REPRESENTATIONS
of
LINEAR TIME-INVARIANT
SYSTEMS

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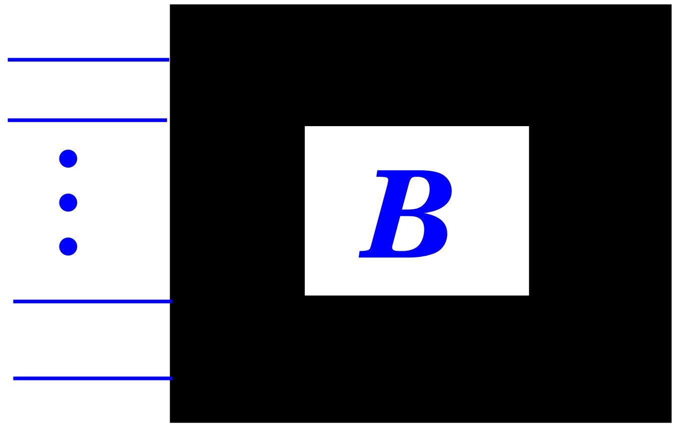
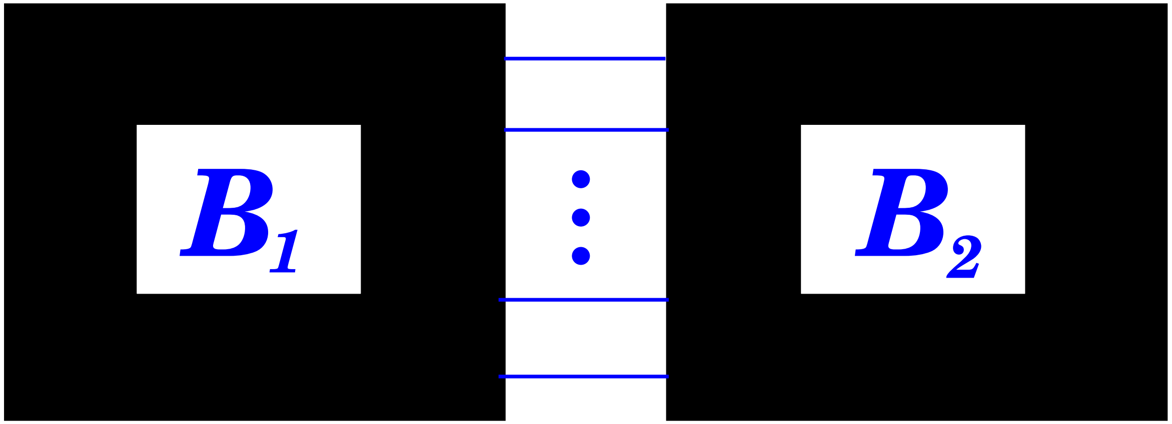
Introduction



Behavior := specifies which trajectories are possible

How do we express these mathematically?



Which representations deal best with robustness?

Representations code properties such as

- **controllability, stabilizability**
- **observability, detectability**
- **...**

Objective of the lecture

Discuss some of the main representations of linear shift-invariant (LSI / LTI) systems

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Competing possibilities:

- **kernel and image representations**
- **state, latent variable representations**
- **i/o, transfer functions**
- **left-prime representations over various rings**
- **...**

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Discuss some of the main representations of linear shift-invariant (LSI / LTI) systems

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- **...**

We use discrete- & continuous-time interchangeably

Formalization

A system as a behavior

A system $:\Leftrightarrow$ $(\mathbb{T}, \mathbb{W}, \mathfrak{B})$

$\mathbb{T}$ **'set of independent variables'**

$\mathbb{W}$ **'set of dependent variables'**

$\mathfrak{B} \subseteq \mathbb{W}^{\mathbb{T}}$ **the 'behavior'**

a family of trajectories mapping $\mathbb{T} \longrightarrow \mathbb{W}$

$w : \mathbb{T} \longrightarrow \mathbb{W} \in \mathfrak{B}$: **'$w$ is compatible with the model'**

$w : \mathbb{T} \longrightarrow \mathbb{W} \notin \mathfrak{B}$: **'the model forbids w'**

Typically, $\mathbb{T} = \mathbb{R}, \mathbb{R}_+, \mathbb{Z}, \mathbb{N}, \mathbb{R}^{\mathtt{n}}, \mathbb{Z}^{\mathtt{n}}$, $\mathbb{W} = \mathbb{R}^{\mathtt{w}}$, **etc.**

A system as a behavior

The system $(\mathbb{T}, \mathbb{R}^{\mathtt{w}}, \mathfrak{B})$ $\quad \mathbb{T} = \mathbb{R}, \mathbb{R}^{\mathtt{n}}, \mathbb{Z}, \mathbb{Z}^{\mathtt{n}} \quad \rightsquigarrow \; \mathfrak{B}$
is

linear $:\Leftrightarrow w_1, w_2 \in \mathfrak{B}, \alpha \in \mathbb{R}$
implies $\alpha w_1 + w_2 \in \mathfrak{B}$

shift-invariant (time-invariant)
$:\Leftrightarrow w \in \mathfrak{B}$, σ **any multi-shift, implies** $\sigma w \in \mathfrak{B}$

Kernel representations

LSIDS

Equivalent for $(\mathbb{Z}^{\mathtt{n}}, \mathbb{R}^{\mathtt{w}}, \mathfrak{B})$, $\quad \mathfrak{B} \subseteq (\mathbb{R}^{\mathtt{w}})^{\mathbb{R}^{\mathtt{n}}}$

1. $\mathfrak{B}$ **is linear, shift-invariant, and closed**
2. $\mathfrak{B}$ **is linear, shift-inv., and prefix determined**
3. $\exists$ **polynomial matrix** $R(\xi_1, \cdots, \xi_{\mathtt{n}})$ **such that** $\mathfrak{B}$ **consists of the sol'ns of**

$$R(\sigma_1, \ldots, \sigma_{\mathtt{n}})\, w = 0$$

'kernel representation'

LSIDS

$$\boldsymbol{R}\left(\boldsymbol{\sigma}_1, \ldots, \boldsymbol{\sigma}_{\mathtt{n}}\right) \boldsymbol{w} = \mathbf{0}$$

Continuous analogue

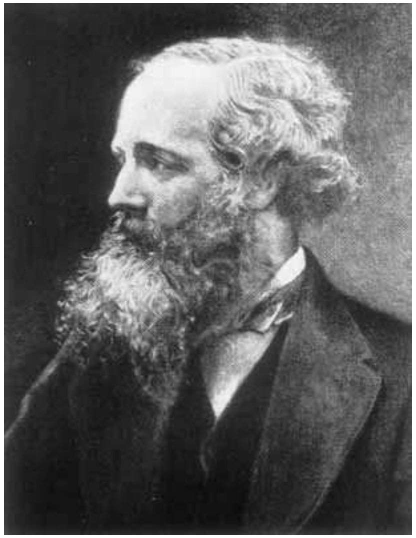
$$\boldsymbol{R}\left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathtt{n}}}\right) \boldsymbol{w} = \mathbf{0}$$

$$\boldsymbol{R}\left(\frac{d}{dt}\right) \boldsymbol{w} = \mathbf{0}$$

Notation: $\mathfrak{L}_{\mathtt{n}}^{\mathtt{w}}$

PDEs: example

Maxwell's equations for EM fields in free space



$$
\begin{aligned}
\nabla \cdot \vec{E} &= \frac{1}{\varepsilon_0}\rho\,, \\
\nabla \times \vec{E} &= -\frac{\partial}{\partial t}\vec{B}, \\
\nabla \cdot \vec{B} &= 0\,, \\
c^2 \nabla \times \vec{B} &= \frac{1}{\varepsilon_0}\vec{j} + \frac{\partial}{\partial t}\vec{E}.
\end{aligned}
$$

independent variables: (t, x, y, z) **time and space**

dependent variables: $(\vec{E}, \vec{B}, \vec{j}, \rho)$

electric field, magnetic field, current density, charge density

PDEs: example

Example: **Maxwell's eq'ns** $(\mathbb{R}^4, \mathbb{R}^{10}, \mathfrak{B})$

4 independent variables, (t, x, y, z)

$\mathtt{w} = 10$, $w = (\vec{E}, \vec{B}, \vec{j}, \rho)$

8 equations, $R \rightsquigarrow 8 \times 10$**, sparse, first order**

Relation with modules

$\boldsymbol{n} \in \mathbb{R}^{\mathtt{w}}(\boldsymbol{\xi}_1, \cdots, \boldsymbol{\xi}_{\mathtt{n}})$ **is an annihilator of** $\mathfrak{B}$ **:**$\Leftrightarrow$

$$\boldsymbol{n}^\top \left(\frac{\partial}{\partial \boldsymbol{x}_1}, \cdots, \frac{\partial}{\partial \boldsymbol{x}_{\mathtt{n}}} \right) \mathfrak{B} = 0$$

The annihilators form an $\mathbb{R}(\boldsymbol{\xi}_1, \cdots, \boldsymbol{\xi}_{\mathtt{n}})$ **module**

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Theorem:

$\exists\, 1 \leftrightarrow 1$ **relation between** $\mathfrak{L}^{\mathtt{w}}_{\mathtt{n}}$ **and the**

$\mathbb{R}(\xi_1, \cdots, \xi_{\mathtt{n}})$ **submodules of** $\mathbb{R}^{\mathtt{w}}(\xi_1, \cdots, \xi_{\mathtt{n}})$

Elimination theorem

Theorem:

$\mathfrak{L}_{\mathtt{n}}^{\mathtt{w}}$ **is closed under projection**

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$$R_1\left(\frac{\partial}{\partial x_1},\cdots,\frac{\partial}{\partial x_{\mathtt{n}}}\right)w_1 = R_2\left(\frac{\partial}{\partial x_1},\cdots,\frac{\partial}{\partial x_{\mathtt{n}}}\right)w_2 \qquad (*)$$

$\mathfrak{B}_1 := \{w_1 \mid \exists\, w_2$ **such that** (w_1, w_2) **satisfies** $(*)\,\}$

'elimination theorem': $\mathfrak{B}_1 \in \mathfrak{L}_{\mathtt{n}}^{\mathtt{w}_1}$ **!**

Application: state systems, interconnected systems

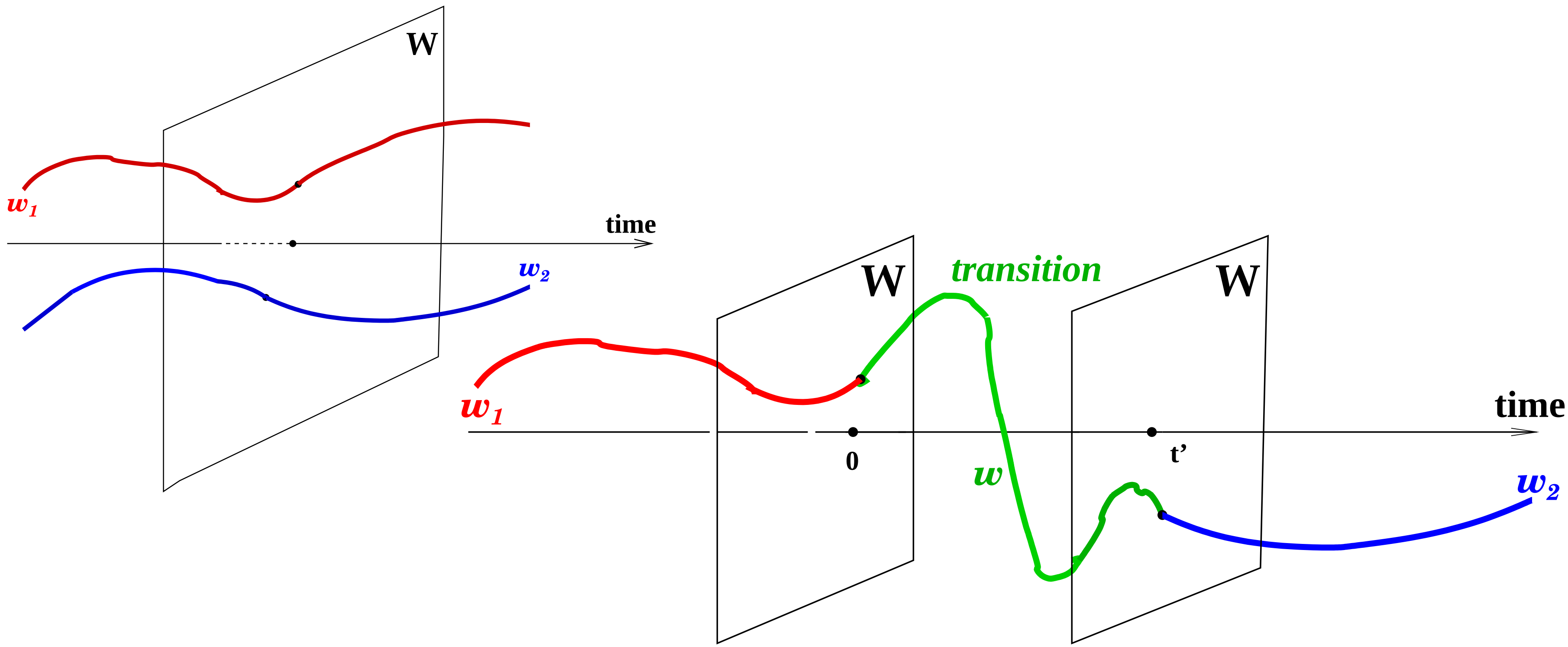
Controllability as a system property

Controllability

The time-invariant system $(\mathbb{R}, \mathbb{R}^{\mathtt{w}}, \mathfrak{B})$ is

controllable $:\Leftrightarrow$

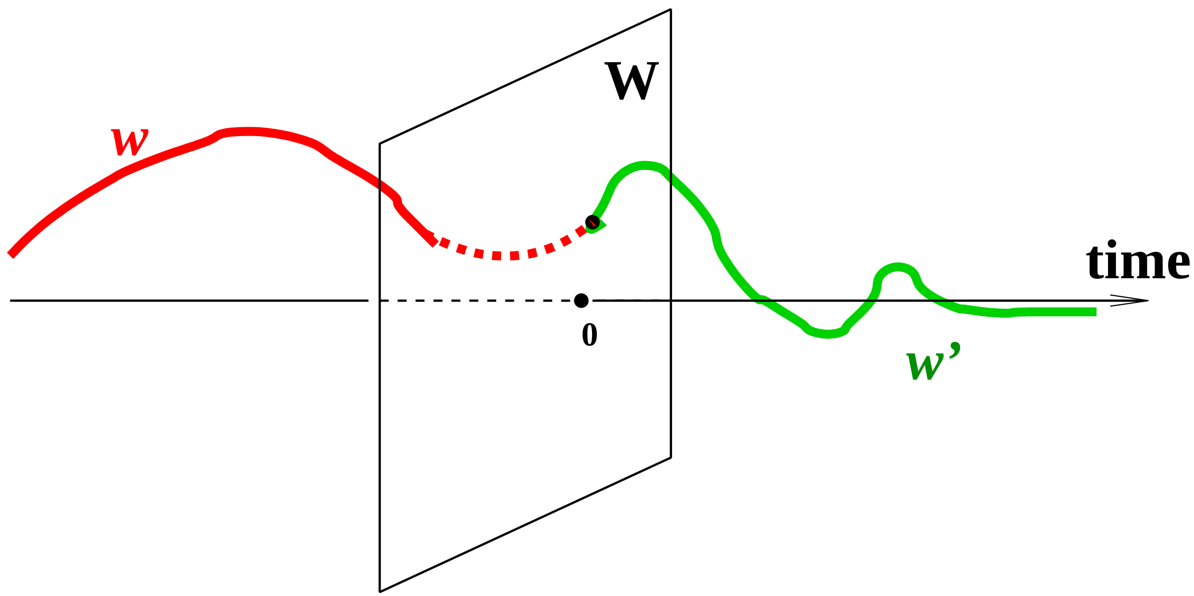
$$\forall\, w_1, w_2 \in \mathfrak{B},\ \exists\, w \in \mathfrak{B} \text{ and } T \geq 0 \text{ such that}$$



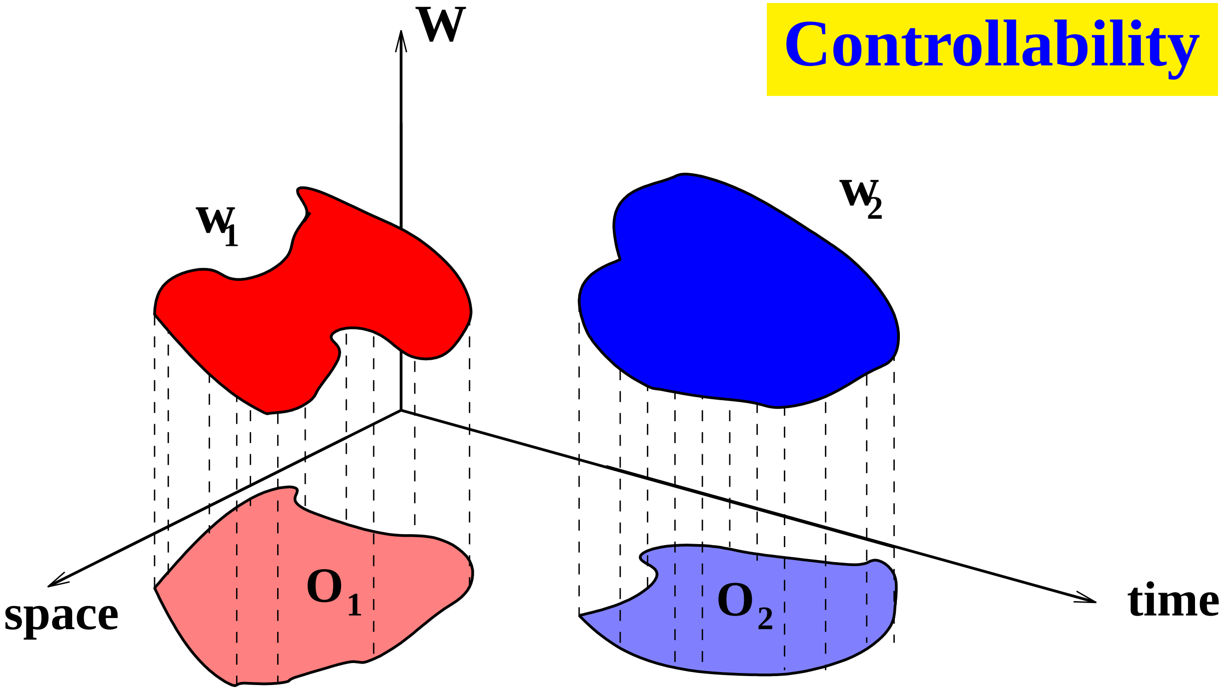
Controllability

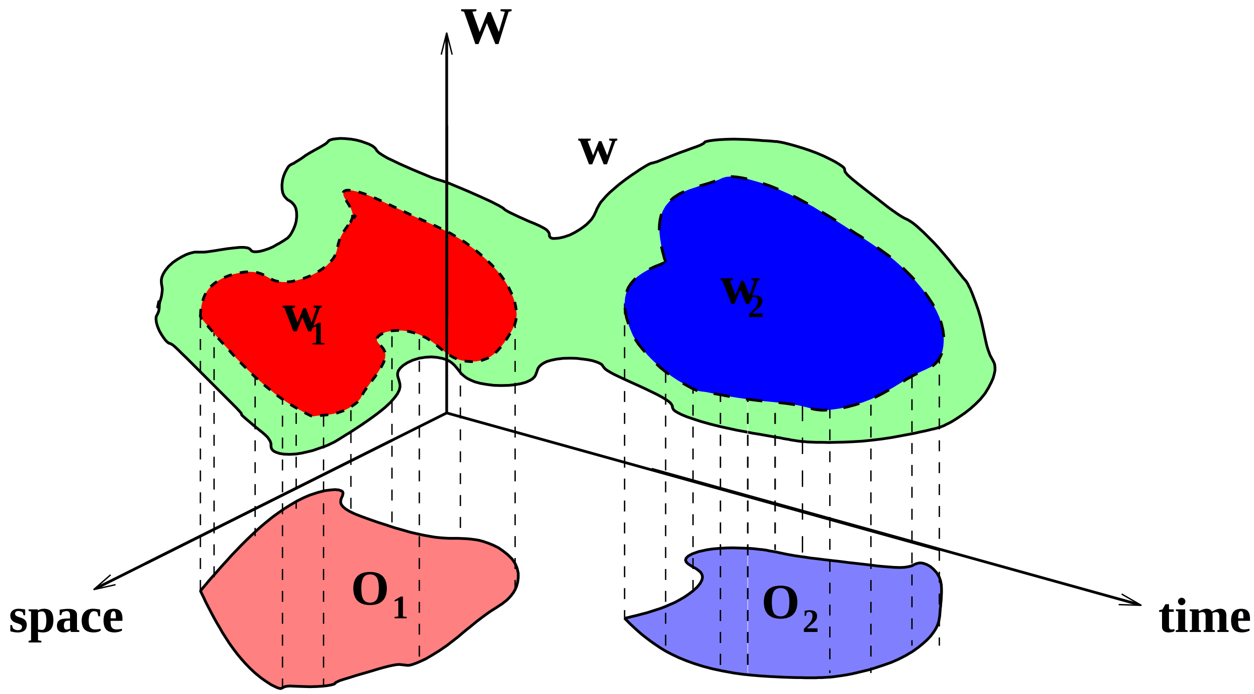
The time-invariant system $(\mathbb{R}, \mathbb{R}^{\mathtt{w}}, \mathfrak{B})$ **is**

stabilizable $:\Leftrightarrow \quad \forall\, w \in \mathfrak{B},\ \exists\, w' \in \mathfrak{B}$ **such that**

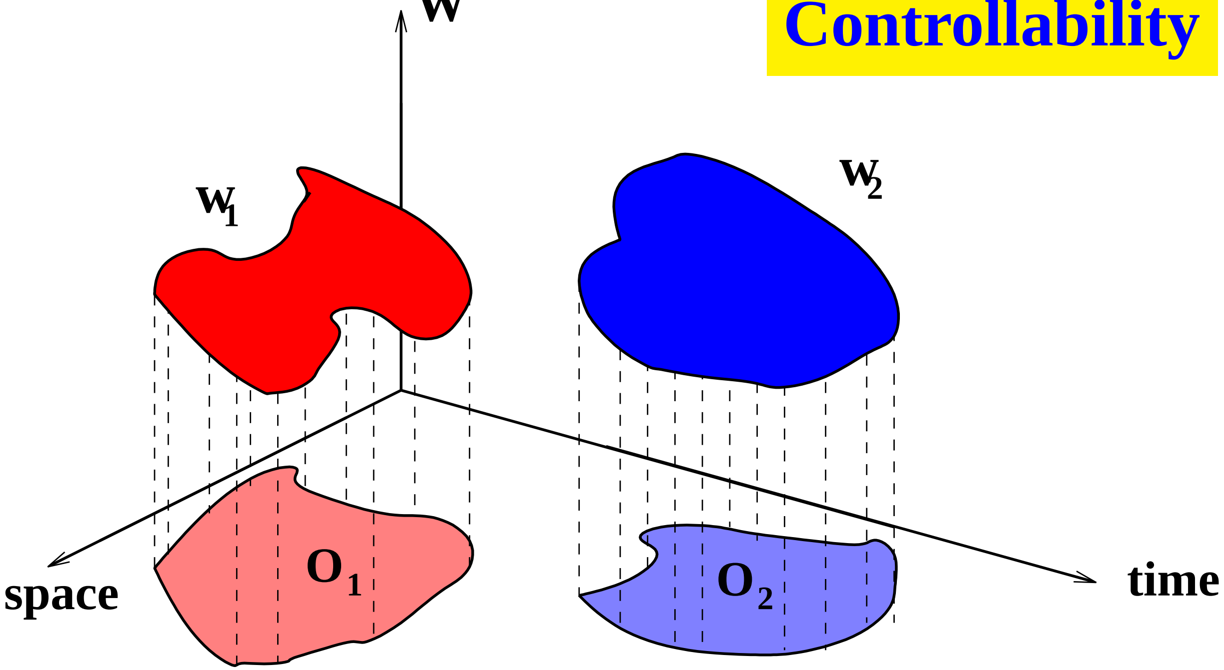


Controllability



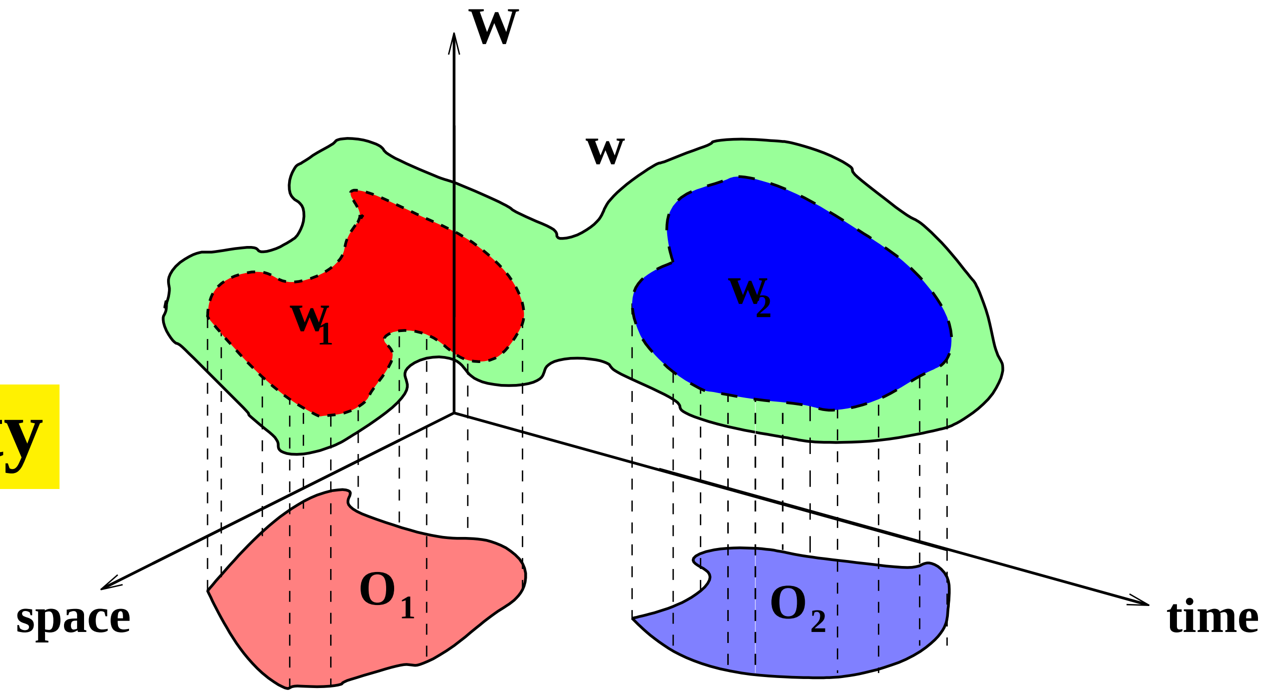


Controllability



Controllability

:= Patchability



Images

<u>Theorem</u>:

$\mathfrak{B} \in \mathfrak{L}_{\mathtt{n}}^{\mathtt{w}}$ is controllable iff it has a representation

$$w = M\left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathtt{n}}}\right)\ell$$

i.e.

$$\mathfrak{B} = \mathtt{image}\left(M\left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathtt{n}}}\right)\right)$$

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Is an image a kernel ? Always ! $\Leftarrow$ Elimination th'm

Is a kernel an image ? Iff the kernel is controllable !

Images

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For $\mathtt{n} > 1$, ℓ observable from w may be impossible.

Images may require hidden variables.

Are EM fields controllable ?

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The following eq'ns in

scalar potential $\phi : \mathbb{R} \times \mathbb{R}^3 \longrightarrow \mathbb{R}$

vector potential $\vec{A} : \mathbb{R} \times \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

generate exactly the solutions to MEs:

$$
\begin{aligned}
\vec{E} &= -\frac{\partial}{\partial t}\vec{A} - \nabla\phi, \\
\vec{B} &= \nabla \times \vec{A}, \\
\vec{j} &= \varepsilon_0 \frac{\partial^2}{\partial t^2}\vec{A} - \varepsilon_0 c^2 \nabla^2 \vec{A} + \varepsilon_0 c^2 \nabla\left(\nabla \cdot \vec{A}\right) + \varepsilon_0 \frac{\partial}{\partial t}\nabla\phi, \\
\rho &= -\varepsilon_0 \frac{\partial}{\partial t}\nabla \cdot \vec{A} - \varepsilon_0 \nabla^2 \phi.
\end{aligned}
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Are EM fields controllable ?

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\end{aligned}
$$

Proves controllability of EM fields.

Not observable, cannot be !

controllability $\Leftrightarrow$ $\exists$ potential!

Rational representations

i/o or i/s/o representations

We do not dwell on the ubiquitous representations

$$P\left(\frac{d}{dt}\right)y = Q\left(\frac{d}{dt}\right)u \qquad w = \begin{bmatrix} u \\ y \end{bmatrix}$$

$$\frac{d}{dt}x = Ax + Bu, \;\; y = Cx + Du \qquad w = \begin{bmatrix} u \\ y \end{bmatrix}$$

Rational representations

Let $G \in \mathbb{R}(\xi)^{\bullet \times \mathtt{w}}$, and consider the 'differential equation'

$$G\left(\frac{d}{dt}\right) w = 0$$

What do we mean by the solutions, i.e. by the behavior?

Rational representations

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What do we mean by the solutions, i.e. by the behavior?

Let (P, Q) be a left coprime polynomial factorization of G

$P, Q \in \mathbb{R}[\xi]^{\bullet \times \bullet}, \det(P) \neq 0, G = P^{-1}Q, [P \vdots Q]$ left-prime.

E.g., in scalar case, means P and Q have no common roots.

$$G(\frac{d}{dt})w = 0 :\Leftrightarrow Q\left(\frac{d}{dt}\right) w = 0$$

Rational representations

Let (P, Q) be a **left coprime** polynomial factorization of G

$$G(\frac{d}{dt})w = 0 :\Leftrightarrow Q\left(\frac{d}{dt}\right)w = 0$$

Justification:

1. G proper. $G(s) = C(Is - A)^{-1}B + D$ controllable realization. Consider output nulling inputs:

$$\frac{d}{dt}x = Ax + Bw, \;\; 0 = Cx + Dw$$

This set of w's are exactly those that satisfy $G\left(\frac{d}{dt}\right)w = 0$.

Same for

$$\frac{d}{dt}x = Ax + Bw, 0 = Cx + D\left(\frac{d}{dt}\right)w = 0, \; D \in \mathbb{R}[\xi]^{\bullet\times\bullet}$$

Rational representations

Let (P, Q) be a **left coprime** polynomial factorization of G

$$G(\frac{d}{dt})w = 0 :\Leftrightarrow Q\left(\frac{d}{dt}\right)w = 0$$

Justification:

2. Consider $y = G(s)u$. View G as a transfer f'n.
Take your usual favorite definition of input/output pairs.

The output nulling inputs are exactly those that satisfy
$G\left(\frac{d}{dt}\right)w = 0$.

Representations

LTIDS

$\mathfrak{B} = \mathrm{kernel}\left(R\left(\frac{d}{dt}\right)\right)$ **for some** $R \in \mathbb{R}[\xi]^{\bullet \times \mathtt{w}}$, **by def.**

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LTIDS

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But we may as well take the representation $G\left(\frac{d}{dt}\right)w = 0$ **for some** $G \in \mathbb{R}(\xi)^{\bullet \times \mathtt{w}}$ **as the definition.**

Representations

LTIDS

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But we may as well take the representation $G\left(\frac{d}{dt}\right)w = 0$ **for some** $G \in \mathbb{R}(\xi)^{\bullet \times \mathtt{w}}$ **as the definition.**
R**: all poles at** ∞**, we can take** G **with no poles at** ∞**, or more generally with all poles in some non-empty set - symmetric w.r.t.** $\mathbb{R}$**. In particular:**

Theorem: Every LTIDS has a representation

$$G\left(\frac{d}{dt}\right)w = 0$$

with $G \in \mathbb{R}(\xi)^{\bullet \times \mathtt{w}}$ **strictly proper stable rational.**

Subrings of $\mathbb{R}(\xi)$

$\mathbb{R}(\xi)$: **real rational functions.**

Consider 3 subrings:

1. $\mathbb{R}[\xi]$: **polynomials with real coefficients**
2. $\mathbb{R}(\xi)_{\mathcal{P}}$: **proper rational functions**
3. $\mathbb{R}(\xi)_{\mathcal{S}}$: **stable proper rational functions**

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   no poles in RHP or ∞

Subrings of $\mathbb{R}(\xi)$

$\mathbb{R}(\xi)$: real rational functions.

Consider 3 subrings:

1. $\mathbb{R}[\xi]$: polynomials with real coefficients — all poles at ∞
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3. $\mathbb{R}(\xi)_{\mathcal{S}}$: stable proper rational functions — no poles in RHP or ∞

Each of these rings has $\mathbb{R}(\xi)$ as its field of fractions.

Unimodular elements (invertible in ring)

1. Non-zero constants
2. bi-proper
3. bi-proper and mini-phase

miniphase:$\Leftrightarrow$ poles & zeros in LHP

Prime representations

Theorem: an LTIDS admits a representation

$$G\left(\frac{d}{dt}\right)w = 0$$

with

1. $G \in \mathbb{R}(\xi)_{\mathcal{P}}^{\bullet\times\mathtt{w}}$ **left prime over** $\mathbb{R}(\xi)_{\mathcal{P}}$ **always**
2. $G \in \mathbb{R}[\xi]^{\bullet\times\mathtt{w}}$ **left prime over** $\mathbb{R}[\xi] \Leftrightarrow$ **it is controllable**
3. $G \in \mathbb{R}(\xi)_{\mathcal{S}}^{\bullet\times\mathtt{w}}$ **left prime over** $\mathbb{R}(\xi)_{\mathcal{S}} \Leftrightarrow$ **it is stabilizable**

The proof of case 3 is not easy!

Image-like representations

Elimination

Consider

$$G_1\left(\frac{d}{dt}\right)w_1 = G_2\left(\frac{d}{dt}\right)w_2$$

$G_1, G_2 \in \mathbb{R}(\xi)^{\bullet\times\bullet}$. **Behavior** $\mathfrak{B}$. **Eliminate** $w_2 \rightsquigarrow$

$$\mathfrak{B}_1 = \{w_1 \mid \exists\, w_2 \text{ such that } (w_1, w_2) \in \mathfrak{B}\}$$

$\mathfrak{B}_1$ **is also a LTID behavior.**

In particular

$$w = H\left(\frac{d}{dt}\right)\ell, \quad H \in \mathbb{R}(\xi)^{\mathtt{w}\times\bullet}.$$

w**-behavior is LTID. Image-like representation.**

Representations of controllable systems

Theorem: The following are equivalent for LTID systems

1. $\mathfrak{B}$ **is controllable**
2. $\mathfrak{B}$ **admits an image-like representation**

$$w = M\left(\frac{d}{dt}\right)\ell \text{ with } H \in \mathbb{R}[\xi]^{\mathtt{w}\times\bullet}$$

3. $\mathfrak{B}$ **admits an image-like representation**

$$w = H\left(\frac{d}{dt}\right)\ell \text{ with } H \in \mathbb{R}(\xi)^{\mathtt{w}\times\bullet}$$

4. **with observability (ℓ can be deduced from w) added**
5. **with $M \in \mathbb{R}[\xi]^{\mathtt{w}\times\bullet}$ right prime over $\mathbb{R}[\xi]$**
6. **with $H \in \mathbb{R}(\xi)_{\mathcal{S}}^{\mathtt{w}\times\bullet}$ right prime over $\mathbb{R}(\xi)_{\mathcal{S}}$**

SUMMARY

- **LSIDS in one-to-one relation with modules**

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- **controllability $\Leftrightarrow$ image representation**

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- **LSIDS in one-to-one relation with modules**
- **controllability $\Leftrightarrow$ image representation**
- **Extends readily to rational functions**

SUMMARY

- **LSIDS in one-to-one relation with modules**
- **controllability $\Leftrightarrow$ image representation**
- **Extends readily to rational functions**
- **Irrelevance of Laplace transforms**

Details & copies of the lecture frames are available from/at

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Thank you

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