OPEN STOCHASTIC SYSTEMS

JAN C. WILLEMS

K.U. Leuven, Flanders, Belgium

CDC, Orlando, Florida **December 12, 2011**

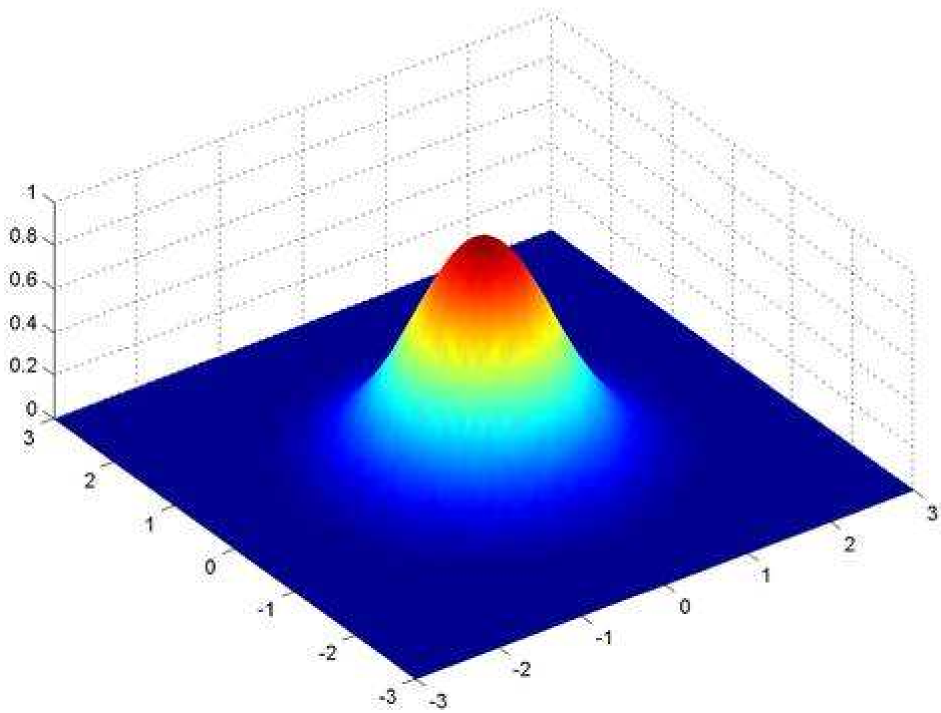
Orthodox probability

Probability (as commonly taught)

Model a phenomenon stochastically; outcomes in $\mathbb{R}^n$.

Usual framework:

- **probability distributions, probability density functions;**
- $\rightsquigarrow$ **'Every' subset of $\mathbb{R}^n$ is assigned a probability.**



for $A \subseteq \mathbb{R}^n$

$$P(A) = \int_A p(x)\, dx$$

Probability (as commonly taught)

Model a phenomenon stochastically; outcomes in $\mathbb{R}^n$.

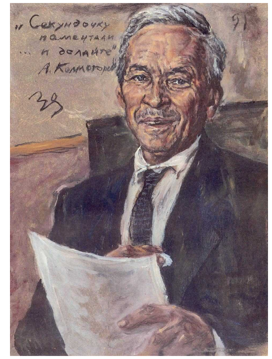
Usual framework:

- **probability distributions, probability density functions;**
- $\rightsquigarrow$ **'Every' subset of $\mathbb{R}^n$ is assigned a probability.**

Thesis

This is unduly restrictive,
even for elementary applications.

Mathematical probability



A.N. Kolmogorov
1903 – 1987

A *stochastic system* is a triple $(\mathbb{W}, \mathscr{E}, P)$

- $\mathbb{W}$ **the *outcome space*,**
- $\mathscr{E}$ **a class of subsets of $\mathbb{W}$,**
  with elements called *events*,
- $P : \mathscr{E} \to [0,1]$ **a *probability measure*.**

$\mathscr{E}$**: the sets that are assigned a probability.**

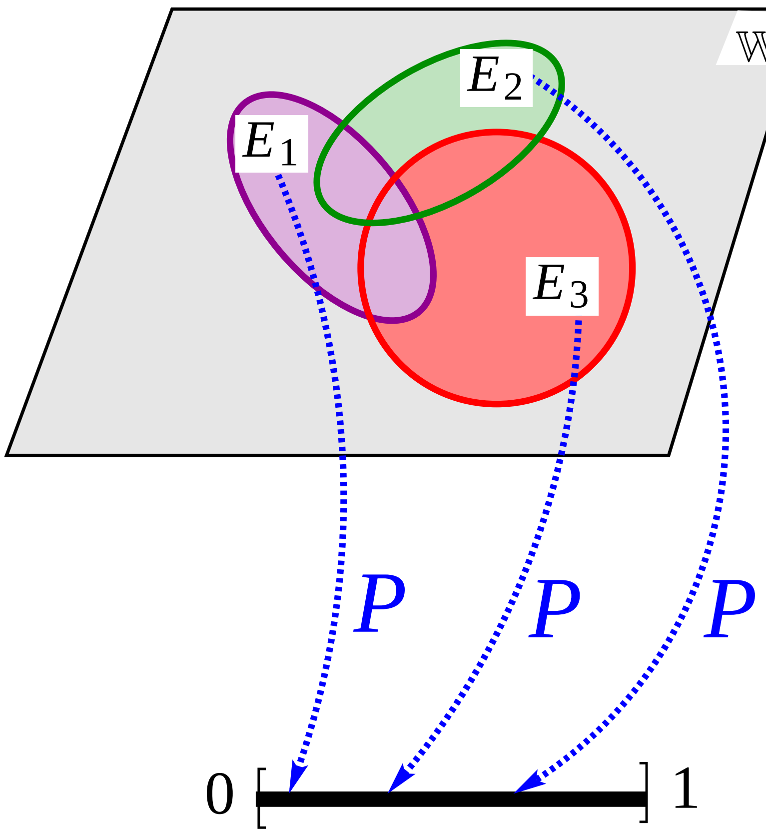
Probability that outcome $\in E$, $E \in \mathscr{E}$, is $P(E)$.

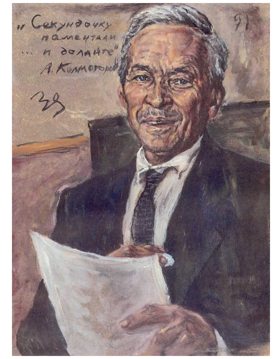
Model $\cong$ $\mathscr{E}$ and P;

$\mathscr{E}$ is an essential part.

$\mathscr{E}$ should not be taken for granted!

Events





A.N. Kolmogorov
1903 – 1987

$\mathscr{E}$ = the sets that are assigned a probability,

:= the class of ‘measurable’ subsets of $\mathbb{W}$.

Main (not all) axioms

The events $\mathscr{E}$ form a "σ-algebra" $:\Rightarrow$

- $$[\![E \in \mathscr{E}]\!] \Rightarrow [\![E^{\text{complement}} \in \mathscr{E}]\!]$$
- $$[\![E_1, E_2 \in \mathscr{E}]\!] \Rightarrow [\![E_1 \cap E_2 \in \mathscr{E},\;\; E_1 \cup E_2 \in \mathscr{E}]\!]$$

$P : \mathscr{E} \to [0,1]$ **is a probability measure** $:\Rightarrow$

- $P(\mathbb{W}) = 1$,
- P **is additive** $:\Leftrightarrow$

$$[\![E_1, E_2 \in \mathscr{E} \text{ and } E_1 \cap E_2 = \emptyset]\!]$$
$$\Rightarrow [\![P(E_1 \cup E_2) = P(E_1) + P(E_2)]\!].$$

Borel

Émile Borel
1871 – 1956

For expositions, both introductory and advanced, $\mathbb{W} = \mathbb{R}^n$ the events are often taken to consist of the *"Borel σ-algebra"*.

$\mathscr{E}$ then contains 'basically every' subset of $\mathbb{R}^n$.

Allows to take probability distributions and pdf's as the primitive concepts, and avoids modeling of $\mathscr{E}$.

<u>Thesis</u>

Borel is unduly restrictive
for system theoretic applications.

Borel

Émile Borel
1871 – 1956

For expositions, both introductory and advanced, $\mathbb{W} = \mathbb{R}^n$ the events are often taken to consist of the *"Borel σ-algebra"*.

$\mathscr{E}$ then contains 'basically every' subset of $\mathbb{R}^n$.

'Classical' stochastic system:
$\mathbb{W} = \mathbb{R}^n$, $\mathscr{E}$ = the Borel σ-algebra $\cong$ 'all' subsets of $\mathbb{R}^n$. P specified by a probability distribution or a pdf.

$\mathscr{E}$ is inherited from the topology of the outcome space, it does not involve the randomness.

Borel

Émile Borel
1871 – 1956

For expositions, both introductory and advanced, $\mathbb{W} = \mathbb{R}^n$ the events are often taken to consist of the *"Borel σ-algebra".*

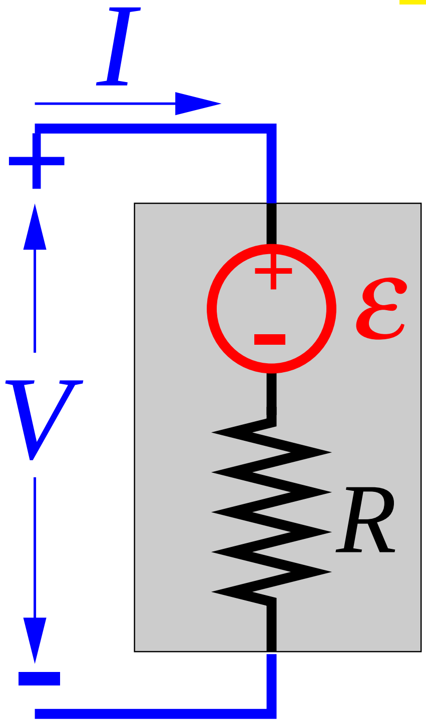
$\mathscr{E}$ then contains 'basically every' subset of $\mathbb{R}^n$.

Borel is usually assumed for many basic concepts, as

- **random variable, random vector,**
- **independence of random variables,**
- **marginal measure, conditioning,**
- **random process,**
- **Brownian motion, Markov process, etc.**

Examples

Noisy (or 'hot') resistor



$$\boxed{V = RI + \varepsilon}$$

ε gaussian

zero mean

variance $\sim RT$

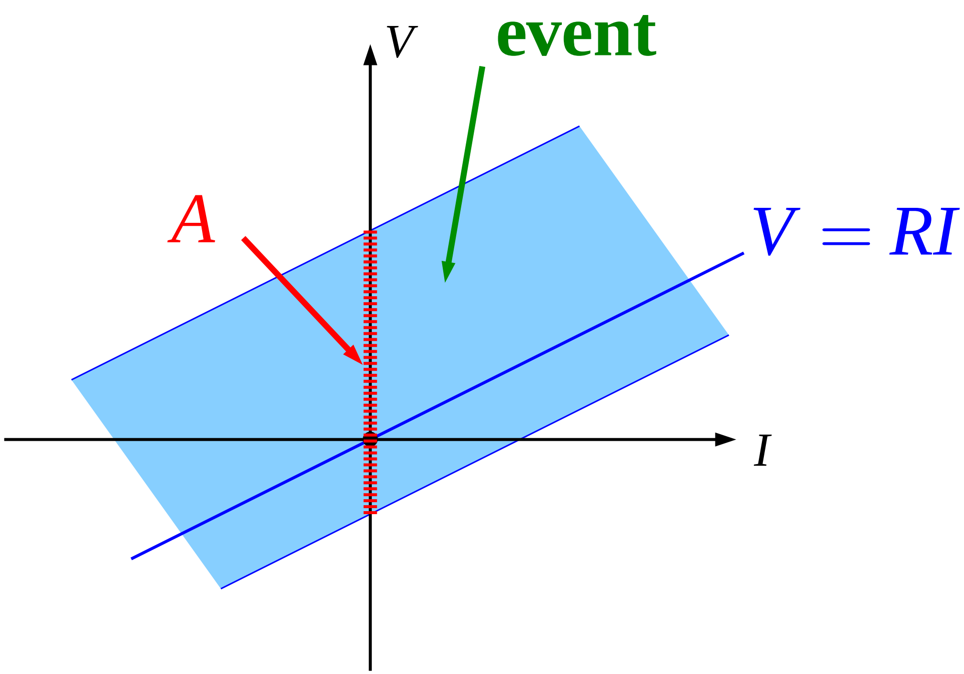
'Johnson-Nyquist resistor'

What is $\begin{bmatrix} V \\ I \end{bmatrix}$ as a mathematical entity?

Noisy resistor

Outcomes $\left[\begin{smallmatrix} V \\ I \end{smallmatrix}\right]$, $\mathbb{W} = \mathbb{R}^2$; **events: subsets of** $\mathbb{R}^2$ **as**

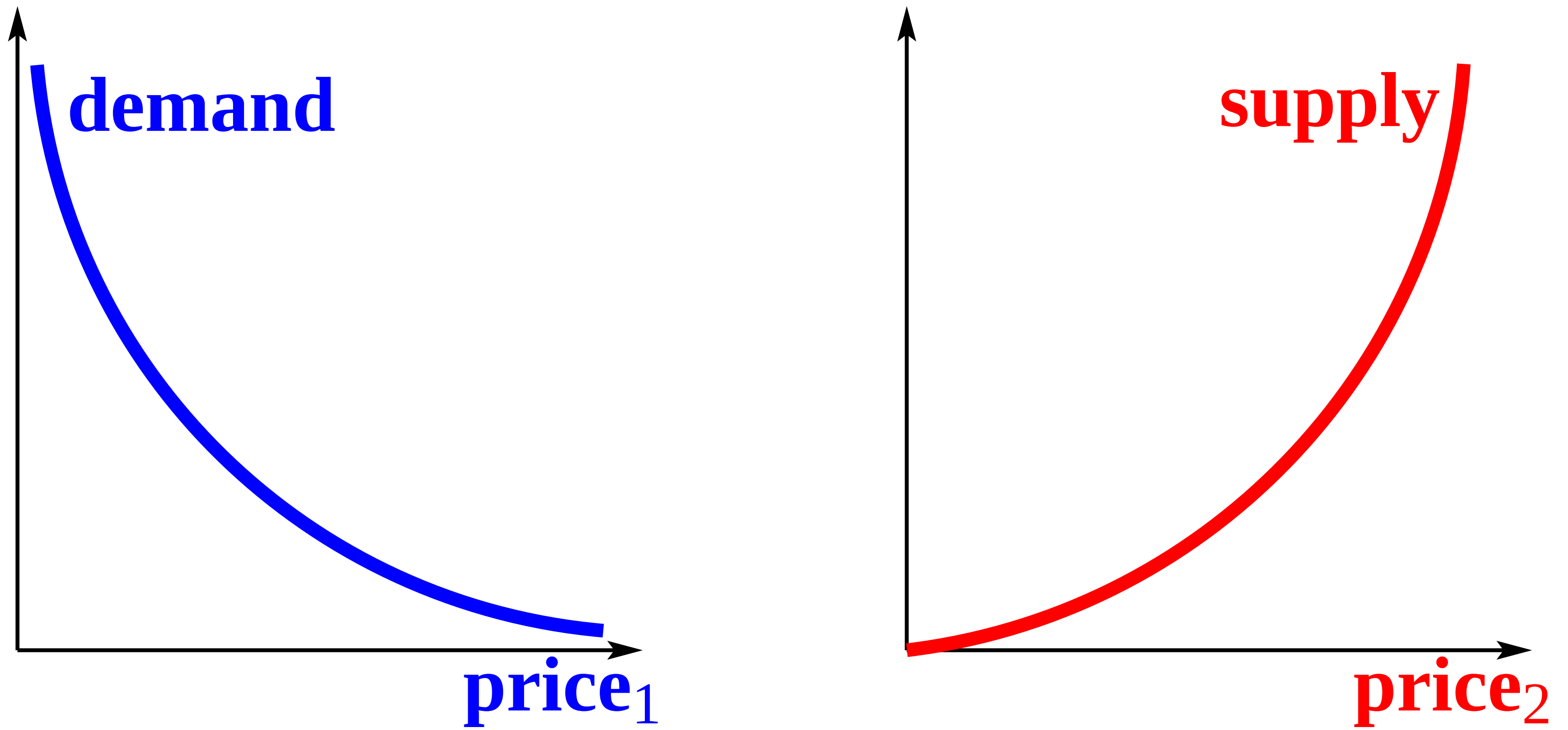
$\left\{ \left[\begin{smallmatrix} V \\ I \end{smallmatrix}\right] \in \mathbb{R}^2 \mid V - RI \in A\right.$ **with** A **a Borel subset of** $\left.\mathbb{R}\right\}$.



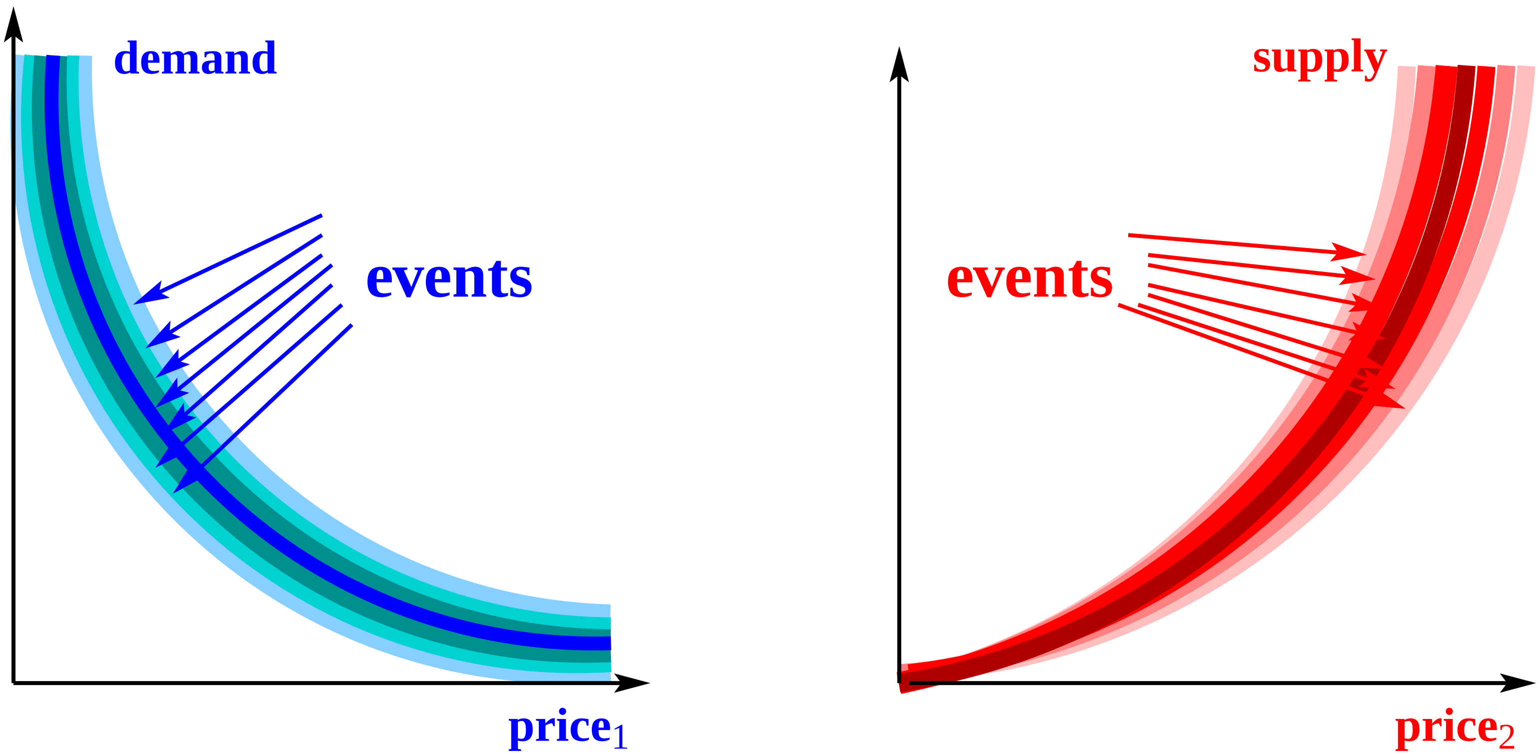
$P(\textbf{event}) =$ **gaussian measure of** $\varepsilon \in A$.

Neither $\left[\begin{smallmatrix} V \\ I \end{smallmatrix}\right]$, **nor** I, **nor** V **possess a pdf.**

Deterministic price/demand/supply



Stochastic price/demand/supply



$\mathcal{E}, \mathcal{E}'$ **= the regions that are assigned a probability.**

p_1, p_2, d, s **are not classical real random variables.**

Linearity

Linear stochastic system

linear stochastic system

$:\Leftrightarrow$ **Borel probability on** $\mathbb{R}^{\mathtt{n}}/\mathbb{L}$**,**

with $\mathbb{L} \subseteq \mathbb{R}^{\mathtt{n}}$ **a linear subspace, the ‘fiber’.**

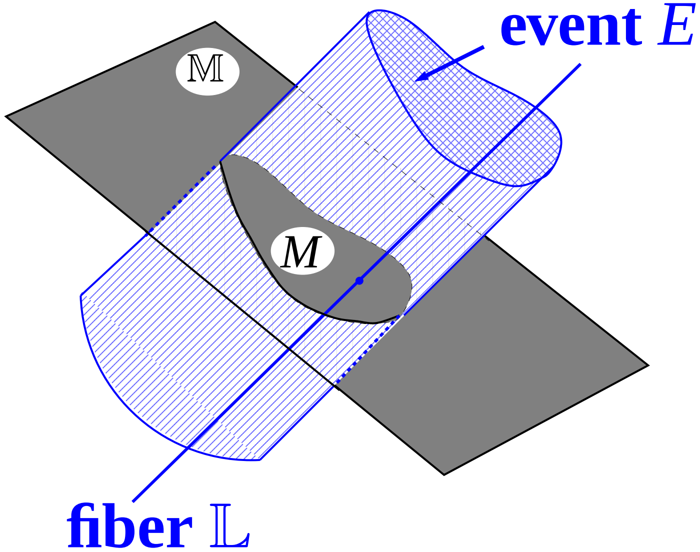
$\mathbb{R}^{\mathtt{n}}/\mathbb{L}$ **real vector space of dimension** $\mathtt{n} - \mathtt{dim}(\mathbb{L})$**.**

Events: **cylinders with sides parallel to** $\mathbb{L}$**.**

Subsets of $\mathbb{R}^{\mathtt{n}}$ **as** $A + \mathbb{L}, \;\; A \subseteq \mathbb{R}^{\mathtt{n}}$ **Borel.**

Linearity

linear stochastic system :$\Leftrightarrow$



$\mathbb{L} \oplus \mathbb{M} = \mathbb{R}^n, \mathbb{M} \cong \mathbb{R}^n/\mathbb{L}$.

Borel probability on $\mathbb{M}$.

Example: the noisy resistor.

Classical $\Rightarrow$ linear!

gaussian :$\Leftrightarrow$ **linear, probability on $\mathbb{R}^n/\mathbb{L}$ gaussian.**

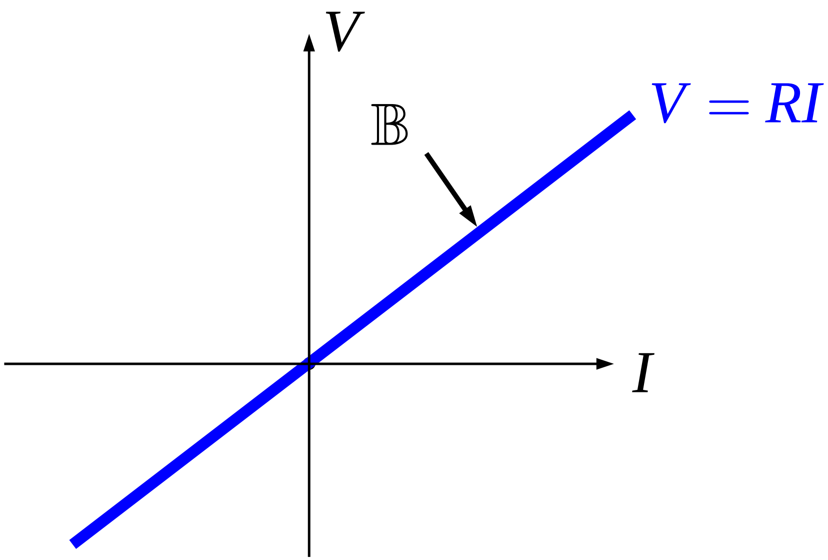
Deterministic system

$(\mathbb{W}, \mathscr{E}, P)$ **is said to be** ***deterministic*** **if**

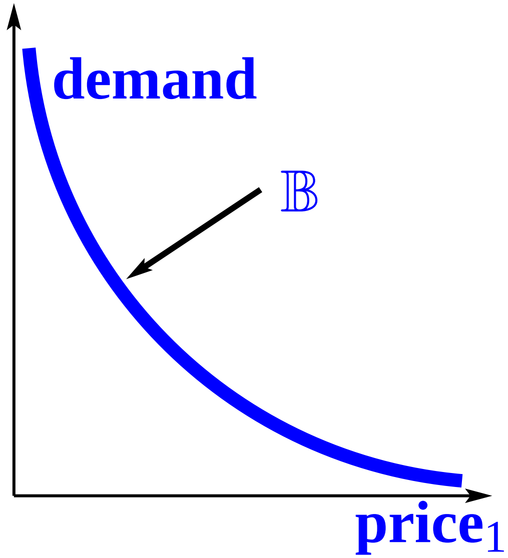
$$\mathscr{E} = \{\emptyset, \mathbb{B}, \mathbb{B}^{complement}, \mathbb{W}\} \textbf{ and } P(\mathbb{B}) = 1.$$

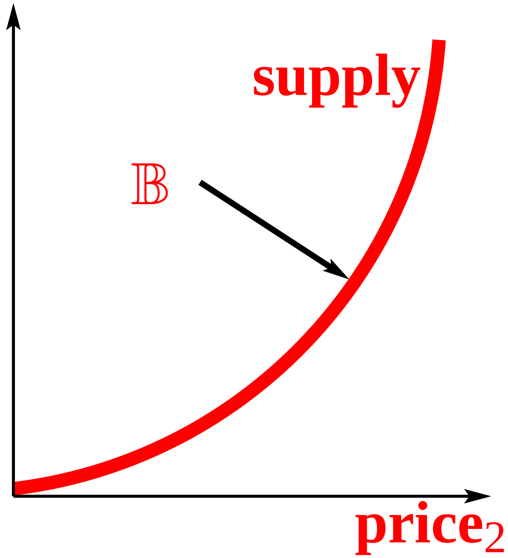
Deterministic examples

Ohmic resistor:



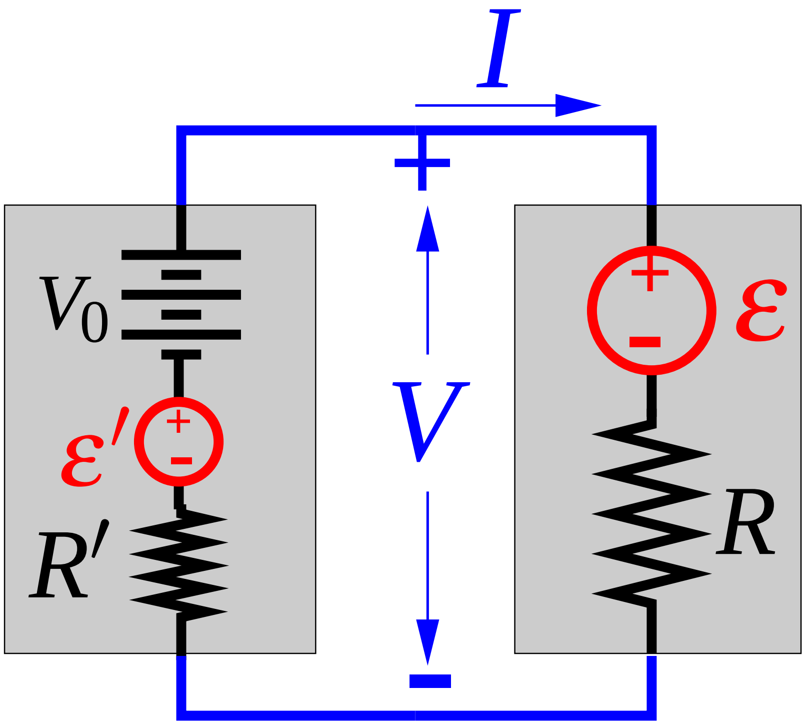
Economic example:





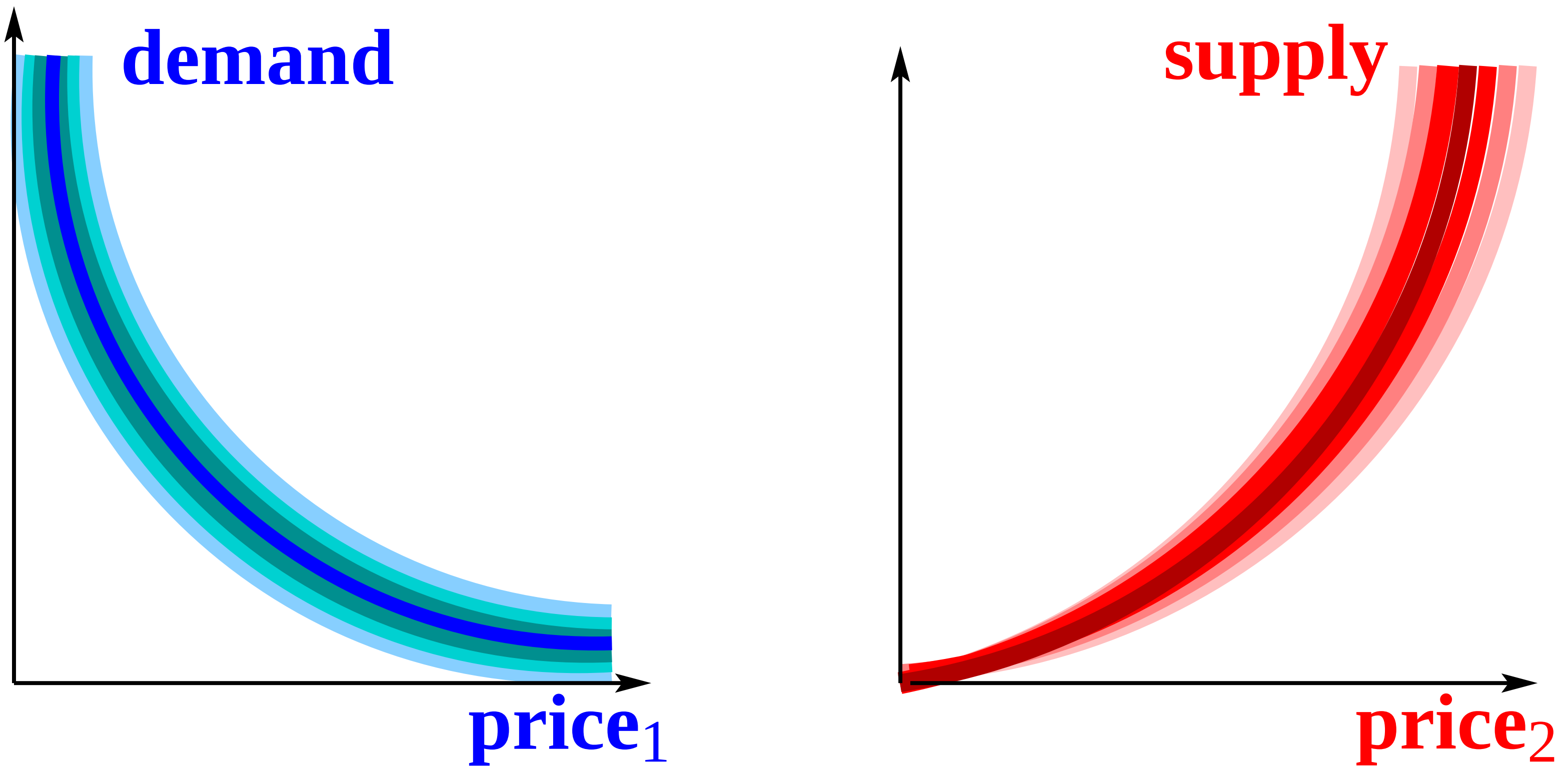
Interconnection

Noisy resistor terminated by a voltage source



How do we deal with interconnection?

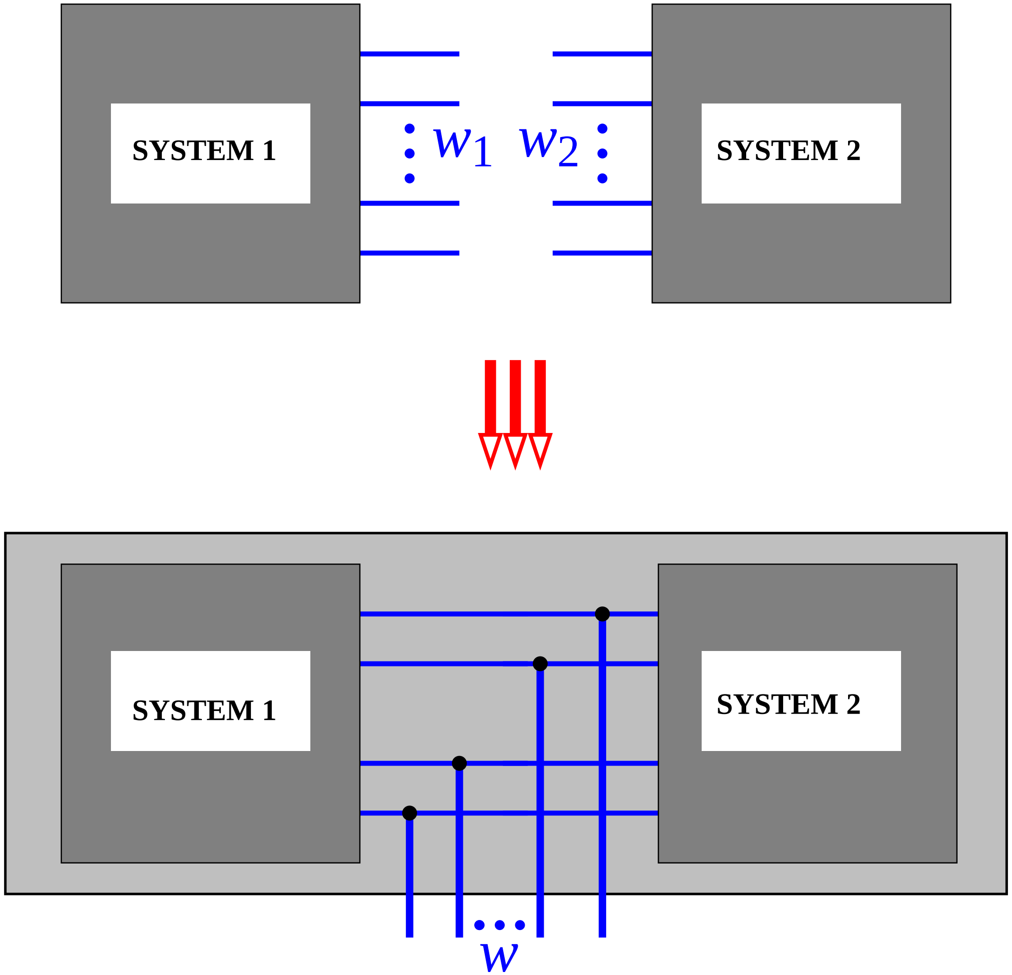
Stochastic price/demand/supply



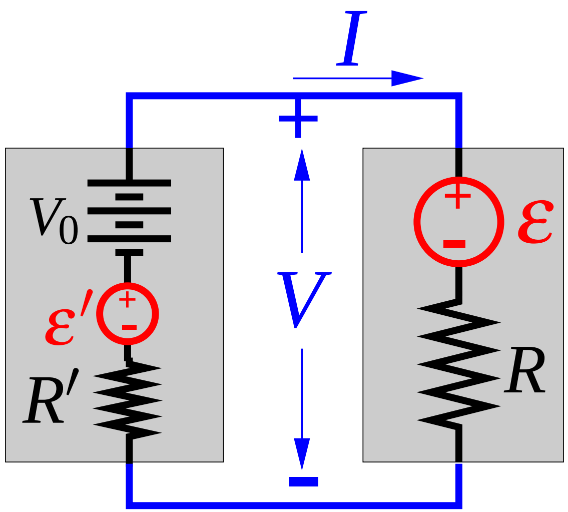
How do we deal with equilibrium?

Equilibrium: price$_1$ = price$_2$, supply = demand.

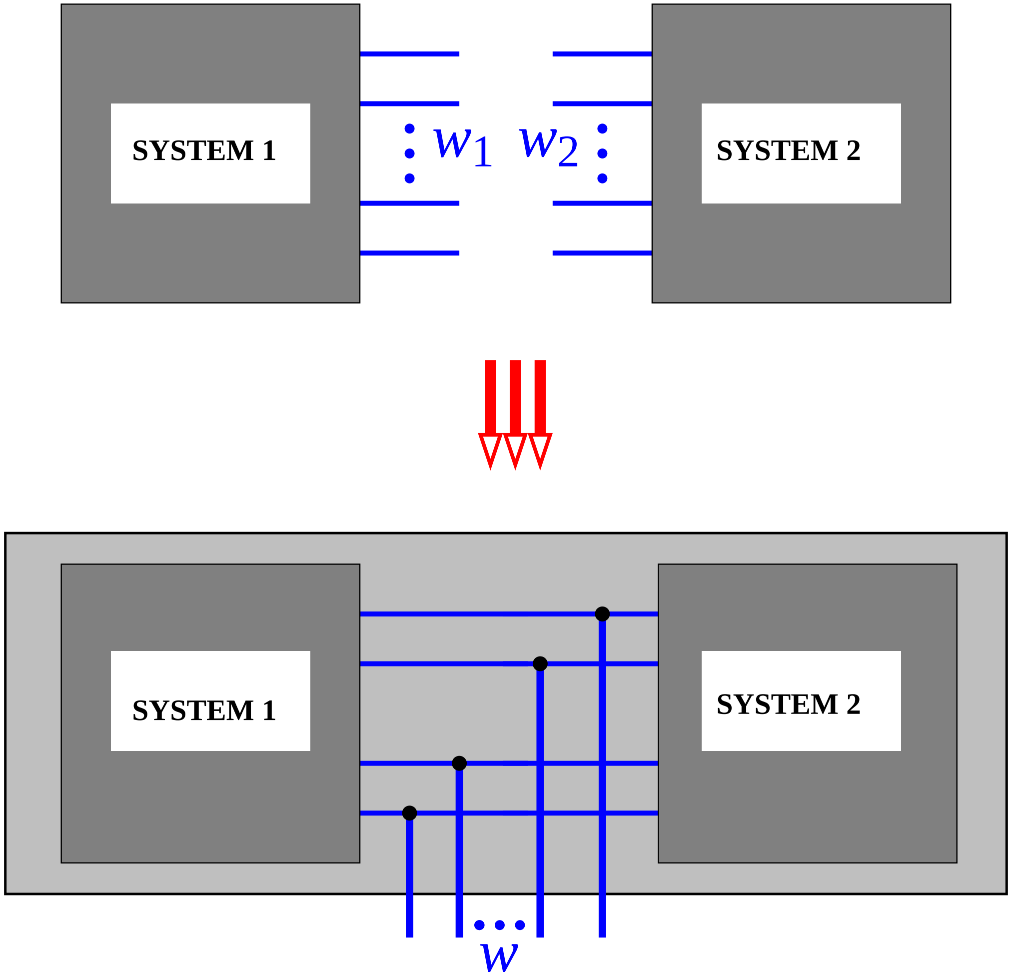
Interconnection



Example:



Interconnection

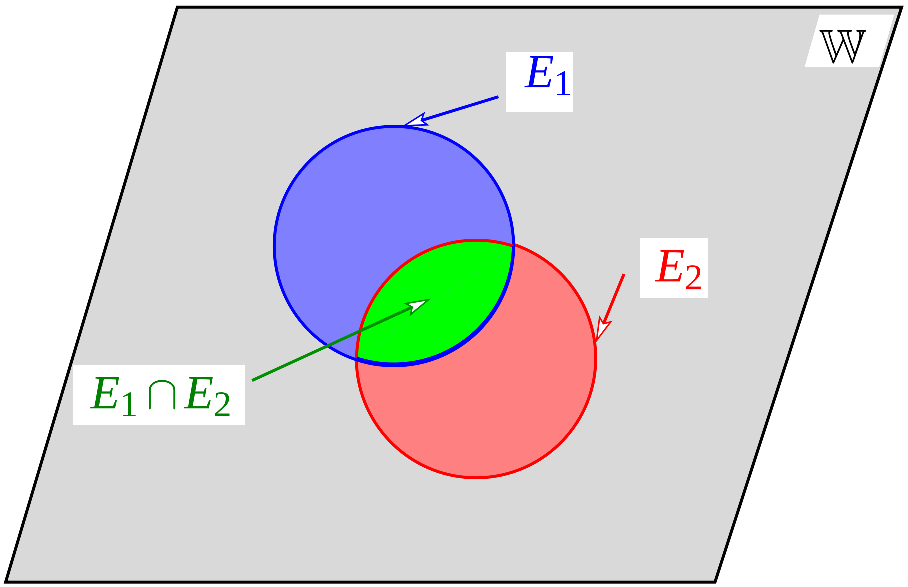


Can two distinct probabilistic laws be imposed on the same set of variables?

Complementarity of σ-algebras

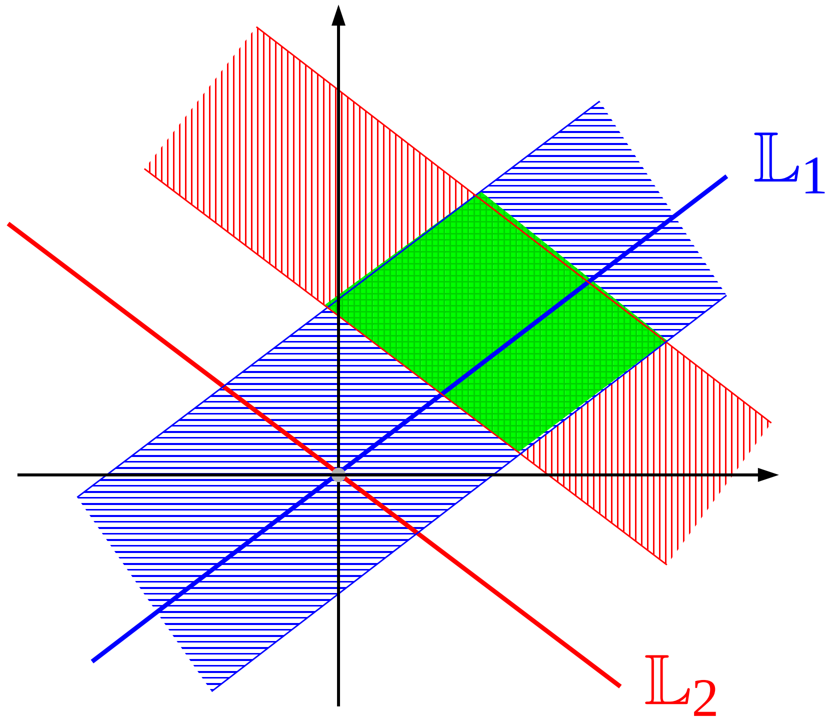
$\mathscr{E}_1$ **and** $\mathscr{E}_2$ **are** ***complementary σ-algebras*** **:⇔**
for all nonempty sets $E_1, E_1' \in \mathscr{E}_1, E_2, E_2' \in \mathscr{E}_2$

$$[\![E_1 \cap E_2 = E_1' \cap E_2']\!] \Rightarrow [\![E_1 = E_1' \textbf{ and } E_2 = E_2']\!].$$



The intersection determines the intersectants.

Linear example



complementarity $\Leftrightarrow$ $\boxed{\mathbb{L}_1 + \mathbb{L}_2 = \mathbb{R}^n}$

Interconnection of complementary systems

Let $(\mathbb{W}, \mathscr{E}_1, P_1)$ **and** $(\mathbb{W}, \mathscr{E}_2, P_2)$ **be stochastic systems (independent). Assume complementarity.**

Their ***interconnection*** **is defined as**

$$(\mathbb{W}, \mathscr{E}, P)$$

with $\mathscr{E} :=$ **the** σ**-algebra generated by 'rectangles'**

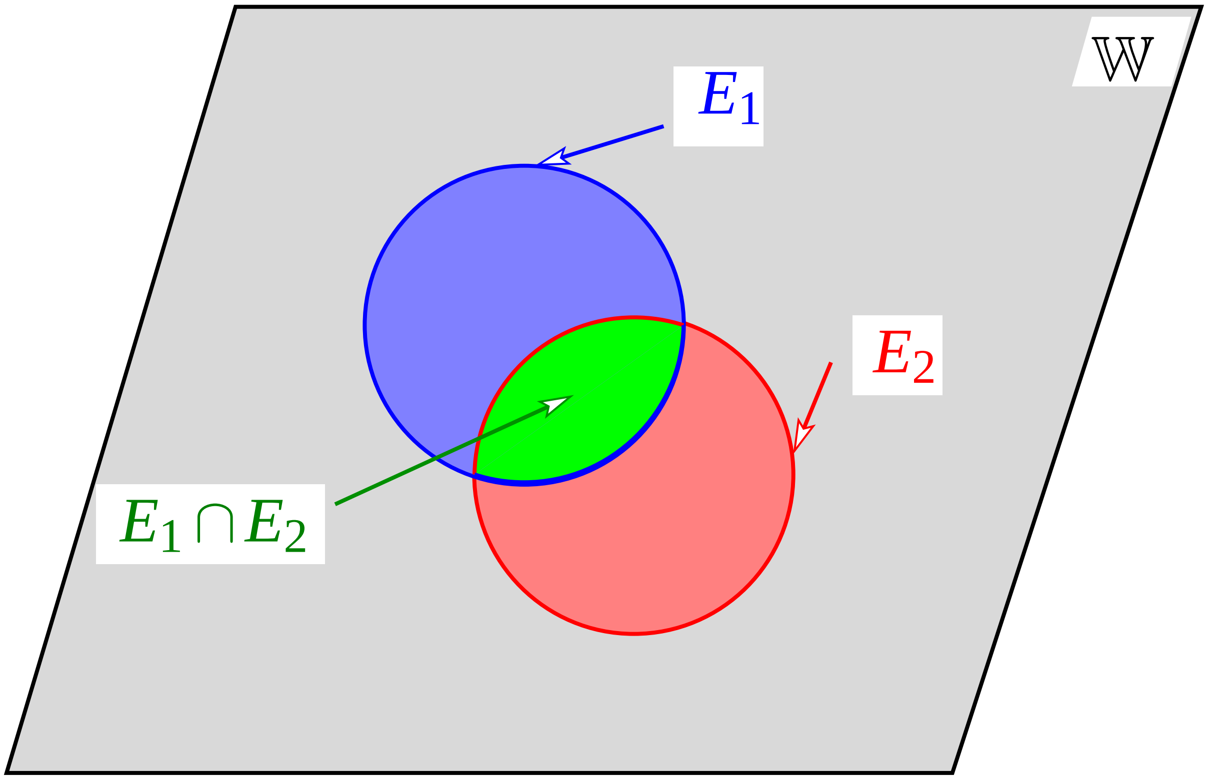
$$\{E_1 \cap E_2 \mid E_1 \in \mathscr{E}_1, E_2 \in \mathscr{E}_2\},$$

and P **defined through the rectangles by**

$$P(E_1 \cap E_2) := P_1(E_1) P_2(E_2).$$

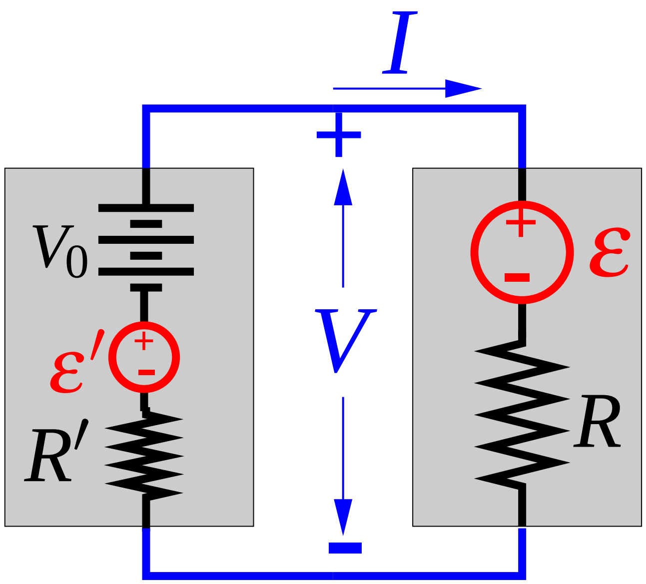
for $E_1 \in \mathscr{E}_1, E_2 \in \mathscr{E}_2$**.**

Interconnection of complementary systems



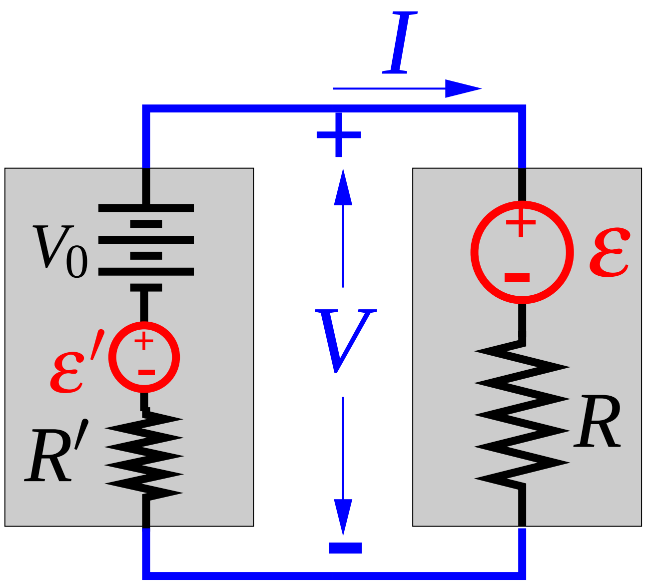
$$P(E_1 \cap E_2) := P_1(E_1)P_2(E_2).$$

Noisy resistor terminated by a voltage source



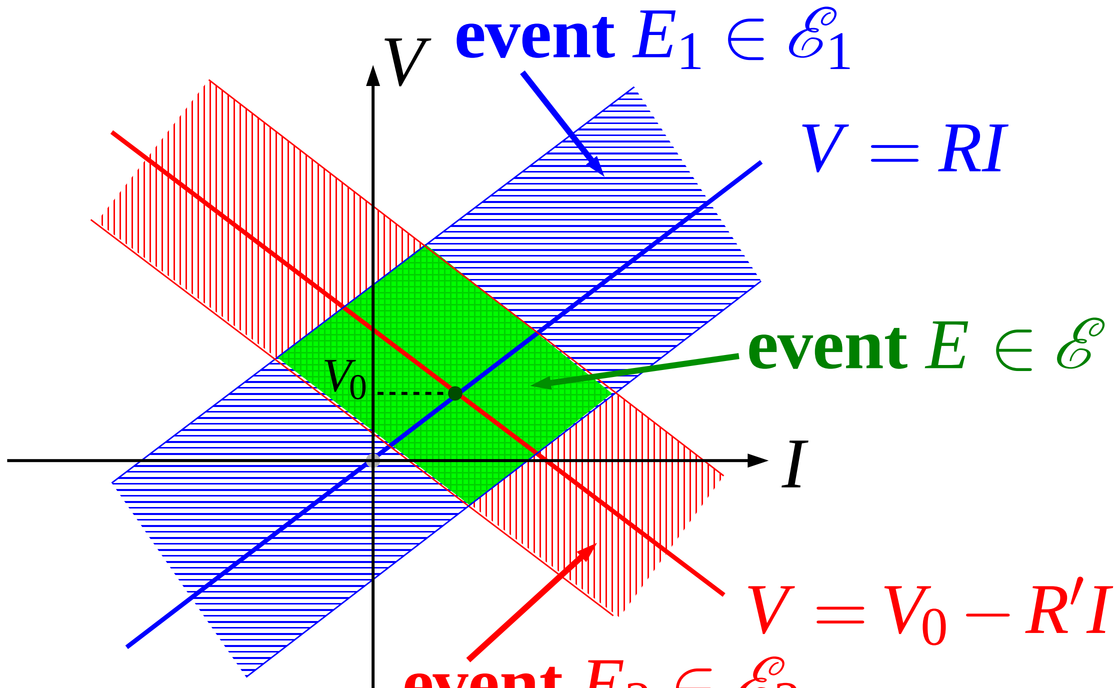
Probability of $\begin{bmatrix} V \\ I \end{bmatrix}$**?**

Noisy resistor terminated by a voltage source

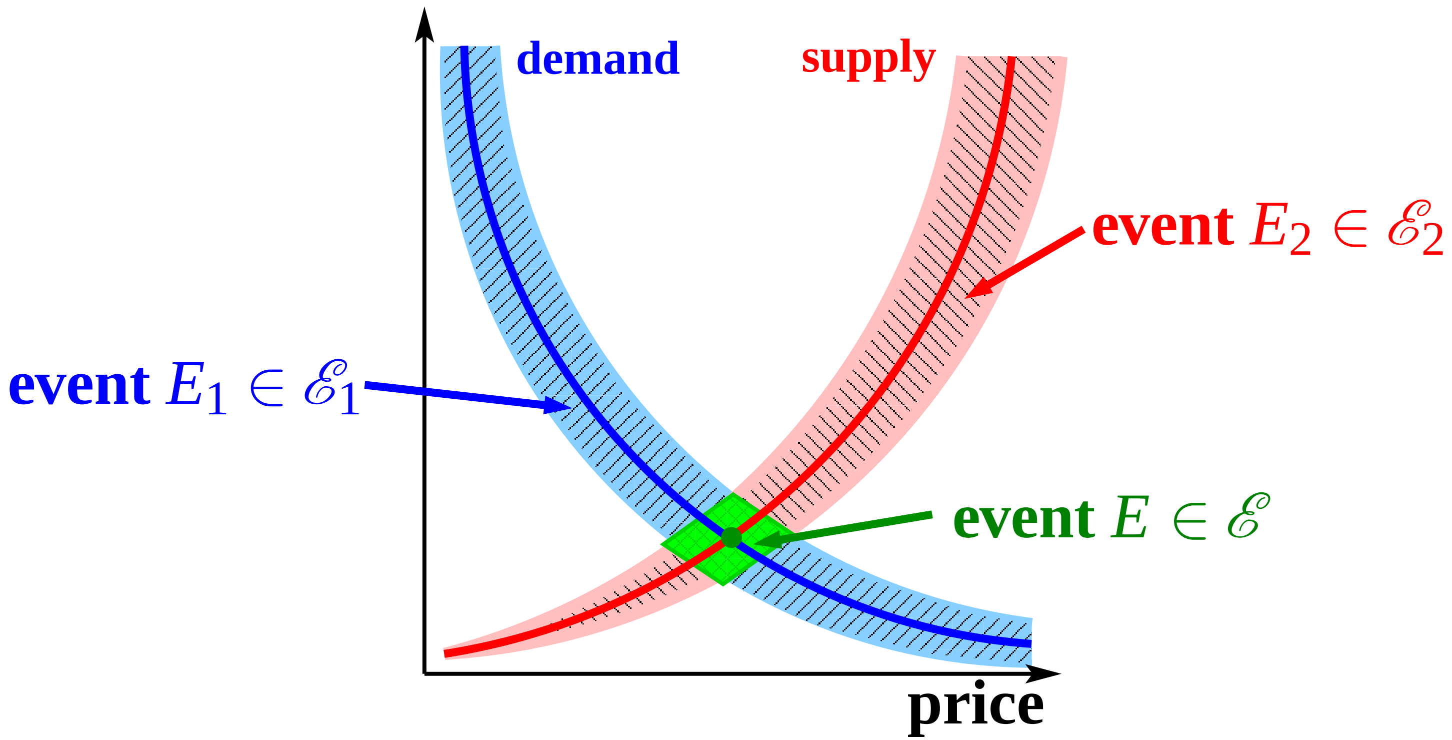


Probability of $\begin{bmatrix} V \\ I \end{bmatrix}$?

$$P(E) = P_1(E_1)P_2(E_2)$$



Equilibrium price/demand/supply



$$P(E) = P_1(E_1)P_2(E_2).$$

Constrained probability

Impose $w \in \mathbb{S}$

Constrained probability

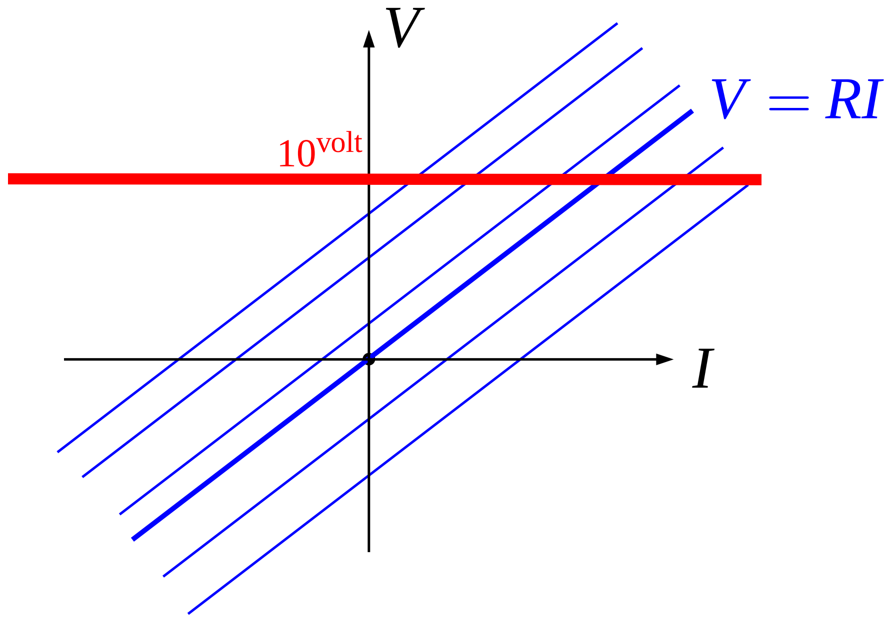
Let $\Sigma = (\mathbb{W}, \mathscr{E}, P)$**.**

Impose the constraint $\boxed{w \in \mathbb{S}}$ **with** $\mathbb{S} \subset \mathbb{W}$.

What is the stochastic nature of the outcomes in $\mathbb{S}$ ***?***

Is this a meaningful question?

Noisy resistor

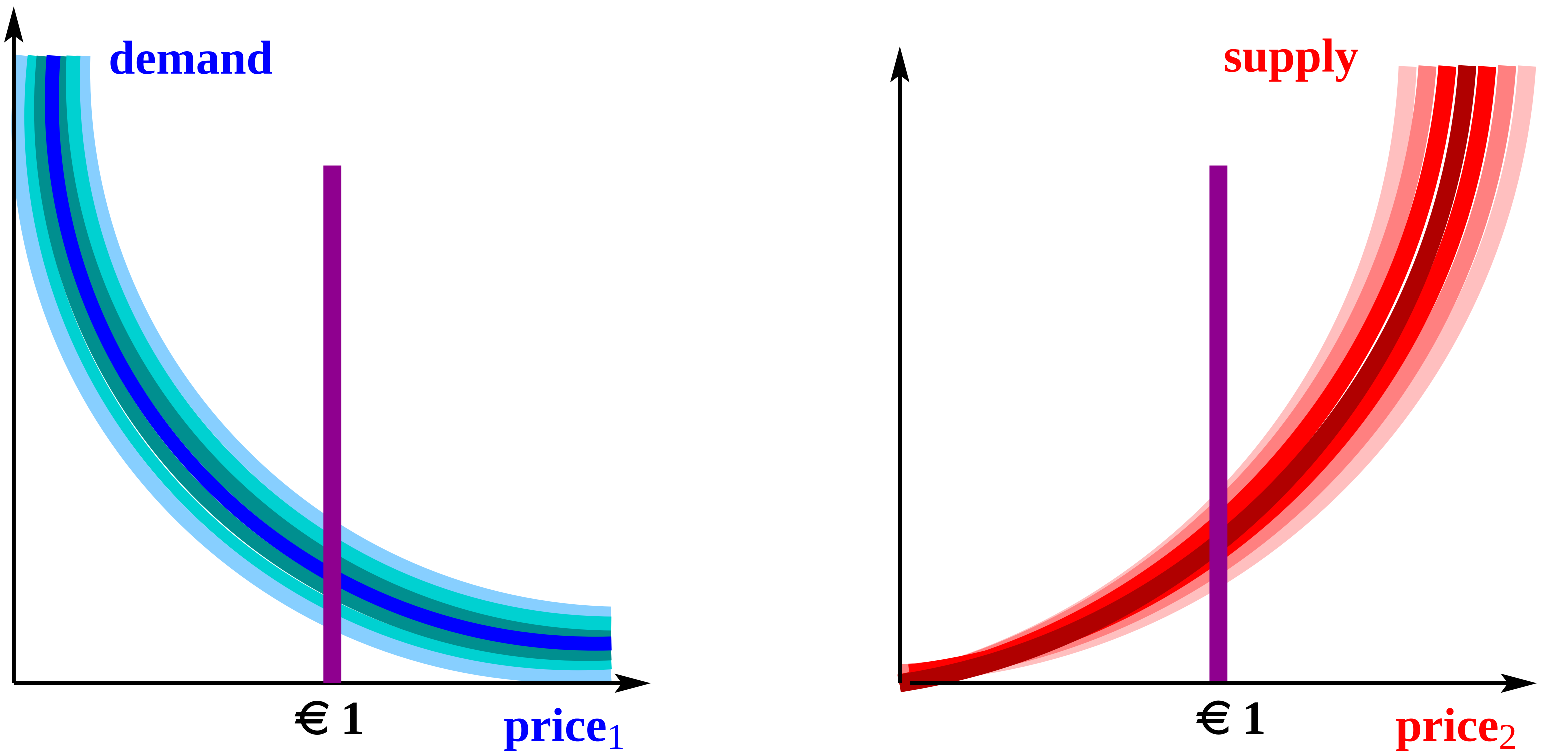


Impose $V = 10^{\text{volt}}$. What is the distribution of I?

$$V = RI + \varepsilon, V = 10^{\text{volt}} \quad \Rightarrow \quad I = \frac{V_0}{10} - \frac{\varepsilon}{10}.$$

I is a well-defined random variable!

Price/demand/supply example



Impose price = € 1. Probability of demand, supply?

Constrained probability

Let $\Sigma = (\mathbb{W}, \mathscr{E}, P)$**.**

Impose the constraint $\boxed{w \in \mathbb{S}}$ **with** $\mathbb{S} \subset \mathbb{W}$**.**

What is the stochastic nature of the outcomes in $\mathbb{S}$ ***?***

Is this a meaningful question? ***Yes, it is!***

Constrained probability

Constraining $\simeq$ **interconnection of** $\Sigma = (\mathbb{W}, \mathscr{E}, P)$ **and the deterministic system with behavior** $\mathbb{S}$**.**

Require complementarity:

$$[\![E_1, E_2 \in \mathscr{E} \text{ and } E_1 \cap \mathbb{S} = E_2 \cap \mathbb{S}]\!] \Rightarrow [\![E_1 = E_2]\!]$$

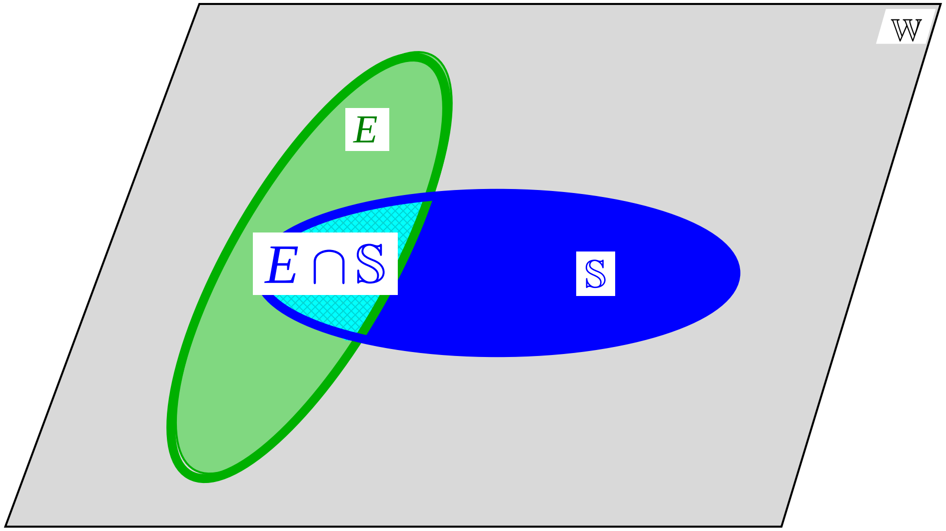
Note: complementarity implies $\boxed{\mathbb{S} \notin \mathscr{E}!}$

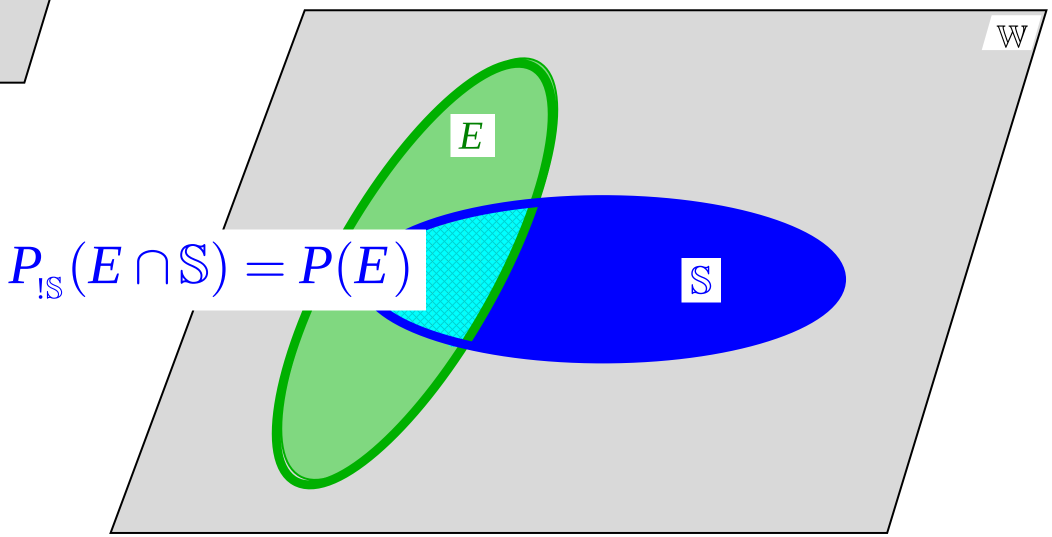
Interconnection $\rightsquigarrow$

$$\Sigma_{!\mathbb{S}} = (\mathbb{S}, \mathscr{E} \cap \mathbb{S}, P_{!\mathbb{S}}) \quad \textbf{with} \quad P_{!\mathbb{S}}(E \cap \mathbb{S}) := P(E).$$

$P_{!\mathbb{S}}$ **= "probability of** w **constrained by** $w \in \mathbb{S}$**".**

Constrained probability





Open stochastic systems

Open versus closed

Consider $\Sigma_1 = (\mathbb{R}^n, \mathscr{E}_1, P_1)$**.**

If $\mathscr{E}_1$ **= the Borel** σ**-algebra, then** Σ_1 **is interconnectable only with the free system** $(\mathbb{R}^n, \mathscr{E}_2, P_2)$**,** $\mathscr{E}_2 = \{\emptyset, \mathbb{R}^n\}$**.**

$\Rightarrow$ **classical** Σ_1 **= 'closed' system.**

It don't mean a thing, if it ain't interconnecting!

Open versus closed

Consider $\Sigma_1 = (\mathbb{R}^{\mathtt{n}}, \mathscr{E}_1, P_1)$**.**

If $\mathscr{E}_1$ **= the Borel** σ**-algebra, then** Σ_1 **is interconnectable only with the free system** $(\mathbb{R}^{\mathtt{n}}, \mathscr{E}_2, P_2)$**,** $\mathscr{E}_2 = \{\emptyset, \mathbb{R}^{\mathtt{n}}\}$**.**

$\Rightarrow$ **classical** Σ_1 **= 'closed' system.**

Coarse $\mathscr{E}_1$

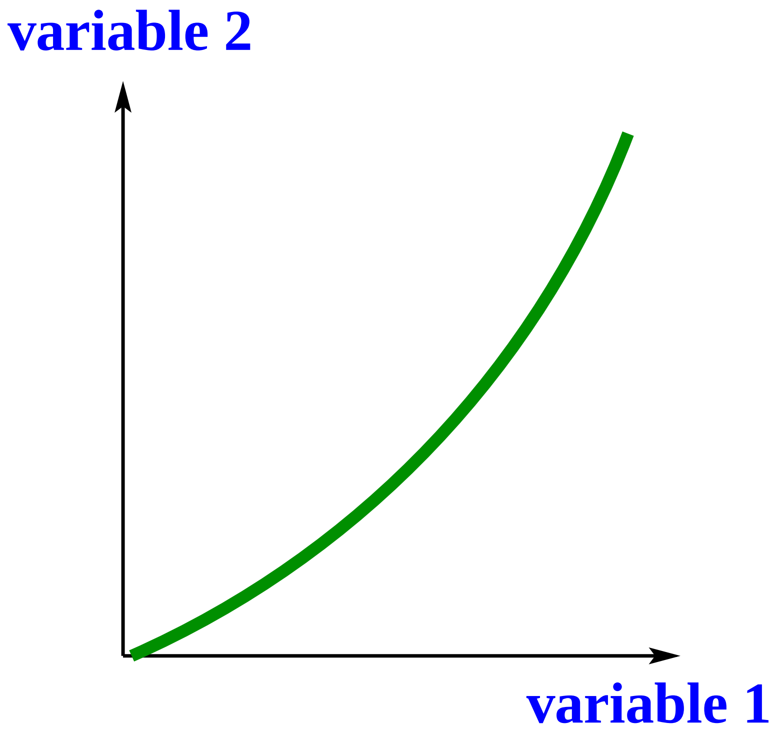
$\Rightarrow$ Σ_1 **is interconnectable.**

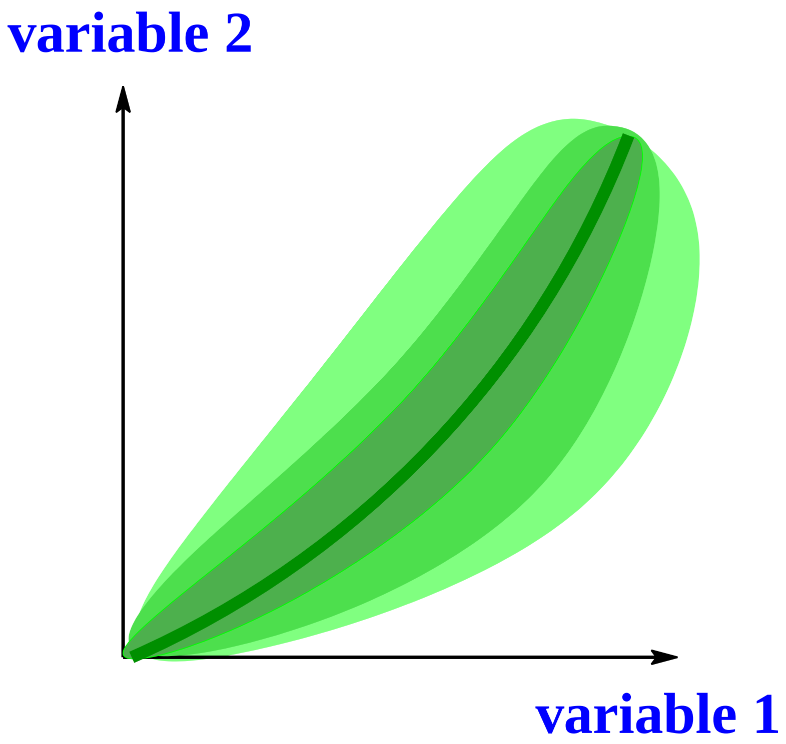
$\Rightarrow$ **'open' stochastic system.**

Conclusions

Stochastic systems

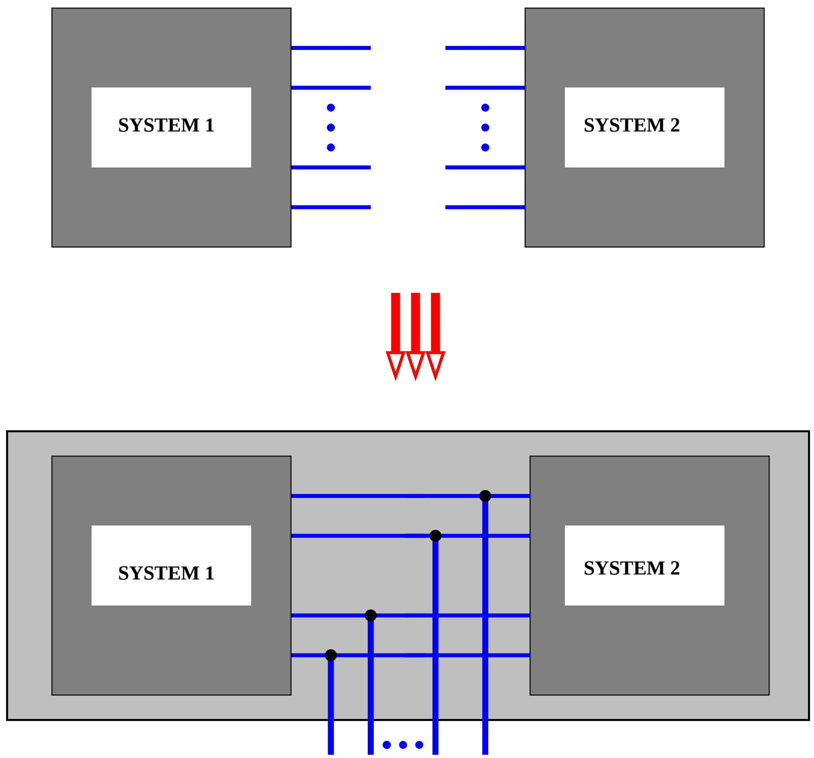
- The Borel σ-algebra is inadequate even for elementary applications.





Stochastic systems

- The Borel σ-algebra is inadequate even for elementary applications.
- Complementary stochastic systems can be interconnected: two distinct laws imposed on one set of variables.



Stochastic systems

- **The Borel σ-algebra is inadequate even for elementary applications.**
- **Complementary stochastic systems can be interconnected: two distinct laws imposed on one set of variables.**
- **Open stochastic systems require a coarse σ-algebra. Classical random vectors imply closed systems.**

Stochastic systems

- **The Borel σ-algebra is inadequate even for elementary applications.**
- **Complementary stochastic systems can be interconnected: two distinct laws imposed on one set of variables.**
- **Open stochastic systems require a coarse σ-algebra. Classical random vectors imply closed systems.**
- $\leadsto$ **Notion of 'constrained probability'.**

Thank you

Thank you

Thank you

Thank you

Thank you

Thank you

Thank you

Thank you