RATIONAL SYMBOLS

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Joint research with

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Outline

I. **Behaviors defined by rational symbols**

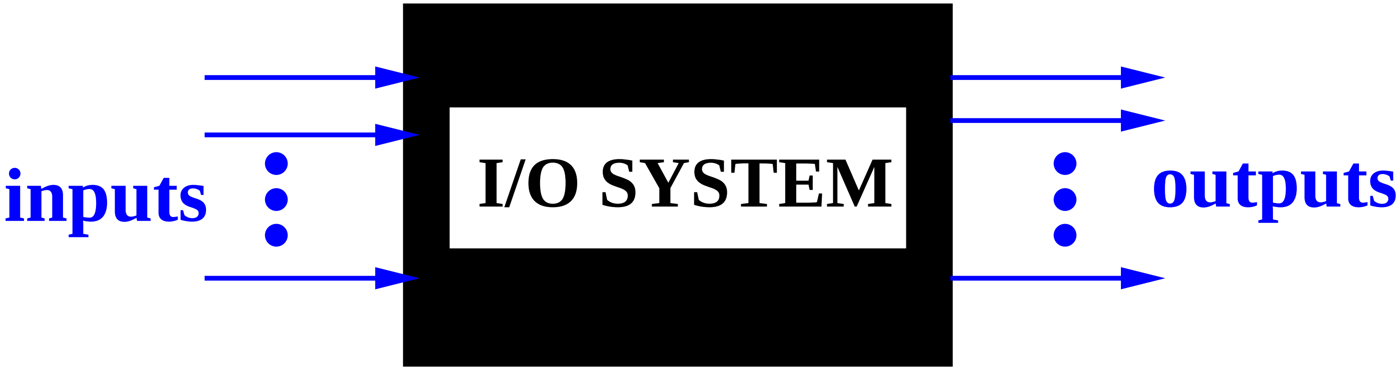
II. **Model reduction**

(III. Parametrization of the stabilizing controllers)

Introduction

Motivation

In system theory, it is customary to think of dynamical models in terms of inputs and outputs.



$\leadsto$ **say,**

$$p_0 y + p_1 \frac{d}{dt} y + \cdots + p_{\mathtt{n}} \frac{d^{\mathtt{n}}}{dt^{\mathtt{n}}} y = q_0 u + q_1 \frac{d}{dt} u + \cdots + q_{\mathtt{n}} \frac{d^{\mathtt{n}}}{dt^{\mathtt{n}}} u$$

i.e., $p(\frac{d}{dt})y = q(\frac{d}{dt})u,$

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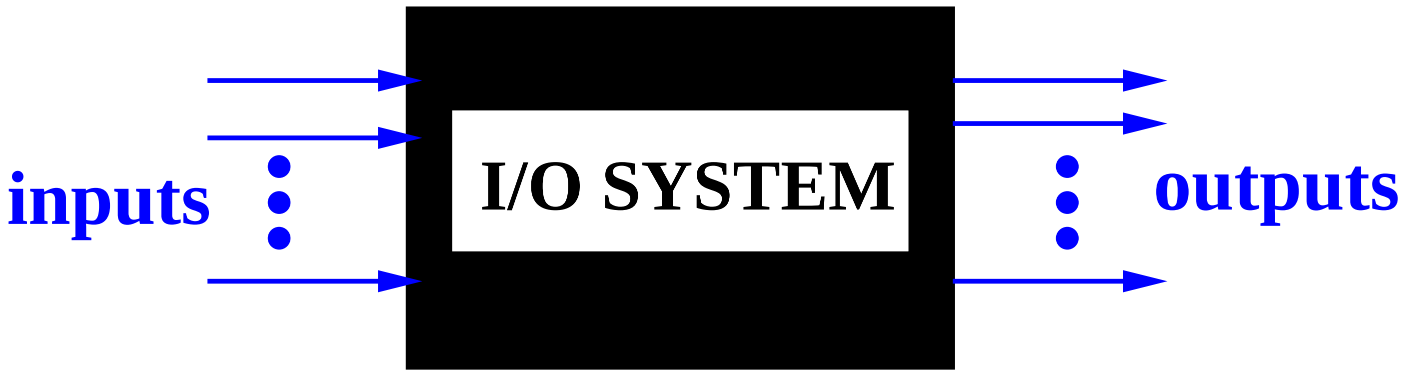

$$p(\tfrac{d}{dt})y = q(\tfrac{d}{dt})u, \qquad \textbf{or} \quad y = F(s)u$$

with p, q **polynomials, or** F **a rational transfer function.**

Motivation

In system theory, it is customary to think of dynamical models in terms of inputs and outputs.

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with p, q polynomials, or F a rational transfer function.

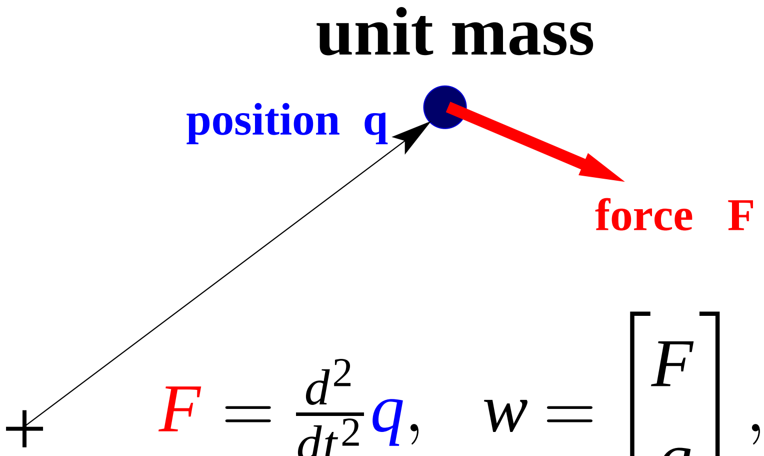
In the present talk, we will

- **(for good reasons) make no distinction between u and y**

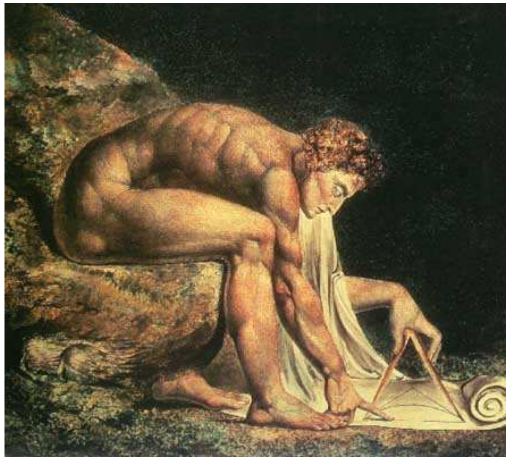
  $$\rightsquigarrow \text{ system variables } \quad w = \begin{bmatrix} u \\ y \end{bmatrix}$$

- **interpret F, not in terms of Laplace transforms, but in terms of differential equations.**
  Important for, among other things, pedagogical reasons.

Example

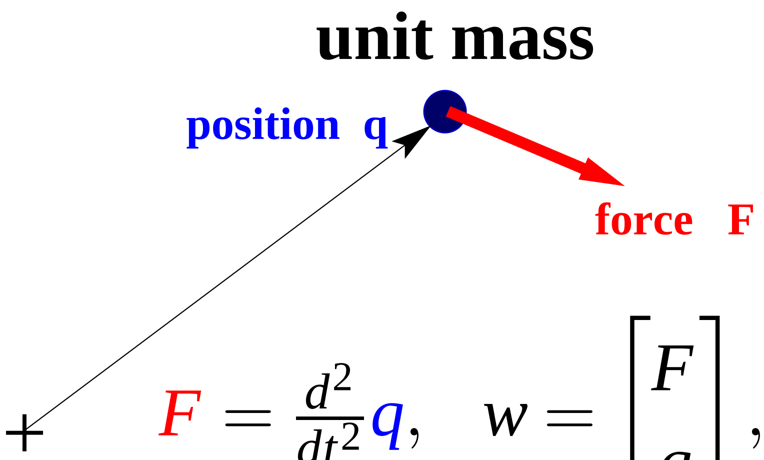


$$F = \frac{d^2}{dt^2} q, \quad w = \begin{bmatrix} F \\ q \end{bmatrix}, \; F, q \in \mathbb{R}^3, w \in \mathbb{R}^6$$



Isaac Newton by William Blake

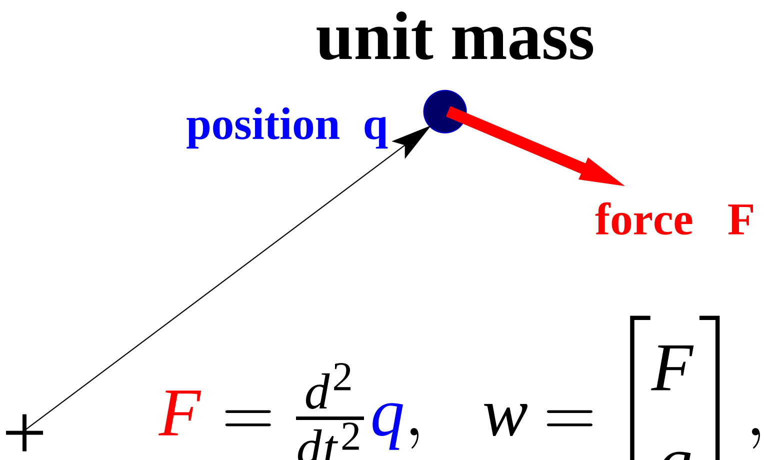
Example



$$F = \frac{d^2}{dt^2} q, \quad w = \begin{bmatrix} F \\ q \end{bmatrix}, \; F, q \in \mathbb{R}^3, w \in \mathbb{R}^6$$

$$\left[I_{3\times 3} \; \vdots \; -(\tfrac{d}{dt})^2 I_{3\times 3} \right] w = 0 \;\rightsquigarrow\; q = \frac{1}{(\frac{d}{dt})^2} F \;\rightsquigarrow\; \left[-\frac{1}{(\frac{d}{dt})^2} I_{3\times 3} \; \vdots \; I_{3\times 3} \right] w = 0$$

Example



$$F = \frac{d^2}{dt^2} q, \quad w = \begin{bmatrix} F \\ q \end{bmatrix}, \; F, q \in \mathbb{R}^3, w \in \mathbb{R}^6$$

$$\begin{bmatrix} I_{3\times 3} \vdots -(\frac{d}{dt})^2 I_{3\times 3} \end{bmatrix} w = 0 \rightsquigarrow q = \frac{1}{(\frac{d}{dt})^2} F \rightsquigarrow \begin{bmatrix} -\frac{1}{(\frac{d}{dt})^2} I_{3\times 3} \vdots I_{3\times 3} \end{bmatrix} w = 0$$

In the scalar case with simple polynomials, it is easy to see how to proceed, but with general multivariable rational functions, less obvious. Today's pbm: What do we mean by

$$y = \frac{q(\frac{d}{dt})}{p(\frac{d}{dt})} u, \qquad \textbf{or} \qquad G(\tfrac{d}{dt}) w = 0 \quad \textbf{with } G \textbf{ rational?}$$

PART I

Linear time-invariant differential systems

LTIDSs

defined by rational symbols

LTIDSs

A system $\rightarrow$ $(\mathbb{T}, \mathbb{W}, \mathscr{B})$ **where**

- $\mathbb{T}$ **= set of independent variables**
  $\mathbb{T}$ **= time** $\leadsto$ **dynamical systems**
  $\mathbb{T}$ **= time & space** $\leadsto$ **distributed systems**

- $\mathbb{W}$ **= set of dependent variables; 'signal space'**

- $\mathscr{B}$ **the *behavior*** $\rightarrow$ $\mathscr{B} \subseteq \mathbb{W}^{\mathbb{T}}$**,**
  set of trajectories $w : \mathbb{T} \rightarrow \mathbb{W}$

$w : \mathbb{T} \rightarrow \mathbb{W}$ **belongs to** $\mathscr{B}$ **means:**
the model 'accepts' the trajectory w

LTIDSs

A dynamical system $\rightarrow$ $(\mathbb{R}, \mathbb{R}^{\mathtt{w}}, \mathscr{B})$ where

- $\mathbb{T}$ = set of independent variables $\rightsquigarrow$ $\mathbb{T} = \mathbb{R}$ 'time'
- $\mathbb{W}$ = set of dependent variables; $\rightsquigarrow$ $\mathbb{W} = \mathbb{R}^{\mathtt{w}}$
- $\mathscr{B}$ the ***behavior*** $\rightarrow$ $\mathscr{B} \subseteq \mathbb{W}^{\mathbb{T}}$, time-trajectories $w : \mathbb{T} \rightarrow \mathbb{W}$

$\mathscr{B}$ = the solutions of a set of

linear constant coefficient ODEs

LTIDSs

A dynamical system → $(\mathbb{R}, \mathbb{R}^{\mathtt{w}}, \mathscr{B})$ **where**

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- $\mathbb{W}$ **= set of dependent variables;** $\rightsquigarrow$ $\mathbb{W} = \mathbb{R}^{\mathtt{w}}$
- $\mathscr{B}$ **the *behavior*** → $\mathscr{B} \subseteq \mathbb{W}^{\mathbb{T}}$,

$\mathscr{B}$ **= the solutions of**

$$R_0 w + R_1 \frac{d}{dt} w + \cdots + R_{\mathtt{L}} \frac{d^{\mathtt{L}}}{dt^{\mathtt{L}}} w = 0, \quad R_0, R_1, \ldots \textbf{ matrices}$$

Polynomial matrix notation $\rightsquigarrow$ $R\left(\frac{d}{dt}\right) w = 0$ $\quad R \in \mathbb{R}[\xi]^{\bullet \times \mathtt{w}}$

LTIDSs

A dynamical system $\rightarrow$ $(\mathbb{R}, \mathbb{R}^{\mathtt{w}}, \mathscr{B})$ **where**

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- $\mathbb{W}$ **= set of dependent variables;** $\rightsquigarrow$ $\mathbb{W} = \mathbb{R}^{\mathtt{w}}$
- $\mathscr{B}$ **the *behavior*** $\rightarrow$ $\mathscr{B} \subseteq \mathbb{W}^{\mathbb{T}}$**,**

$\mathscr{B}$ **= the** $\mathscr{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$**-solutions of**

$$R_0 w + R_1 \frac{d}{dt} w + \cdots + R_{\mathtt{L}} \frac{d^{\mathtt{L}}}{dt^{\mathtt{L}}} w = 0$$

Polynomial matrix notation $\rightsquigarrow$ $R\left(\frac{d}{dt}\right) w = 0$

Representations

Behaviors of LTIDSs allow many useful representations

- **As the set of solutions of** $R\left(\frac{d}{dt}\right)w = 0$ $\quad R \in \mathbb{R}[\xi]^{\bullet\times\mathtt{w}}$

Representations

Behaviors of LTIDSs allow many useful representations

- **As the set of solutions of** $R\left(\frac{d}{dt}\right)w = 0$ $\quad R \in \mathbb{R}[\xi]^{\bullet\times \mathtt{w}}$
- **With input/output partition**

$$P\left(\frac{d}{dt}\right)y = Q\left(\frac{d}{dt}\right)u \quad w \cong \begin{bmatrix} u \\ y \end{bmatrix} \quad \det(P) \neq 0, P^{-1}Q \textbf{ proper}$$

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- **As the set of solutions of** $R\left(\frac{d}{dt}\right)w = 0 \quad R \in \mathbb{R}[\xi]^{\bullet \times \mathtt{w}}$
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- **Input/state/output representation**

  $\exists$ **matrices** A, B, C, D **such that**

  $\mathscr{B}$ **consists of all** $w's$ **generated by**

$$\frac{d}{dt}x = Ax + Bu,\ y = Cx + Du \quad w \cong \begin{bmatrix} u \\ y \end{bmatrix}$$

Rudolf E. Kalman

- ...

Rational Symbols

Rational representations

In signal processing, control, etc., we often meet models that involve rational functions, instead of ODEs. Cfr. transfer functions,

$$y = F(\text{‘s’})u$$

etc. $\rightsquigarrow$

Let $G \in \mathbb{R}(\xi)^{\bullet \times \mathtt{w}}$**, and consider the 'differential equation'**

$$G\left(\frac{d}{dt}\right)w = 0 \qquad G \text{ is called the ‘symbol’}$$

What do we mean by its solutions, i.e. by the behavior?

Rational representations

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What do we mean by its solutions, i.e. by the behavior?

$[\![M$ **left prime** $]\!] :\Leftrightarrow [\![\; [\![M = FM']\!] \Rightarrow [\![F$ **unimodular** $]\!] \;]\!]$
$\Leftrightarrow \quad \exists\, H$ **such that** $MH = I$**.**
In scalar case, $M = \begin{bmatrix} m_1 & m_2 & \cdots & m_{\mathtt{n}} \end{bmatrix}$**, this means:**

$m_1, m_2, \cdots, m_{\mathtt{n}}$ **have no common root.**

Rational representations

Let $G \in \mathbb{R}(\xi)^{\bullet \times \mathtt{w}}$**, and consider the 'differential equation'**

$$G\left(\frac{d}{dt}\right)w = 0$$ G **is called the 'symbol'**

What do we mean by its solutions, i.e. by the behavior?

Let (P, Q) **be a left coprime polynomial factorization of** G

i.e. $P, Q \in \mathbb{R}[\xi]^{\bullet \times \bullet}, \det(P) \neq 0, G = P^{-1}Q, [P \vdots Q]$ **left-prime.**

E.g., in scalar case, means P **and** Q **have no common roots.**

Rational representations

Let (P, Q) **be a left coprime polynomial factorization of** G

$$[\![G(\frac{d}{dt})w = 0]\!] \Leftrightarrow [\![P^{-1}Q(\frac{d}{dt})w = 0]\!] :\Leftrightarrow [\![Q\left(\frac{d}{dt}\right)w = 0]\!]$$

By definition, therefore, the behavior of $G(\frac{d}{dt})w = 0$ **is equal to the behavior of** $Q(\frac{d}{dt})w = 0$**.**

Rational representations

Let (P,Q) be a left coprime polynomial factorization of G

$$[\![G(\frac{d}{dt})w = 0]\!] \Leftrightarrow [\![P^{-1}Q(\frac{d}{dt})w = 0]\!] :\Leftrightarrow [\![\, Q\left(\frac{d}{dt}\right) w = 0 \,]\!]$$

Justification:

1. G **proper.** $G(\xi) = C(I\xi - A)^{-1}B + D$ **controllable realization. Consider output nulling inputs:**

$$\frac{d}{dt}x = Ax + Bw, \quad 0 = Cx + Dw$$

This set of w**'s are exactly those that satisfy** $G\left(\frac{d}{dt}\right)w = 0$.

Analogous for $\frac{d}{dt}x = Ax + Bw, 0 = Cx + D\left(\frac{d}{dt}\right)w$**,** $\quad D \in \mathbb{R}[\xi]^{\bullet\times\bullet}$**.**

Rational representations

Let (P, Q) be a left coprime polynomial factorization of G

$$[\![G(\frac{d}{dt})w = 0]\!] \Leftrightarrow [\![P^{-1}Q(\frac{d}{dt})w = 0]\!] :\Leftrightarrow [\![\; Q\left(\frac{d}{dt}\right)w = 0 \;]\!]$$

Justification:

2. Consider $y = G(s)w$. View $G(s)$ as a transfer f'n.
Take your favorite definition of input/output pairs.

Output nulling inputs exactly those that satisfy $G\left(\frac{d}{dt}\right)w = 0$.

3. ...

Rational representations

Let (P,Q) be a left coprime polynomial factorization of G

$$[\![G(\frac{d}{dt})w = 0]\!] \Leftrightarrow [\![P^{-1}Q(\frac{d}{dt})w = 0]\!] :\Leftrightarrow [\![\; Q\left(\tfrac{d}{dt}\right)w = 0 \;]\!]$$

Note! With this def., we can deal with transfer functions,

$$y = F(\frac{d}{dt})u, \quad \textbf{i.e.} \quad \begin{bmatrix} F(\frac{d}{dt}) & \vdots & -I \end{bmatrix} \begin{bmatrix} u \\ y \end{bmatrix} = 0$$

with F a matrix of rational functions, while completely avoiding Laplace transforms, domains of convergence, and such mathematical traps.

Caveats

$F\left(\frac{d}{dt}\right)$ is not a map!

Consider

$$y = F\left(\frac{d}{dt}\right) u$$

We now know what it means that $(u, y) \in \mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^\bullet)$ **satisfies this 'ODE'.**

Is there a unique y **for a given** u**?**

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Consider

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Is there a unique y for a given u?

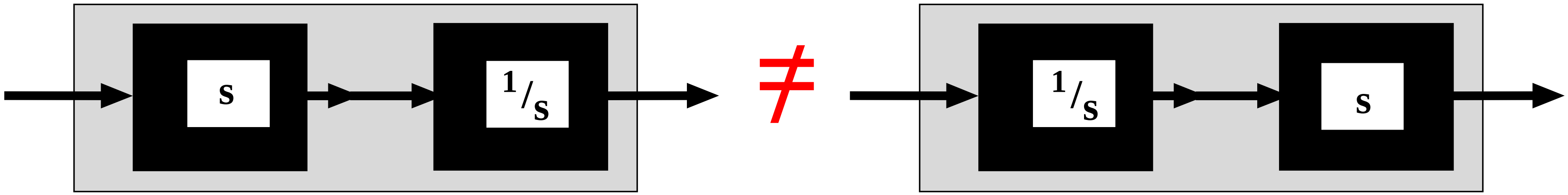
$F = P^{-1}Q$ **coprime fact.** $\Leftrightarrow P^{-1}\begin{bmatrix} P & -Q \end{bmatrix}$ **coprime fact.**

$$F = P^{-1}Q \quad \rightsquigarrow \quad y = F\left(\tfrac{d}{dt}\right)u \Leftrightarrow P\left(\frac{d}{dt}\right)y = Q\left(\frac{d}{dt}\right)u$$

If $P \neq I$ (better, not unimodular), there are many sol'ns y of this ODE for a given u.

$$y = y_{\textbf{particular}} + y_{\textbf{homogeneous}} \qquad P\left(\frac{d}{dt}\right)y_{\textbf{homogeneous}} = 0$$

$G_1\left(\frac{d}{dt}\right)$ and $G_2\left(\frac{d}{dt}\right)$ do not commute



$$G_1(s) = \frac{1}{s} \text{ and } G_2(s) = s$$

$$y = \frac{1}{\frac{d}{dt}} v, \;\; v = \frac{d}{dt} u \quad \Rightarrow \quad y(t) = u(t) + \text{ constant}$$

$$y = \frac{d}{dt} v, \;\; v = \frac{1}{\frac{d}{dt}} u \quad \Rightarrow \quad y(t) = u(t)$$

Raison d'être

LTIDSs are defined in terms of polynomial symbols

$$R\left(\frac{d}{dt}\right)w = 0 \qquad R \in \mathbb{R}[\xi]^{\bullet\times\mathtt{w}}$$

(behavior $\mathscr{B}$:= the $\mathscr{C}^\infty\left(\mathbb{R},\mathbb{R}^{\mathtt{w}}\right)$ solutions)

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$$G\left(\frac{d}{dt}\right)w = 0 \qquad G \in \mathbb{R}(\xi)^{\bullet \times \mathtt{w}}$$

Behavior := the set of solutions of

$$Q\left(\frac{d}{dt}\right)w = 0 \qquad Q \in \mathbb{R}[\xi]^{\bullet \times \mathtt{w}}$$

where $G = P^{-1}Q, \quad P, Q \in \mathbb{R}[\xi]^{\bullet \times \bullet}, \quad P$ **and** Q **left coprime**

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$$G\left(\frac{d}{dt}\right)w = 0 \qquad G \in \mathbb{R}(\xi)^{\bullet\times\mathtt{w}}$$

This added flexibility $\leadsto$ better adapted to certain applications,

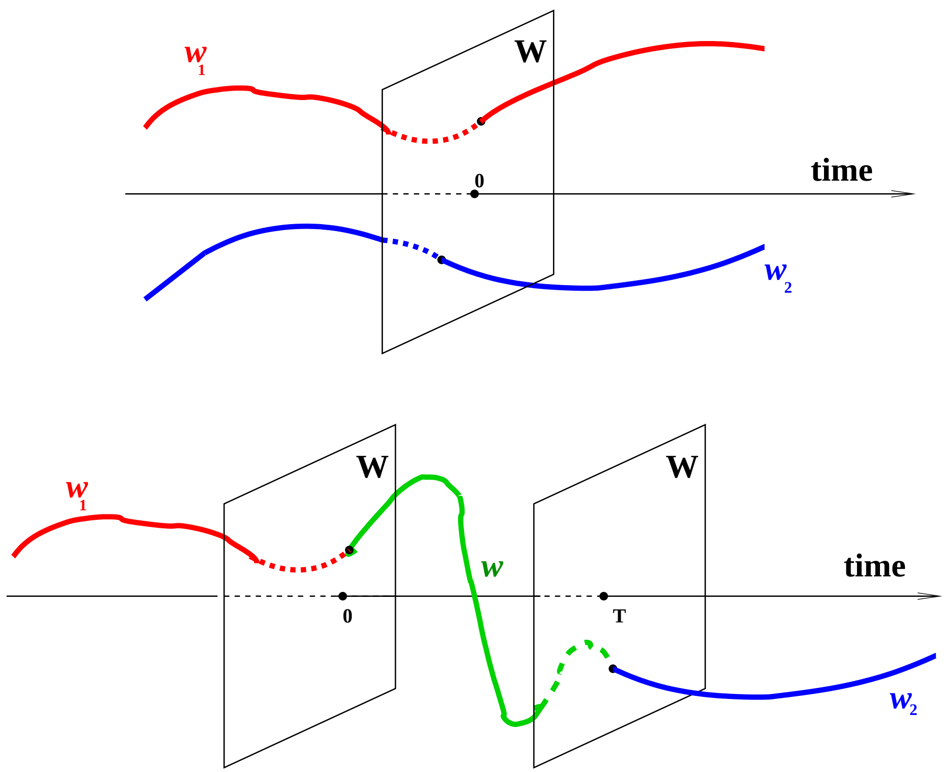
e.g. distance between systems
e.g. behavioral model reduction
e.g. parametrization of the set of stabilizing controllers

Controllability c.s.

Controllability and stabilizability

$\mathscr{B}$ **is said to be controllable** $:\Leftrightarrow$

$\forall\, w_1, w_2 \in \mathscr{B},\ \exists\, T \geq 0$ **and** $w \in \mathscr{B}$ **such that ...**

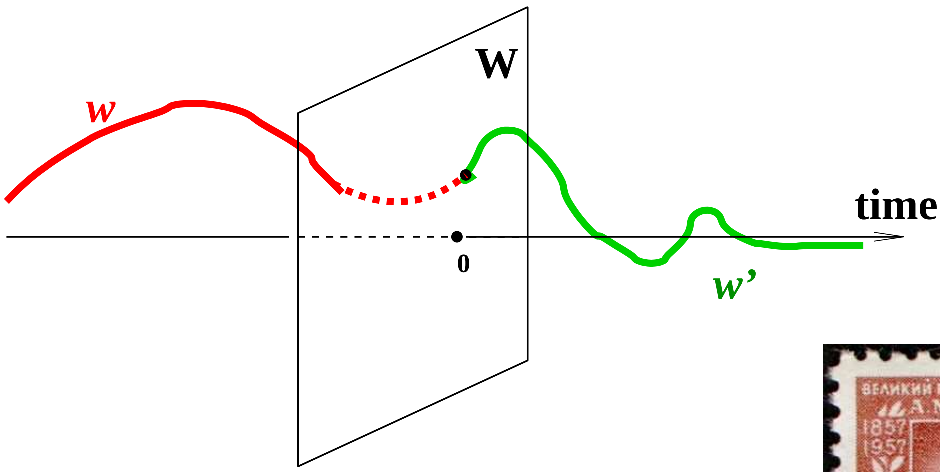


Controllability and stabilizability

$\mathscr{B}$ **is said to be controllable** $:\Leftrightarrow$

$\mathscr{B}$ **is said to be stabilizable** $:\Leftrightarrow$

$\forall\, w \in \mathscr{B},\ \exists\, w' \in \mathscr{B}$ **such that ...**



Stability in the sense of Lyapunov

Representations

What properties on G imply that the system with rational representation

$$G\left(\frac{d}{dt}\right)w = 0 \qquad G \in \mathbb{R}(\xi)^{\bullet \times \mathtt{w}}$$

has any of these properties?

Representations

What properties on G imply that the system with rational representation

$$G\left(\frac{d}{dt}\right)w = 0 \qquad G \in \mathbb{R}(\xi)^{\bullet \times \mathtt{w}}$$

has any of these properties?

Under what conditions on G does $G\left(\frac{d}{dt}\right)w = 0$ define a controllable or a stabilizable system?

Can a rational representation be used to put one of these properties in evidence?

Tests

Theorem: The LTIDS

$$G\left(\frac{d}{dt}\right)w = 0 \qquad G \in \mathbb{R}(\xi)^{\bullet \times \mathtt{w}}$$

is controllable if and only if

$$G(\lambda) \text{ has the same rank } \forall \lambda \in \mathbb{C}$$

Interpret carefully in cases like

$$G(s) = \begin{bmatrix} s & 0 \\ 0 & \frac{1}{s} \end{bmatrix}, G(s) = \begin{bmatrix} s \\ \frac{1}{s} \end{bmatrix}, G(s) = \begin{bmatrix} s & \frac{1}{s} \end{bmatrix}$$

Tests

Theorem: **The LTIDS**

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Theorem: **The LTIDS**

$$G\left(\frac{d}{dt}\right)w = 0 \qquad G \in \mathbb{R}(\xi)^{\bullet \times \mathtt{w}}$$

is stabilizable if and only if

$$G(\lambda) \text{ has the same rank } \forall \lambda \in \mathbb{C} \text{ with realpart}(\lambda) \geq 0$$

Image representation

For example,

Theorem: A LTIDS is controllable if and only if its behavior allows an image representation

$$w = M\left(\frac{d}{dt}\right)\ell \qquad M \in \mathbb{R}(\xi)^{\mathtt{w}\times\bullet}$$

Module & vector spaces

Take a LTIDS $\mathscr{B}$.

$n \in \mathbb{R}(\xi)^{1\times \mathtt{w}}$ **is an annihilator** $:\Leftrightarrow n(\frac{d}{dt})\mathscr{B} = 0$**, i.e.,**

$$n(\frac{d}{dt})w = 0 \;\; \forall\, w \in \mathscr{B}$$

What structure does the set of annihilators of a given $\mathscr{B}$ have?

Module & vector spaces

Take a LTID behavior $\mathscr{B}$.

$n \in \mathbb{R}[\xi]^{1\times \mathtt{w}}$ **is a polynomial annihilator** $:\Leftrightarrow n(\frac{d}{dt})\mathscr{B} = 0$

The polynomial annihilators form a $\mathbb{R}[\xi]$-module:
n_1, n_2 **polynomial annihilators,** $p \in \mathbb{R}[\xi]$
$\Rightarrow n_1 + pn_2$ **polynomial annihilator.**

Module & vector spaces

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n_1, n_2 **polynomial annihilators,** $p \in \mathbb{R}[\xi]$
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$n \in \mathbb{R}(\xi)^{1\times \mathtt{w}}$ **is a rational annihilator** $:\Leftrightarrow n(\frac{d}{dt})\mathscr{B} = 0$

The rational annihilators of a controllable $\mathscr{B}$ form a $\mathbb{R}(\xi)$-vector space:
n_1, n_2 **rational annihilators,** $p \in \mathbb{R}(\xi)$
$\Rightarrow n_1 + pn_2$ **rational annihilator.**

Module & vector spaces

By identifying a system with its polynomial annihilators, we obtain the one-to-one relation between LTIDSs with $\mathtt{w}$ variables and the

$$\mathbb{R}[\xi]\text{-} \textbf{submodules of } \mathbb{R}[\xi]^{\mathtt{w}}$$

By identifying a system with its rational annihilators, we obtain the one-to-one relation between the controllable LTIDSs with $\mathtt{w}$ variables and the

$$\mathbb{R}(\xi)\text{-} \textbf{subspaces of } \mathbb{R}(\xi)^{\mathtt{w}}$$

LTIDS $\cong$ finite dimensional $\mathbb{R}[\xi]$-modules

Controllable LTIDS $\cong$ finite dimensional $\mathbb{R}(\xi)$-subspaces.

PART II

Model reduction

Reducing the state dimension

What is a good, computable, definition for the distance between two LTIDS?

Basic issue underlying model reduction, robustness, etc.

- **Approximate a system by a simpler one.**
- **If a system has a particular property (e.g., stabilized by a controller), will this also hold for close by systems?**

What is meant by ‘approximate’, by ‘close by’?

Reducing the state dimension

There is an elegant theory for reducing the state space dimension of stable LTI input/output systems.
Let $\mathscr{B}$ be described by

$$\frac{d}{dt}x = Ax + Bu,\; y = Cx + Du \quad w \cong \begin{bmatrix} u \\ y \end{bmatrix}$$

with A Hurwitz(:⇔ eigenvalues in left half plane).

There are effective methods (balancing, AAK) with good error bounds (in terms of the $\mathscr{H}_\infty$ norm) for approximating $\mathscr{B}$ by a (stable) system with a lower dimensional state space.

Keith Glover

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Let $\mathscr{B}$ be described by

$$\frac{d}{dt}x = Ax + Bu, \; y = Cx + Du$$

with A Hurwitz. T'f f'n $F(s) = C(Is - A)^{-1}B + D$

proper stable rational. Reduced system

$$\frac{d}{dt}x_{\text{reduced}} = A_{\text{reduced}}x_{\text{reduced}} + B_{\text{reduced}}u, \; y = C_{\text{reduced}}x_{\text{reduced}} + Du$$

T'f f'n $F_{\text{reduced}}(s) = C_{\text{reduced}}(Is - A_{\text{reduced}})^{-1}B_{\text{reduced}} + D$
proper stable rational. Balanced model reduction $\Rightarrow$

$$||F(i\omega) - F_{\text{reduced}}(i\omega)|| \leq 2\left(\sum_{\textbf{neglected Hankel SVs}} \sigma_k\right) \quad \forall \omega \in \mathbb{R}$$

Reducing the state dimension

There is an elegant theory for reducing the state space dimension of stable LTI input/output systems.
Let $\mathscr{B}$ be described by

$$\frac{d}{dt}x = Ax + Bu, \; y = Cx + Du$$

with A Hurwitz.

$F(s)$ **proper stable rational** $\Rightarrow$ **reducible !**

Extend this to situations where we do not make a distinction between inputs and outputs, and to unstable systems!

Distance between systems

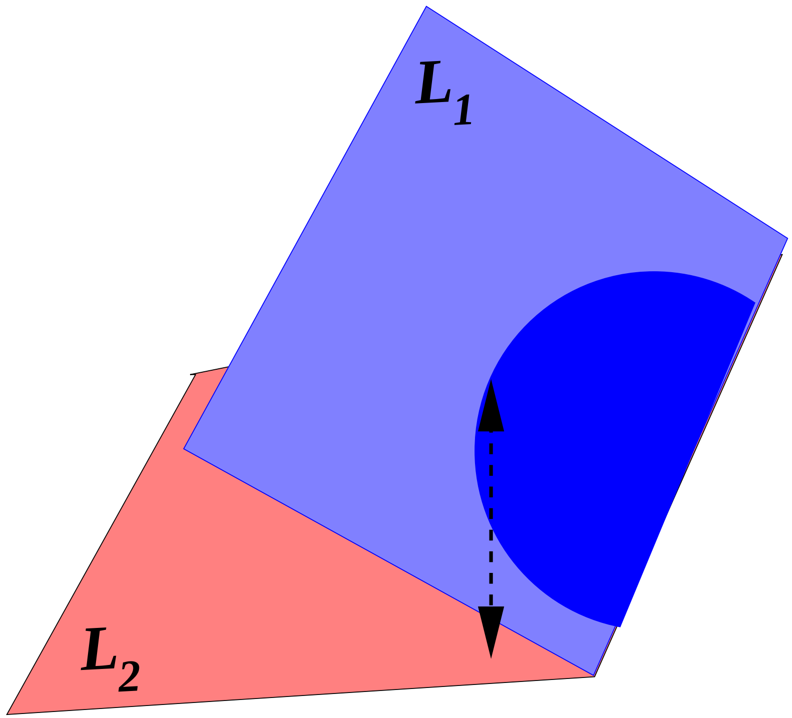
Distance between linear subspaces

In the behavioral theory, we identify a dynamical system with its behavior, a subspace $\mathscr{B} \subseteq \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$. We are hence led to study the distance between linear subspaces of a vector space.

Linear subspaces of $\mathbb{R}^n$

$\mathcal{L}_1, \mathcal{L}_2 \subseteq \mathbb{R}^n$, **linear subspaces**

$$\overrightarrow{d}(\mathcal{L}_1, \mathcal{L}_2) \cong \max_{x_1 \in \mathcal{L}_1, ||x_1||=1} \; \min_{x_2 \in \mathcal{L}_2} ||x_1 - x_2||$$



Linear subspaces of $\mathbb{R}^{\mathtt{n}}$

$\mathscr{L}_1, \mathscr{L}_2 \subseteq \mathbb{R}^{\mathtt{n}}$, **linear subspaces**

$$d(\mathscr{L}_1, \mathscr{L}_2) :=$$

$$\max\{ \max_{x_1 \in \mathscr{L}_1, ||x_1||=1} \min_{x_2 \in \mathscr{L}_2} ||x_1 - x_2||, \max_{x_2 \in \mathscr{L}_1, ||x_2||=1} \min_{x_1 \in \mathscr{L}_1} ||x_1 - x_2|| \}$$

$$0 \leq d(\mathscr{L}_1, \mathscr{L}_2) \leq 1$$

$= 1$ **if** $\mathtt{dimension}(\mathscr{L}_1) \neq \mathtt{dimension}(\mathscr{L}_2)$

Linear subspaces of $\mathbb{R}^{\mathtt{n}}$

$\mathscr{L}_1, \mathscr{L}_2 \subseteq \mathbb{R}^{\mathtt{n}}$, **linear subspaces**
$P_{\mathscr{L}}$ $\perp$ **projection onto** $\mathscr{L}$
S_1, S_2 **matrices, columns orthonormal basis for** $\mathscr{L}_1, \mathscr{L}_2$
$S_1 S_1^\top, S_2 S_2^\top$ **orthogonal projectors**

$$
\begin{aligned}
d(\mathscr{L}_1, \mathscr{L}_2) &= ||P_{\mathscr{L}_1} - P_{\mathscr{L}_2}|| \qquad \text{'gap', 'aperture'} \\
&= ||S_1 S_1^\top - S_2 S_2^\top|| \\
&= \min_{\text{matrices } U} ||S_1 - S_2 U|| \\
&= \min_{U \text{ such that } U\mathscr{L}_1 = \mathscr{L}_2} ||I - U||
\end{aligned}
$$

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'`gap`**', 'aperture'**

Note

$$d(\mathscr{L}_1, \mathscr{L}_2) = \|S_1 S_1^\top - S_2 S_2^\top\| \leq \|S_1 - S_2\|$$

Distance between controllable behaviors

$\min \rightarrow \inf, \max \rightarrow \sup$**, etc., readily generalized to closed subspaces of Hilbert space.**

For LTIDS, behaviors $\mathscr{B} \mapsto \mathscr{B} \cap \mathscr{L}_2(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$**. Keep notation.**
So, we consider only $\mathscr{L}_2$**-behavior for measuring distance.**

$$d(\mathscr{B}_1, \mathscr{B}_2) := \mathtt{gap}(\mathscr{B}_1, \mathscr{B}_2)$$

$\forall w_1 \in \mathscr{B}_1, \exists w_2 \in \mathscr{B}_2$ **such that** $||w_1 - w_2|| \leq \mathtt{gap}(\mathscr{B}_1, \mathscr{B}_2)||w_1||$

and vice-versa. **Small gap** $\Rightarrow$ **the models are 'close'.**

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and vice-versa. Small gap $\Rightarrow$ **the models are 'close'.**

- **How to compute the gap?**
- **Model reduce according to the gap!**

Norm-preserving representations

Let $\mathscr{B}$ be the behavior of a controllable LTIDS. Then it allows a rational symbol based image representation

$$w = M(\frac{d}{dt})\ell \quad \textbf{with} \quad M \in \mathbb{R}(\xi)^{\mathtt{w}\times\bullet} \ \ \textbf{\&} \ \ M(-\xi)^\top M(\xi) = I$$

i.e., $||\ell||^2_{\mathscr{L}_2(\mathbb{R},\mathbb{R}^\bullet)} = ||w||^2_{\mathscr{L}_2(\mathbb{R},\mathbb{R}^{\mathtt{w}})}$ **'norm preserving image repr.'**

$$\int_{-\infty}^{+\infty} ||w(t)||^2\, dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} ||\hat{w}(i\omega)||^2\, d\omega =$$

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} ||M(i\omega)\hat{\ell}(i\omega)||^2\, d\omega = \frac{1}{2\pi}\int_{-\infty}^{+\infty} ||\hat{\ell}(i\omega)||^2\, d\omega = \int_{-\infty}^{+\infty} ||\ell(t)||^2\, dt$$

Note: M cannot be polynomial, it must be rational.
Obviously M must be proper. Can also make it stable.

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Note: M cannot be polynomial, it must be rational.
Obviously M must be proper. Can also make it stable.
Proof: Start with an observable polynomial image representation $w = N(\frac{d}{dt})\ell, N \in \mathbb{R}[\xi]^{\mathtt{w}\times\mathtt{m}(\mathscr{B})}$. Factor

$$N(-\xi)^\top N(\xi) = F(-\xi)^\top F(\xi), F \in \mathbb{R}[\xi]^{\mathtt{m}(\mathscr{B})\times\mathtt{m}(\mathscr{B})}$$

Can make $\mathtt{determinant}(F)$ Hurwitz. Take $M = NF^{-1}$.

Norm-preserving representations

Let $\mathscr{B}$ be the behavior of a controllable LTIDS. Then it allows a rational symbol based image representation

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Obviously M must be proper. Can also make it stable.
Note that

$$f \in \mathscr{L}_2(\mathbb{R},\mathbb{R}^{\mathtt{w}}) \mapsto M(i\omega)M(-i\omega)^\top \hat{f}(i\omega)$$

is the orthogonal projection onto $\mathscr{B} \cap \mathscr{L}_2(\mathbb{R},\mathbb{R}^{\mathtt{w}})$.

Norm-preserving representations

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Note: M **cannot be polynomial, it must be rational.**
Obviously M **must be proper. Can also make it stable.**
$\mathscr{B}_1 \mapsto M_1, \mathscr{B}_2 \mapsto M_2$ **norm preserving, then**

$$\begin{aligned} \mathtt{gap}(\mathscr{B}_1, \mathscr{B}_2) &= ||M_1(i\omega)M_1(-i\omega)^\top - M_2(i\omega)M_2(-i\omega)^\top||_{\mathscr{L}_\infty} \\ &\leq ||M_1(i\omega) - M_2(i\omega)||_{\mathscr{H}_\infty} \end{aligned}$$

Model reduction by balancing

Start with $\mathscr{B}$. Take representatation

$$w = M(\frac{d}{dt})\ell \quad \textbf{with} \quad M \in \mathbb{R}(\xi)^{\mathtt{w}\times\bullet} \quad \textbf{norm preserving, stable}$$

Now model reduce $w = M(\frac{d}{dt})\ell$ (viewed as a stable input/output system) using, for example, balancing

$$\rightsquigarrow \quad w = M_{\mathtt{reduced}}(\frac{d}{dt})\ell$$

and an error bound

$$||M - M_{\mathtt{reduced}}||_{\mathscr{H}_\infty} \leq 2 \left(\Sigma_{\textbf{neglected SVs of } M}\, \sigma_{\mathtt{k}}\right)$$

Behavioral error bound

Start with stable norm preserving representation of $\mathscr{B}$

$$w = M(\frac{d}{dt})\ell \quad \textbf{with} \quad M \in \mathbb{R}(\xi)^{\mathtt{w}\times\bullet}$$

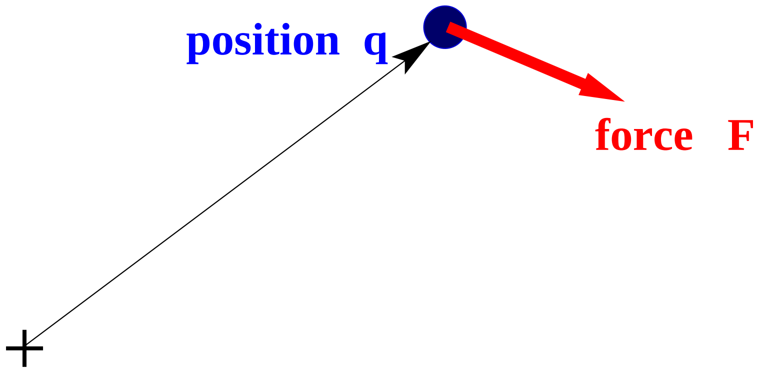
Model reduce using balancing $\rightsquigarrow \quad w = M_{\mathtt{reduced}}(\frac{d}{dt})\ell$.
Call behavior $\mathscr{B}_{\text{reduced}}$. Error bound

$$\begin{aligned} \mathrm{gap}(\mathscr{B}, \mathscr{B}_{\text{reduced}}) &= ||MM^\top - M_{\mathtt{reduced}}M_{\mathtt{reduced}}^\top||_{\mathscr{L}_\infty} \\ &\leq ||M - M_{\mathtt{reduced}}||_{\mathscr{H}_\infty} \\ &\leq 2\ \left(\textstyle\sum_{\textbf{neglected SVs of } M} \sigma_{\mathrm{k}}\right) \end{aligned}$$

$\forall w \in \mathscr{B} \exists w' \in \mathscr{B}_{\text{red}}$ **such that** $||w - w'|| \leq 2(\sum_{\textbf{neglected SVs}} \sigma_{\mathrm{k}})||w||$

and vice-versa.
$\sum_{\textbf{neglected SVs}} \sigma_{\mathrm{k}}$ **small** $\Rightarrow$ **good approximation in the gap.**

Example



$$F = \frac{d^2}{dt^2} q, \qquad w = \begin{bmatrix} F \\ q \end{bmatrix}$$

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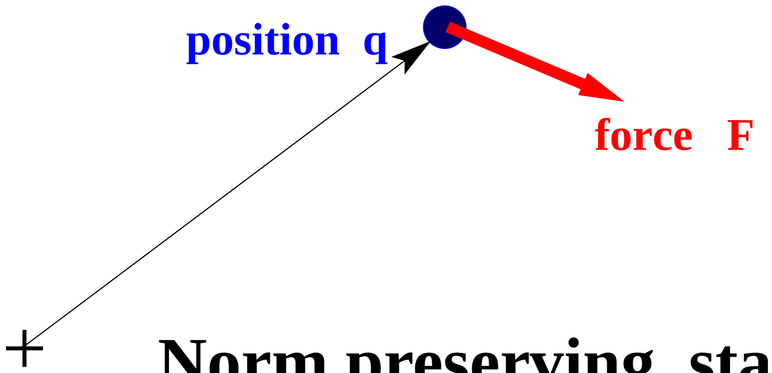

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Norm preserving, stable

$$\begin{bmatrix} F \\ q \end{bmatrix} \cong \begin{bmatrix} \frac{\xi^2}{\xi^2 + \sqrt{2}\xi + 1} \\ \frac{1}{\xi^2 + \sqrt{2}\xi + 1} \end{bmatrix} \ell$$

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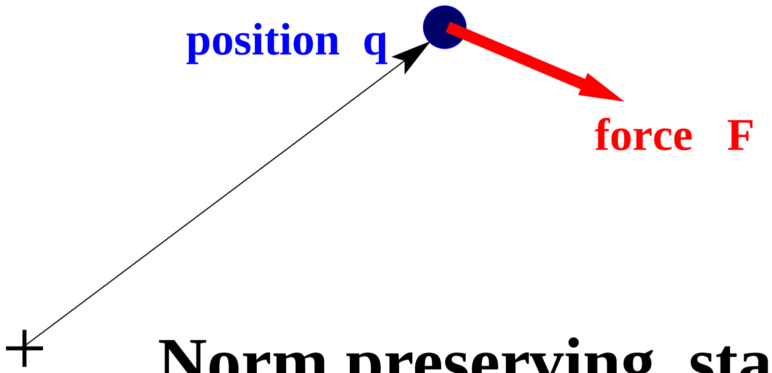

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reduced model

$$\begin{bmatrix} F \\ q \end{bmatrix} \cong \begin{bmatrix} \frac{\xi-\frac{1}{2}}{\xi+\frac{1}{\sqrt{2}}} \\ \frac{\frac{1}{2}}{\xi+\frac{1}{\sqrt{2}}} \end{bmatrix} \ell$$

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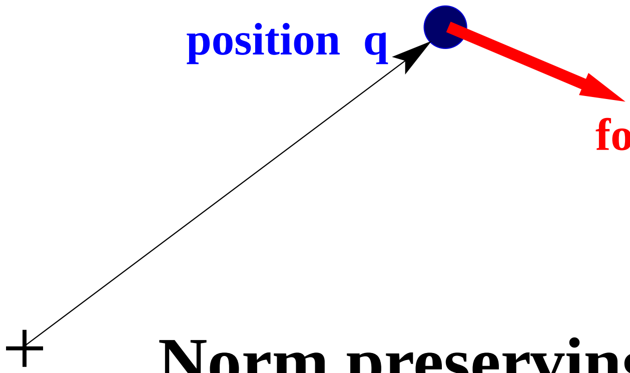

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$F = \frac{d^2}{dt^2} q$ **first order approximation** $\frac{1}{2} F = \frac{d}{dt} q - \frac{1}{2} q$

Summary

Conclusions

- $G(\frac{d}{dt})w = 0$ **defined in terms left-coprime factorization of rational** G**.**

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- **Stable norm preserving representation** $w = M(\frac{d}{dt})\ell$ **leads to model reduction of unstable systems and systems without input/output partition.**

PART III

Parametrization of stabilizing controllers

$\mathbb{R}(\xi)$ and some of its subrings

Relevant rings

Field of (real) rationals

Subrings of interest

- **polynomials**
- **proper rationals**
- **stable rationals**
- **proper stable rationals**

Relevant rings

unimodularity $:\Leftrightarrow$ **invertibility in the ring**

Field of (real) rationals	**nonzero**
Subrings of interest	
polynomials	**nonzero constant**
proper rationals	**biproper**
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unimodularity of square matrices over rings

$\Leftrightarrow$ `determinant` **unimodular**

left primeness of matrices over rings

$$:\Leftrightarrow \left[\!\left[[\![M = FM']\!] \Rightarrow [\![F \text{ unimodular}]\!] \right]\!\right]$$

Representability

The LTIDS $\mathscr{B}$ admits a representation that is left prime over

- **rationals: always**
- **proper rationals: always**

- **stable rationals: iff $\mathscr{B}$ is stabilizable**
- **proper stable rationals: iff $\mathscr{B}$ is stabilizable**

- **polynomials: iff $\mathscr{B}$ is controllable**

Left prime representations over subrings allow to express certain system properties...

Representability

The LTIDS $\mathscr{B}$ admits a representation that is left prime over

- **stable rationals: iff $\mathscr{B}$ is stabilizable**
- **proper stable rationals: iff $\mathscr{B}$ is stabilizable**

$\mathscr{B}$ stabilizable $\Leftrightarrow \exists G$, matrix of rational functions, such that

(i) $\mathscr{B} = \mathtt{kernel}\left(G\left(\frac{d}{dt}\right)\right)$

(ii) G **is proper (no poles at ∞)**

(iii) $G^{\infty} := \mathtt{limit}_{\lambda \to \infty} G(\lambda)$ **has full row rank (no zeros at ∞)**

(iv) G **has no poles in** $\mathbb{C}_{+} := \{\lambda \in \mathbb{C} \mid \mathtt{real}(\lambda \geq 0\}$

(v) $G(\lambda)$ **has full row rank** $\forall\, \lambda \in \mathbb{C}_{+}$ **(no zeros in $\mathbb{C}_{+}$)**

Unimodular completion

Unimodular completion lemma

Let G be a matrix over one of our rings
(polynomial, proper rat., stable rat., proper stable rat.).

¿ Does there exist a unimodular completion G'
i.e. a matrix G' over that same ring such that

$$\begin{bmatrix} G \\ G' \end{bmatrix}$$

is unimodular (`determinant` is invertible in the ring) ?

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¡ There exists a unimodular completion G'
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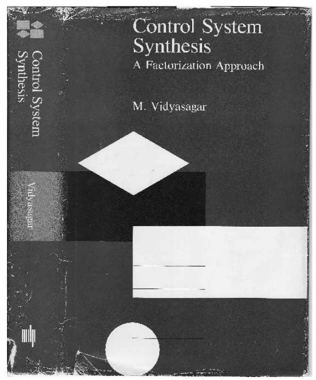
is unimodular

if and only if

G is left prime over the ring !



M. Vidyasagar

Unimodular completion lemma

G: 1 row, 2 columns

$$G = \begin{bmatrix} p & q \end{bmatrix} \qquad G' = \begin{bmatrix} -y & x \end{bmatrix} \qquad \begin{bmatrix} G \\ G' \end{bmatrix} = \begin{bmatrix} p & q \\ -y & x \end{bmatrix}$$

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$\texttt{determinant} = px + qy,$ **unimodularity** $\Leftrightarrow$ $px + qy = 1$

solvable for $x, y \Leftrightarrow p$ **&** q **coprime** $\Leftrightarrow G = \begin{bmatrix} p & q \end{bmatrix}$ **left prime**

Unimodular completion lemma

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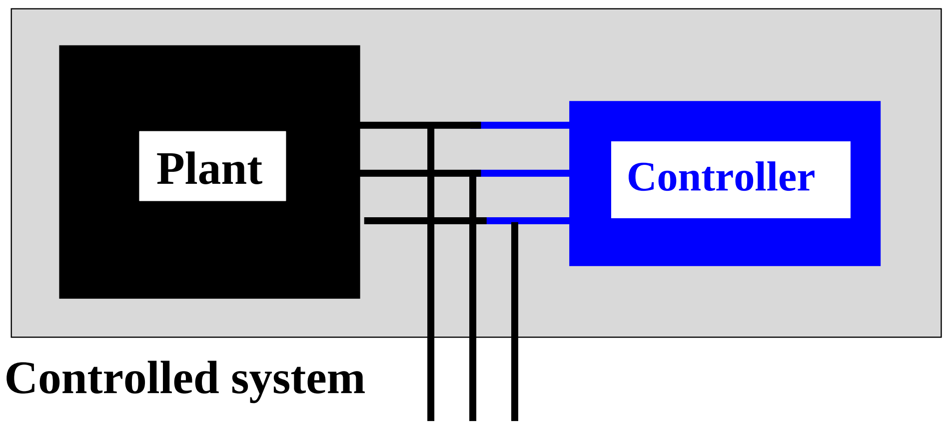
Our rings are Hermite rings

G **left prime** $\Leftrightarrow$ **unimodularly completable** $\Leftrightarrow \exists H : GH = I \Leftrightarrow \cdots$

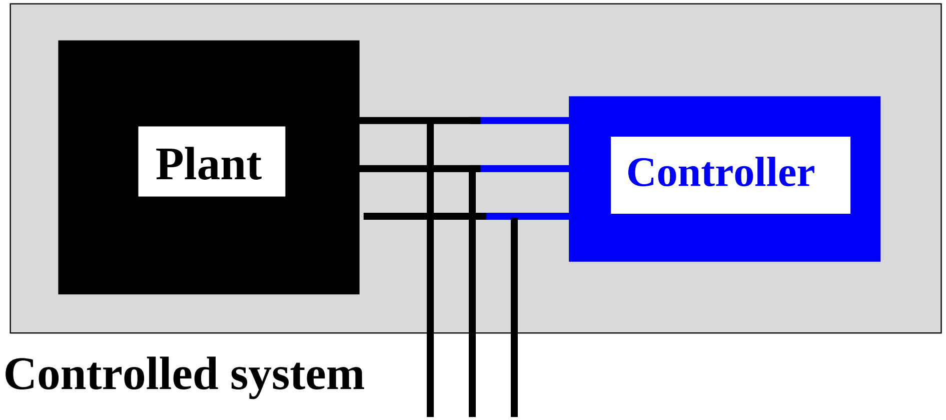
Control

Control



Plant $\mathscr{P}$, controller $\mathscr{C}$, controlled system $\mathscr{P} \cap \mathscr{C}$

Control



Plant $\mathscr{P}$, **controller** $\mathscr{C}$, **controlled system** $\mathscr{P} \cap \mathscr{C}$

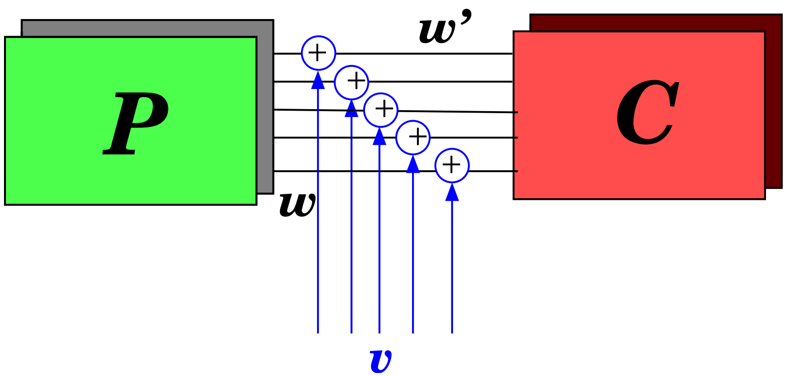
$$[\![\mathscr{C} \text{ is } \textbf{stabilizing}]\!] :\Leftrightarrow [\![\mathscr{P} \cap \mathscr{C} \text{ is stable}]\!]$$

$$\Leftrightarrow \Big[\!\!\Big[[\![w \in \mathscr{P} \cap \mathscr{C}]\!] \Rightarrow [\![w(t) \longrightarrow 0 \textbf{ for } t \longrightarrow \infty]\!] \Big]\!\!\Big]$$

Control

$[\![\mathscr{C}$ **is a regular controller**$]\!] :\Leftrightarrow [\![\mathscr{P} + \mathscr{C} = \mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})]\!]$

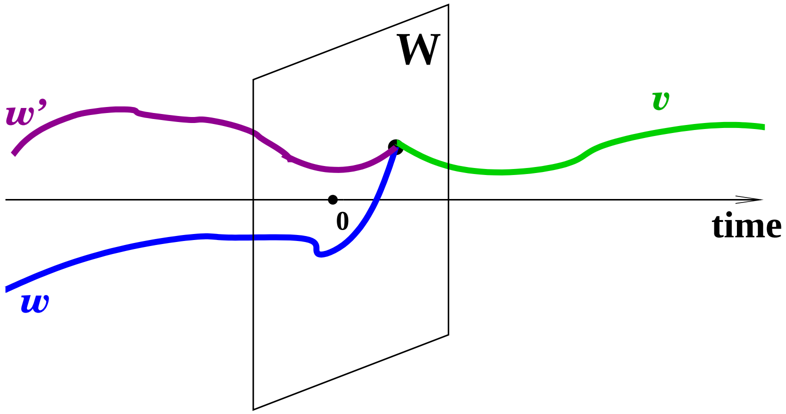
$\forall v \in \mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})\ \exists w \in \mathscr{P}$ **and** $w' \in \mathscr{C}$ **such that** $v = w + w'$



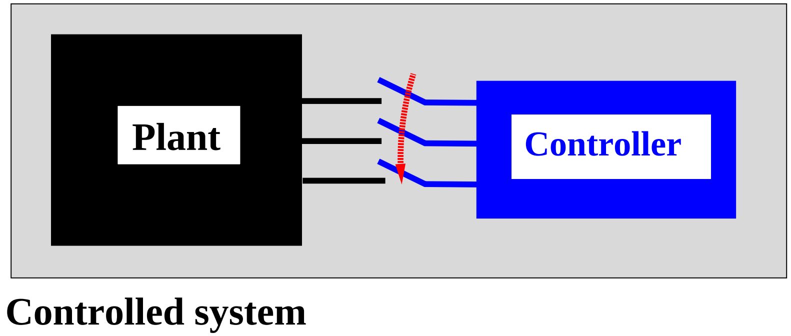
Control

$[\![\mathscr{C}$ **is a superregular controller** $]\!]$ $:\Leftrightarrow$ **in addition,**

$$[\![\forall w \in \mathscr{P}, \forall w' \in \mathscr{C} \;\; \exists v \text{ such that } w \wedge_0 v, w' \wedge_0 v \in \mathscr{P} \cap \mathscr{C}]\!]$$



A superregular controller can be engaged at any time



superregular $\Rightarrow$ controller can be engaged at any time

(Super)regular controllers

Usual feedback controllers are superregular

PID controllers are regular, but not superregular

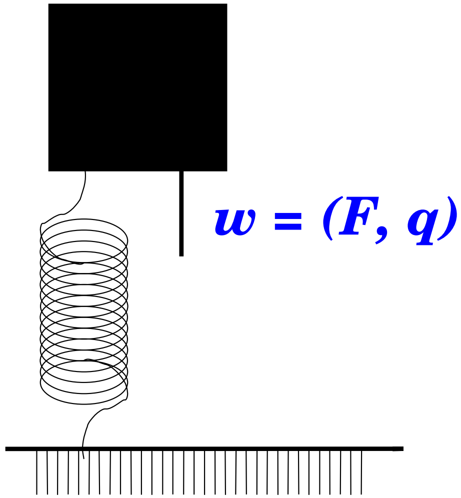
Controllers that are not superregular are relevant:
control is interconnection , not just signal processing

Harry Trentelman

Madhu Belur

A regular, but not superregular, controller

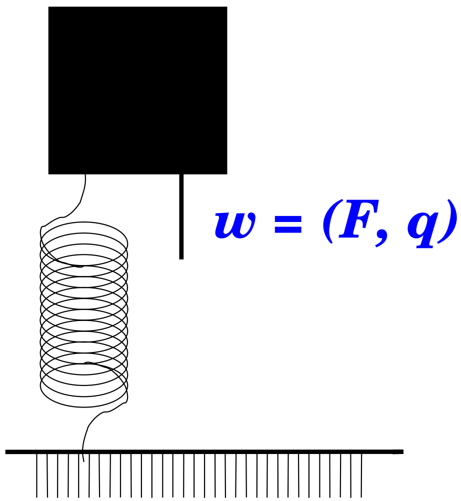
Plant:



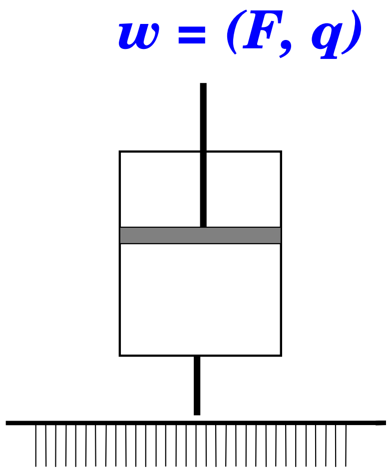
$$M\frac{d^2}{dt^2}q + Kq = F, \quad w = (F, q)$$

A regular, but not superregular, controller

Plant:



Controller:

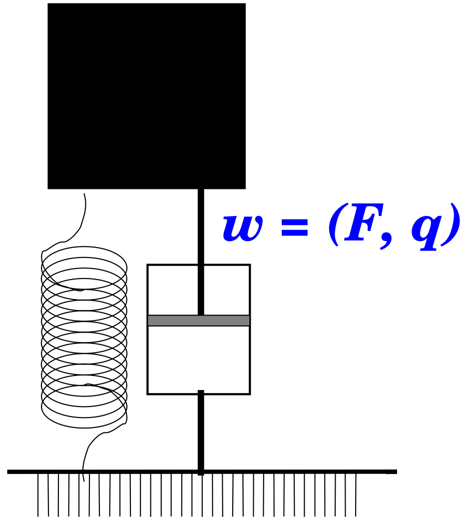


$$M\frac{d^2}{dt^2}q + Kq = F, \quad w = (F,q)$$

$$F = -D\frac{d}{dt}q$$

A regular, but not superregular, controller

Controlled system:



$$M\frac{d^2}{dt^2}q + D\frac{d}{dt}q + Kq = 0, \quad F = -D\frac{d}{dt}q$$

Existence of stabilizing controllers

Existence

Proposition

$\mathscr{P}$ **is stabilizable** $\Leftrightarrow \exists$ **a regular stabilizing controller**

$\Leftrightarrow \exists$ **a superregular stabilizing controller**

Existence

Proposition

$\mathscr{P}$ **is stabilizable** $\Leftrightarrow \exists$ **a regular stabilizing controller**

$\Leftrightarrow \exists$ **a superregular stabilizing controller**

$\mathscr{P}$ **is controllable** $\Leftrightarrow \exists$ **pole placement for** $\mathscr{P} \cap \mathscr{C}$

$\nexists$ **a controller that is superregular**
& $\mathscr{P} \cap \mathscr{C}$ **has a low order characterisitic polynomial.**

Parametrization of stabilizing controllers

Parametrization of superregular stabilizing controllers

Start with $G\left(\frac{d}{dt}\right)w = 0$ a (rational symbol based) representation of the plant

Assume G left prime over proper stable rational functions.
Iff the plant is stabilizable, such a G exists.

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Par'ion of superregular stabilizing controllers $C\left(\frac{d}{dt}\right)w = 0$

$$\boxed{C = F_1 G + F_2 G'}$$

F_1 **free over ring of proper stable rational**

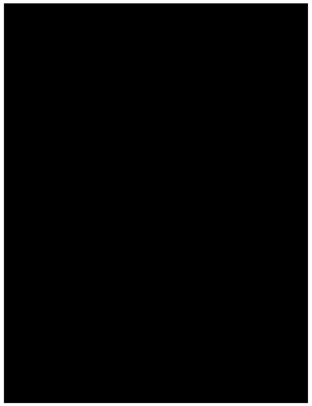
F_2 **unimodular over proper stable rational**

So

Using rational symbol based representations $G\left(\frac{d}{dt}\right)w = 0$ that are left prime over suitable rings, we obtain parametrizations of regular and superregular stabilizing controllers

$\cong$ Kučera-Youla parametrization, with proper attention for the uncontrollable part

Vladimir Kučera



Dante Youla

Margreta Kuijper

Details & copies of the lecture frames are available from/at

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