State space representation of SISO periodic behaviors

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Scope

- **Aim of this talk**

  Construction of minimal periodic state space representations for periodic behavioral systems

- **Methodology**

  The use of suitable state space representations of the associated (time–invariant) lifted behavior

- **Focus**

  SISO 2-periodic behaviors

Outline

1. Periodic behaviors – the lifted system
2. System representations – SISO periodic behaviors
3. 2-periodic state space representations
4. Algorithm 1

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1 Periodic behaviors – the lifted system

2 System representations – SISO periodic behaviors

3 2-periodic state space representations

4 Algorithm 1

Behavioral systems

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- **λ-shift**

$$\sigma^{\lambda} : (\mathbb{R}^q)^{\mathbb{Z}} \to (\mathbb{R}^q)^{\mathbb{Z}} : \left(\sigma^{\lambda} w\right)(k) := w\,(k + \lambda)$$

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  - $\mathfrak{B}$ – P-periodic – $\sigma^P\mathfrak{B} = \mathfrak{B}$

Lifted system

Consider the linear map:

$$\begin{array}{rcccccc} L: & \left(\mathbb{R}^q\right)^{\mathbb{Z}} & \longrightarrow & \left(\mathbb{R}^{Pq}\right)^{\mathbb{Z}} & & & \\ & w & \mapsto & Lw: & \mathbb{Z} & \longrightarrow & \mathbb{R}^{Pq} \\ & & & & k & \mapsto & \begin{bmatrix} w\left(Pk\right) \\ \vdots \\ w\left(Pk+P-1\right) \end{bmatrix} \end{array}$$

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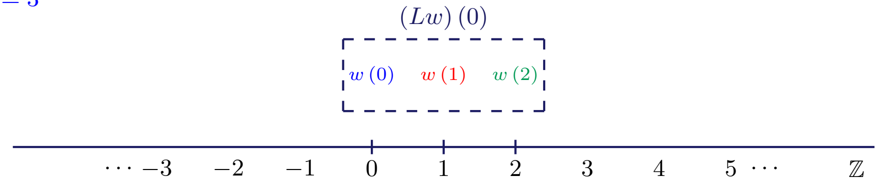

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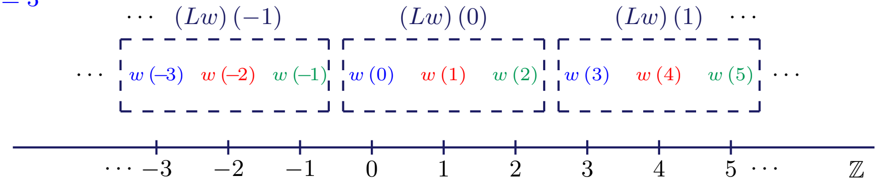

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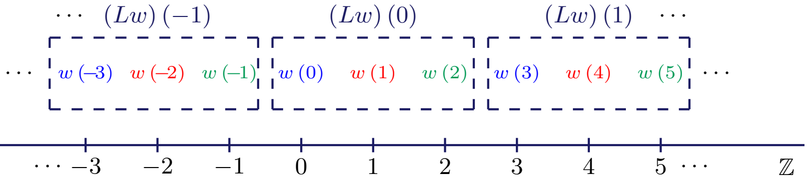

Then we can associate

$$\Sigma = (\mathbb{Z}, \mathbb{R}^q, \mathfrak{B}) \text{ } P\text{-periodic} \longleftrightarrow \Sigma^L = \left(\mathbb{Z}, \mathbb{R}^{Pq}, L\mathfrak{B}\right) \text{ time-invariant}$$

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Kernel representations

Invariant case

Thm [Jan C. Willems, 91]

$\mathfrak{B}$ time-invariant, linear, closed subspace of $(\mathbb{R}^q)^{\mathbb{Z}}$

Periodic case

Thm [M. Kuijper and Jan C. Willems, 97] – Reformulated

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$$\Leftrightarrow$$

$$\mathfrak{B} \sim \left(R\left(\sigma, \sigma^{-1}\right) w\right)(k) = 0, \forall k \in \mathbb{Z}$$

with

$$R\left(\xi, \xi^{-1}\right) = R_{-M}\xi^{-M} + \cdots + R_N \xi^N$$

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$\mathfrak{B} = \ker R\left(\sigma, \sigma^{-1}\right)$ $\quad$ R – Kernel representation (matrix)

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R – P-periodic kernel representation (matrix) – PPKR (matrix)

Kernel-type representation

Unique decomposition

$$R\left(\xi,\xi^{-1}\right) = R_0^L\left(\xi^P,\xi^{-P}\right) + \cdots + \xi^{P-1}R_{P-1}^L\left(\xi^P,\xi^{-P}\right)$$

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$$=: R^L\left(\xi^P,\xi^{-P}\right)\Omega_{P,q}\left(\xi\right)$$

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Thm [M. Kuijper and Jan C. Willems, 97]: $\mathfrak{B}$, P-periodic, is given by the PPKR $R\left(\xi,\xi^{-1}\right)$

$$\Leftrightarrow$$

The associated lifted behavior $L\mathfrak{B}$ is given by the kernel representation $R^L\left(\xi,\xi^{-1}\right)$

SISO P-Periodic behaviors

SISO P-periodic behavior: $\mathfrak{B} \sim \left(p\left(\sigma, \sigma^{-1}\right)y\right)(Pk) = \left(q\left(\sigma, \sigma^{-1}\right)u\right)(Pk)$

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Induced representations

- **2-periodic state space system of Σ**

$$\Sigma_s \sim \begin{cases} (\sigma x)(k) = A(k)\,x(k) + B(k)\,u(k) \\ y(k) = C(k)\,x(k) + D(k)\,u(k) \end{cases}$$

where $A(k+2) = A(k)$

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with

$$F = A(1)A(0) \qquad G = \begin{bmatrix} A(1)B(0) & B(1) \end{bmatrix}$$

$$H = \begin{bmatrix} C(0) \\ C(1)A(0) \end{bmatrix} \qquad J = \begin{bmatrix} D(0) & 0 \\ C(1)B(0) & D(1) \end{bmatrix}$$

When is a representation induced?

Proposition

Let $\mathfrak{B}$ be a SISO 2–periodic behavior and $\mathfrak{B}^L$ the lifted behavior associated to $\mathfrak{B}$. Then a n–dimensional state space representation $\Sigma_s^L = (F, G, H, J)$ of $\mathfrak{B}^L$, with

$$F \in \mathbb{R}^{n \times n} \qquad G = \begin{bmatrix} G_1 & G_2 \end{bmatrix} \in \mathbb{R}^{n \times 2}$$

$$H = \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} \in \mathbb{R}^{2 \times n} \qquad J = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} \in \mathbb{R}^{2 \times 2} ,$$

is induced, by a state space representation Σ_s of Σ, if and only if

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Observe that for the induced state space representation in $(\star)$ we will have

$$\mathcal{M} = \begin{bmatrix} A(1)A(0) & A(1)B(0) \\ C(1)A(0) & C(1)B(0) \end{bmatrix} = \underbrace{\begin{bmatrix} A(1) \\ C(1) \end{bmatrix}}_{n} \left.\begin{bmatrix} A(0) & B(0) \end{bmatrix}\right\} n \quad .$$

Example: Not every state space representation of $\mathfrak{B}^L$ is an induced one!

Consider the 2–periodic input–output behavior $\mathfrak{B}$ described by

$$\left(\begin{bmatrix} \sigma^2 - 1 \\ \sigma \end{bmatrix} y\right)(2k) = \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} u\right)(2k)$$

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for which a minimal state space representation, of dimension 1, is

$$\begin{cases} \sigma z(k) = z(k) \\ y^L(k) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} z(k) + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} u^L(k) \end{cases}$$

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$\therefore \mathcal{M} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ which is clearly of full rank and hence not decomposable!

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$$\widetilde{\mathcal{M}} = \left[\begin{array}{cc|c} F & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} & G_1 \\ 0 \cdots 0 & 0 & 0 \cdots 0 \\ \hline H_2 & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} & J_{21} \end{array}\right]$$

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$$\begin{cases} \sigma \overline{x}^{\mathrm{e}}(k) = \left[\begin{array}{c|c} F & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ \hline 0 \cdots 0 & 0 \end{array}\right] \overline{x}^{\mathrm{e}}(k) + \left[\begin{array}{c} G \\ \hline 0 \end{array}\right] u^L(k) \\ y^L(k) = \left[\begin{array}{c|c} H & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \end{array}\right] \overline{x}^{\mathrm{e}}(k) + J u^L(k) \end{cases}$$

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and clearly $\widetilde{\mathcal{M}}$ is an $(n+2)\times(n+2)$ matrix with rank $n+1$ that can be decomposed as

$$\widetilde{\mathcal{M}} = \left[\begin{array}{cc} F & G_1 \\ 0 & 0 \\ \hline H_2 & J_{21} \end{array}\right] \left[\begin{array}{cc|c} I & 0 & 0 \\ 0 & 0 & I \end{array}\right]$$

Construction of induced representations

This yields a $(n+1)$–dimensional state space representation $\Sigma(\cdot)$ of $\mathfrak{B}$, given by:

$$A(0) = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \qquad B(0) = \begin{bmatrix} 0 \\ I \end{bmatrix}$$

$$A(1) = \begin{bmatrix} F & G_1 \\ 0 & 0 \end{bmatrix} \qquad B(1) = \begin{bmatrix} G_2 \\ 0 \end{bmatrix}$$

$$C(0) = \begin{bmatrix} H_1 & 0 \end{bmatrix} \qquad D(0) = J_{11}$$

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Thus, it is always possible to construct a periodic state space representation Σ for a SISO 2–periodic behavior $\mathfrak{B}$ starting from a state space representation Σ^L of $\mathfrak{B}^L$. Moreover $\dim(\Sigma) = \dim\left(\Sigma^L\right)$ or $\dim(\Sigma) = \dim\left(\Sigma^L\right) + 1$.

Example revisited

Adding a zero row and a zero column to matrix $\mathcal{M} = I_2$ given in the example illustrated previously, we obtain a new 3×3 matrix $\widetilde{\mathcal{M}}$, which can be decomposed as

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Example revisited

Adding a zero row and a zero column to matrix $\mathcal{M} = I_2$ given in the example illustrated previously, we obtain a new 3×3 matrix $\widetilde{\mathcal{M}}$, which can be decomposed as

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This factorization allows us to obtain a 2–dimensional state space representation $\Sigma(\cdot)$ of $\mathfrak{B}$, given by

$$A(0) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad B(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$A(1) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad B(1) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$C(0) = \begin{bmatrix} 1 & 0 \end{bmatrix} \qquad D(0) = 0$$

$$C(1) = \begin{bmatrix} 0 & 1 \end{bmatrix} \qquad D(1) = 0$$

Main results

Lemma

Let $\mathfrak{B}^L$ be the lifted behavior associated to a SISO 2–periodic behavior $\mathfrak{B}$. If $\mathfrak{B}^L$ has one minimal state space representation which is induced, then all its minimal representations are induced.

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Theorem

Let $\mathfrak{B}$ be a SISO 2–periodic behavior and let $\mathfrak{B}^L$ be the corresponding lifted behavior. Then:

(i) $\mathfrak{B}$ has a 2–periodic state space representation $\Sigma(\cdot) = (A(\cdot), B(\cdot), C(\cdot), D(\cdot))$;

(ii) The dimensions $n_{\mathfrak{B}}$ and $n_{\mathfrak{B}^L}$ of the minimal state space representations of $\mathfrak{B}$ and of $\mathfrak{B}^L$, respectively, are such that:

$$n_{\mathfrak{B}^L} \leqslant n_{\mathfrak{B}} \leqslant n_{\mathfrak{B}^L} + 1;$$

(iii) A minimal periodic state space representation of $\mathfrak{B}$ can be obtained by Algorithm 1.

Outline

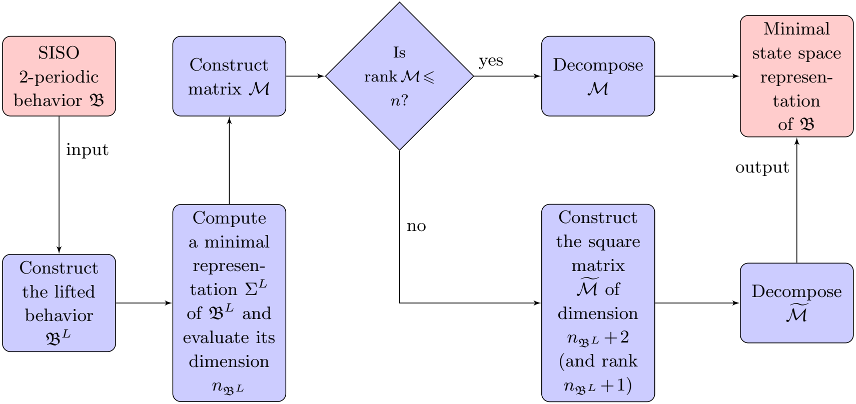
1 Periodic behaviors – the lifted system

2 System representations – SISO periodic behaviors

3 2-periodic state space representations

4 Algorithm 1

Algorithm 1



SISO 2-periodic behavior $\mathfrak{B}$
input
Construct the lifted behavior $\mathfrak{B}^L$
Compute a minimal represen-tation Σ^L of $\mathfrak{B}^L$ and evaluate its dimension $n_{\mathfrak{B}^L}$
Construct matrix $\mathcal{M}$
Is rank $\mathcal{M} \leqslant n$?
yes
no
Decompose $\mathcal{M}$
Construct the square matrix $\widetilde{\mathcal{M}}$ of dimension $n_{\mathfrak{B}^L}+2$ (and rank $n_{\mathfrak{B}^L}+1$)
Decompose $\widetilde{\mathcal{M}}$
output
Minimal state space represen-tation of $\mathfrak{B}$