FOUNDATIONS

of

(LINEAR) SYSTEM THEORY

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K.U. Leuven, Flanders, Belgium

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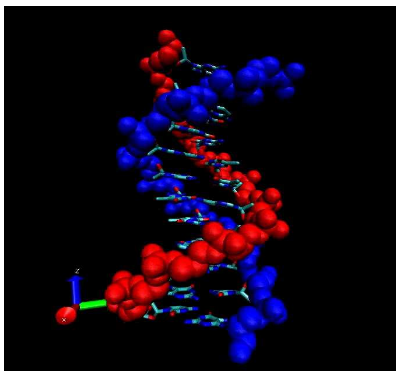
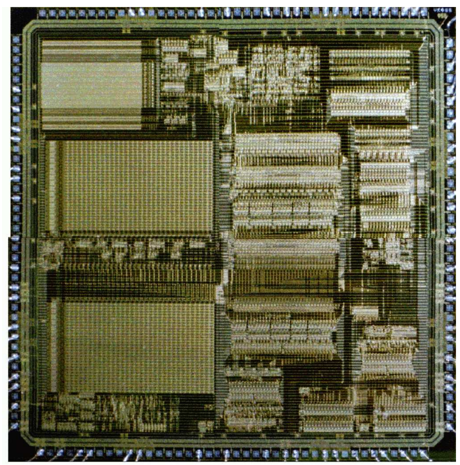
Systems




ZYGOTE



Features

- open
- interconnected
- modular
- dynamic

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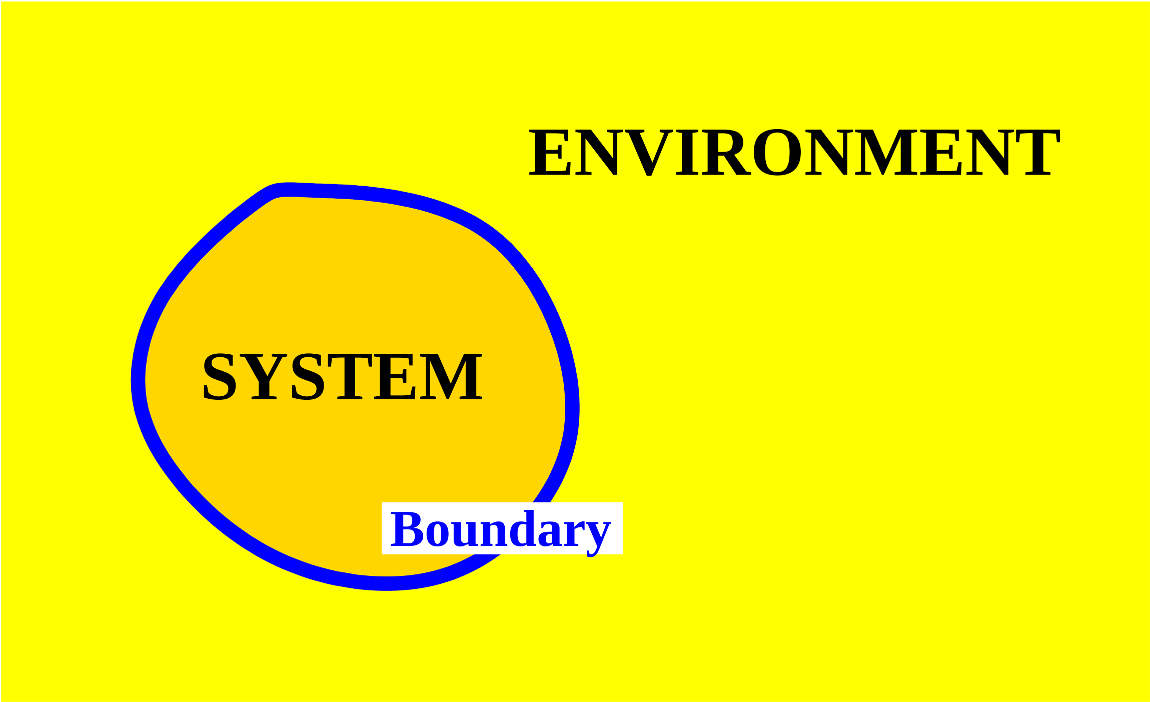
Theme of this seminar:

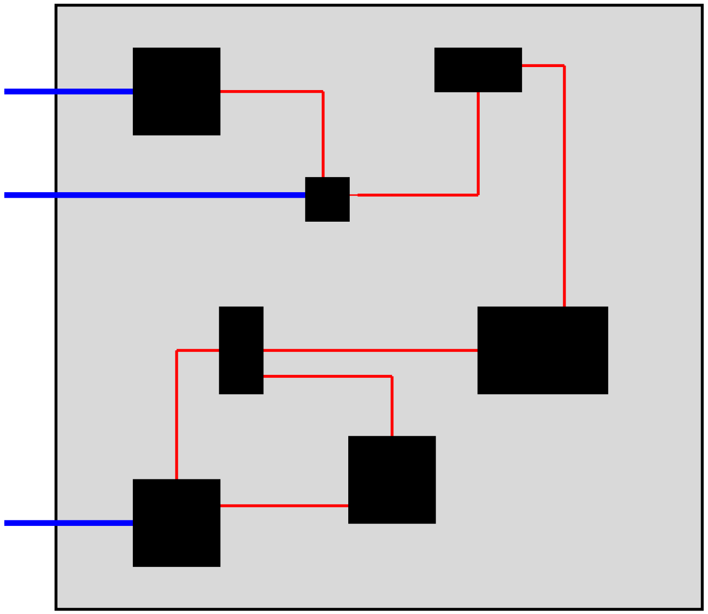
develop a suitable mathematical framework

Open and Connected

Open



Connected



Architecture with subsystems

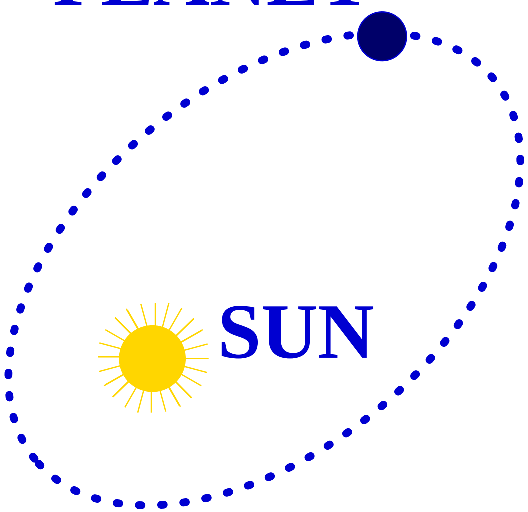
Historical introduction

How it all began ...

How, for heaven's sake, does it move?

Kepler's laws

Johannes Kepler (1571-1630)



Kepler's laws:

Ellipse, sun in focus; = areas in = times;
$(\text{period})^2 \cong (\text{diameter})^3$

The equation of the planet

Consequence:

acceleration = function of position and velocity

$$\frac{d^2}{dt^2}w(t) = A(w(t), \frac{d}{dt}w(t))$$

$\rightsquigarrow$ **via calculus and calculation**

$$\frac{d^2}{dt^2}w(t) + \frac{1_{w(t)}}{|w(t)|^2} = 0$$

Isaac Newton (1643-1727)

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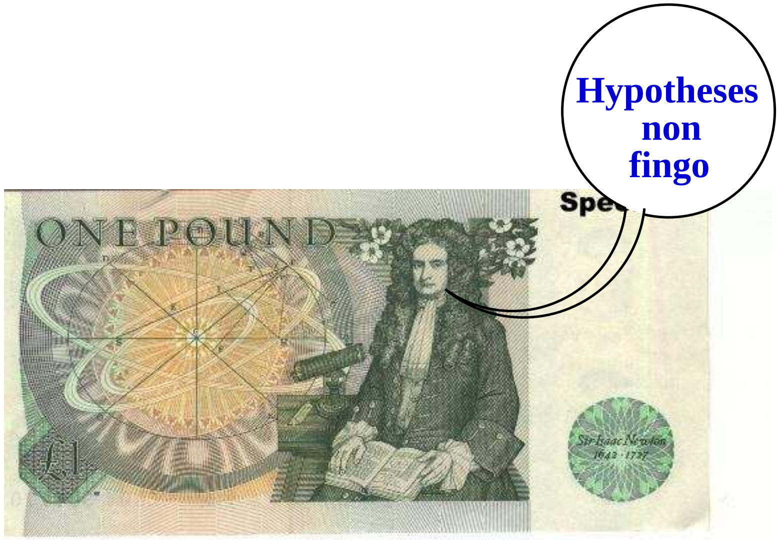


Isaac Newton (1643-1727)

Newton's laws

tearing, zooming, linking

2-nd law $F'(t) = m\frac{d^2}{dt^2}w(t)$

gravity $F''(t) = m\frac{1_{w(t)}}{|w(t)|^2}$

3-rd law $F'(t) + F''(t) = 0$

$$\Downarrow$$

$$\boxed{\frac{d^2}{dt^2}w(t) + \frac{1_{w(t)}}{|w(t)|^2} = 0}$$

The paradigm of *closed* systems

‘Axiomatization’

K.1, K.2, & K.3

$$\rightsquigarrow \quad \frac{d^2}{dt^2}w(t) + \frac{1_{w(t)}}{|\frac{d}{dt}w(t)|^2} = 0$$

$$\rightsquigarrow \quad \frac{d}{dt}x = f(x)$$

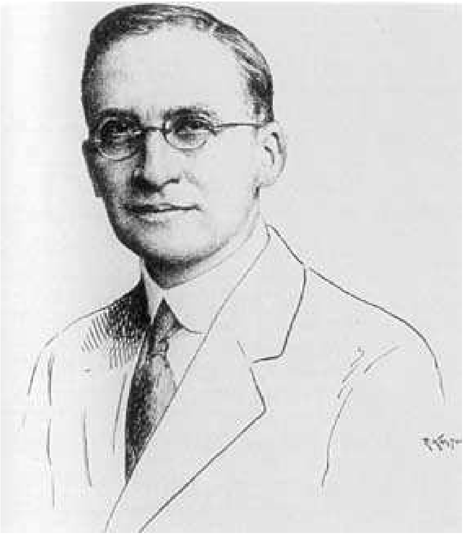
$\rightsquigarrow$ **‘dynamical systems’, flows**

$\rightsquigarrow$ **flows as paradigm of dynamics: closed systems**

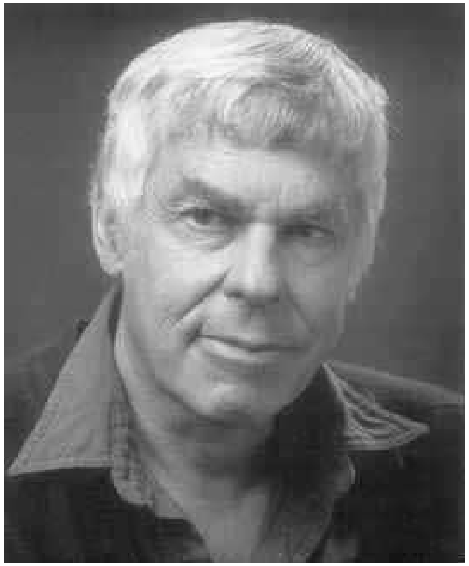
Motion determined by internal initial conditions.

‘Axiomatization’

Henri Poincaré (1854-1912)



George Birkhoff (1884-1944)



Stephen Smale (1930-   )

‘Axiomatization’

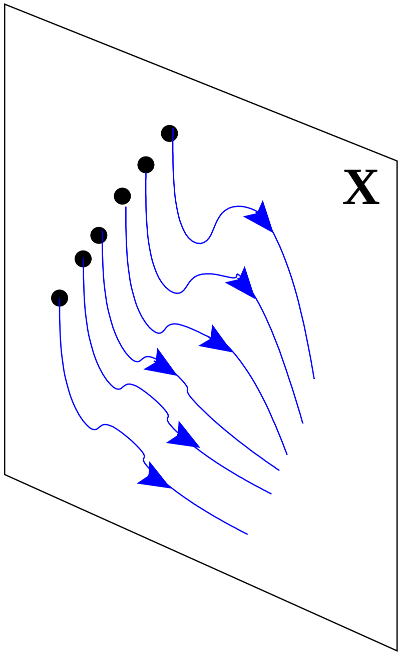
A *dynamical system* is defined by

a **state space** X and

a **state transition function**

ϕ : $\cdots$ such that $\cdots$

$\phi(t, \mathrm{x})$ = state at time t starting from state x



This framework of closed systems

is universally used for dynamics

in mathematics and physics

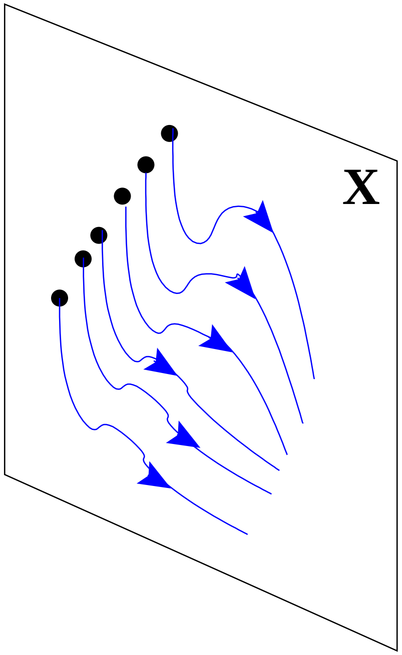
‘Axiomatization’

A *dynamical system* is defined by

a state space X and

a state transition function

$\phi :$ $\cdots$ such that $\cdots$



$\phi(t, \mathrm{x})$ = state at time t starting from state x

How could they forget Newton’s 2$^{\text{nd}}$ law,

about Maxwell’s eq’ns,

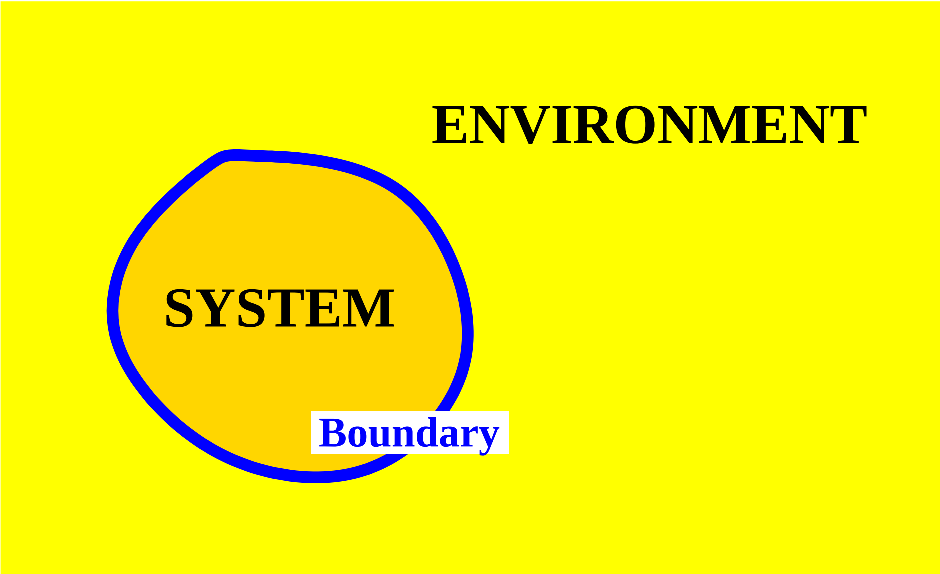
about thermodynamics,

about interconnection,

about tearing & zooming & linking, ...?

‘Axiomatization’

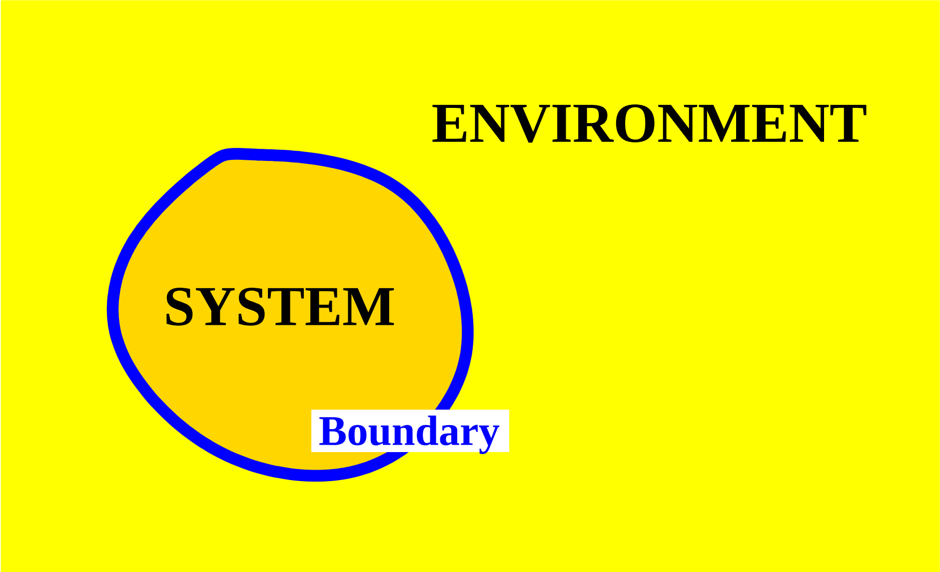
Reply: assume ‘fixed boundary conditions’



⇝ to model a system,

we have to model also the environment!

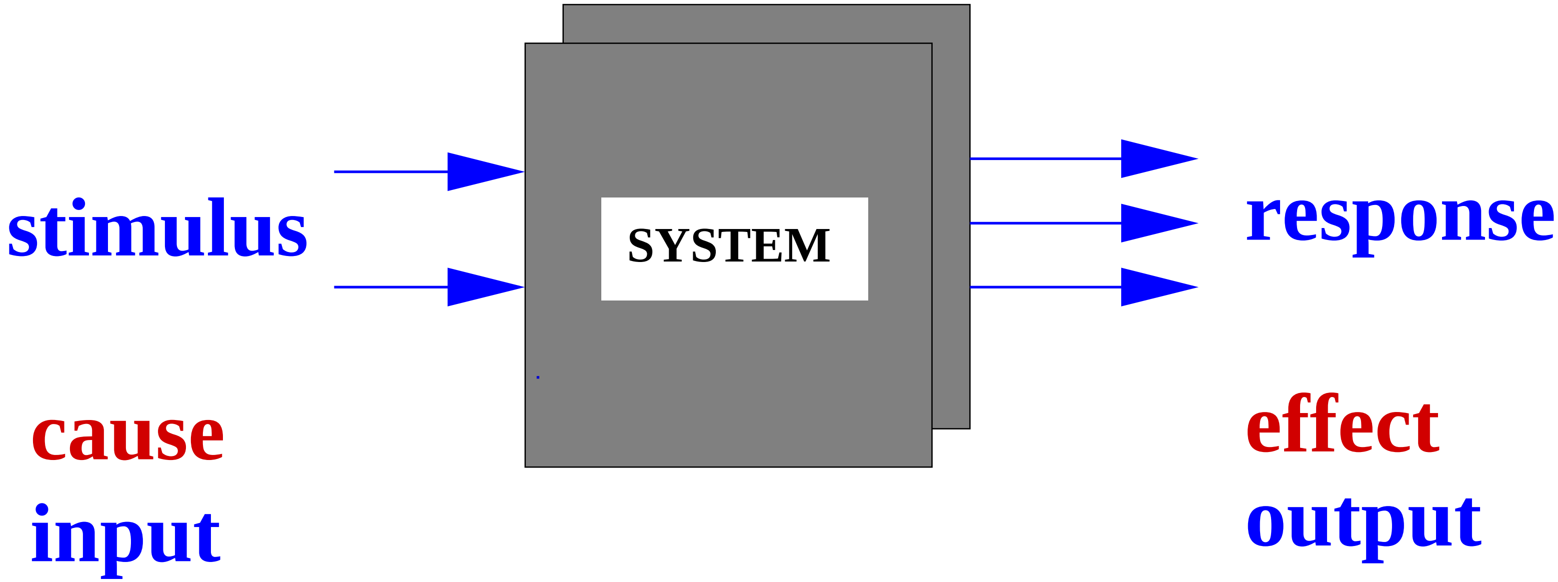
'Axiomatization'

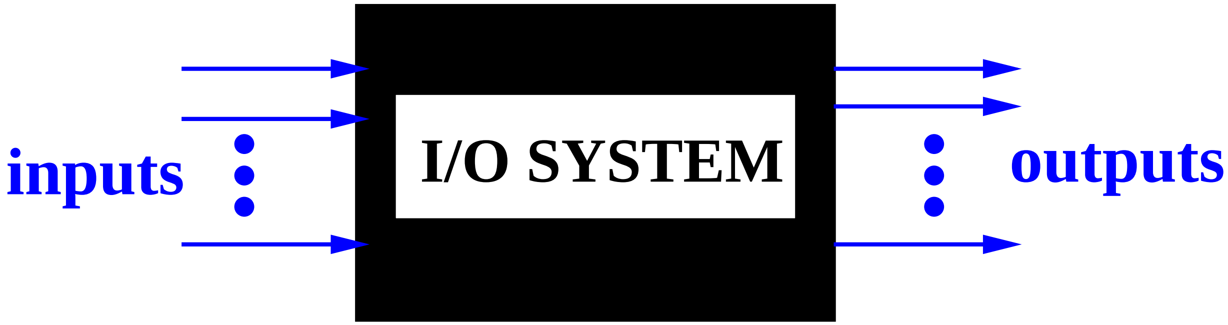


Chaos theory, cellular automata, sync, etc.,
function in this framework ...

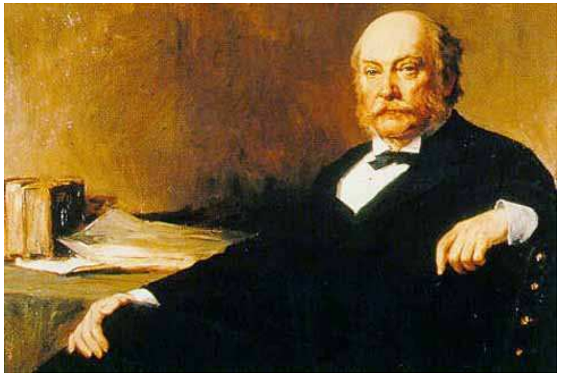
Meanwhile, in engineering, ...

Input/output systems

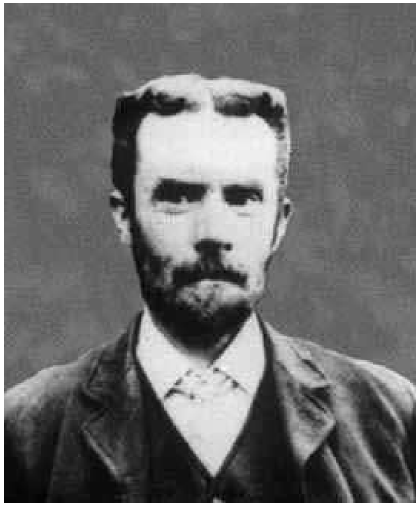




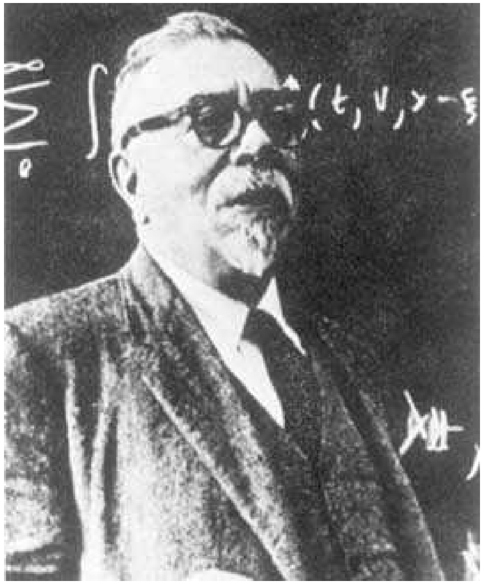
The originators



Lord Rayleigh (1842-1919)



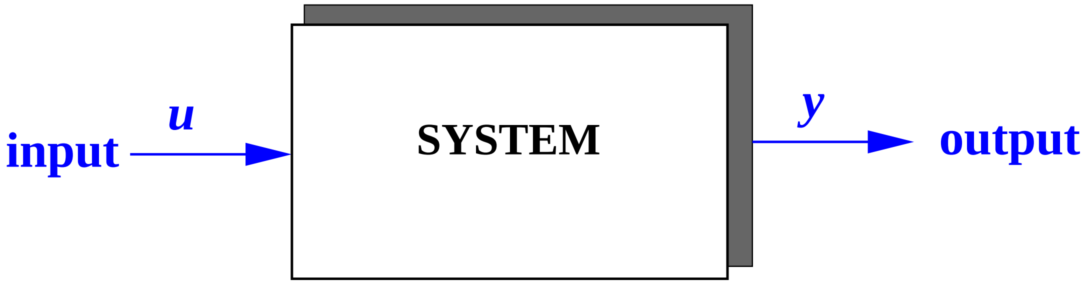
Oliver Heaviside (1850-1925)



Norbert Wiener (1894-1964)

and the many electrical circuit theorists ...

Mathematical description



Classical control $$p\left(\frac{d}{dt}\right) y = q\left(\frac{d}{dt}\right) u$$

u: input, y: output, p and q polynomials

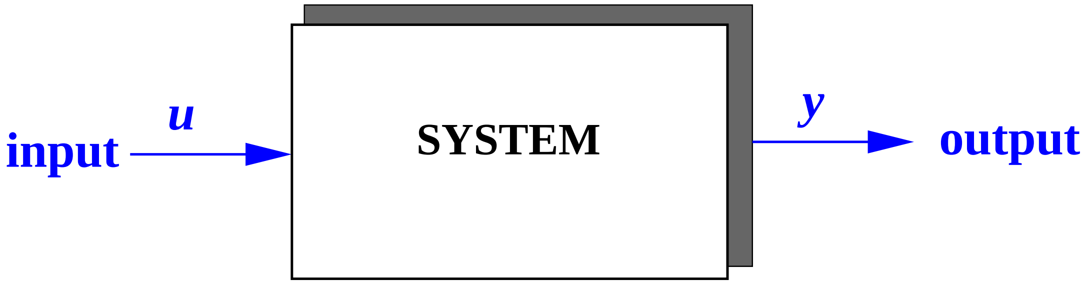
$G(s) = \frac{q(s)}{p(s)}$ **transfer functions, impedances, admittances.**

PID rules. Bode, Nyquist, Nichols. Lead-lag. Root-locus.

Mathematical framework:

Laplace t'fms, instead of symbolic calculus

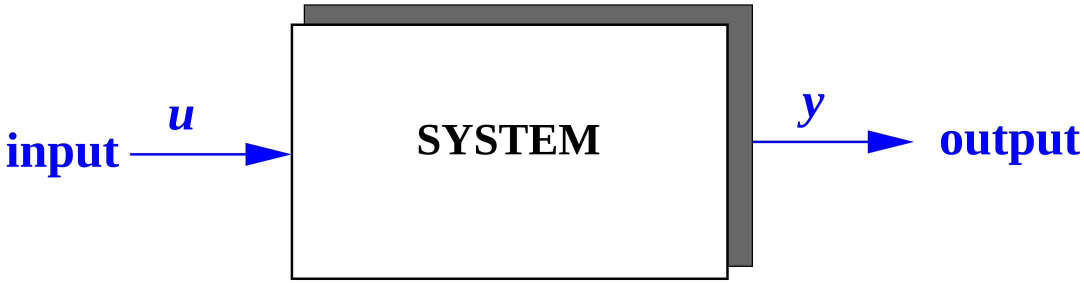
Mathematical description



$$y(t) = \int_{0 \text{ or } -\infty}^{t} H(t-t')u(t')\,dt'$$

$$y(t) = H_0(t) + \int_{-\infty}^{t} H_1(t-t')u(t')\,dt' + \int_{-\infty}^{t}\int_{-\infty}^{t'} H_2(t-t', t'-t'')u(t')u(t'')\,dt'dt'' + \cdots$$

Mathematical description



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Far from the physics

Fail to deal with 'initial conditions'.

A system is seldom an i/o map

Input/state/output systems

Around 1960: a paradigm shift to

$$\frac{d}{dt}x = f(x,u), \; y = g(x,u)$$

Rudolf Kalman (1930-)

Input/state/output systems

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$$\frac{d}{dt}x = f(x,u), \; y = g(x,u)$$

Rudolf Kalman (1930-)

1. **open, flows special case**
2. **ready to be interconnected**

   outputs of one system $\mapsto$ inputs of another
3. **deals with initial conditions**
4. **incorporates nonlinearities, time-variation**
5. **models many physical phenomena**
6. **...**

Input/state/output systems

The input/state/output view turned out to be very effective and fruitful

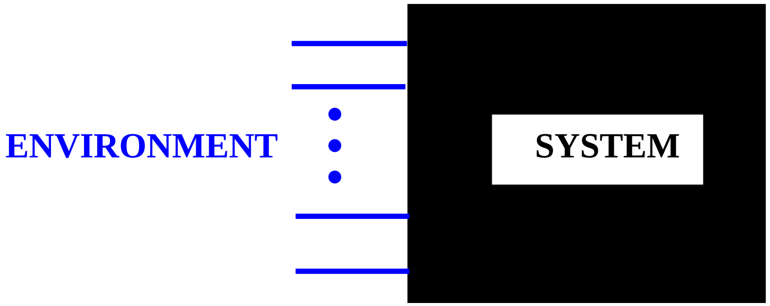
- **for modeling**
- **for control (stabilization, robustness, ...)**

Input/state/output systems

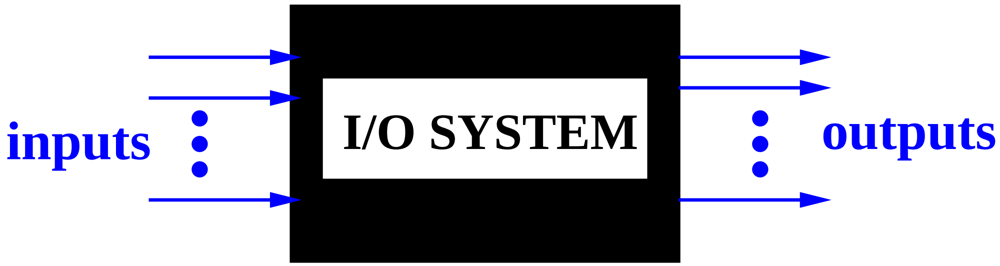
The input/state/output view turned out to be very effective and fruitful

- **for modeling**
- **for control (stabilization, robustness, ...)**
- **prediction of one signal from another, filtering**
- **understanding system representations**
  (transfer f'n, input/state/output, etc.)
- **model simplification, reduction**
- **system ID: models from data**
- **etc., etc., etc.**

Open and Connected



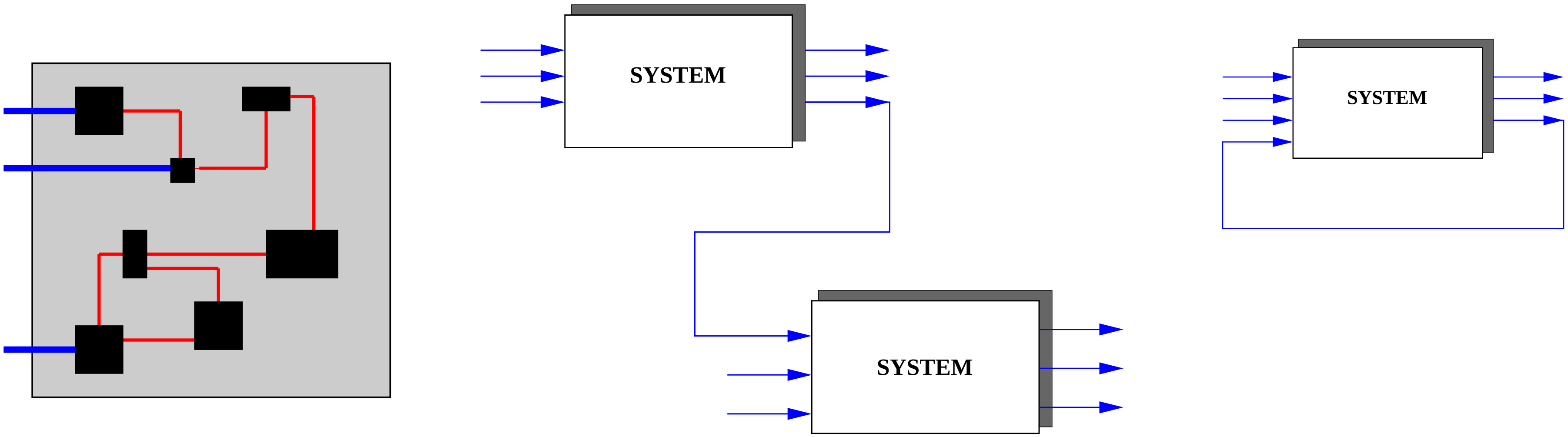
In system theory, we are accustomed to view an open dynamical system as an input/output structure



Open and Connected

In system theory, we are accustomed to view an open dynamical system as an input/output structure

& interconnection as output-to-input assignment.



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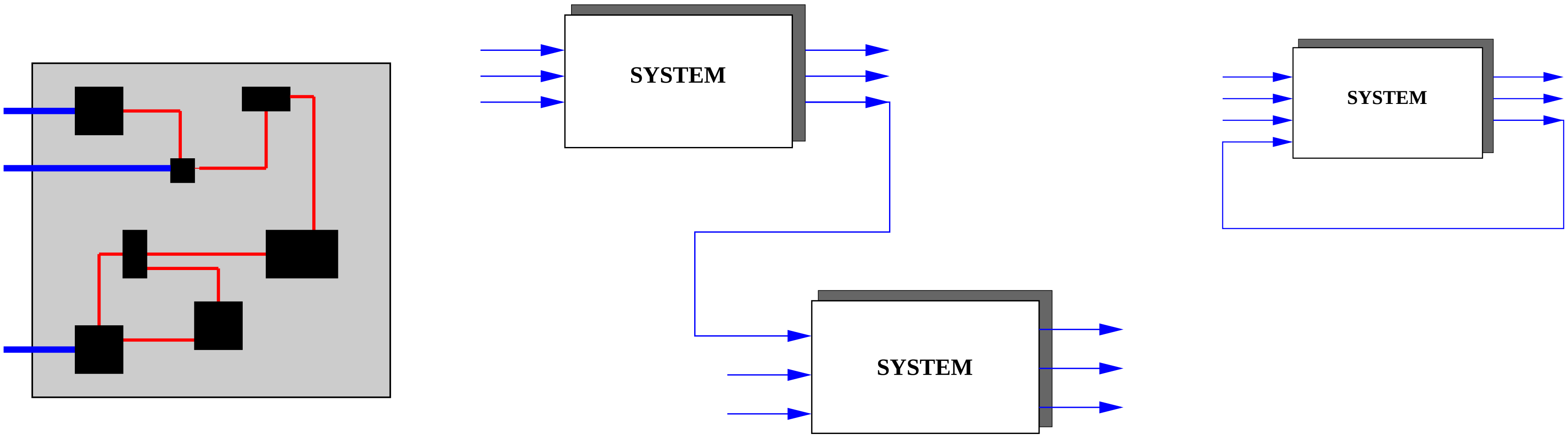
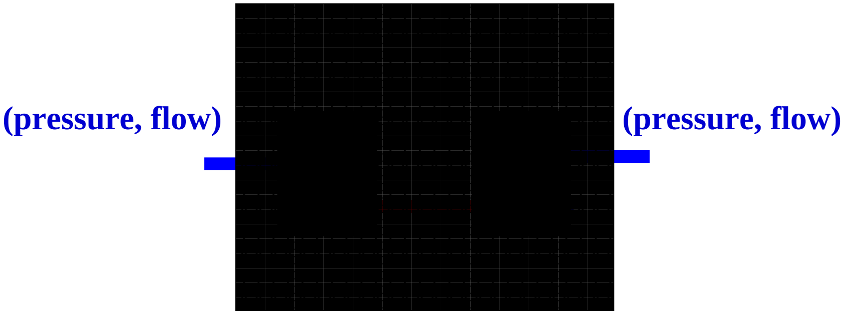
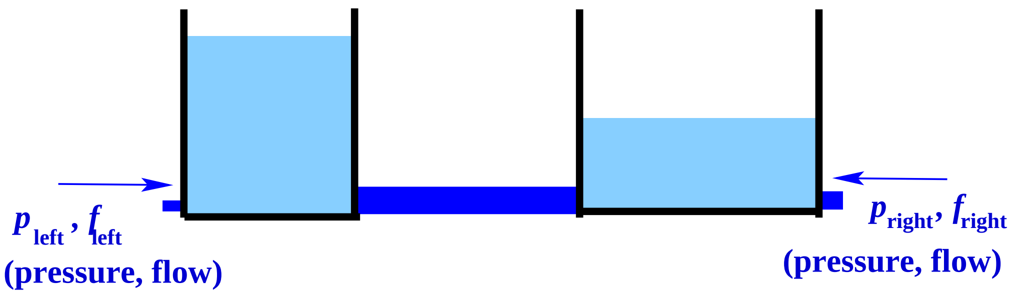


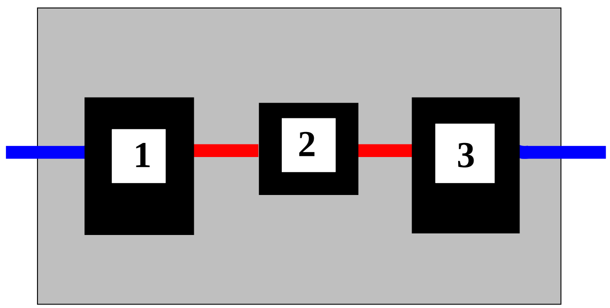
Is this appropriate for modeling physical systems?

Interconnection in physical systems

Example

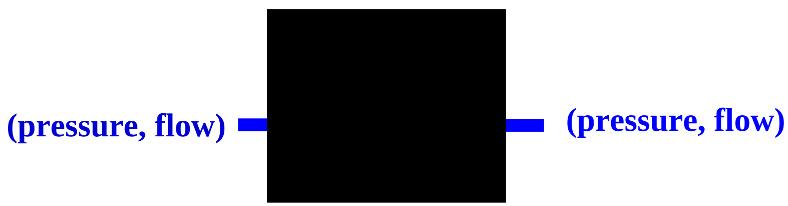

(pressure, flow)
(pressure, flow)

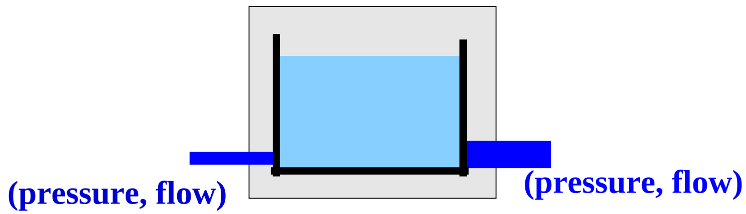

p_{left}, f_{left}
(pressure, flow)
p_{right}, f_{right}
(pressure, flow)

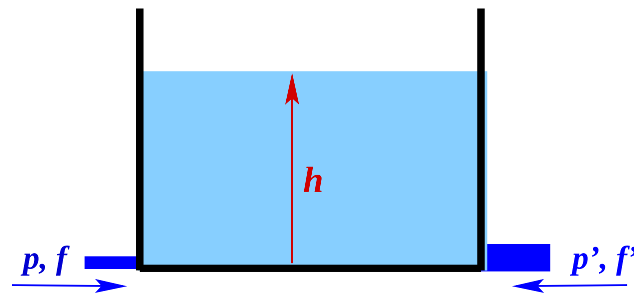

1
2
3

Example

Subsystems 1 and 3:

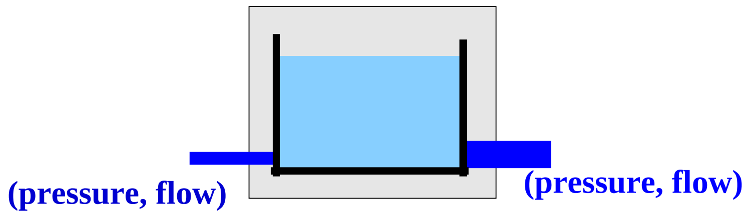


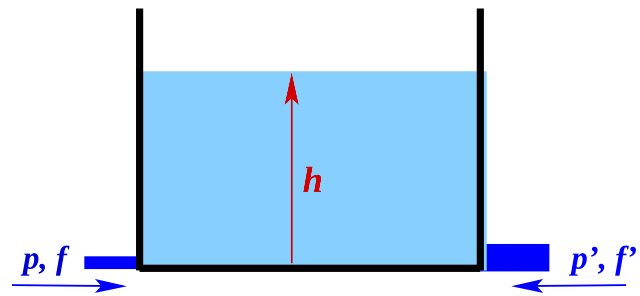




Example

Subsystems 1 and 3:

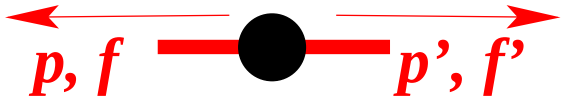




Subsystem 2:

Example

Interconnection laws:



$$p = p', \qquad f + f' = 0.$$

$$A_1 \frac{d}{dt} h_1 = f_1 + f_1',$$

$$B_1 f_1 = \begin{cases} \sqrt{|p_1 - p_0 - \rho h_1|} & \text{if } p_1 - p_0 \geq \rho h_1, \\ -\sqrt{|p_1 - p_0 - \rho h_1|} & \text{if } p_1 - p_0 \leq \rho h_1, \end{cases} \tag{1}$$

$$C f_1' = \begin{cases} \sqrt{|p_1' - p_0 - \rho h_1|} & \text{if } p_1' - p_0 \geq \rho h_1, \\ -\sqrt{|p_1' - p_0 - \rho h_1|} & \text{if } p_1' - p_0 \leq \rho h_1, \end{cases}$$

$$f_2 = -f_2', \quad p_2 - p_2' = \alpha f_2, \tag{2}$$

$$A_3 \frac{d}{dt} h_3 = f_3 + f_3',$$

$$C f_3 = \begin{cases} \sqrt{|p_3 - p_0 - \rho h_3|} & \text{if } p_3 - p_0 \geq \rho h_3, \\ -\sqrt{|p_3 - p_0 - \rho h_3|} & \text{if } p_3 - p_0 \leq \rho h_3, \end{cases} \tag{3}$$

$$C_3 f_3' = \begin{cases} \sqrt{|p_3' - p_0 - \rho h_3|} & \text{if } p_3' - p_0 \geq \rho h_3, \\ -\sqrt{|p_3' - p_0 - \rho h_3|} & \text{if } p_3' - p_0 \leq \rho h_3, \end{cases}$$

$$p_1' = p_2, \; f_1' + f_2 = 0, \; p_2' = p_3, \; f_2' + f_3 = 0. \tag{4}$$

$$p_{\text{left}} = p_1, \; f_{\text{left}} = f_1, \; p_{\text{right}} = p_3', \; f_{\text{right}} = f_3'. \tag{5}$$

- **Unclear input/output structure for terminal variables**
- **Many variables, indivisibly, at the same terminal**
- **Interconnection = variable sharing**
- **No signal flows, no output-to-input assignment**
- **...**

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- **...**

"Block diagrams unsuitable for serious physical modeling

- the control/physics barrier"

"Behavior based (declarative) modeling is a good alternative"

from K.J. Åström, ***Present Developments in Control Applications***

IFAC 50-th Anniversary Celebration

Heidelberg, September 12, 2006.

Behavioral systems

A dynamical system

:⇔ a family of time functions, *'the behavior'*

Interconnection :⇔ *'variable sharing'*.

Control :⇔ *interconnection*.

Modeling of interconnected physical systems is the strongest case for 'behaviors'.

Models

A dynamical system $:\Leftrightarrow \; (\mathbb{T}, \mathbb{W}, \mathfrak{B})$

$\mathbb{T} \subseteq \mathbb{R}$ **'time set'**

$\mathbb{W}$ **'signal space'**

$\mathfrak{B} \subseteq \mathbb{W}^{\mathbb{T}}$ **the 'behavior'**

a family of trajectories $\mathbb{T} \to \mathbb{W}$

mostly, today, $\mathbb{T} = \mathbb{R}, \mathbb{W} = \mathbb{R}^{\mathtt{w}}$**, hence** $\mathfrak{B}$ **is a family of vector-valued continuous-time trajectories**

$w : \mathbb{R} \to \mathbb{R}^{\mathtt{w}} \in \mathfrak{B}$: **'$w$ is compatible with the model'**

$w : \mathbb{R} \to \mathbb{R}^{\mathtt{w}} \notin \mathfrak{B}$: **'the model forbids w'**

Models

The dynamical system $(\mathbb{R}, \mathbb{R}^{\mathtt{w}}, \mathfrak{B})$ $\quad \rightsquigarrow \quad \mathfrak{B}$

linear $:\Leftrightarrow w_1, w_2 \in \mathfrak{B}, \alpha \in \mathbb{R}$

$$\Rightarrow \alpha w_1 + w_2 \in \mathfrak{B}$$

time-invariant $:\Leftrightarrow w \in \mathfrak{B}, \sigma$ any shift,

$$\Rightarrow \sigma w \in \mathfrak{B}$$

differential $:\Leftrightarrow$ 'described' by an ODE.

Models

For example,

$$\frac{d}{dt}x = Ax + Bu, y = Cx + Du, \quad w = \begin{bmatrix} u \\ y \\ x \end{bmatrix} \text{ or } w = \begin{bmatrix} u \\ y \end{bmatrix}$$

DAE's $\quad F\frac{d}{dt}x + Gx + Hw = 0$

$P\left(\frac{d}{dt}\right)y = Q\left(\frac{d}{dt}\right)u, \quad w = \begin{bmatrix} u \\ y \end{bmatrix}, \quad P, Q$ **polynomial matrices**

$y = G\left(\frac{d}{dt}\right)u, w = \begin{bmatrix} u \\ y \end{bmatrix}, \quad G$ **matrix of rational f'ns**

etc., and their nonlinear analogues

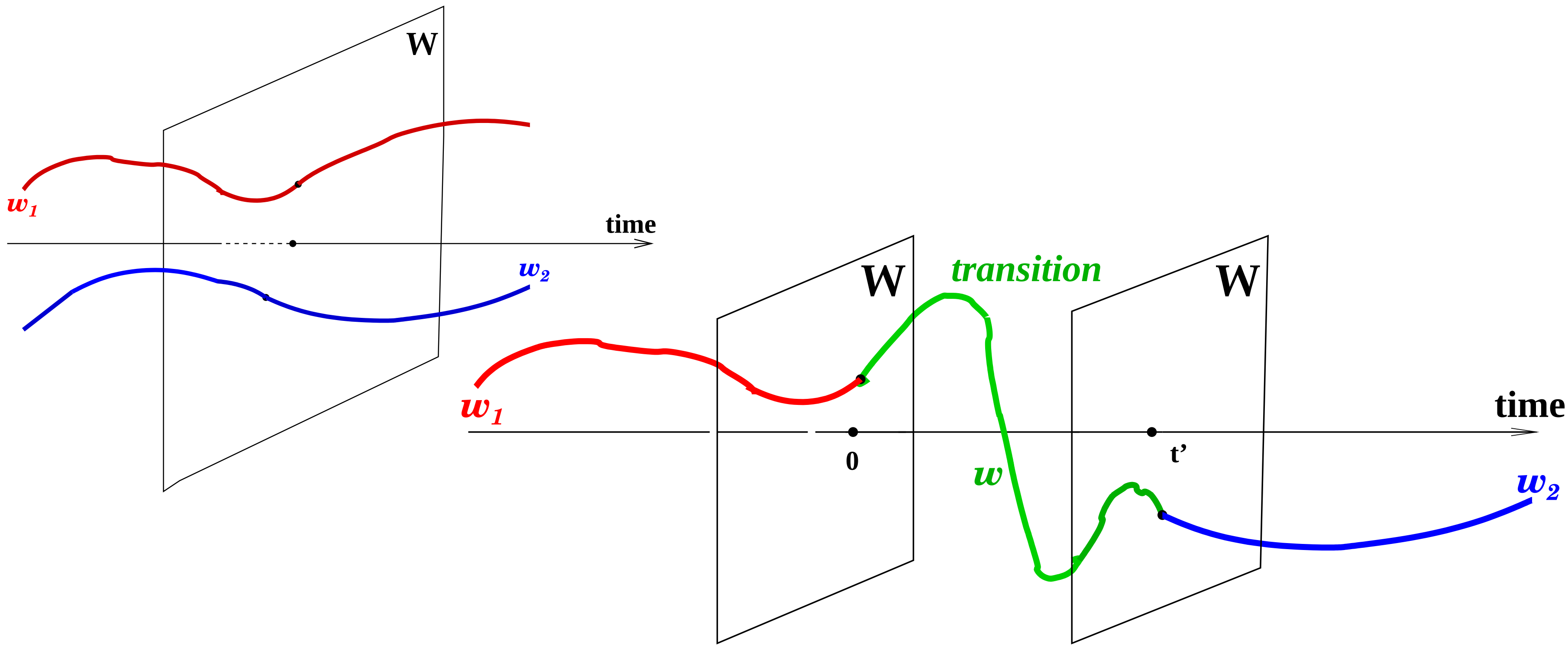
Controllability as a system property

Controllability

The time-invariant system $(\mathbb{R}, \mathbb{R}^{\mathtt{w}}, \mathfrak{B})$ is

controllable $:\Leftrightarrow$

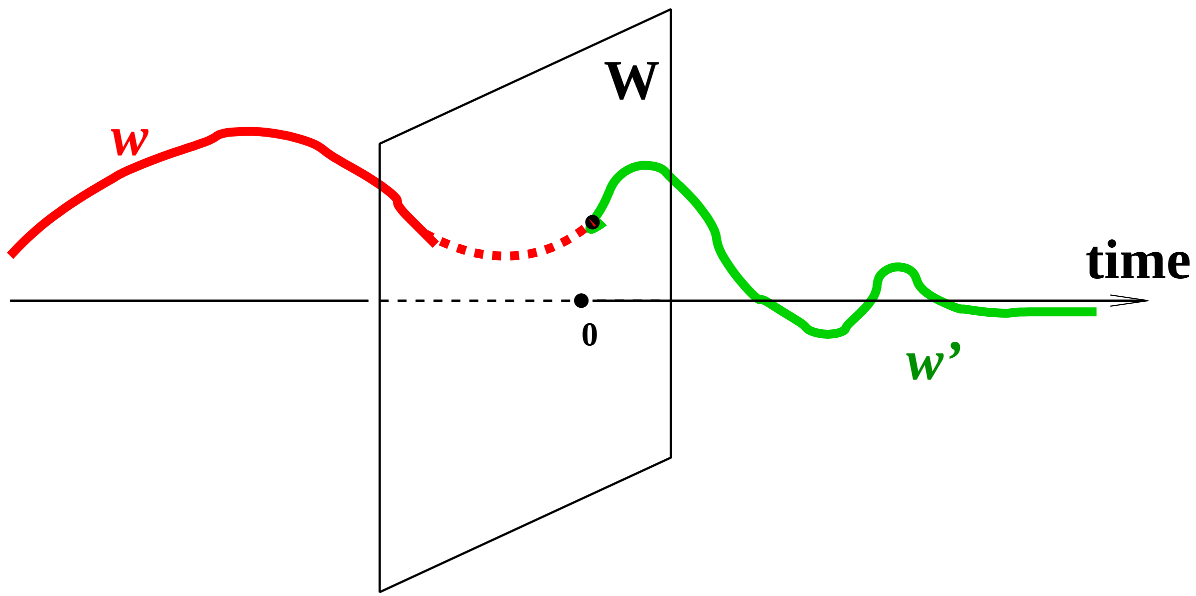
$$\forall\, w_1, w_2 \in \mathfrak{B}, \exists\, w \in \mathfrak{B} \text{ and } T \geq 0 \text{ such that}$$



Controllability

The time-invariant system $(\mathbb{R}, \mathbb{R}^{\mathtt{w}}, \mathfrak{B})$ **is**

stabilizable $:\Leftrightarrow \quad \forall\, w \in \mathfrak{B},\ \exists\, w' \in \mathfrak{B}$ **such that**



Controllability

Makes controllability into a bona fide, genuine, true system property, instead of of merely a (state space) representation.

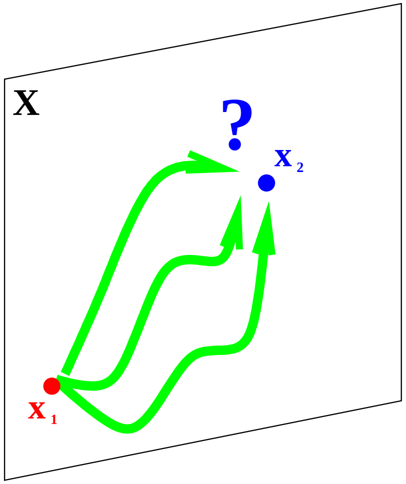
State Controllability

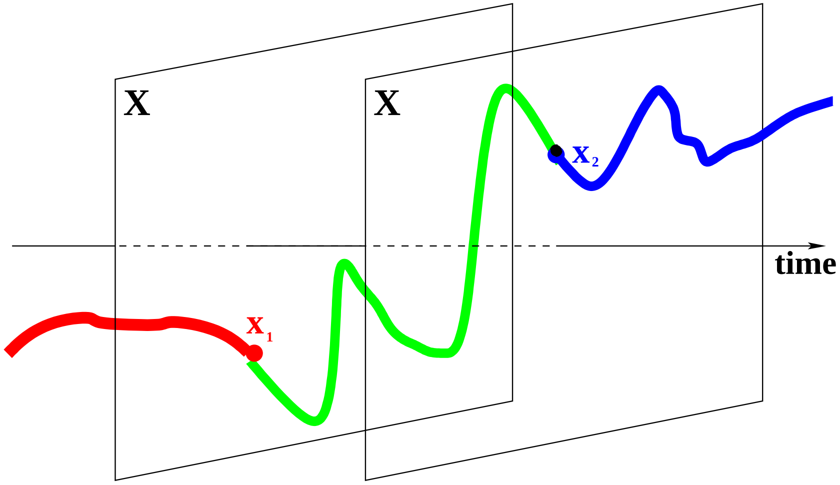
Special case: classical Kalman definitions for

$$\frac{d}{dt}x = f(x, u).$$

controllability: variables = state or (input, state)

This is a special case of our controllability:





State Controllability

Special case: classical Kalman definitions for

$$\frac{d}{dt}x = f\,(x, u)\,.$$

controllability: variables = state or (input, state)

Why should we be so concerned with the state?

If a system is not (state) controllable, why is it?

Insufficient influence of the control?
Or bad choice of the state?
Or not properly editing the equations?

Kalman's definition addresses a rather special situation.

The basics of LTIDS

LTIDS

The dynamical system $(\mathbb{R}, \mathbb{R}^{\mathtt{w}}, \mathfrak{B})$ is

linear, time-invariant, and differential (LTIDS)

$:\Leftrightarrow$ 'described' by an ODE

$$R_0 w + R_1 \frac{d}{dt} w + \cdots + R_{\mathtt{n}} \frac{d^{\mathtt{n}}}{dt^{\mathtt{n}}} w = 0.$$

$R_0, R_1, \cdots, R_{\mathtt{n}}$ real matrices.

LTIDS

$$R_0 w + R_1 \frac{d}{dt} w + \cdots + R_{\mathrm{n}} \frac{d^{\mathrm{n}}}{dt^{\mathrm{n}}} w = 0.$$

$R_0, R_1, \cdots, R_{\mathrm{n}}$ **real matrices.**

$$\rightsquigarrow \qquad R\left(\frac{d}{dt}\right) w = 0$$

$R = R_0 + R_1 \xi + \cdots + R_{\mathrm{n}} \xi^{\mathrm{n}}$ **polynomial matrix.**

R **typically 'wide'** 

LTIDS

$$R_0 w + R_1 \frac{d}{dt} w + \cdots + R_{\mathtt{n}} \frac{d^{\mathtt{n}}}{dt^{\mathtt{n}}} w = 0.$$

$R_0, R_1, \cdots, R_{\mathtt{n}}$ **real matrices.**

$$\rightsquigarrow \quad R\left(\frac{d}{dt}\right) w = 0$$

$R = R_0 + R_1 \xi + \cdots + R_{\mathtt{n}} \xi^{\mathtt{n}}$ **polynomial matrix.**

Defines $\mathfrak{B} = \mathrm{kernel}\left(R\left(\frac{d}{dt}\right)\right)$ ***'kernel representation'***

Notation: $\mathfrak{L}^{\mathtt{w}}, \mathfrak{L}^{\bullet}$.

$\mathfrak{L}^{\mathtt{w}}$ as a module

Let $\mathfrak{B} = \text{kernel}\left(R\left(\frac{d}{dt}\right)\right)$

R **determines** $\mathfrak{B}$**, but** $\mathfrak{B}$ **does not determine** R **!**

What is the mathematical structure of $\mathfrak{L}^{\mathtt{w}}$**?**

$\mathfrak{L}^{\mathtt{w}}$ as a module

Consider $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$. Call $n \in \mathbb{R}[\xi]^{\mathtt{w}}$ an annihilator

$:\Leftrightarrow$

$$n\left(\frac{d}{dt}\right) w = 0 \text{ for all } w \in \mathfrak{B}$$

Examples:

Transposes of rows of R

$$\text{for } \mathfrak{B} = \text{kernel}\left(R\left(\frac{d}{dt}\right)\right)$$

$\mathfrak{L}^{\mathtt{w}}$ as a module

Consider $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$. Call $n \in \mathbb{R}[\xi]^{\mathtt{w}}$ an annihilator :⇔

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Note

- n_1, n_2 **annihilators** $\Rightarrow n_1 + n_2$ **annihilator**
- n **annihilator,** p **polynomial** $\Rightarrow pn$ **annihilator**

$\Rightarrow$ **Annihilators of $\mathfrak{B}$ form a submodule of $\mathbb{R}[\xi]^{\mathtt{w}}$.**

$\mathfrak{L}^{\mathtt{w}}$ as a module

Consider $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$. Call $n \in \mathbb{R}[\xi]^{\mathtt{w}}$ an **annihilator** :$\Leftrightarrow$

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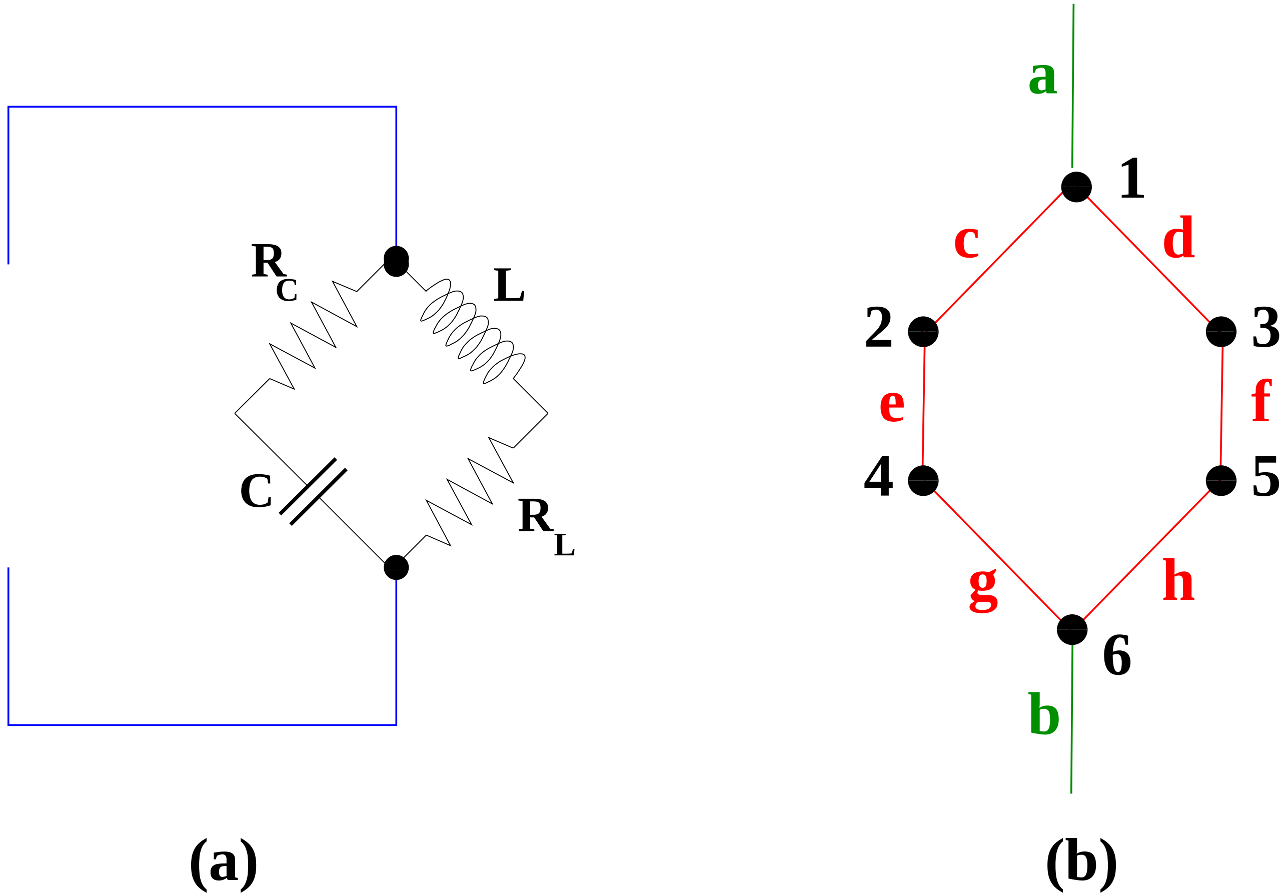
- n annihilator, p polynomial $\Rightarrow pn$ annihilator

$\Rightarrow$ Annihilators of $\mathfrak{B}$ form a submodule of $\mathbb{R}[\xi]^{\mathtt{w}}$.

Thm. 1: $\exists\ 1 \leftrightarrow 1$ relation between $\mathfrak{L}^{\mathtt{w}}$ and submodules of $\mathbb{R}[\xi]^{\mathtt{w}}$

Motivating example: model port behavior of



by tearing, zooming, and linking

Module equations

$$
\begin{aligned}
\textbf{vertex 1}: &\quad V_{\text{connector}_1,1} = V_{\text{connector}_1,2} = V_{\text{connector}_1,3} \\
&\quad I_{\text{connector}_1,1} + I_{\text{connector}_1,2} + I_{\text{connector}_1,3} = 0 \\
\textbf{vertex 2}: &\quad V_{R_C,1} - V_{R_C,2} = R_C I_{R_C,1},\ I_{R_C,1} + I_{R_C,2} = 0 \\
\textbf{vertex 3}: &\quad L\tfrac{d}{dt} I_{L,1} = V_{L,1} - V_{L,2},\ I_{L,1} + I_{L,2} = 0 \\
\textbf{vertex 4}: &\quad C\tfrac{d}{dt}\left(V_{C,1} - V_{C,2}\right) = I_{C,1},\ I_{C,1} + I_{C,2} = 0 \\
\textbf{vertex 5}: &\quad V_{R_L,1} - V_{R_L,2} = R_L I_{R_L,1} \\
&\quad I_{R_L,1} + I_{R_L,2} = 0 \\
\textbf{vertex 6}: &\quad V_{\text{connector}_2,1} = V_{\text{connector}_2,2} = V_{\text{connector}_2,3} \\
&\quad I_{\text{connector}_2,1} + I_{\text{connector}_2,2} + I_{\text{connector}_2,3} = 0
\end{aligned}
$$

Interconnection equations

edge c :	$V_{R_{C,1}} = V_{\text{connector1}_2}$	$I_{R_{C,1}} + I_{\text{connector1,2}} = 0$
edge d :	$V_{L_1} = V_{\text{connector1}_3}$	$I_{L_1} + I_{\text{connector1}_3} = 0$
edge e :	$V_{R_{C,2}} = V_{C_1}$	$I_{R_{C,2}} + I_{C_1} = 0$
edge f :	$V_{L_2} = V_{R_{C,1}}$	$I_{L_2} + I_{R_{L,1}} = 0$
edge g :	$V_{C_2} = V_{\text{connector2}_1}$	$I_{C_2} + I_{\text{connector2}_1} = 0$
edge h :	$V_{R_{L,2}} = V_{\text{connector2}_2}$	$I_{R_{L,2}} + I_{\text{connector2}_2} = 0$

Elimination

Manifest variable assignment

$$V_{\text{externalport}} = V_{\text{connector}_1,1} - V_{\text{connector}_2,3}$$

$$I_{\text{externalport}} = I_{\text{connector1}_1}$$

Manifest variable assignment

$$V_{\mathrm{externalport}} = V_{\mathrm{connector_1,1}} - V_{\mathrm{connector_2,3}}$$
$$I_{\mathrm{externalport}} = I_{\mathrm{connector1_1}}$$

What equation(s) governs the behavior of

$$(V_{\mathrm{externalport}}, I_{\mathrm{externalport}})$$

Constant-coefficient linear differential equation?

Elimination

$\rightsquigarrow$ $\Sigma = (\mathbb{R}, \mathbb{R}^2, \mathfrak{B})$ **behavior** $\mathfrak{B}$ **specified by:**

Case 1: $CR_C \neq \frac{L}{R_L}$

$$\left(\frac{R_C}{R_L} + \left(1 + \frac{R_C}{R_L}\right) CR_C \frac{d}{dt} + CR_C \frac{L}{R_L} \frac{d^2}{dt^2}\right) V_{\text{externalport}}$$
$$= \left(1 + CR_C \frac{d}{dt}\right) \left(1 + \frac{L}{R_L} \frac{d}{dt}\right) R_C I_{\text{externalport}}$$

Case 2: $CR_C = \frac{L}{R_L}$

$$\left(\frac{R_C}{R_L} + CR_C \frac{d}{dt}\right) V_{\text{externalport}} = (1 + CR_C) \frac{d}{dt} R_C I_{\text{externalport}}$$

Elimination Theorem

Thm. 2: $\mathfrak{L}^\bullet$ is closed under projection

Elimination Theorem

Thm. 2: $\mathfrak{L}^\bullet$ is closed under projection

Consider

$$R_1\left(\frac{d}{dt}\right)w_1 + R_2\left(\frac{d}{dt}\right)w_2 = 0 \ \rightsquigarrow \ \textbf{behavior } \mathfrak{B}$$

Define

$$\mathfrak{B}_1 := \{w_1 \mid \exists\, w_2 \textbf{ such that } (w_1, w_2) \in \mathfrak{B}\}$$

Elimination thm $\Rightarrow \ \exists\, R$ **such that** $\mathfrak{B}_1 = \mathrm{kernel}\left(R\left(\frac{d}{dt}\right)\right)$**!**

E.g. $\frac{d}{dt}x = Ax + Bu, y = Cx + Du \ \Rightarrow \ P(\frac{d}{dt})y = Q(\frac{d}{dt})u$
linear DAE's allow elimination of nuisance variables

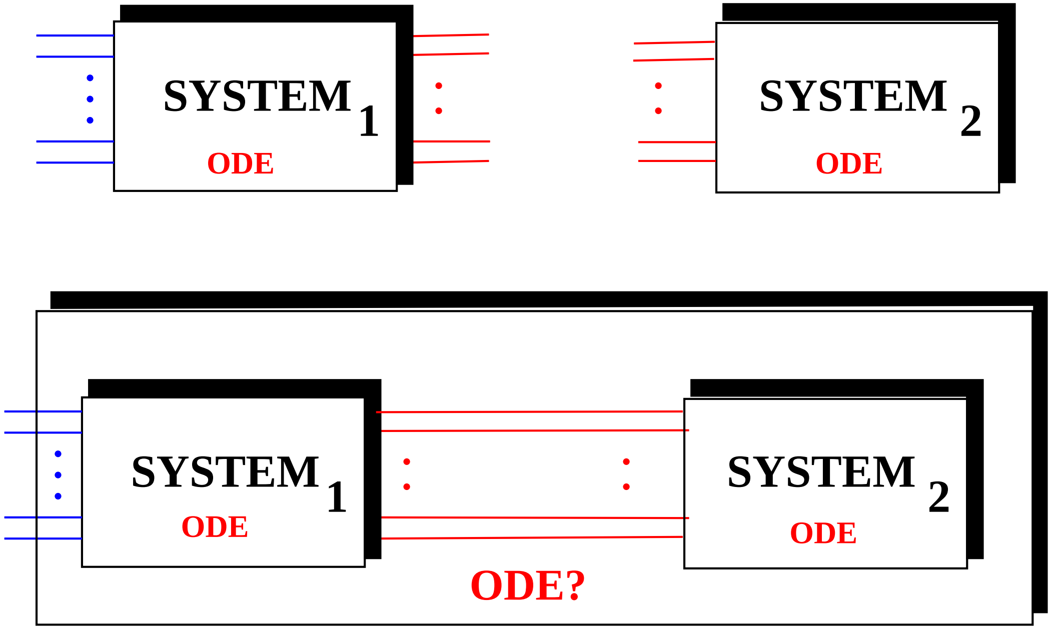
Elimination Theorem

Thm. 2: $\mathfrak{L}^\bullet$ is closed under projection

$\mathfrak{L}^\bullet$ is closed under intersection, addition, projection

Elimination Theorem

Thm. 2: $\mathfrak{L}^\bullet$ is closed under projection



In LTIDS systems 1 and 2 ODEs $\Rightarrow$ interconnection ODE. In nonlinear case, very unlikely described by ODE, even if systems 1 and 2 are!

Why are ODE models so common?

Image representation

$$w = M\left(\frac{d}{dt}\right)\ell \quad \rightsquigarrow w\text{-behavior } \mathfrak{B} = \mathrm{image}\left(M\left(\frac{d}{dt}\right)\right)$$

Elimination thm $\Rightarrow \mathfrak{B} = \mathrm{kernel}\left(R\left(\frac{d}{dt}\right)\right)$**, for some** R**.**

So, all images are kernels, but what kernels are images?

$\Leftrightarrow$ $\mathfrak{B}$ **is controllable**

Image representation

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$\Leftrightarrow \mathfrak{B}$ **is controllable**

Thm. 3: $\mathfrak{B} \in \mathfrak{L}^\bullet$ **controllable** $\Leftrightarrow$ **admits image representation**

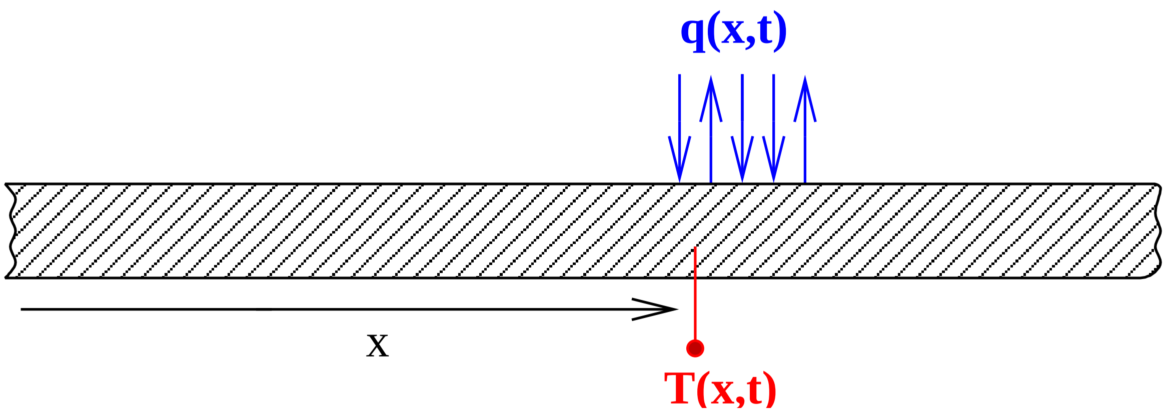
Leads to controllability tests.
There are various other tests for controllability, e.g.

$$\mathrm{rank}\,(R(\lambda)) \;\textbf{ the same } \forall\, \lambda \in \mathbb{C}$$

Partial differential equations

PDEs: Examples

Diffusion



$$\frac{\partial}{\partial t}T = \frac{\partial^2}{\partial x^2}T + q$$

independent variables: (t, x) **time and space**

dependent variables: (T, q) **temperature and heat**

PDEs: Examples

Wave phenomena

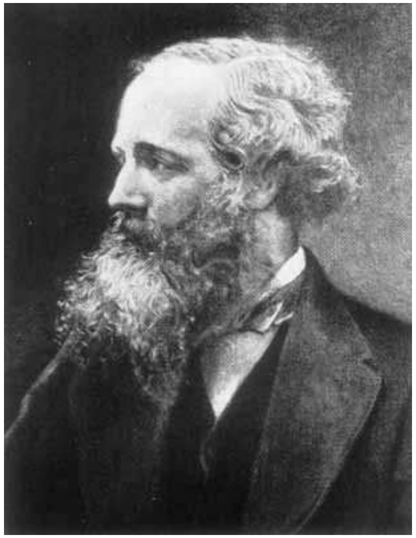
$$\boxed{\frac{\partial^2}{\partial t^2} w = \frac{\partial^2}{\partial x^2} w + \frac{\partial^2}{\partial y^2} w}$$

independent variables: (t, x, y) **time and space**

dependent variables: w **deflection**

PDEs: Examples

Maxwell's equations for EM fields in free space



$$
\begin{aligned}
\nabla \cdot \vec{E} &= \frac{1}{\varepsilon_0}\rho\,, \\
\nabla \times \vec{E} &= -\frac{\partial}{\partial t}\vec{B}, \\
\nabla \cdot \vec{B} &= 0\,, \\
c^2 \nabla \times \vec{B} &= \frac{1}{\varepsilon_0}\vec{j} + \frac{\partial}{\partial t}\vec{E}.
\end{aligned}
$$

independent variables: (t, x, y, z) **time and space**

dependent variables: $(\vec{E}, \vec{B}, \vec{j}, \rho)$

electric field, magnetic field, current density, charge density

PDEs: Notation

$\mathbb{R}[\xi_1, \ldots, \xi_{\mathtt{n}}]$: **polynomials, $\mathtt{n}$ indet., real coeff.**

$\mathbb{R}[\xi_1, \ldots, \xi_{\mathtt{n}}]^{\bullet \times \mathtt{w}}$, $\mathbb{R}[\xi_1, \ldots, \xi_{\mathtt{n}}]^{\bullet \times \bullet}$ **matrices of ...**

PDEs: Notation

$$\boldsymbol{R} \in \mathbb{R}\left[\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_{\mathtt{n}}\right]^{\bullet \times \mathtt{w}} \rightsquigarrow \boldsymbol{R}\left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathtt{n}}}\right) \boldsymbol{w} = 0$$

linear constant coefficient PDEs with

$\mathtt{n}$ **independent variables,** $x_1, \ldots, x_{\mathtt{n}}$

$\mathtt{w}$ **dependent variables,** $w_1, \ldots, w_{\mathtt{w}}$

$\mathrm{rowdim}(R)$ **= number of equations**

PDEs: Notation

$$\boldsymbol{R} \in \mathbb{R}[\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_{\mathtt{n}}]^{\bullet \times \mathtt{w}} \rightsquigarrow \boldsymbol{R}\left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathtt{n}}}\right) \boldsymbol{w} = 0$$

Ex.: Diffusion eq'n $\frac{\partial}{\partial t} T = \frac{\partial^2}{\partial x^2} T + q$

2 indep. variables, (t, x)**,** $\mathtt{w} = 2$**,** $w = (T, q)$**, 1 eq'n.**

$$R(\xi_t, \xi_x) = [\, \xi_t - \xi_x^2 \mid -1 \,]$$

PDEs: Notation

$$\boldsymbol{R} \in \mathbb{R}\left[\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_{\mathtt{n}}\right]^{\bullet \times \mathtt{w}} \rightsquigarrow \boldsymbol{R}\left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathtt{n}}}\right) \boldsymbol{w} = 0$$

Example: Maxwell's eq'ns

4 independent variables, (t, x, y, z)

$\mathtt{w} = 10$, $w = (\vec{E}, \vec{B}, \vec{j}, \rho)$

8 equations, R 8×10**, sparse, first order**

PDEs: Notation

$$R \in \mathbb{R}[\xi_1, \ldots, \xi_{\mathtt{n}}]^{\bullet \times \mathtt{w}} \rightsquigarrow R\left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathtt{n}}}\right) w = 0$$

Behavior:

$$\mathfrak{B} = \{w \in \mathfrak{C}^\infty(\mathbb{R}^{\mathtt{n}}, \mathbb{R}^{\mathtt{w}}) \mid R\left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathtt{n}}}\right) w = 0\}$$

Notation:

$$\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\mathtt{n}}, \quad \mathfrak{B} = \mathtt{kernel}\left(R\left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathtt{n}}}\right)\right)$$

PDEs: Notation

$$R \in \mathbb{R}[\xi_1, \ldots, \xi_{\mathtt{n}}]^{\bullet \times \mathtt{w}} \rightsquigarrow R\left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathtt{n}}}\right) w = 0$$

Linear constant coefficient PDEs

$\mathfrak{C}^\infty$-solutions

infinite domain, no boundary conditions

'everything' valid for convex, open domain $\Omega \subseteq \mathbb{R}^{\mathtt{n}}$

$\mathfrak{L}_{\mathtt{n}}^{\mathtt{w}}$: the basics

$\mathfrak{L}^{\mathtt{w}}_{\mathtt{n}}$

<u>Thm. 1</u>:

$\mathfrak{L}^{\mathtt{w}}_{\mathtt{n}} \quad \leftrightarrow \quad$ **the submodules of** $\mathbb{R}\left[\xi_1, \ldots, \xi_{\mathtt{n}}\right]^{\mathtt{w}}$

<u>Thm. 2</u>: Elimination thm

$\mathfrak{L}^{\mathtt{w}}_{\mathtt{n}}$ **is closed under projection**

<u>Thm. 3</u>:

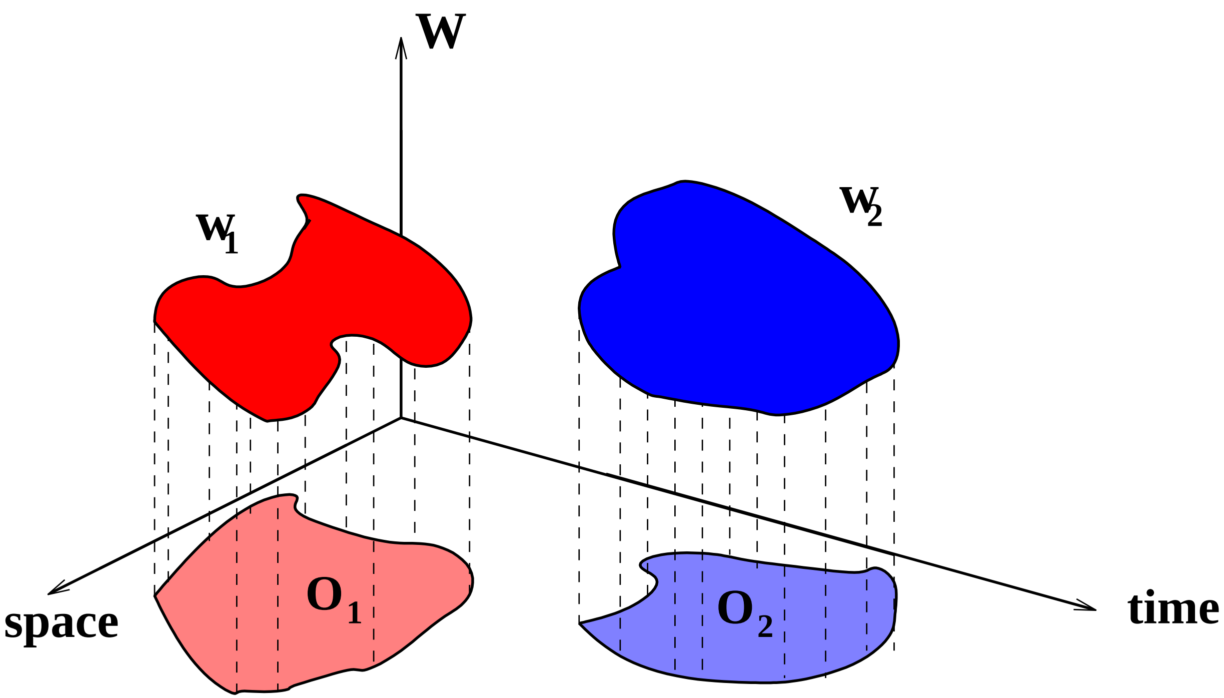
$\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\mathtt{n}}$ **is controllable** $\Leftrightarrow$ $\mathfrak{B}$ **is an image**

Describe $(\rho, \vec{E}, \vec{j})$ in Maxwell's equations

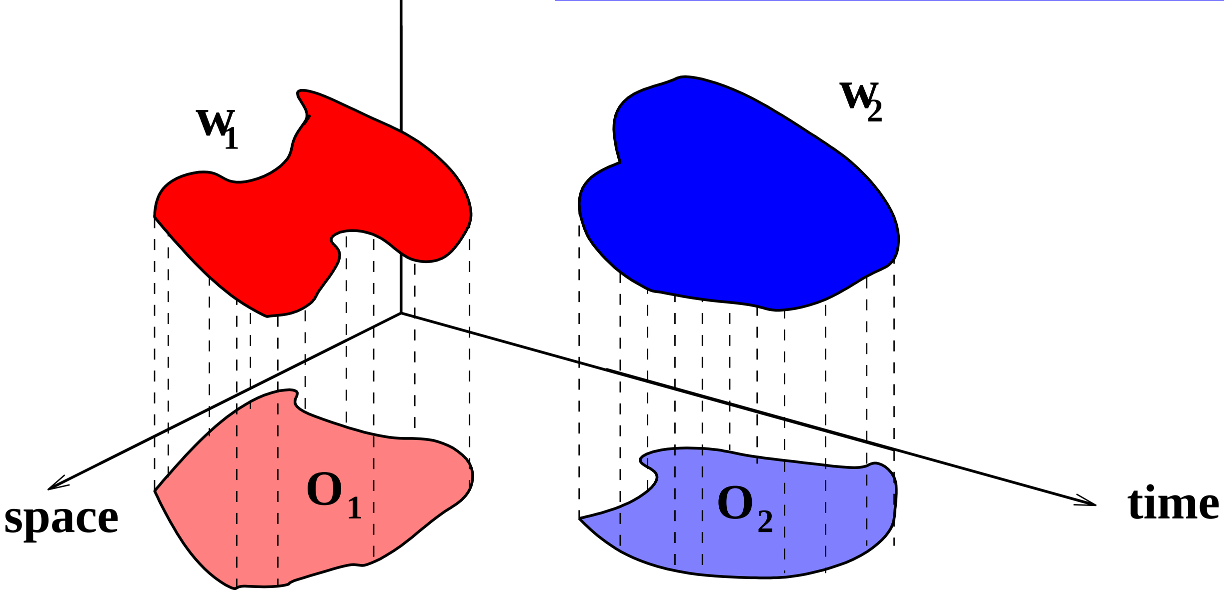
Eliminate $\vec{B}$ from Maxwell's equations $\rightsquigarrow$

$$\begin{aligned}
\nabla \cdot \vec{E} &= \frac{1}{\varepsilon_0}\rho\,, \\
\varepsilon_0 \frac{\partial}{\partial t}\nabla \cdot \vec{E} + \nabla \cdot \vec{j} &= 0, \\
\varepsilon_0 \frac{\partial^2}{\partial t^2}\vec{E} + \varepsilon_0 c^2 \nabla \times \nabla \times \vec{E} + \frac{\partial}{\partial t}\vec{j} &= 0.
\end{aligned}$$

Controllability on $\mathtt{n}$D systems

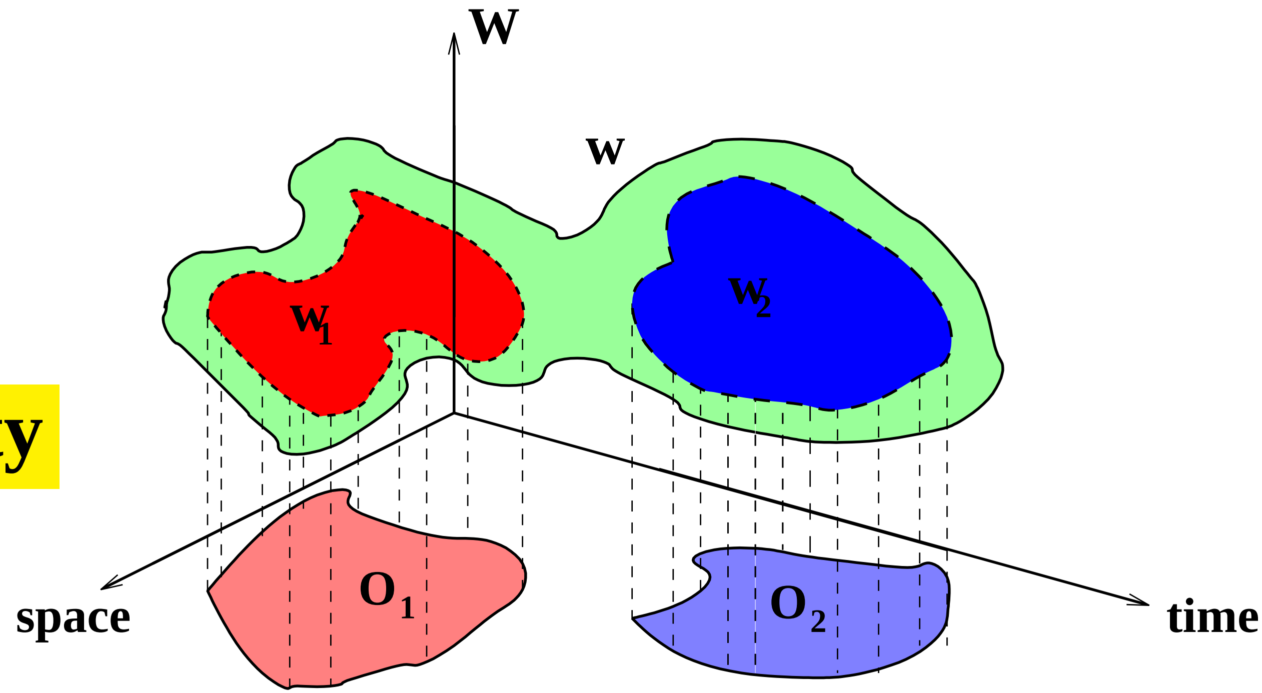


Controllability on nD systems



Controllability

:= Patchability



Controllability on $\mathtt{n}$D systems

$\mathfrak{B} \in \mathfrak{L}_{\mathtt{n}}^{\mathtt{w}}$ **controllable if and only if it has a repr.**

$$w = M\left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathtt{n}}}\right)\ell$$

$$\mathfrak{B} = \mathtt{image}\left(M\left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathtt{n}}}\right)\right)$$

Is an image a kernel ? Always ! $\Leftarrow$ Elimination th'm

Is a kernel an image ? Iff the kernel is controllable !

Controllability on $\mathtt{n}$D systems

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$$\mathfrak{B} = \mathtt{image}\left(M\left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathtt{n}}}\right)\right)$$

But, for $\mathtt{n} > 1$, this image may not be observable.

Images may require hidden variables.

Are EM fields controllable ?

Are EM fields controllable ?

The following eq'ns in

scalar potential $\phi : \mathbb{R} \times \mathbb{R}^3 \longrightarrow \mathbb{R}$

vector potential $\vec{A} : \mathbb{R} \times \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

generate exactly the solutions to MEs:

$$
\begin{aligned}
\vec{E} &= -\frac{\partial}{\partial t}\vec{A} - \nabla\phi, \\
\vec{B} &= \nabla \times \vec{A}, \\
\vec{j} &= \varepsilon_0 \frac{\partial^2}{\partial t^2}\vec{A} - \varepsilon_0 c^2 \nabla^2 \vec{A} + \varepsilon_0 c^2 \nabla\left(\nabla \cdot \vec{A}\right) + \varepsilon_0 \frac{\partial}{\partial t}\nabla\phi, \\
\rho &= -\varepsilon_0 \frac{\partial}{\partial t}\nabla \cdot \vec{A} - \varepsilon_0 \nabla^2 \phi.
\end{aligned}
$$

Are EM fields controllable ?

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\rho &= -\varepsilon_0 \frac{\partial}{\partial t}\nabla \cdot \vec{A} - \varepsilon_0 \nabla^2 \phi.
\end{aligned}$$

Proves controllability of EM fields.

Not observable, cannot be !

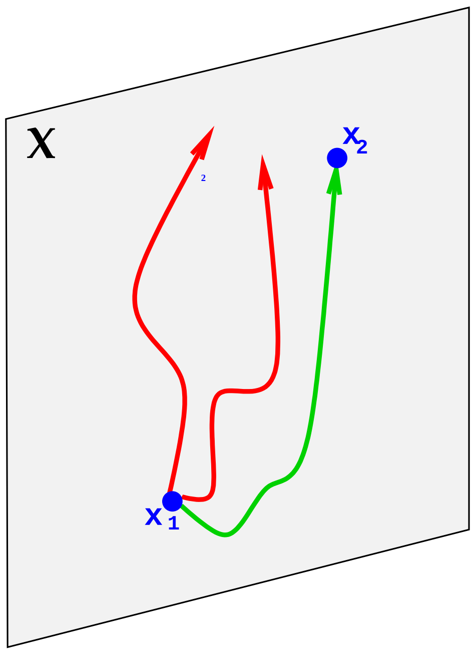
controllability $\Leftrightarrow$ $\exists$ potential!

Control

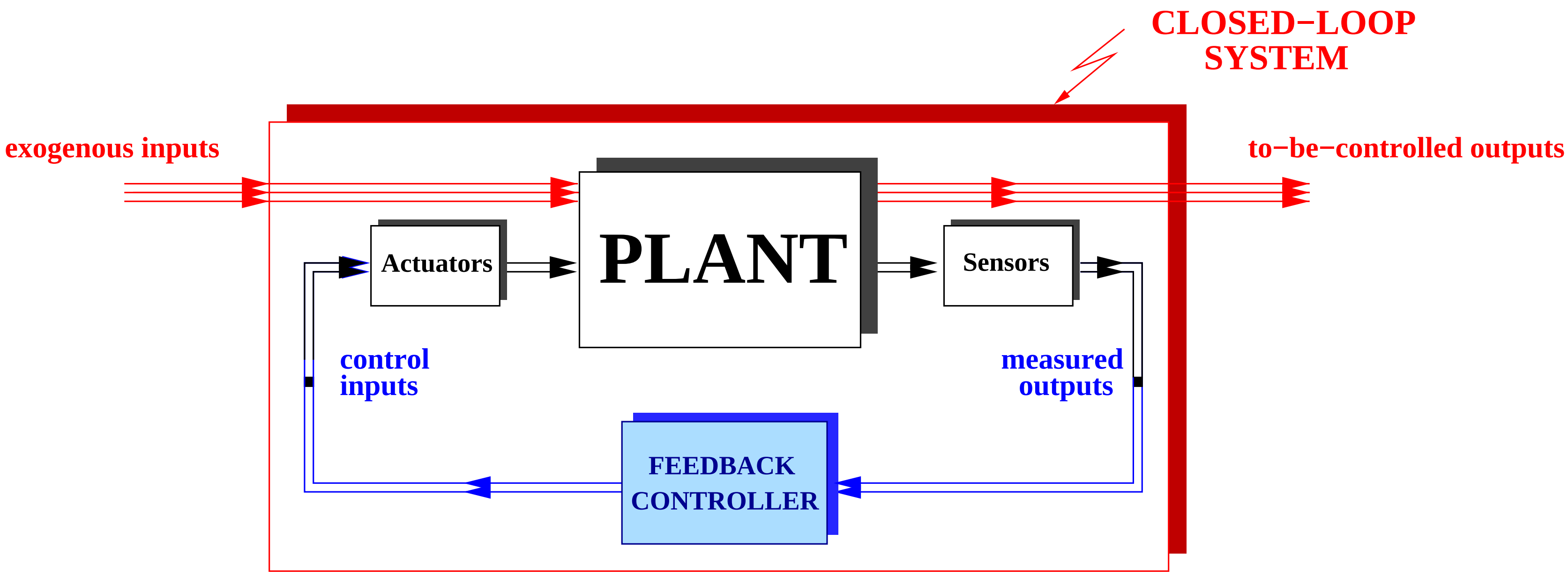
Path planning

$$\frac{d}{dt}x = f\left(x, u\right)$$

Choose time-function $u(\cdot) : [0, T] \longrightarrow \mathbb{U}$ so as to achieve (optimal) state transfer. **'open loop control'**



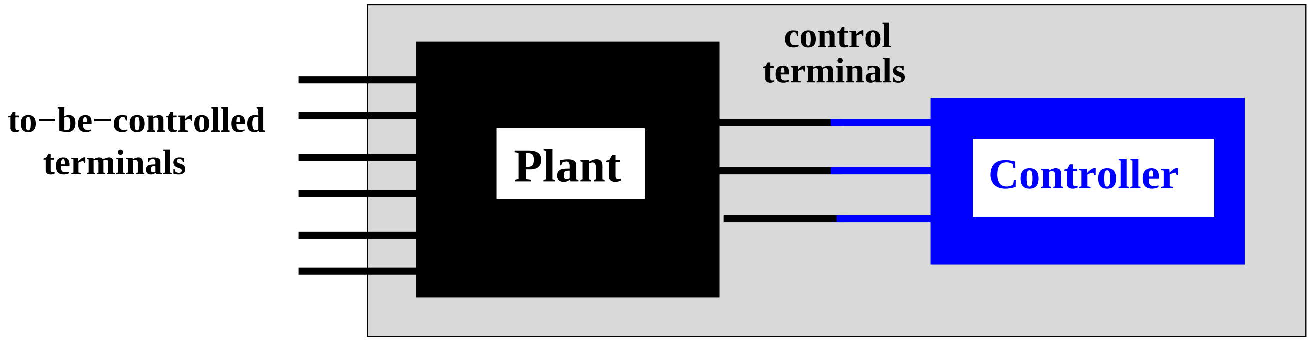
Decision making



Choose map from sensor outputs to actuator inputs so as to achieve good (optimal) performance.

'feedback control'
'closed loop control'
'intelligent control'

Embedded systems



Controlled system

Choose controller so as to achieve good (optimal) performance of the interconnected system

'control as interconnection'
'integrated system design'

Look back

Behavioral systems

- **Gets the physics right**

Behavioral systems

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- **Deals faithfully with interconnections: variable sharing**

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- **Deals faithfully with interconnections: variable sharing**
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- **i/o and i/s/o are special cases**
- **Extends seamlessly to PDEs**
- **Views control as interconnection**
- **Advantages in SYSID, etc.**
- **Far easier pedagogically**
- **...**

Details & copies of frames are available from/at

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`http://www.esat.kuleuven.be/~jwillems`

Thank you

Thank you

Thank you

Thank you

Thank you

Thank you

Thank you

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