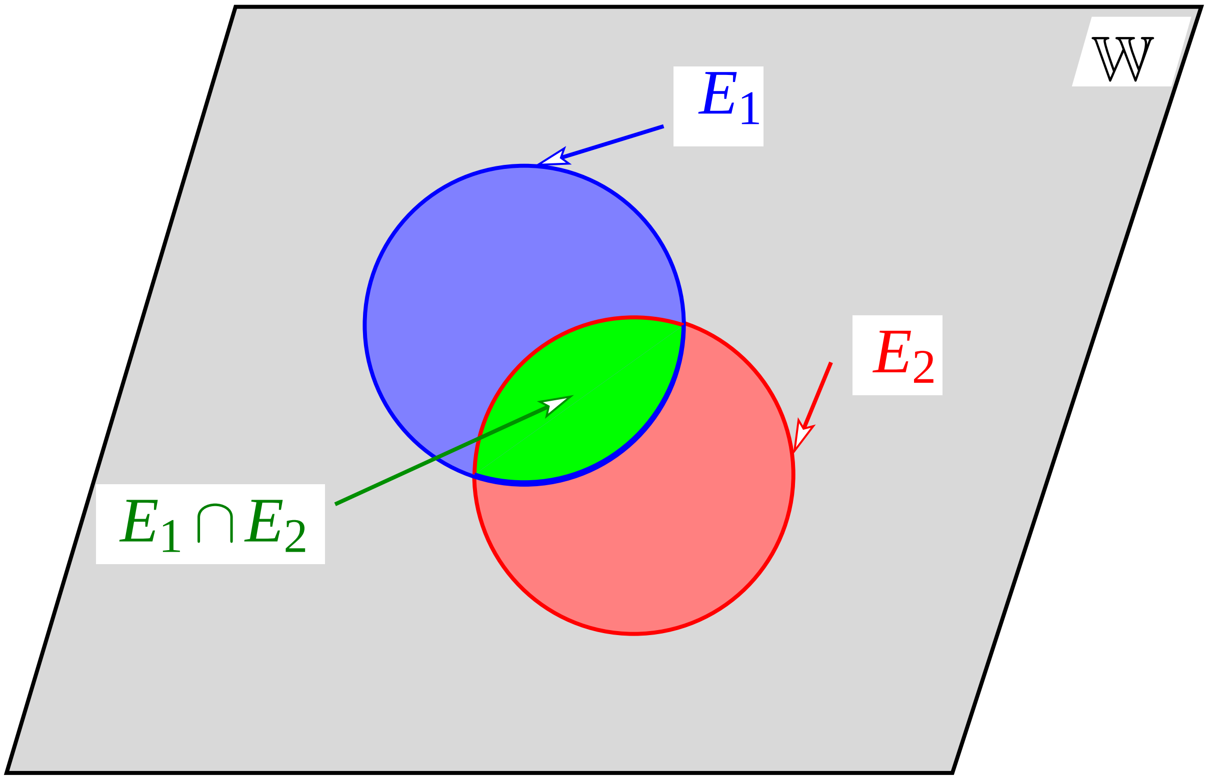
with $\mathscr{E}$:= the σ-algebra generated by the 'rectangles'

$$\{E_1 \cap E_2 \mid E_1 \in \mathscr{E}_1, E_2 \in \mathscr{E}_2\},$$

and P defined through the rectangles by

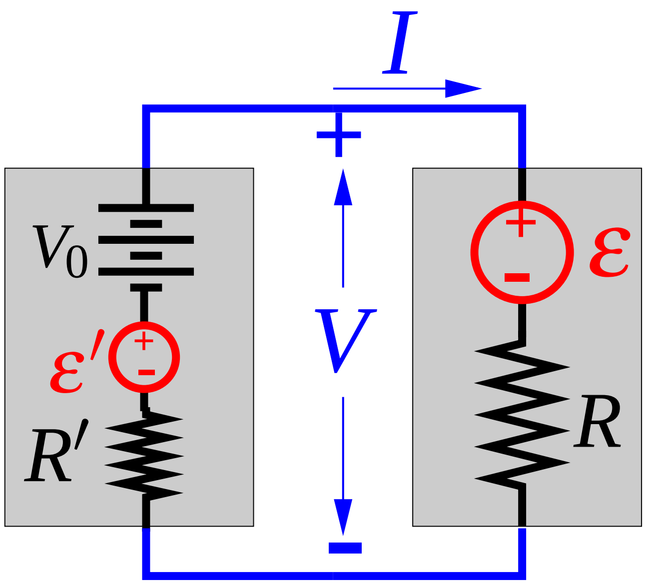
$$\boxed{P(E_1 \cap E_2) := P_1(E_1) P_2(E_2).}$$

Interconnection of complementary systems



$$P(E_1 \cap E_2) := P_1(E_1)P_2(E_2).$$

Noisy resistor terminated by voltage source



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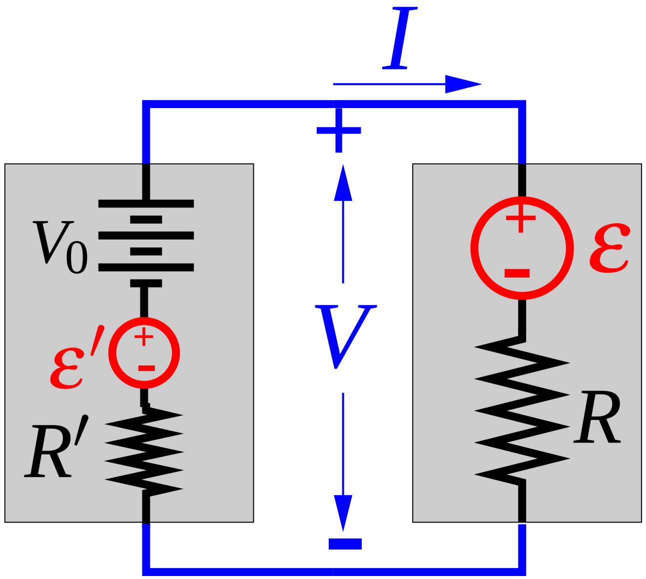
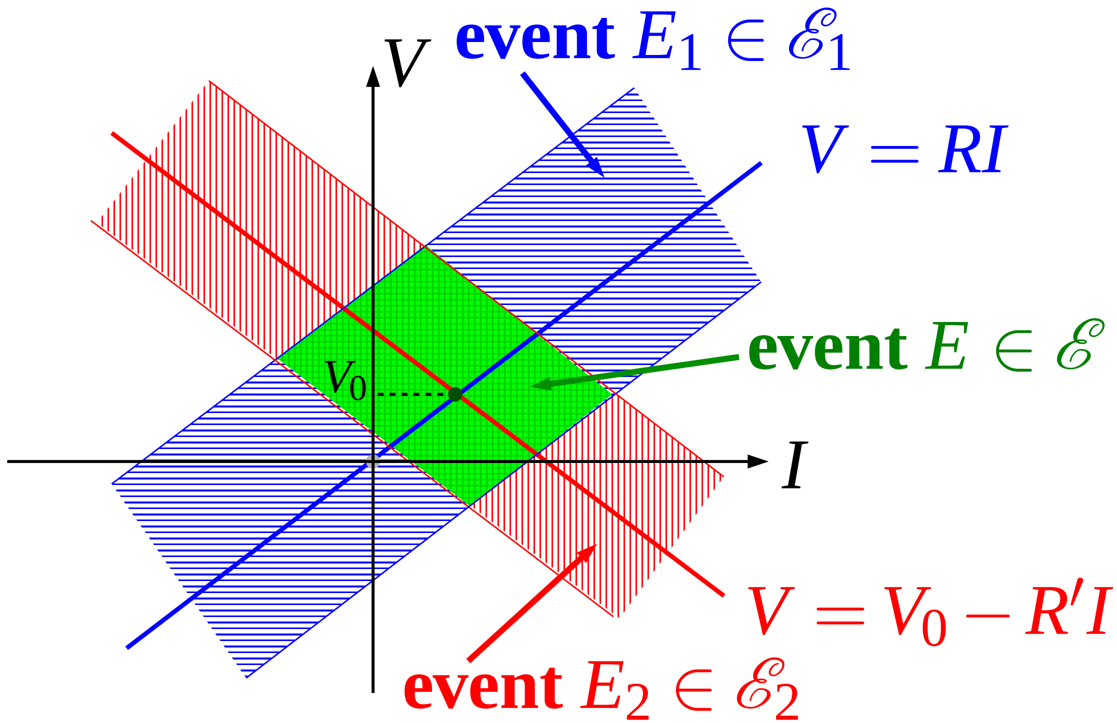
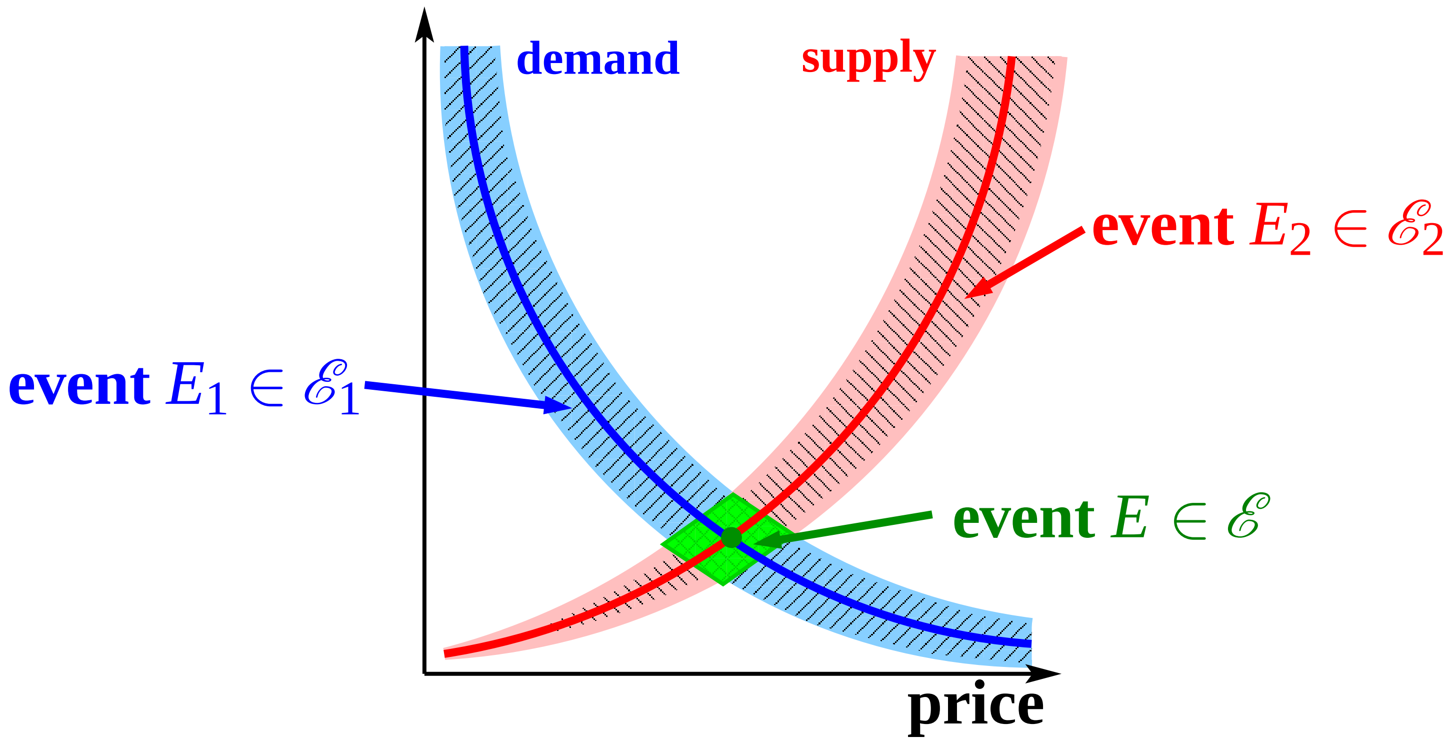


$$P(E) = P_1(E_1)P_2(E_2)$$



Equilibrium price/demand/supply



$$P(E) = P_1(E_1)P_2(E_2).$$

Open stochastic systems

Open versus closed

$\Sigma_1 = (\mathbb{R}^{\mathtt{n}}, \mathscr{E}_1, P_1)$.

If $\mathscr{E}_1$ = the Borel σ-algebra, and $\mathtt{support}(P_1) = \mathbb{R}^{\mathtt{n}}$, then Σ_1 is interconnectable only with the free system $\Sigma_2 = (\mathbb{R}^{\mathtt{n}}, \mathscr{E}_2, P_2)$, $\mathscr{E}_2 = \{\emptyset, \mathbb{R}^{\mathtt{n}}\}$.

$\Rightarrow$ **classical Σ_1 = ‘closed’ system.**

Open versus closed

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$\Rightarrow$ **classical Σ_1 = 'closed' system.**

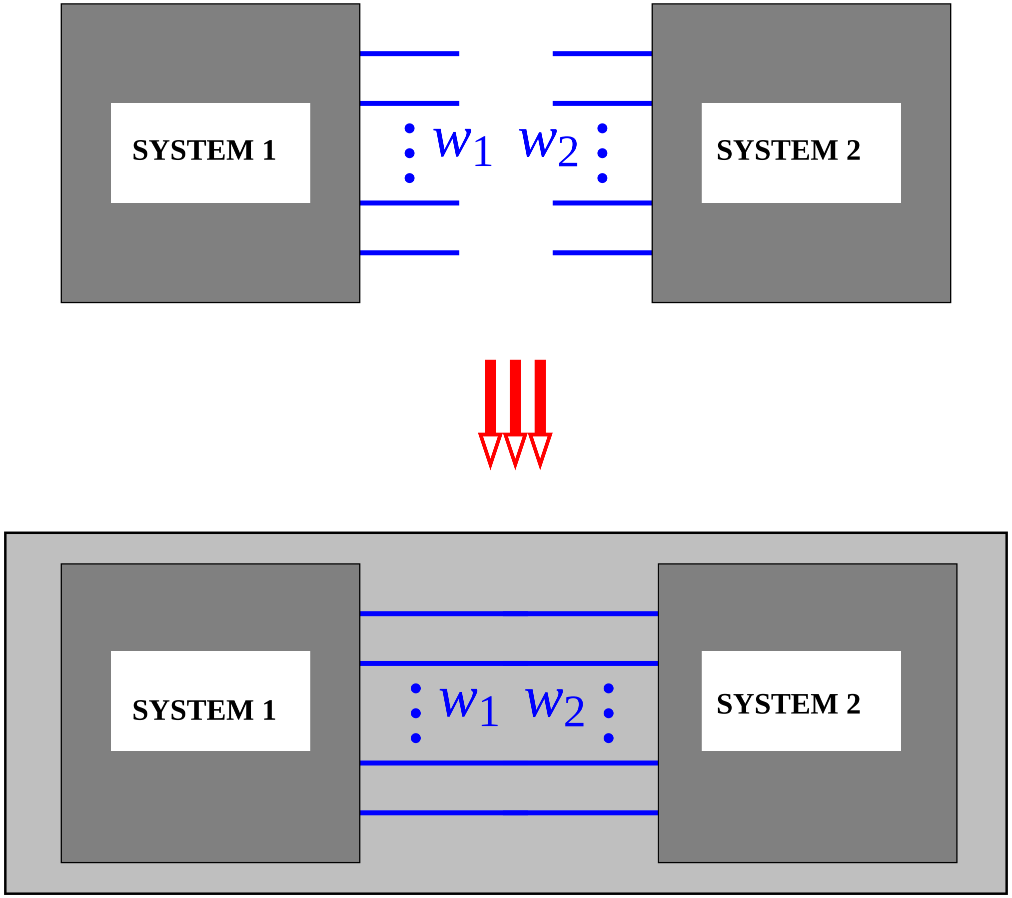
Parsimonious $\mathscr{E}_1$

$\Rightarrow \Sigma_1$ **is interconnectable.**

$\Rightarrow$ **'open' system.**

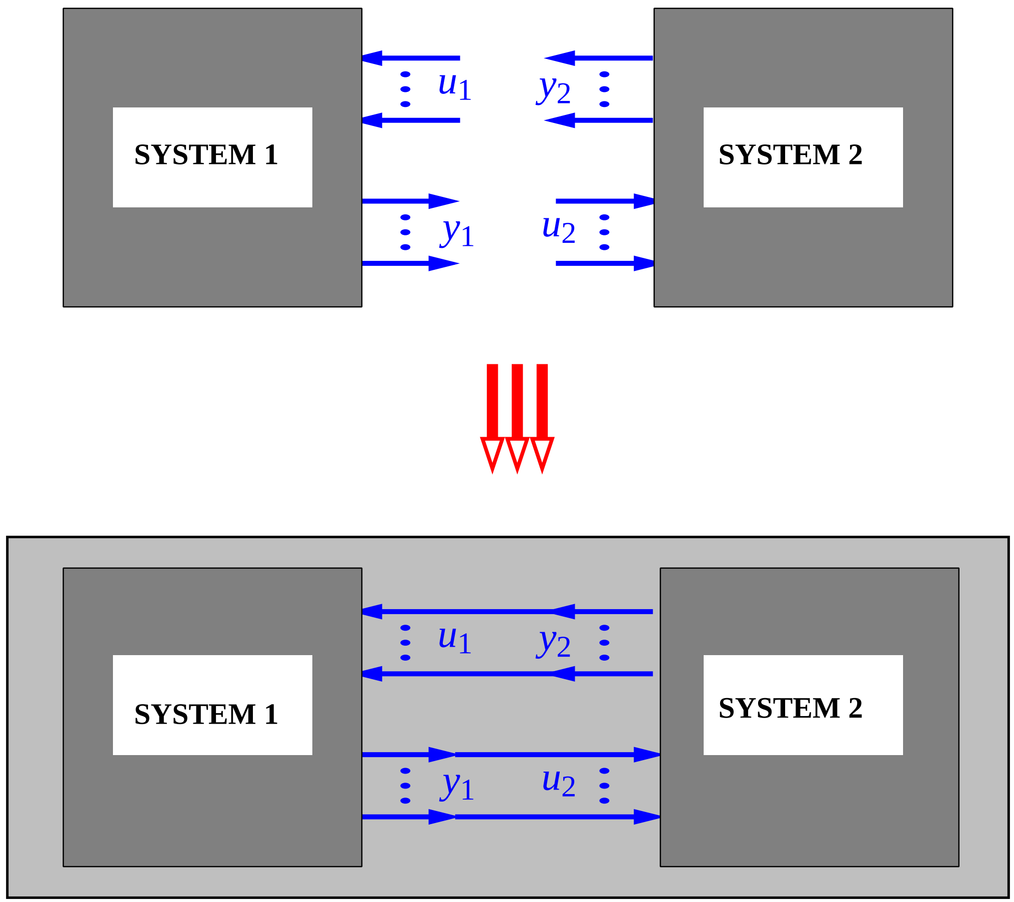
Interconnection $\Leftrightarrow$ variable sharing

Variable sharing



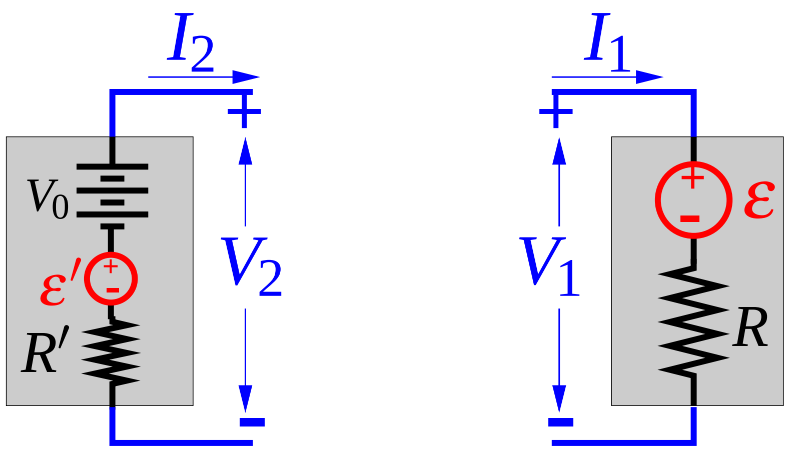
$$w_1 = w_2$$

Output-to-input assignment



$$u_1 = y_2, \quad u_2 = y_1$$

Resistor interconnection



$$V_1 = V_2, \quad I_1 = I_2$$

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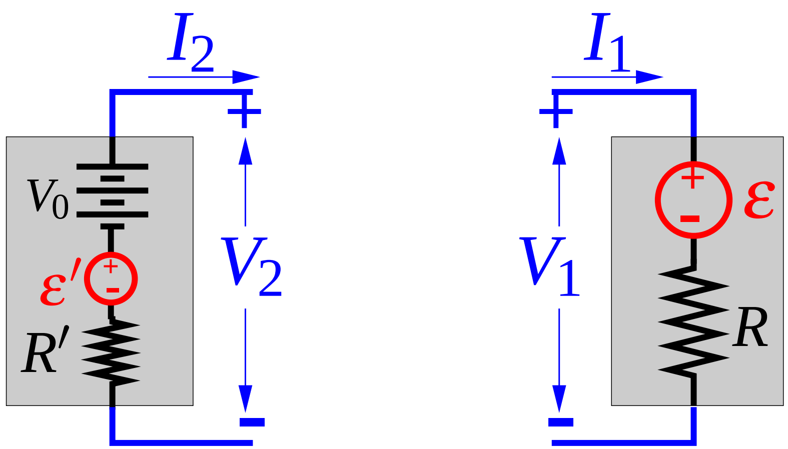

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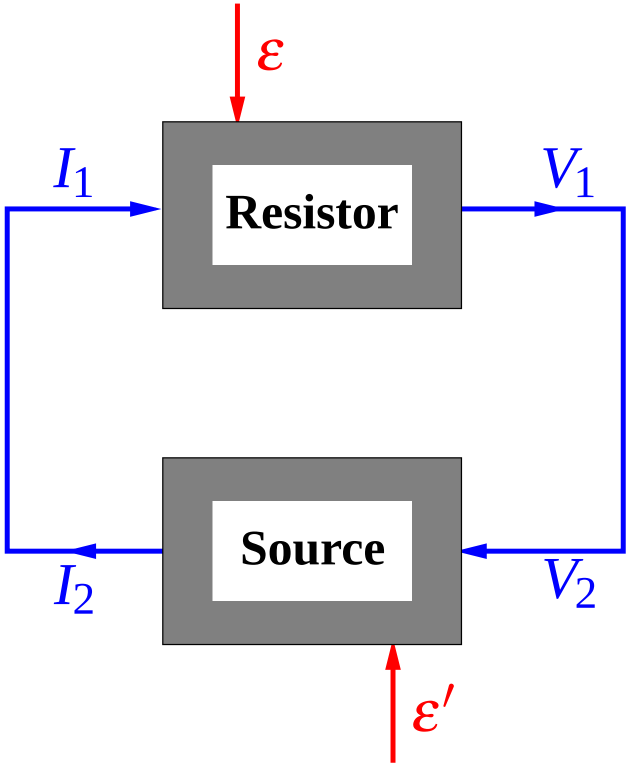
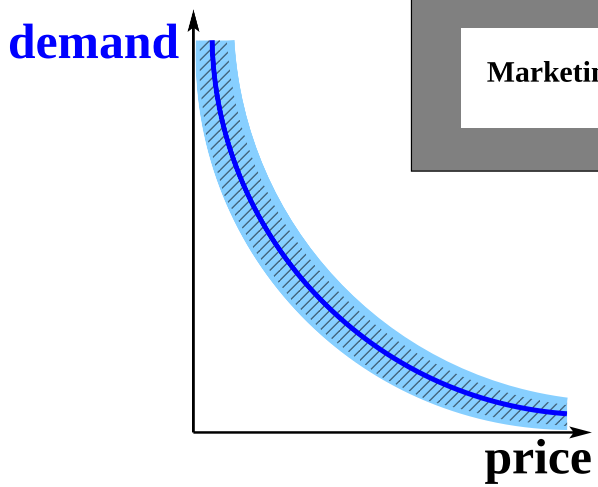
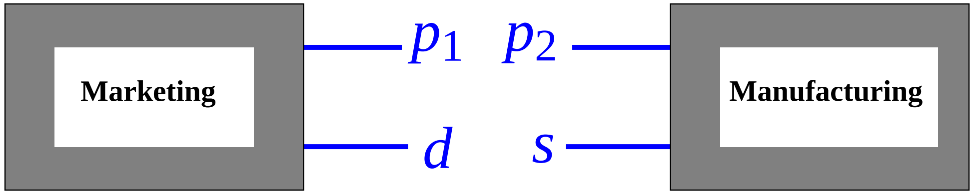
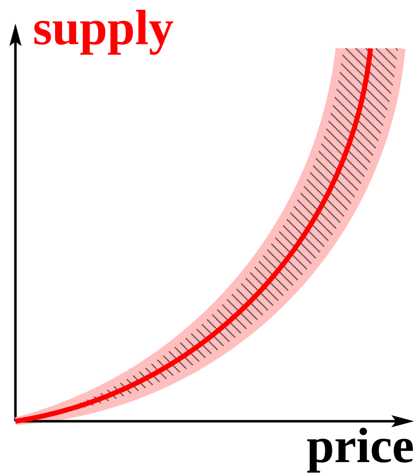


Price/demand/supply interconnection

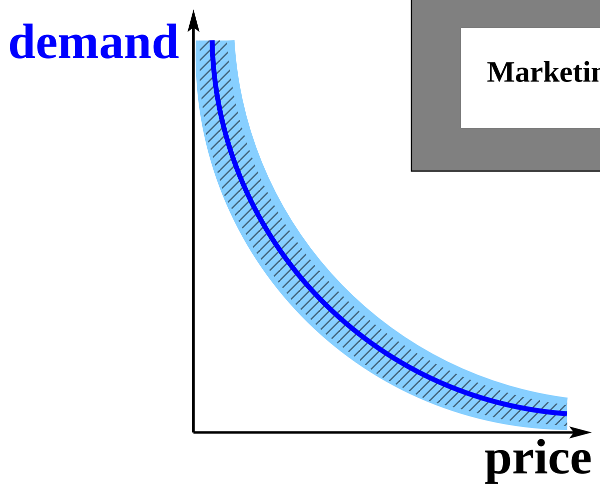


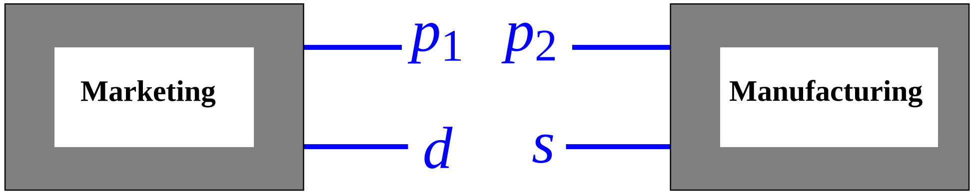


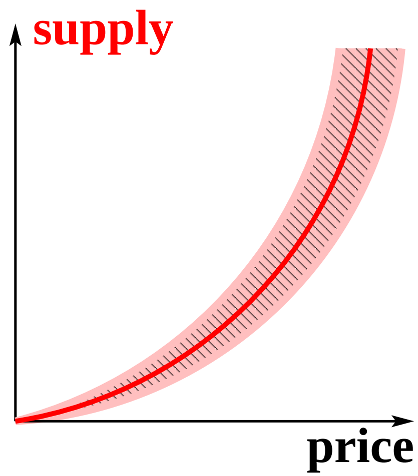


$$p_1 = p_2, \quad d = s$$

Price/demand/supply interconnection







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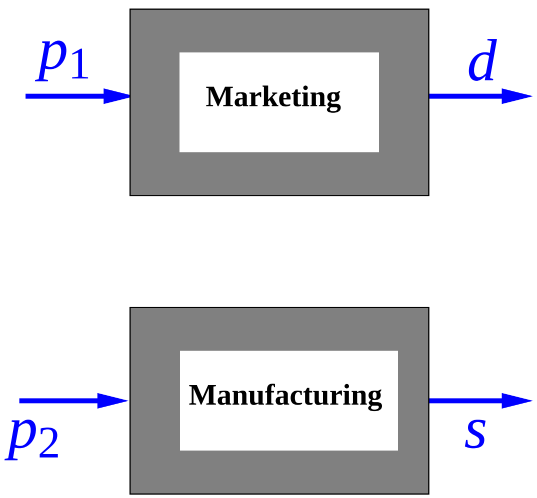

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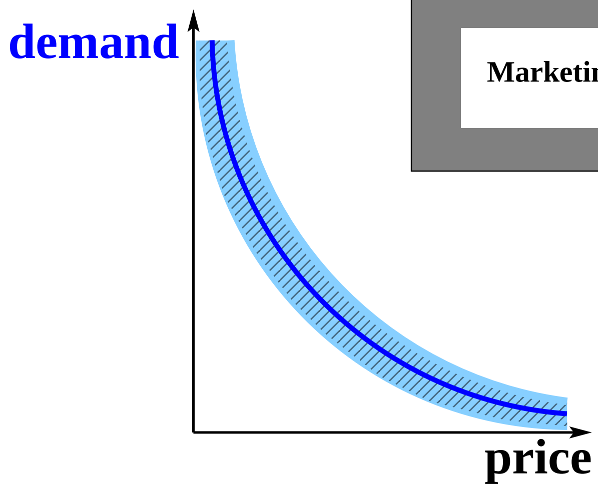

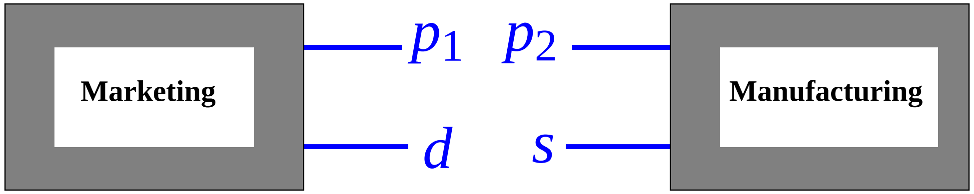

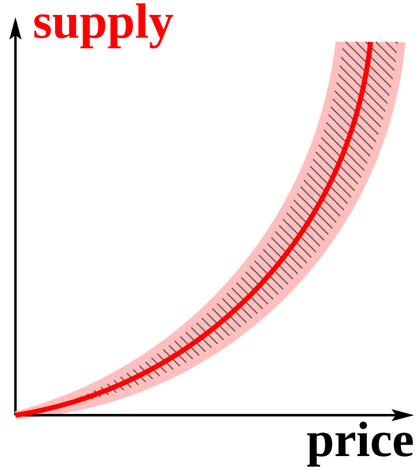

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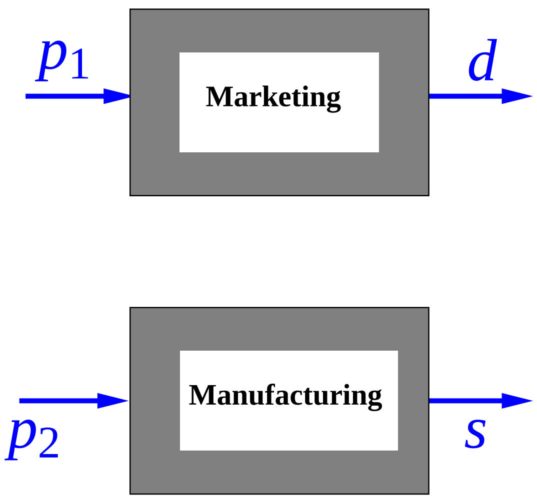
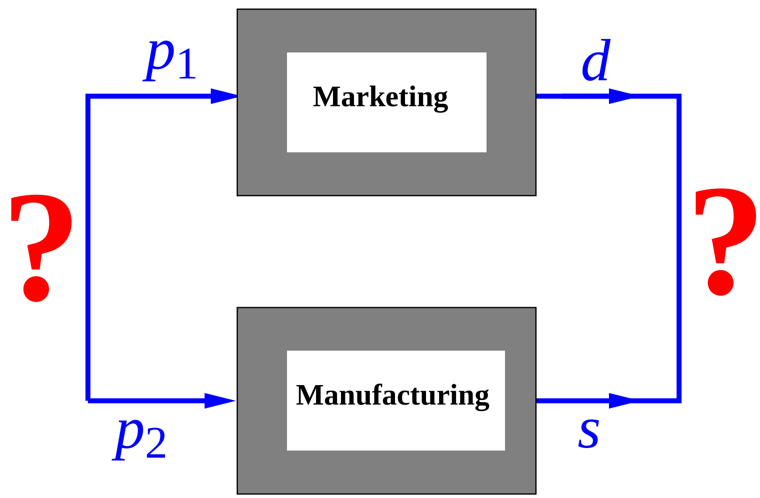
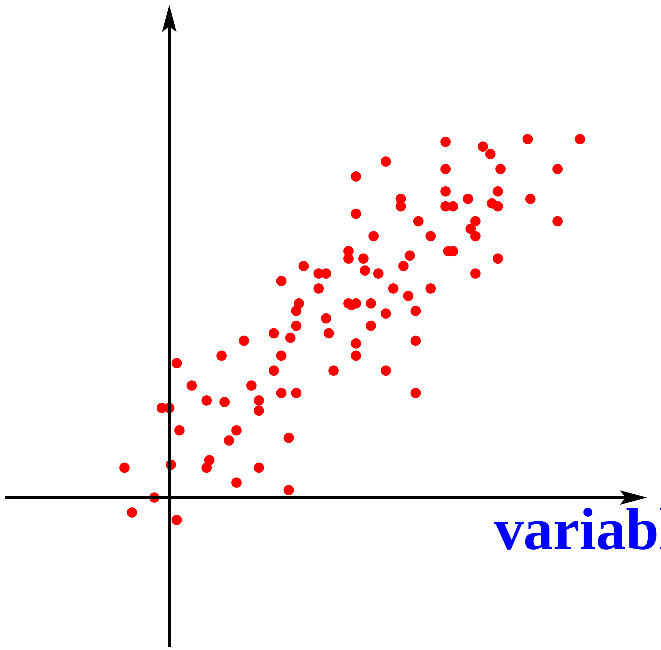




Identification

Sampling

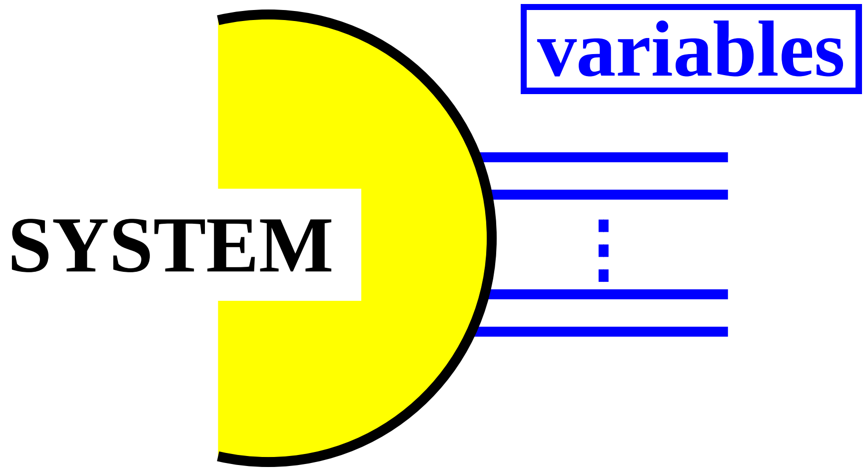


System identification: deduce the stochastic model

$\mathscr{E}$ and P

from the samples.

Measurements



Data collection implies observing a stochastic system

in interaction with an environment.

Measurements

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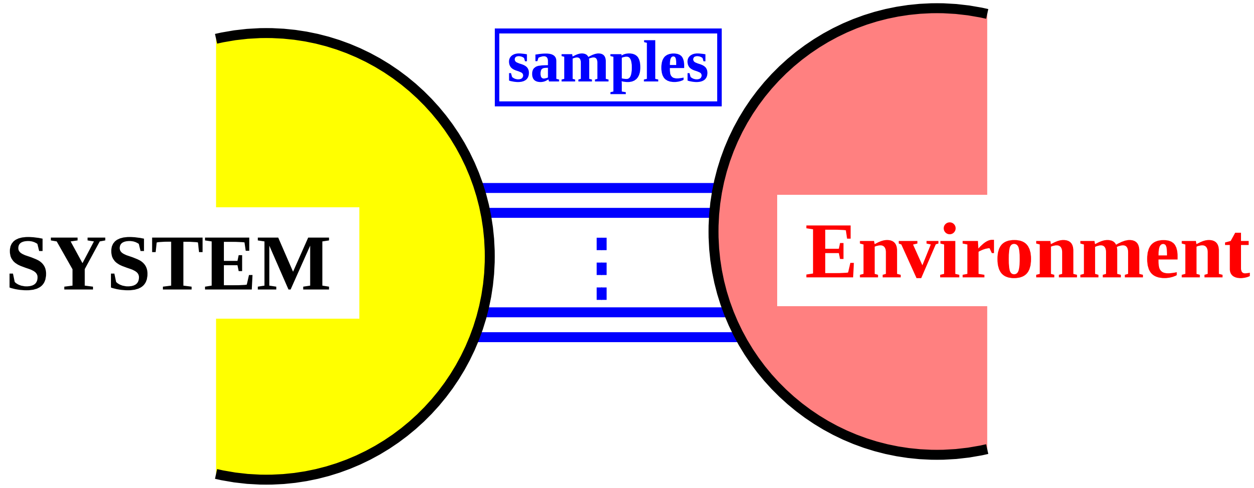


Measurements

Data collection implies observing a stochastic system *in interaction with an environment*.

Is it possible to disentangle the laws of a system from the laws of the environment?

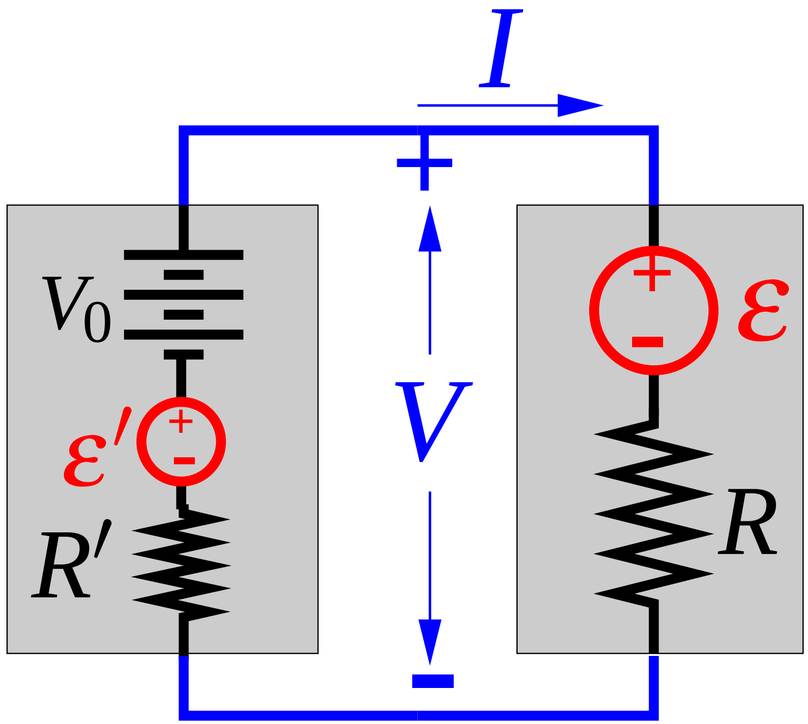
Measurements

Data collection implies observing a stochastic system *in interaction with an environment*.

Is it possible to disentangle the laws of a system from the laws of the environment?

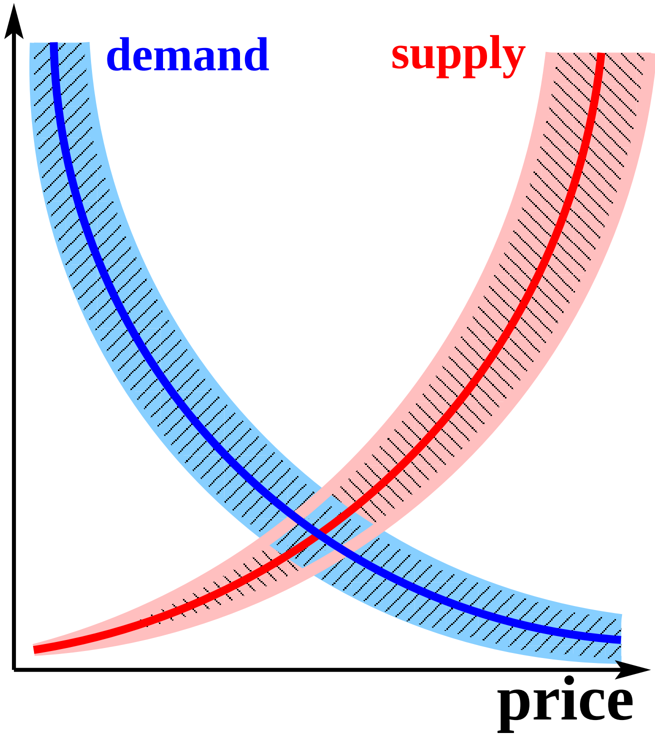
In engineering, it may be possible to set the experimental conditions.
In economics and the social sciences (and biology?), data often gathered passively 'in vivo'.

Disentangling



Can R and σ be deduced by sampling (V, I)?

Disentangling



Can the price/demand characteristic be deduced by sampling (p, d) in equilibrium?

SYSID for gaussian systems

Let Σ_1 and Σ_2 be complementary gaussian systems and assume that the interconnection $\Sigma_1 \wedge \Sigma_2$ is a classical random system.

Sampling $\leadsto$ the mean and covariance of $\Sigma_1 \wedge \Sigma_2$. Assume that the covariance is nonsingular.

SYSID for gaussian systems

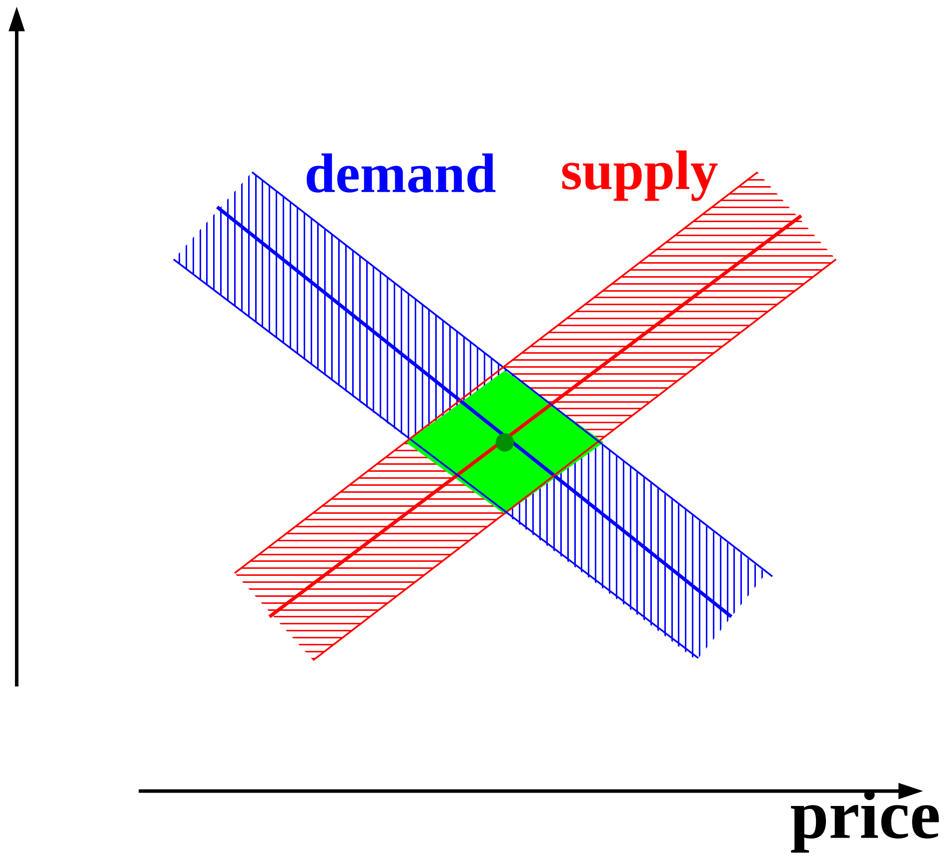
Let Σ_1 and Σ_2 be complementary gaussian systems and assume that the interconnection $\Sigma_1 \wedge \Sigma_2$ is a classical random system.

Sampling $\rightsquigarrow$ the mean and covariance of $\Sigma_1 \wedge \Sigma_2$. Assume that the covariance is nonsingular.

Given the fiber of either Σ_1 or Σ_2, then all the other parameters of Σ_1 and Σ_2 can be deduced from $\Sigma_1 \wedge \Sigma_2$.

The fiber of Σ_1 can be chosen freely.

Linearized gaussian price/demand/supply



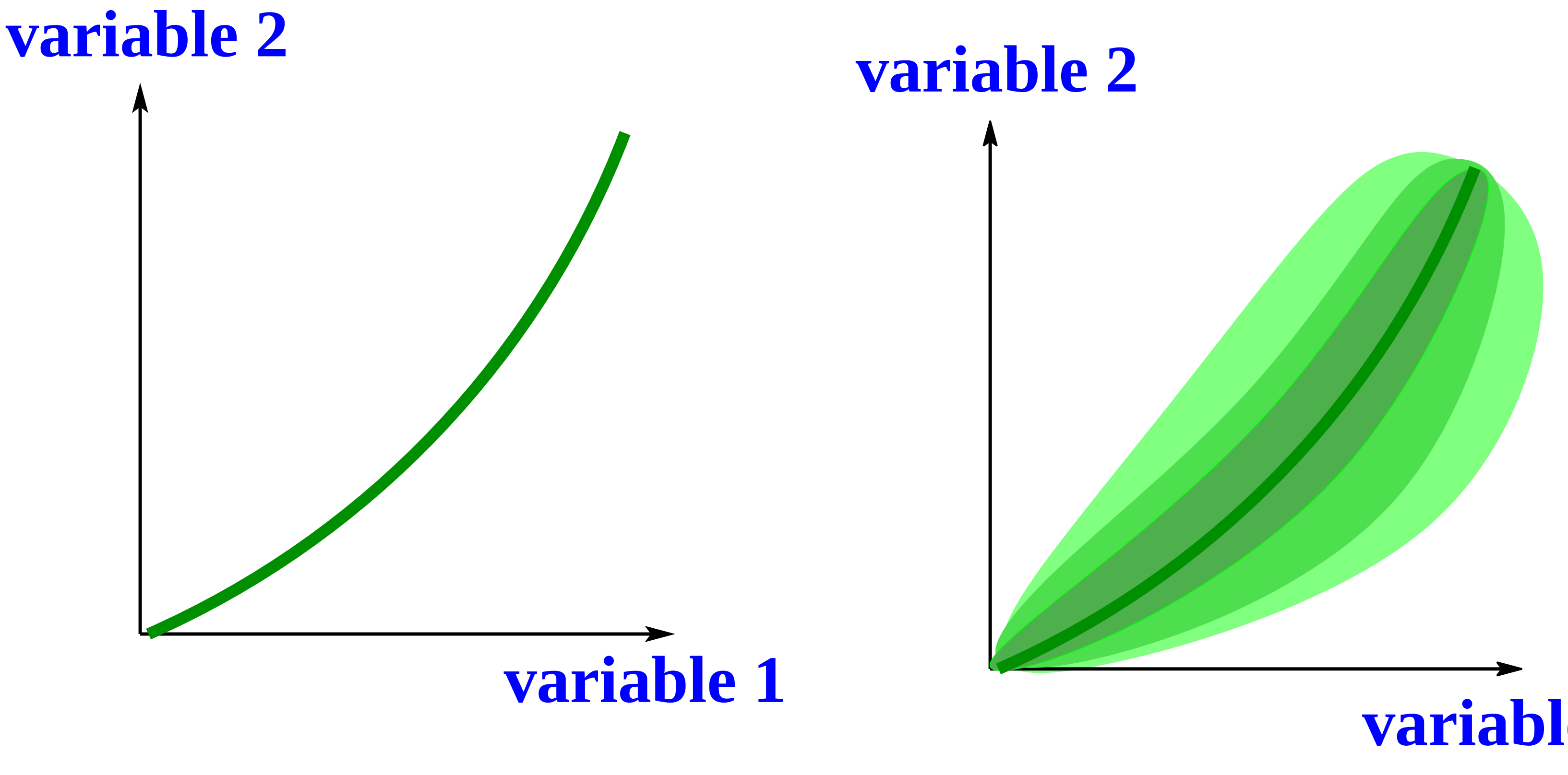
Identifiability provided one of the fibers is known.

Sampling alone does not give these elasticities.

Conclusions

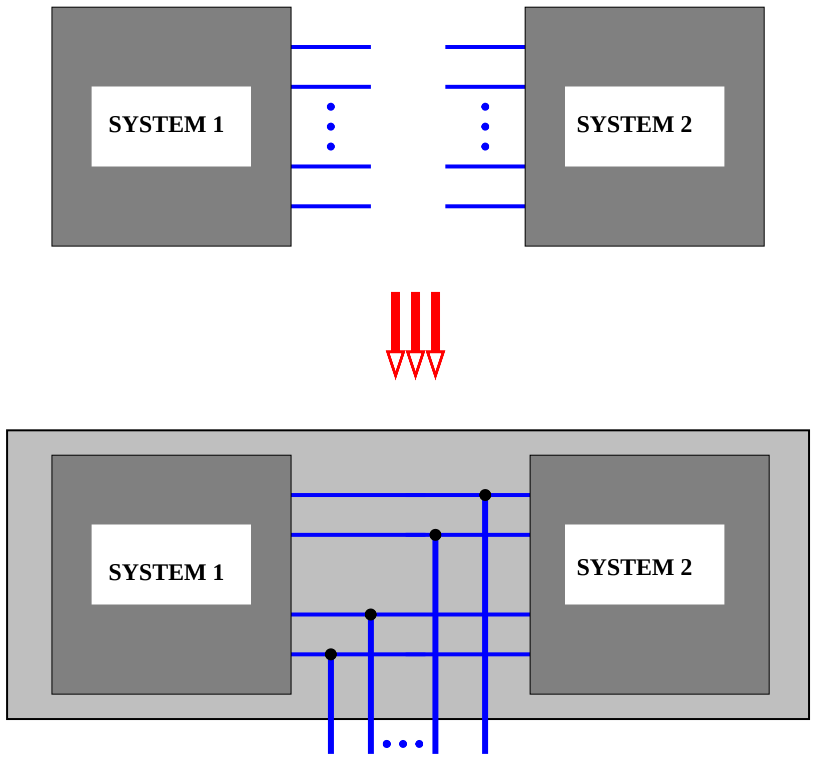
Stochastic systems

- **The Borel σ-algebra is inadequate even for elementary applications.**



Stochastic systems

- Complementary stochastic systems can be interconnected: two distinct laws imposed on one set of variables.

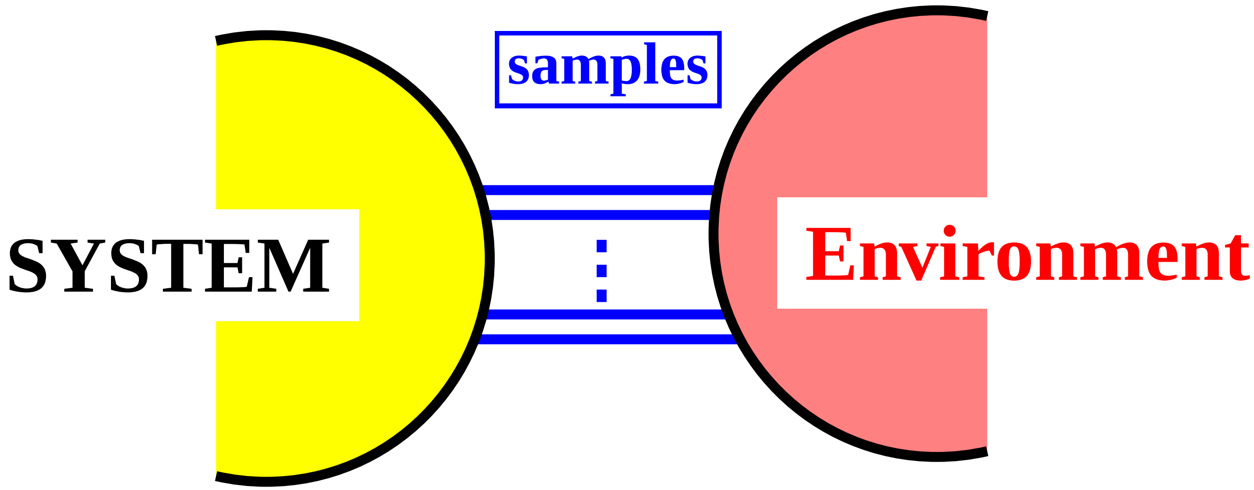


Stochastic systems

- **Open stochastic systems require a parsimonious σ-algebra.**

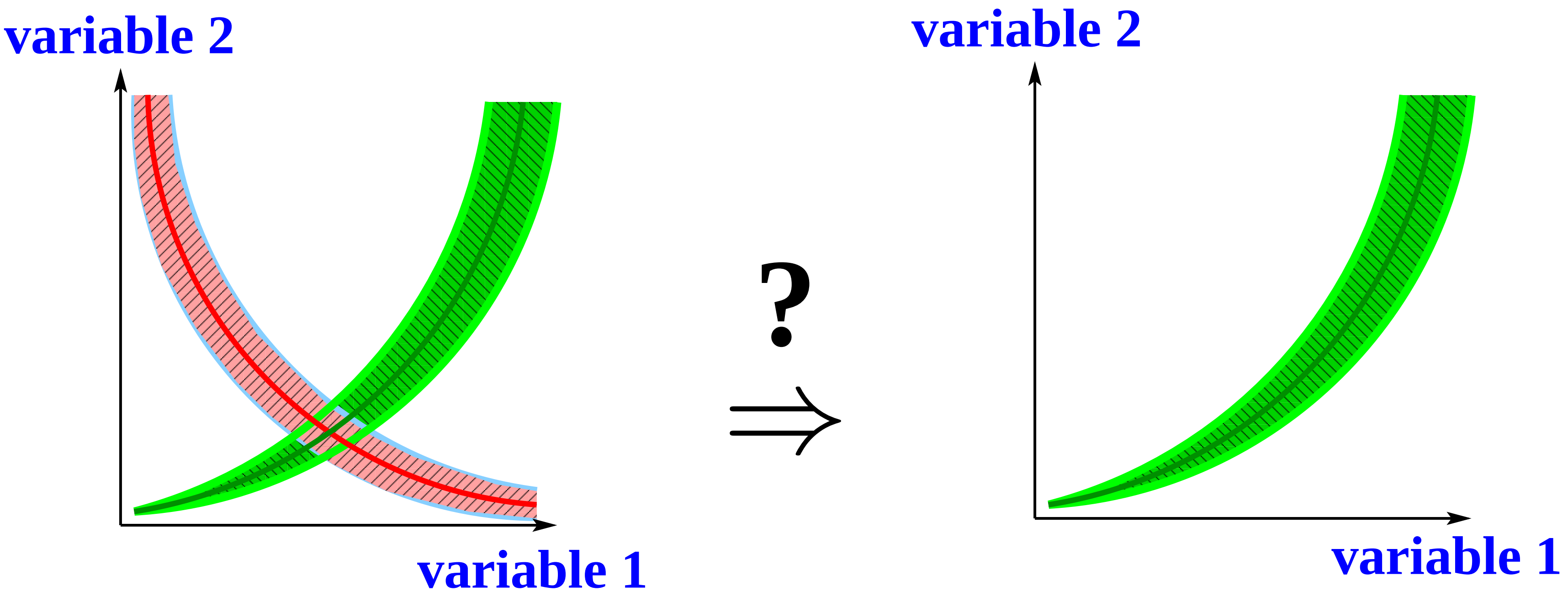
  Classical random vectors imply closed systems.

- **Measurements are the result of interaction with an environment.**



Modeling from data requires disentanglement.

- **Modeling from data requires disentanglement.**
  The data alone are insufficient for identifiability.



Future work

Urgent:

Generalization to stochastic processes.

<u>Reference</u>: *Open stochastic systems*, IEEE AC, submitted.

Copies of the lecture frames available from/at

`http://www.esat.kuleuven.be/∼jwillems`

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Thank you

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