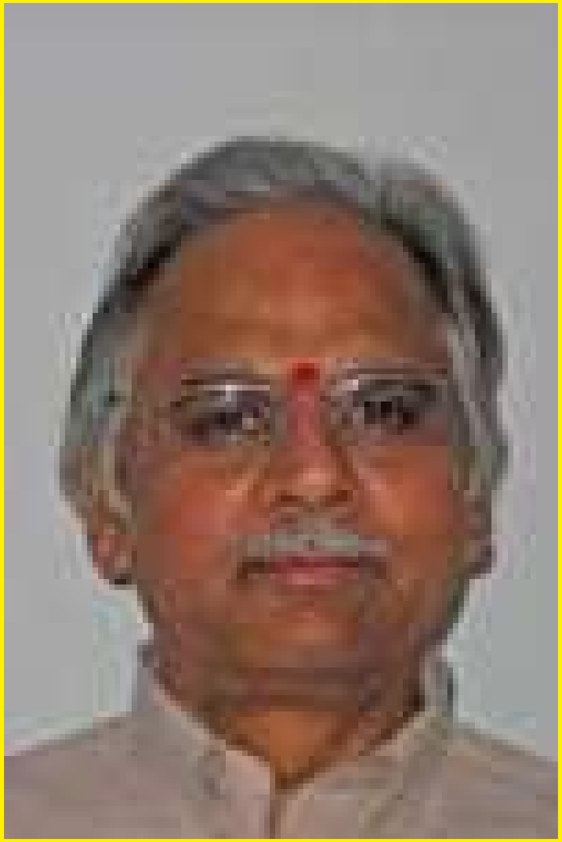
On the occasion of Sagar's 60-th



PARAMETRIZATION

of

STABILIZING CONTROLLERS

Jan Willems, K.U. Leuven, Flanders, Belgium

&

Yataka Yamamoto, Kyoto University, Japan

SagarFest, Hyderabad, India **January 7, 2008**


BELGIUM
NETHERLANDS
GERMANY
FRANCE
LUXEMBOURG
Zeebrugge
Knokke
Blankenberge
Ostend
Bruges
Veurne
Diksmuide
Ghent
Ypres
Kortrijk
Mouscron
Tournai
Ath
Antwerp
Turnhout
Herentals
Lier
Mechelen
Aarschot
Hasselt
Brussels
Leuven
Tienen
Tongeren
Liège
Verviers
Nivelles
Namur
Charleroi
Hautes Fagnes NP
Chimay
The Ardennes
Bastogne
Bouillon
Arlon
0 40 km
0 24 miles


BELGIUM
NETHERLANDS
GERMANY
FRANCE
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Zeebrugge
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Ostend
Bruges
Turnhout
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Herentals
Veurne
Diksmuide
Ghent
Lier
Mechelen
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Hasselt
Ypres
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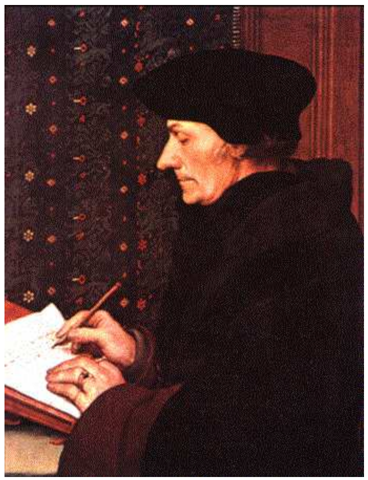
Verviers
Charleroi
Namur
Hautes Fagnes NP
Chimay
The Ardennes
Bastogne
Bouillon
Arlon
0 40 km
0 24 miles

Adrianus VI
1459–1523



Erasmus
1469–1536

de la Valleé Poussin
1866–1962

Lemaître
1894–1966

Reminders

Linear time-invariant differential systems

Rational symbols

LTIDS are defined in terms of polynomial symbols

$$R\left(\frac{d}{dt}\right)w = 0 \qquad R \in \mathbb{R}[\xi]^{\bullet\times\mathtt{w}}$$

(behavior $\mathscr{B}$:= the $\mathscr{C}^{\infty}(\mathbb{R},\mathbb{R}^{\mathtt{w}})$ solutions)

Rational symbols

LTIDS are defined in terms of polynomial symbols

$$R\left(\frac{d}{dt}\right)w = 0 \qquad R \in \mathbb{R}[\xi]^{\bullet\times\mathtt{w}}$$

(behavior $\mathscr{B}$:= the $\mathscr{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$ solutions) but can also be represented by rational symbols

$$G\left(\frac{d}{dt}\right)w = 0 \qquad G \in \mathbb{R}(\xi)^{\bullet\times\mathtt{w}}$$

Rational symbols

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(behavior $\mathscr{B}$:= the $\mathscr{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}})$ solutions) but can also be represented by rational symbols

$$G\left(\frac{d}{dt}\right)w = 0 \qquad G \in \mathbb{R}(\xi)^{\bullet\times\mathtt{w}}$$

Behavior := the set of solutions of

$$Q\left(\frac{d}{dt}\right)w = 0 \qquad Q \in \mathbb{R}[\xi]^{\bullet\times\mathtt{w}}$$

where $G = P^{-1}Q, \quad P,Q \in \mathbb{R}[\xi]^{\bullet\times\bullet}, \quad P$ **and** Q **left coprime**

Rational symbols

LTIDS are defined in terms of polynomial symbols

$$R\left(\frac{d}{dt}\right)w = 0 \qquad R \in \mathbb{R}[\xi]^{\bullet \times \mathtt{w}}$$

(behavior $\mathscr{B}$:= the $\mathscr{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$ solutions) but can also be represented by rational symbols

$$G\left(\frac{d}{dt}\right)w = 0 \qquad G \in \mathbb{R}(\xi)^{\bullet \times \mathtt{w}}$$

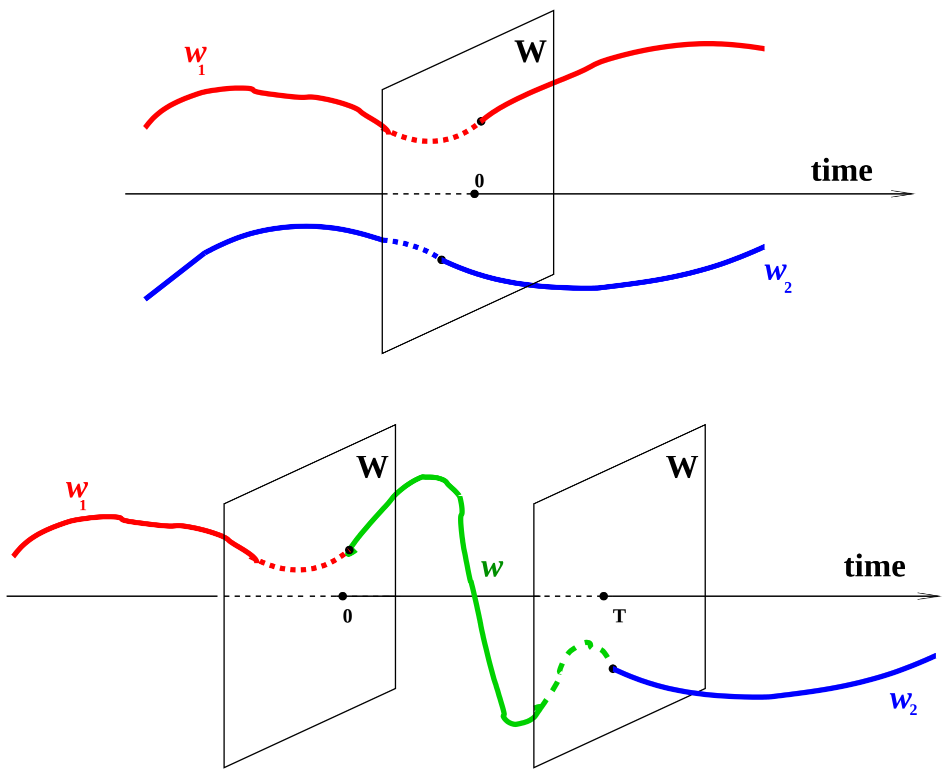
This added flexibility $\rightsquigarrow$ better results in certain problems, e.g. parametrization of the set of stabilizing controllers

Controllability c.s.

Controllability and stabilizability

$\mathscr{B}$ **is said to be controllable** $:\Leftrightarrow$

$\forall\, w_1, w_2 \in \mathscr{B},\ \exists\, T \geq 0$ **and** $w \in \mathscr{B}$ **such that ...**

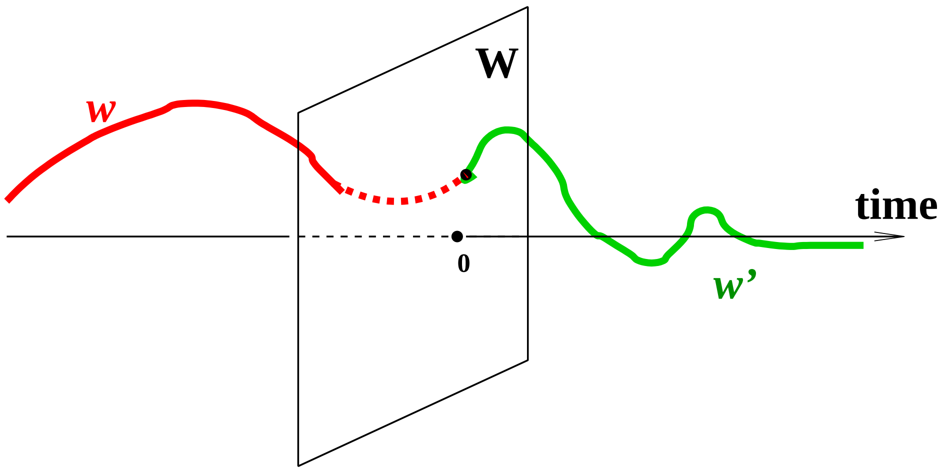


Controllability and stabilizability

$\mathscr{B}$ **is said to be controllable** $:\Leftrightarrow$

$\mathscr{B}$ **is said to be stabilizable** $:\Leftrightarrow$

$\forall\, w \in \mathscr{B},\ \exists\, w' \in \mathscr{B}$ **such that ...**



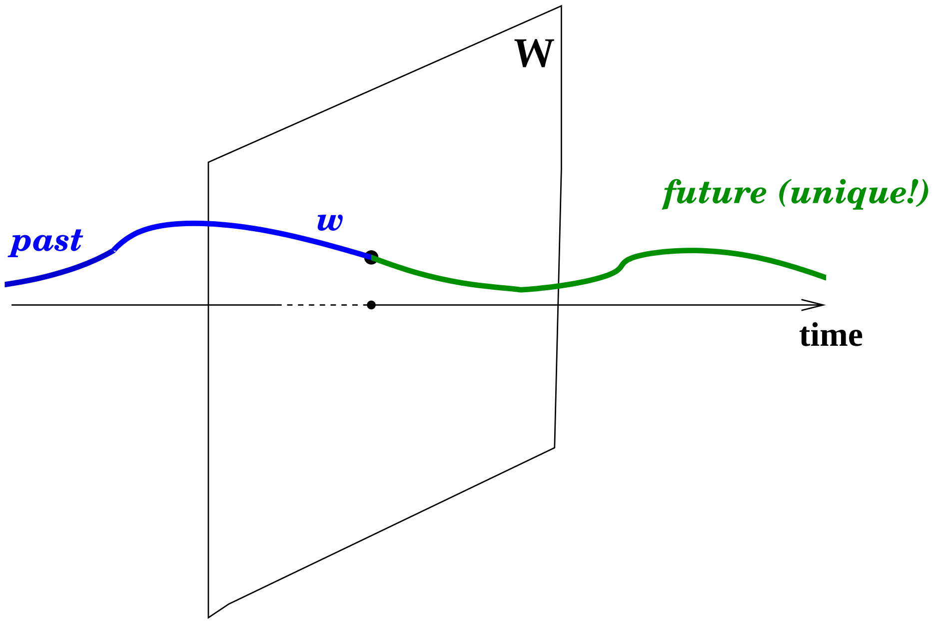
Controllability and stabilizability

$\mathscr{B}$ **is said to be controllable** $:\Leftrightarrow$

$\mathscr{B}$ **is said to be stabilizable** $:\Leftrightarrow$

$\mathscr{B}$ **is said to be autonomous** $:\Leftrightarrow$

$$\forall\, w_- \in \mathscr{B}_-,\ \exists\ (!)\ w_+ \in \mathscr{B}_+ \text{ such that ...}$$



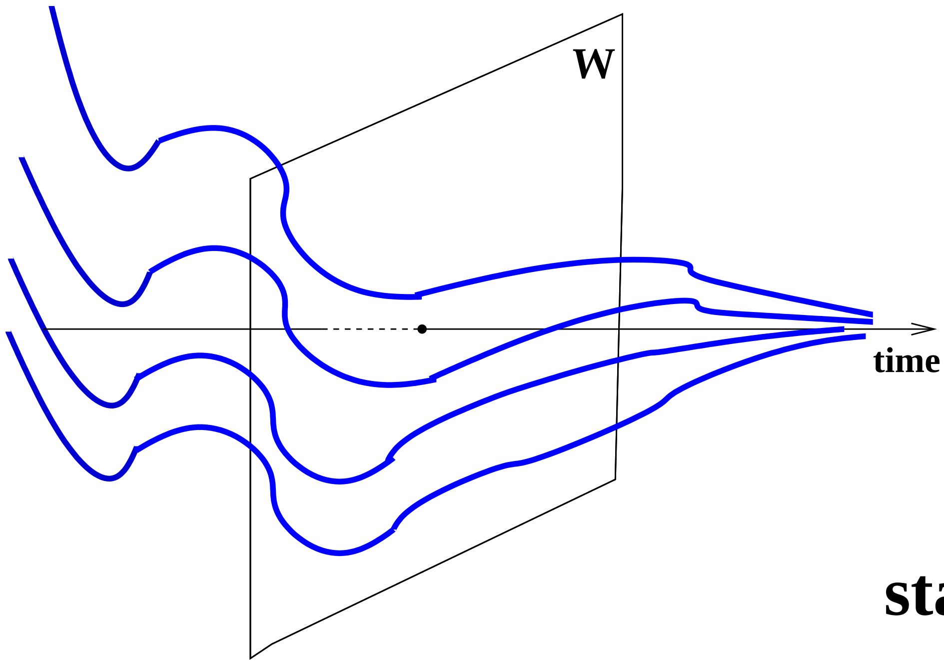
Controllability and stabilizability

$\mathscr{B}$ **is said to be controllable** $:\Leftrightarrow$

$\mathscr{B}$ **is said to be stabilizable** $:\Leftrightarrow$

$\mathscr{B}$ **is said to be autonomous** $:\Leftrightarrow$

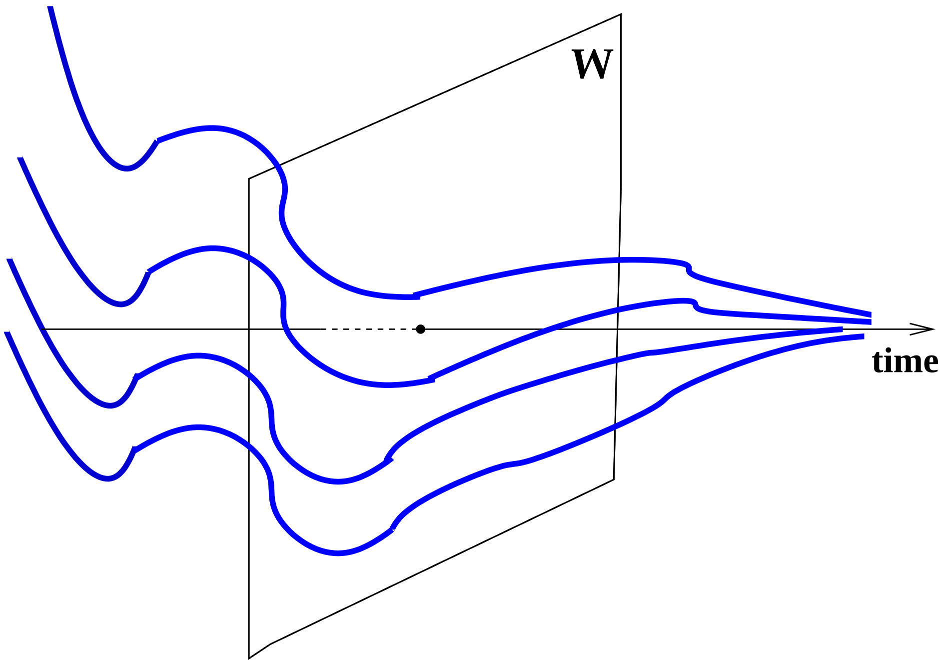
$\mathscr{B}$ **is said to be stable** $:\Leftrightarrow$ $[\![w \in \mathscr{B}]\!] \Rightarrow [\![w(t) \to 0$ **as** $t \to \infty]\!]$



stable $\Rightarrow$ **autonomous**

Stability

$\mathscr{B}$ **is said to be stable** $:\Leftrightarrow \quad [\![w \in \mathscr{B}]\!] \ \Rightarrow \ [\![w(t) \longrightarrow 0$ **as** $t \longrightarrow \infty]\!]$



Stability in the sense of Lyapunov

$\mathbb{R}(\xi)$ and some of its subrings

Relevant rings

Field of (real) rationals

Subrings of interest

- **polynomials**
- **proper rationals**
- **stable rationals**
- **proper stable rationals**

Relevant rings

unimodularity :⇔ **invertibility in the ring**

Field of (real) rationals	**nonzero**

Subrings of interest

polynomials	**nonzero constant**
proper rationals	**biproper**
stable rationals	**miniphase**
proper stable rationals	**biproper & miniphase**

Relevant rings

unimodularity $:\Leftrightarrow$ **invertibility in the ring**

Field of (real) rationals **nonzero**

Subrings of interest

polynomials	**nonzero constant**
proper rationals	**biproper**
stable rationals	**miniphase**
proper stable rationals	**biproper & miniphase**

unimodularity of square matrices over rings

$\Leftrightarrow$ `determinant` **unimodular**

left primeness of matrices over rings

$$:\Leftrightarrow \Big[\!\Big[[\![F = UF']\!] \Rightarrow [\![U \text{ unimodular}]\!] \Big]\!\Big]$$

Representability

The LTIDS $\mathscr{B}$ admits a representation that is left prime over

- **rationals: always**
- **proper rationals: always**

- **stable rationals: iff $\mathscr{B}$ is stabilizable**
- **proper stable rationals: iff $\mathscr{B}$ is stabilizable**

- **polynomials: iff $\mathscr{B}$ is controllable**

Representability

The LTIDS $\mathscr{B}$ admits a representation that is left prime over

- **stable rationals: iff $\mathscr{B}$ is stabilizable**
- **proper stable rationals: iff $\mathscr{B}$ is stabilizable**

$\mathfrak{B}$ **stabilizable** $\Leftrightarrow \exists G$**, matrix of rational functions, such that**

(i) $\mathfrak{B} = \mathtt{kernel}\left(G\left(\frac{d}{dt}\right)\right)$

(ii) G **is proper (no poles at** ∞**)**

(iii) $G^{\infty} := \mathtt{limit}_{\lambda \to \infty} G(\lambda)$ **has full row rank (no zeros at** ∞**)**

(iv) G **has no poles in** $\mathbb{C}_+ := \{\lambda \in \mathbb{C} \mid \mathtt{real}(\lambda \geq 0\}$

(v) $G(\lambda)$ **has full row rank** $\forall\, \lambda \in \mathbb{C}_+$ **(no zeros in** $\mathbb{C}_+$**)**

Representability

The LTIDS $\mathscr{B}$ admits a representation that is left prime over

- **polynomials: iff $\mathscr{B}$ is controllable**

$\mathfrak{B}$ **controllable** $\Leftrightarrow \exists R$**, matrix of polynomials, such that**

(i) $\mathfrak{B} = \texttt{kernel}\left(R\left(\frac{d}{dt}\right)\right)$

(ii) $R(\lambda)$ **full row rank** $\forall\, \lambda \in \mathbb{C}$

Preliminaries

Unimodular completion

Unimodular completion lemma

Let G be a matrix over one of our rings (polynomial, proper rat., stable rat., proper stable rat.).

¿ Does there exist a unimodular completion G' i.e. a matrix G' over that same ring such that

$$\begin{bmatrix} G \\ G' \end{bmatrix}$$

is unimodular (`determinant` is invertible in the ring) ?

Unimodular completion lemma

Let G **be a matrix over one of our rings (polynomial, proper rat., stable rat., proper stable rat.).**

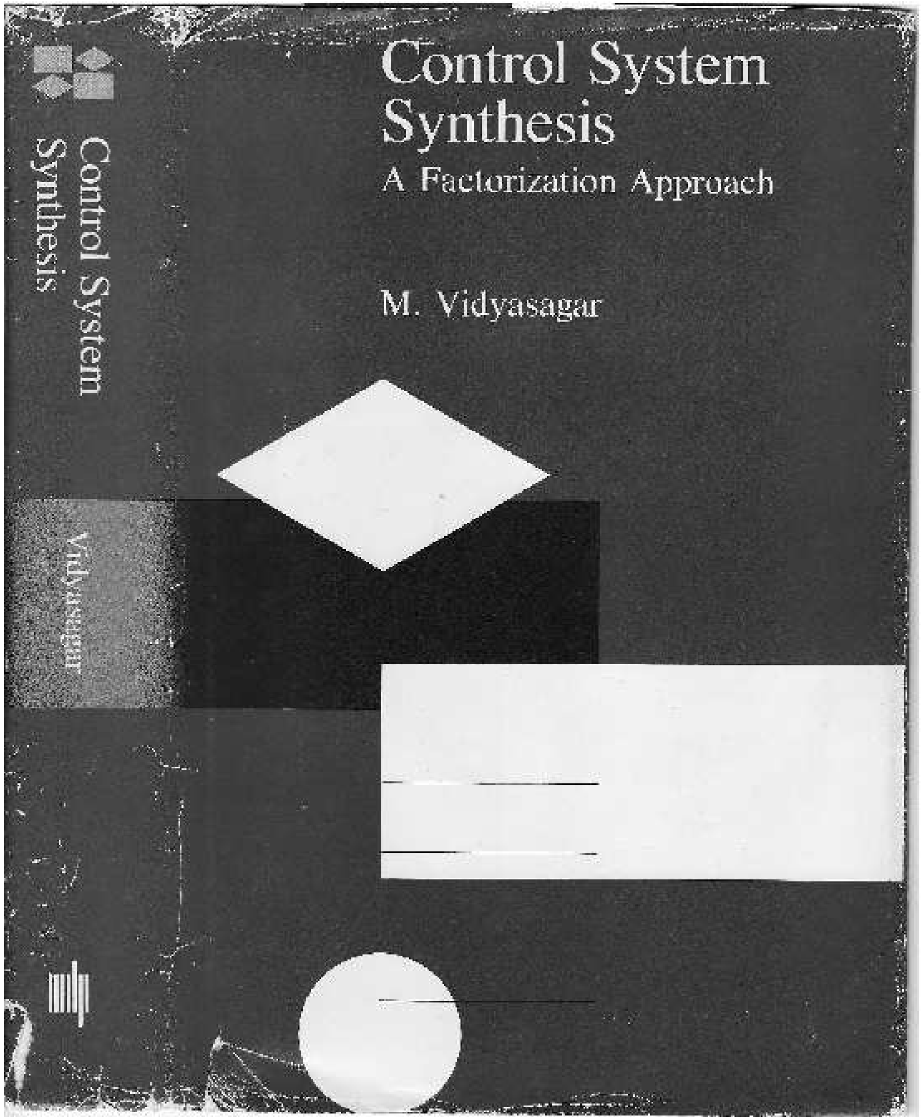
¡ There exists a unimodular completion G' **i.e. a matrix** G' **over that same ring such that**

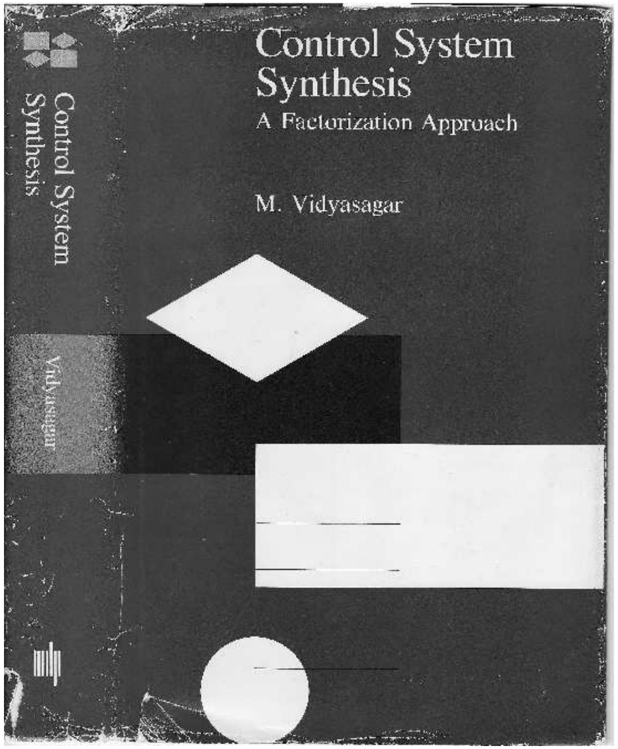
$$\begin{bmatrix} G \\ G' \end{bmatrix}$$

is unimodular (`determinant` **is invertible in the ring)**

if and only if

G **is left prime over the ring !**


Control System
Synthesis
A Factorization Approach
M. Vidyasagar
Control System
Synthesis
Vidyasagar



Work and problems posed by
Kučera, Youla c.s., Desoer c.s., Sontag & Khargonekar, Francis, e.m.a.

Unimodular completion lemma

***G*: 1 row, 2 columns**

$$G = \begin{bmatrix} p & q \end{bmatrix}$$

Unimodular completion lemma

G: 1 row, 2 columns

$$G = \begin{bmatrix} p & q \end{bmatrix} \qquad G' = \begin{bmatrix} -y & x \end{bmatrix} \qquad \begin{bmatrix} G \\ G' \end{bmatrix} = \begin{bmatrix} p & q \\ -y & x \end{bmatrix}$$

Unimodular completion lemma

G**: 1 row, 2 columns**

$$G = \begin{bmatrix} p & q \end{bmatrix} \qquad G' = \begin{bmatrix} -y & x \end{bmatrix} \qquad \begin{bmatrix} G \\ G' \end{bmatrix} = \begin{bmatrix} p & q \\ -y & x \end{bmatrix}$$

$\texttt{determinant} = px + qy$, **unimodularity** $\Leftrightarrow$ $px + qy = 1$

solvable for $x, y \Leftrightarrow p$ **&** q **coprime** $\Leftrightarrow G = \begin{bmatrix} p & q \end{bmatrix}$ **left prime**

Unimodular completion lemma

G**: 1 row, 2 columns**

$$G = \begin{bmatrix} p & q \end{bmatrix} \qquad G' = \begin{bmatrix} -y & x \end{bmatrix} \qquad \begin{bmatrix} G \\ G' \end{bmatrix} = \begin{bmatrix} p & q \\ -y & x \end{bmatrix}$$

`determinant` $= px + qy$, **unimodularity** $\Leftrightarrow$ $px + qy = 1$

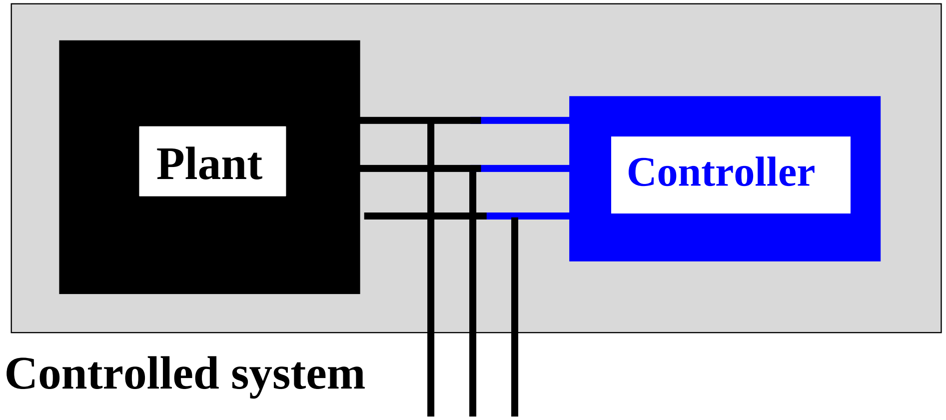
solvable for $x, y \Leftrightarrow p$ **&** q **coprime** $\Leftrightarrow G = \begin{bmatrix} p & q \end{bmatrix}$ **left prime**

Our rings are Hermite rings

G **left prime** $\Leftrightarrow$ **unimodularly completable** $\Leftrightarrow \exists H : GH = I \Leftrightarrow \cdots$

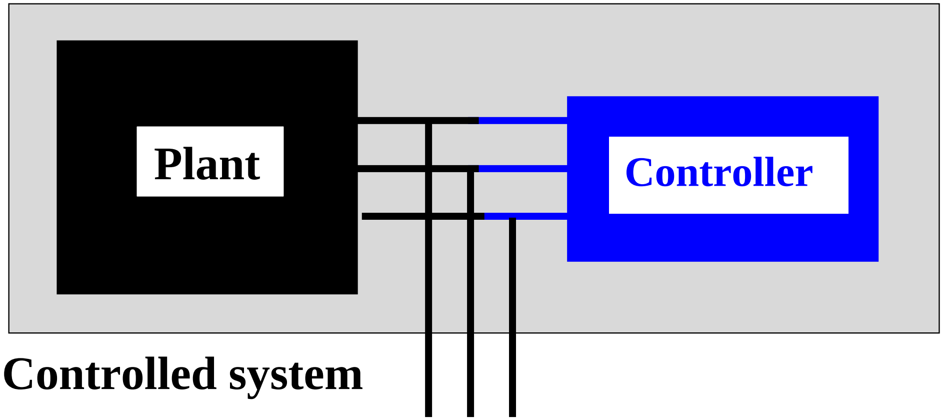
Control

Control



Plant $\mathscr{P}$, controller $\mathscr{C}$, controlled system $\mathscr{P} \cap \mathscr{C}$

Control



Plant $\mathscr{P}$, **controller** $\mathscr{C}$, **controlled system** $\mathscr{P} \cap \mathscr{C}$

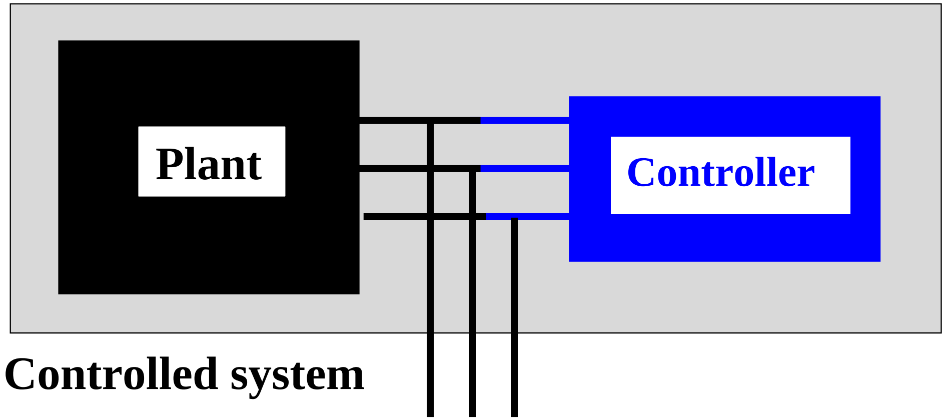
$$[\![\mathscr{C} \textbf{ is stabilizing}]\!] :\Leftrightarrow [\![\mathscr{P} \cap \mathscr{C} \textbf{ is stable}]\!]$$

$$\Leftrightarrow \left[\!\!\left[[\![w \in \mathscr{P} \cap \mathscr{C}]\!] \Rightarrow [\![w(t) \longrightarrow 0 \textbf{ for } t \longrightarrow \infty]\!] \right]\!\!\right]$$

$$[\![\mathscr{C} \textbf{ is deadbeat}]\!] :\Leftrightarrow [\![\mathscr{P} \cap \mathscr{C} = \{0\}]\!]$$

$$\Leftrightarrow [\![w \in \mathscr{P} \cap \mathscr{C}]\!] \Rightarrow [\![w = 0]\!]$$

Control



Plant $\mathscr{P}$, **controller** $\mathscr{C}$, **controlled system** $\mathscr{P} \cap \mathscr{C}$

$[\![\mathscr{C}$ **is a regular controller**$]\!] :\Leftrightarrow [\![\mathscr{P} + \mathscr{C} = \mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})]\!]$

$[\![\mathscr{C}$ **is a superregular controller**$]\!] :\Leftrightarrow$ **in addition,**

$$[\![\forall w \in \mathscr{P}, \forall w' \in \mathscr{C} \ \ \exists v \text{ such that } w \wedge_0 v, w' \wedge_0 v \in \mathscr{P} \cap \mathscr{C}]\!]$$

Control

Plant $\mathscr{P}$, **controller** $\mathscr{C}$, **controlled system** $\mathscr{P} \cap \mathscr{C}$

$[\![\mathscr{C}$ **is a regular controller**$]\!] :\Leftrightarrow [\![\mathscr{P} + \mathscr{C} = \mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})]\!]$

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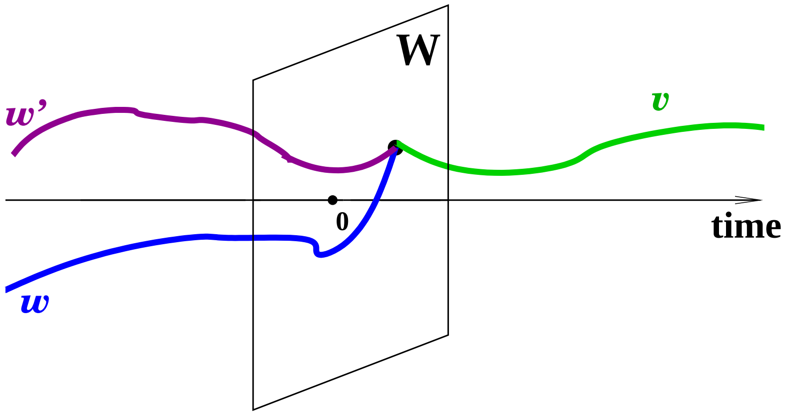
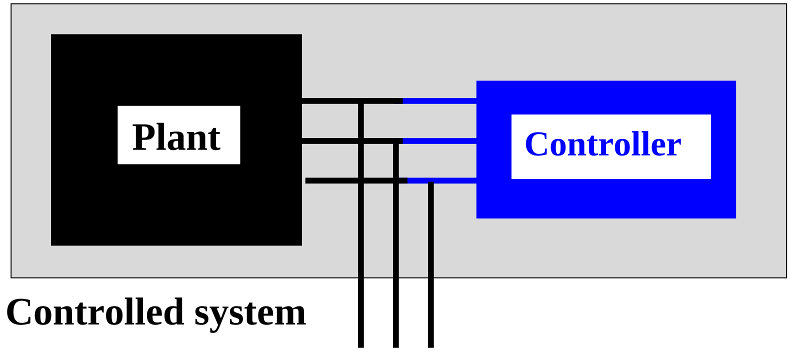
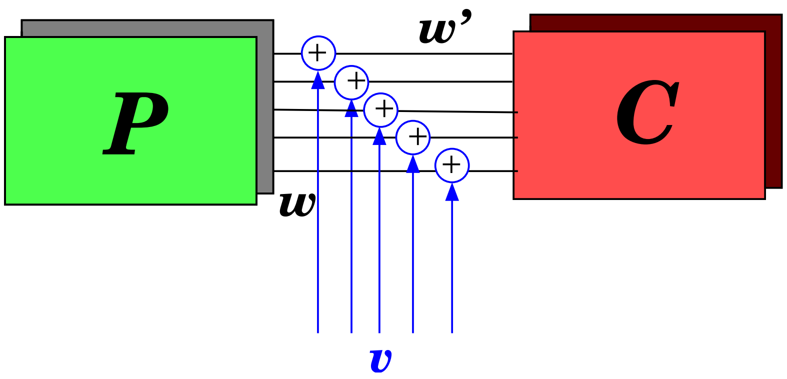


Regular & superregular

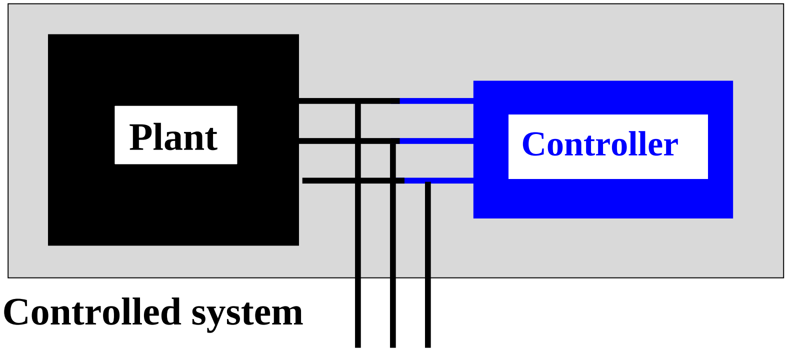
Regular controller

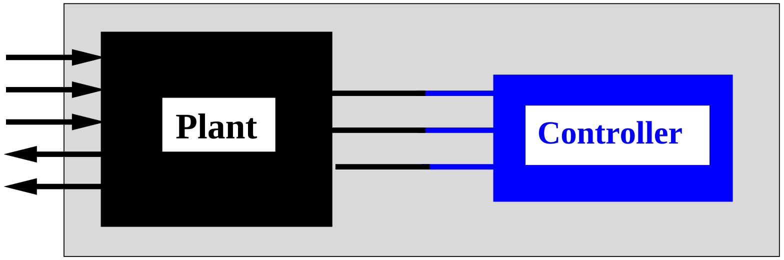


$\forall v \in \mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})\ \exists w \in \mathscr{P}$ **and** $w' \in \mathscr{C}$ **such that** $v = w + w'$



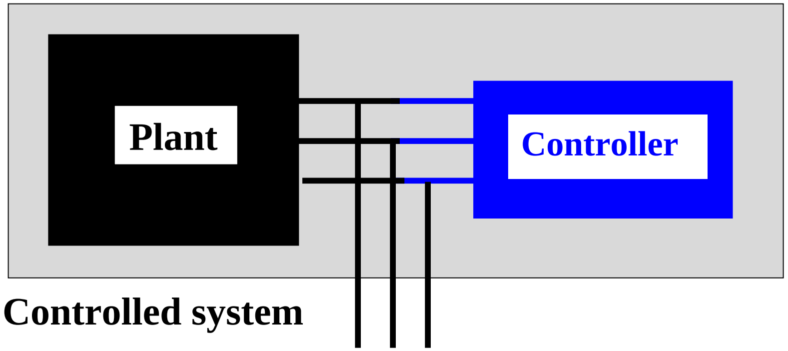
Regular controller

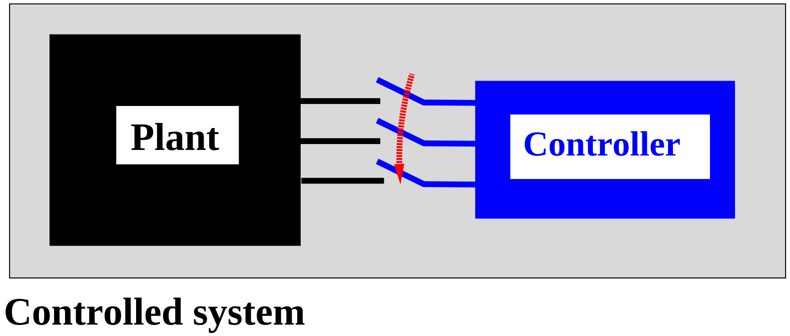




regular $\Rightarrow$ exogenous inputs unchanged after control

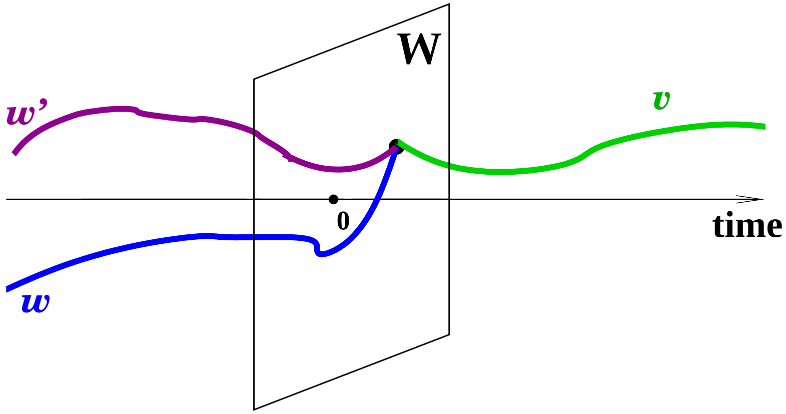
Superregular controllers





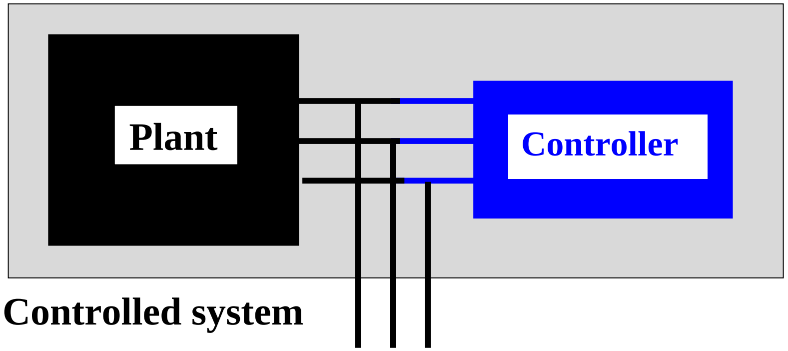
superregular $\Rightarrow$ controller can be engaged at any time

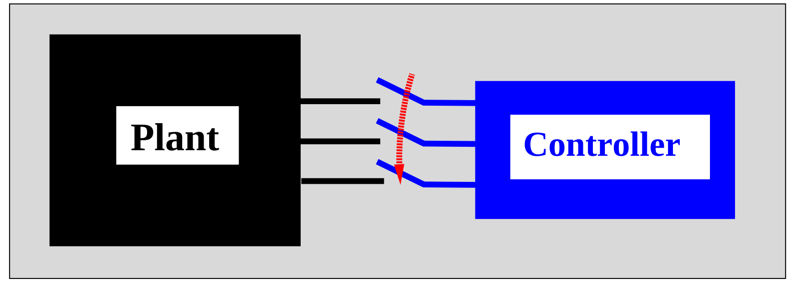
$\forall w \in \mathscr{P}, \forall w' \in \mathscr{C}$ $\exists v$ **such that** $w \wedge_0 v, w' \wedge_0 v \in \mathscr{P} \cap \mathscr{C}$



Interc'ions requiring 'state preparation' $\Rightarrow$ not superregular

Superregular controllers





Controlled system

Usual feedback controllers are superregular

PID controllers are regular, but not superregular

Controllers that are regular, but not superregular, relevant:
control is interconnection, not just signal processing

Cardinalities

Let $\mathscr{B}$ be a LTIDS. Define

$$\mathtt{p}(\mathscr{B}) := \textbf{number of output components}$$

$$= \textbf{number of system equations} = \mathrm{rank}(R) = \mathrm{rank}(G)$$

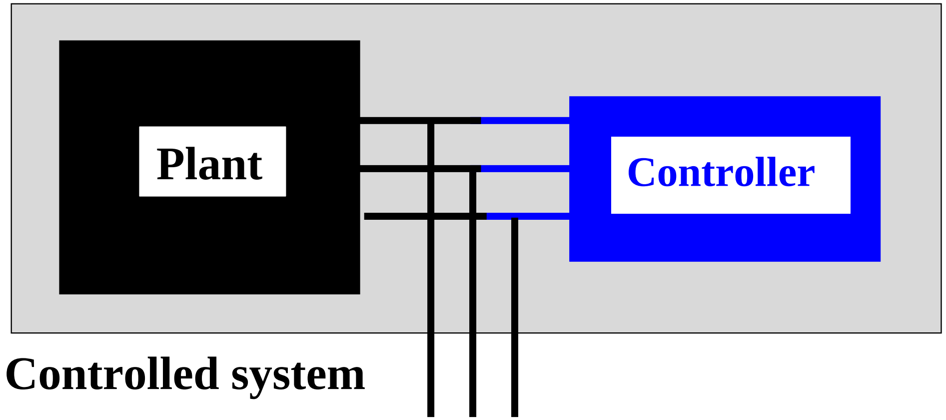
$$\mathtt{n}(\mathscr{B}) := \textbf{number of state components}$$

$$= \textbf{dimension of state space} = \textbf{McMillan degree}$$

$$\mathtt{m}(\mathscr{B}) := \textbf{number of (free) input components}$$

$$= \mathtt{w}(\mathscr{B}) - \mathtt{p}(\mathscr{B})$$

(Super)regular & cardinalities



Plant $\mathscr{P}$, **controller** $\mathscr{C}$, **controlled system** $\mathscr{P} \cap \mathscr{C}$

$$[\![\mathscr{C} \text{ is a } \textbf{regular} \text{ controller}]\!] :\Leftrightarrow [\![\mathscr{P} + \mathscr{C}) = \mathscr{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathtt{w}})]\!]$$
$$\Leftrightarrow [\![\mathtt{p}(\mathscr{P} \cap \mathscr{C}) = \mathtt{p}(\mathscr{P}) + \mathtt{p}(\mathscr{C})]\!]$$

$$[\![\mathscr{C} \text{ is a } \textbf{superregular} \text{ controller}]\!]$$
$$:\Leftrightarrow \text{ in addition, } [\![\mathtt{n}(\mathscr{P} \cap \mathscr{C}) = \mathtt{n}(\mathscr{P}) + \mathtt{n}(\mathscr{C})]\!]$$

Existence of stabilizing controllers

Existence

<u>Proposition</u>

$\mathscr{P}$ **is stabilizable** $\Leftrightarrow \exists$ **a regular stabilizing controller**

$\Leftrightarrow \exists$ **a superregular stabilizing controller**

Existence

Proposition

$\mathscr{P}$ **is stabilizable** $\Leftrightarrow \exists$ **a regular stabilizing controller**

$\Leftrightarrow \exists$ **a superregular stabilizing controller**

$\mathscr{P}$ **is controllable** $\Leftrightarrow \exists$ **a regular deadbeat controller**

$\Leftrightarrow \exists$ **pole placement ...**

$\nexists$ **a controller that is superregular & deadbeat!**

Parametrization of controllers

Parametrization of regular stabilizing controllers

Start with $G\left(\frac{d}{dt}\right)w = 0$ **a (rational symbol based) representation of the plant**

Assume G **left prime over ring of stable rational functions.**
Iff the plant is stabilizable, such a G **exists.**

Parametrization of regular stabilizing controllers

Start with $G\left(\frac{d}{dt}\right)w = 0$ **a (rational symbol based) representation of the plant**

Assume G left prime over ring of stable rational functions. Iff the plant is stabilizable, such a G exists.

$\Rightarrow \exists\, G'$ **such that** $\begin{bmatrix} G \\ G' \end{bmatrix}$ **is unimodular over stable rat. f'ns.**

Parametrization of regular stabilizing controllers

Start with $G\left(\frac{d}{dt}\right)w = 0$ a (rational symbol based) representation of the plant

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$\Rightarrow \exists\, G'$ **such that** $\begin{bmatrix} G \\ G' \end{bmatrix}$ **is unimodular over stable rat. f'ns.**

Par'ion of regular stabilizing controllers $C\left(\frac{d}{dt}\right)w = 0$

$$\boxed{C = F_1 G + F_2 G'}$$

F_1 **free over stable rationals,** F_2 **unimodular over stable rat.**

Parametrization of superregular stabilizing controllers

Start with $G\left(\frac{d}{dt}\right)w = 0$ **a (rational symbol based) representation of the plant**

Assume G **left prime over proper stable rational functions.**
Iff the plant is stabilizable, such a G **exists.**

$\Rightarrow \exists\, G'$ **such that** $\begin{bmatrix} G \\ G' \end{bmatrix}$ **is unimodular over proper stable rat.**

Par'ion of superregular stabilizing controllers $C\left(\frac{d}{dt}\right)w = 0$

$$\boxed{C = F_1 G + F_2 G'}$$

F_1 **free over proper st. rat.,** F_2 **unimodular over pr. st. rat.**

Parametrization of regular deadbeat controllers

$R\left(\frac{d}{dt}\right)w = 0$ **a (polynomial symbol based) repr. of the plant.**

Assume R **left prime over ring of polynomials.**
Iff the plant is controllable, such an R **exists.**

$\Rightarrow \exists\, R'$ **such that** $\begin{bmatrix} R \\ R' \end{bmatrix}$ **is unimodular as a polynomial matrix.**

Parametrization of regular deadbeat controllers $C\left(\frac{d}{dt}\right)w = 0$

$$\boxed{C = FR + R'}$$

F **free over polynomial matrices.**

Simplification

If we consider controllers 'equivalent' if they have the same controllable part ($\cong$ same transfer function)

Par'ion of stabilizing (super)regular controllers $C\left(\frac{d}{dt}\right)w = 0$

$$\boxed{C = FG + G'}$$

F **free over (proper) stable rational.**

Parametrization of regular stabilizing controllers

Start with $R\left(\frac{d}{dt}\right) w = 0$ **a (polynomial symbol based) repr. of the plant, for simplicity assumed controllable.**

Assume R **left prime over ring of polynomials.**
Iff the plant is controllable, such an R **exists.**

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Parametrization of regular stabilizing controllers

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Par'ion of regular stabilizing controllers $C\left(\frac{d}{dt}\right) w = 0$

$$\boxed{C = F_1 R + F_2 R'}$$

F_1 **free over pol. matr.,** F_2 **Hurwitz (i.e.** $\texttt{det}(F_2)$ **Hurwitz)**

A glimpse of the proof

Polynomial case

Start with the plant $R\left(\frac{d}{dt}\right)w = 0$ **assumed controllable**

means $R(\lambda)$ **full row rank** $\forall \lambda \in \mathbb{C}$

i.e. R **is left prime as a polynomial matrix**

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Therefore $\exists R'$ **such that** $\begin{bmatrix} R \\ R' \end{bmatrix}$ **is unimodular**

Polynomial case

$$\begin{bmatrix} R \\ R' \end{bmatrix}$$ **unimodular**

Consider the controller $C\left(\frac{d}{dt}\right)w = 0$

Polynomial case

$$\begin{bmatrix} R \\ R' \end{bmatrix} \text{ unimodular}$$

Consider the controller $C\left(\frac{d}{dt}\right) w = 0$

unimodularity $\Rightarrow \; C = \begin{bmatrix} F_1 & F_2 \end{bmatrix} \begin{bmatrix} R \\ R' \end{bmatrix} = F_1 R + F_2 R'$

$\Rightarrow$ **every controller is of the form** $C = F_1 R + F_2 R'$

Polynomial case

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Consider the controller $C\left(\frac{d}{dt}\right)w = 0$

$\Rightarrow$ **every controller is of the form** $C = F_1R + F_2R'$

$\rightsquigarrow$ **controlled system** $\begin{bmatrix} I & 0 \\ F_1 & F_2 \end{bmatrix} \begin{bmatrix} R \\ R' \end{bmatrix} \left(\frac{d}{dt}\right) w = 0$

Regularity $\Leftrightarrow F_2$ **square,** $\texttt{determinant}(F_2) \neq 0$

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deadbeat $\Leftrightarrow F_2$ **unimodular** $\rightsquigarrow$ **WLOG** $F_2 = I \rightsquigarrow C = FR + R'$

Polynomial case

$$\begin{bmatrix} R \\ R' \end{bmatrix}$$ **unimodular**

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stable $\Leftrightarrow$ $\mathtt{determinant}(F_2)$ **Hurwitz**

Polynomial case

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stable $\Leftrightarrow \texttt{determinant}(F_2)$ **Hurwitz**

... other proofs similar

... superregular

... advantages of rational representations

Examples

A superregular controller

$$\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = 0 \quad \rightsquigarrow \quad w_1 = 0, \;\; w_2 \textbf{ free}$$

A superregular controller

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$$G = \begin{bmatrix} 1 & 0 \end{bmatrix}, \;\; G' = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

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Controller: $w_2 + \alpha \frac{d}{dt} w_2 = 0, \;\; \alpha > 0$ **superregular & stabilizing**

Captured by first, but not by the simplified parametrization.

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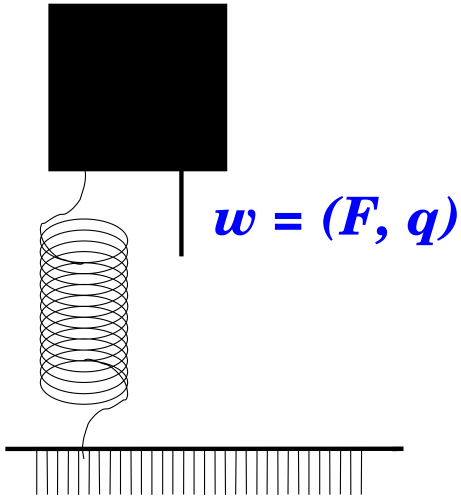
Transfer function thinking has limitations.
It does not capture the uncontrollable part of a behavior.

A regular, but not superregular, controller

Plant:



$$M\frac{d^2}{dt^2}q+Kq=F\,,\quad w=(F,q)$$

A regular, but not superregular, controller

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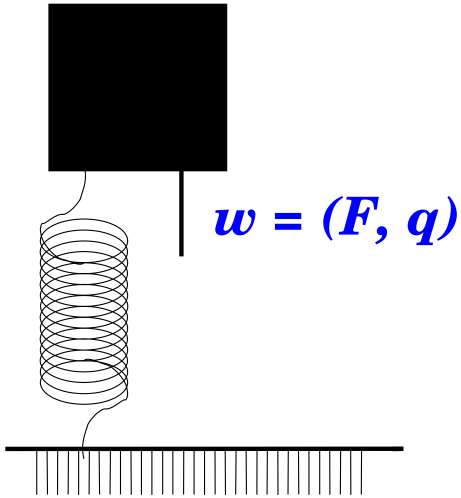
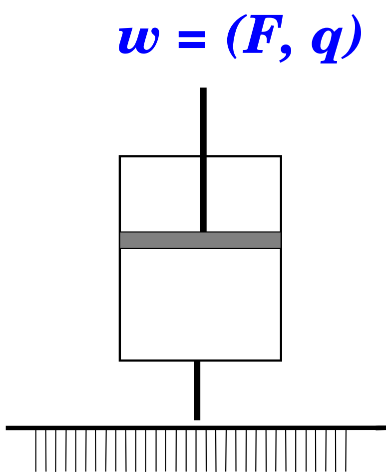


Controller:

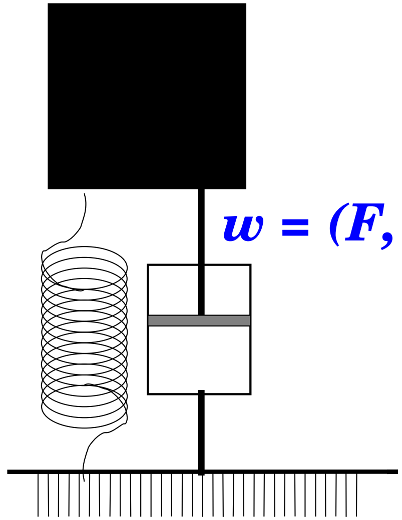


$$M\frac{d^2}{dt^2}q + Kq = F, \quad w = (F, q)$$

$$F = -D\frac{d}{dt}q$$

A regular, but not superregular, controller

Controlled system:



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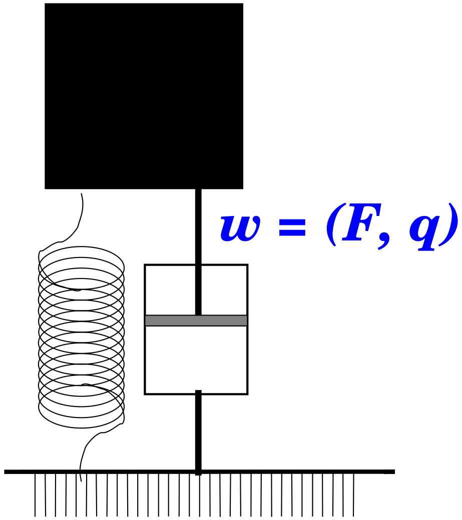


$$M\frac{d^2}{dt^2}q + D\frac{d}{dt}q + Kq = 0, \quad F = -D\frac{d}{dt}q$$

$$w \to \begin{bmatrix} F \\ w \end{bmatrix}, \qquad R \to \left[\, 1 \mid -1-\xi^2 \,\right], \qquad R' \to \left[\, 1 \mid \xi^2 \,\right]$$

Reg. stab. $c \to \left[f(\xi)+h(\xi) \middle| -h(\xi)-\xi^2(f(\xi)+h(\xi))\right]$ h **H'itz**

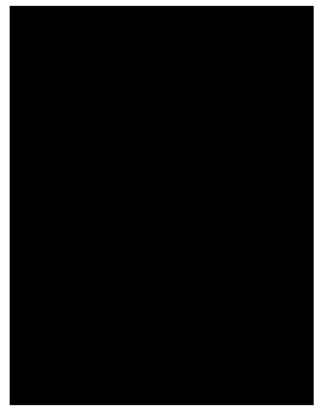
$$f \to -\xi-\xi^2, \quad h \to 1+\xi+\xi^2, \qquad c \to \left[\, 1 \mid \xi \,\right]$$

Summary

Conclusion

Using rational symbol based representations $G\left(\frac{d}{dt}\right)w = 0$ that are left prime over suitable rings, we obtain parametrizations of regular and superregular stabilizing controllers

$\cong$ Kučera-Youla parametrization, with proper attention for the uncontrollable part



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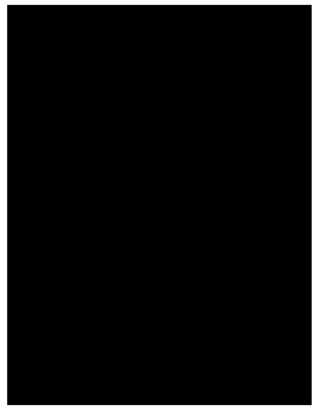

Other applications where rational symbols are indispensable: $\mathscr{L}_2$ **unitary representations and behavioral model reduction.**

Thank you for your attention

Happy Birthday, Sagar !!!