DISTANCE BETWEEN LINEAR SYSTEMS

and

ORDER REDUCTION

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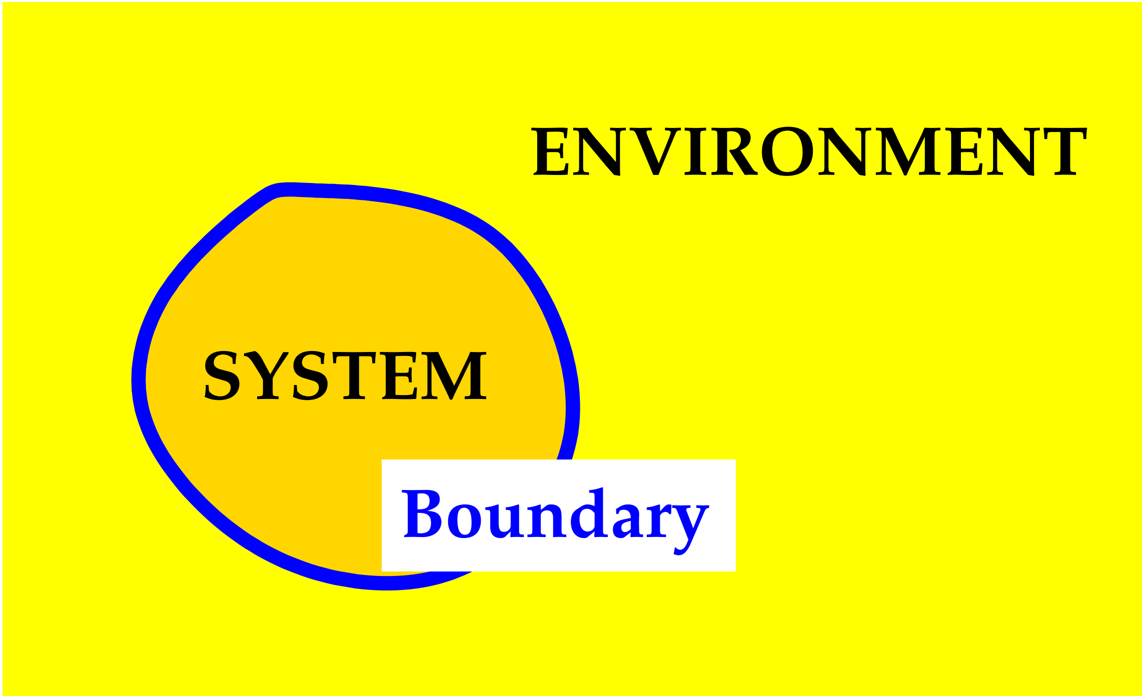
In honor of Adhemar Bultheel on the occasion of his 60th birthday

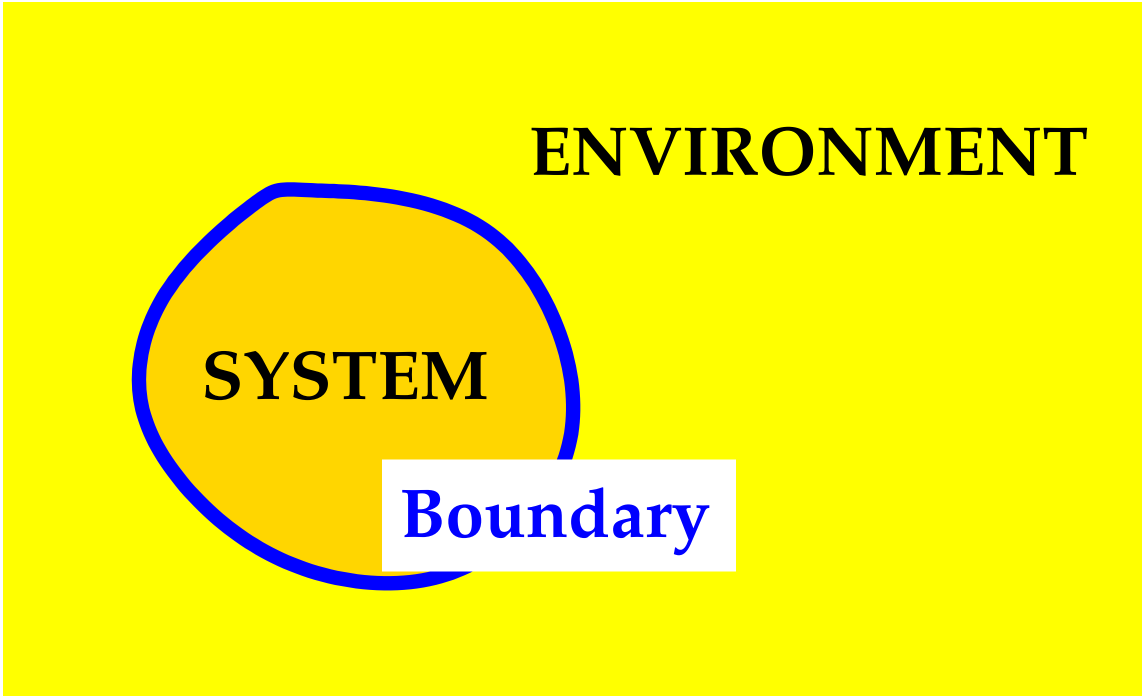
Open systems

Open systems



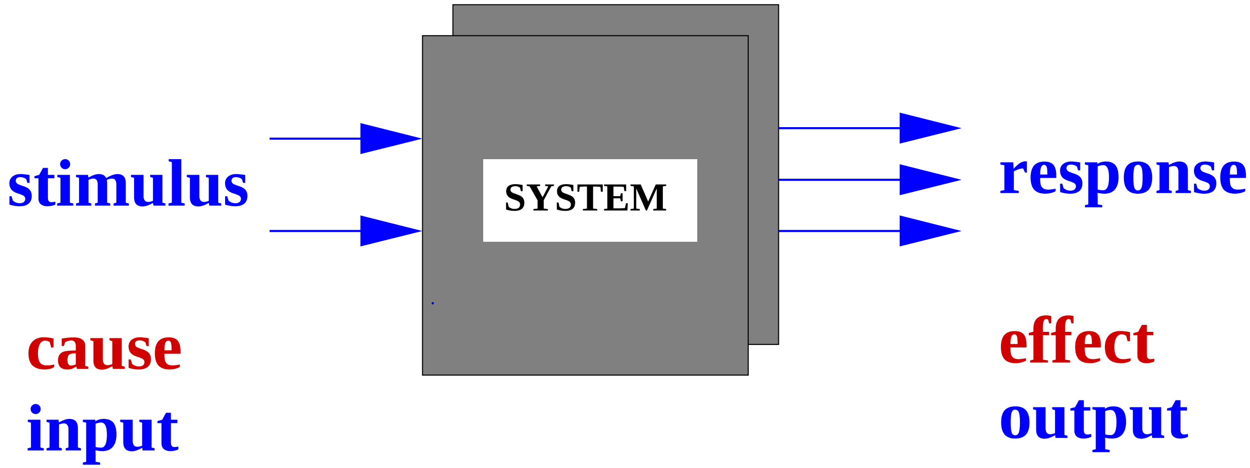
Systems interact with their environment

Open systems



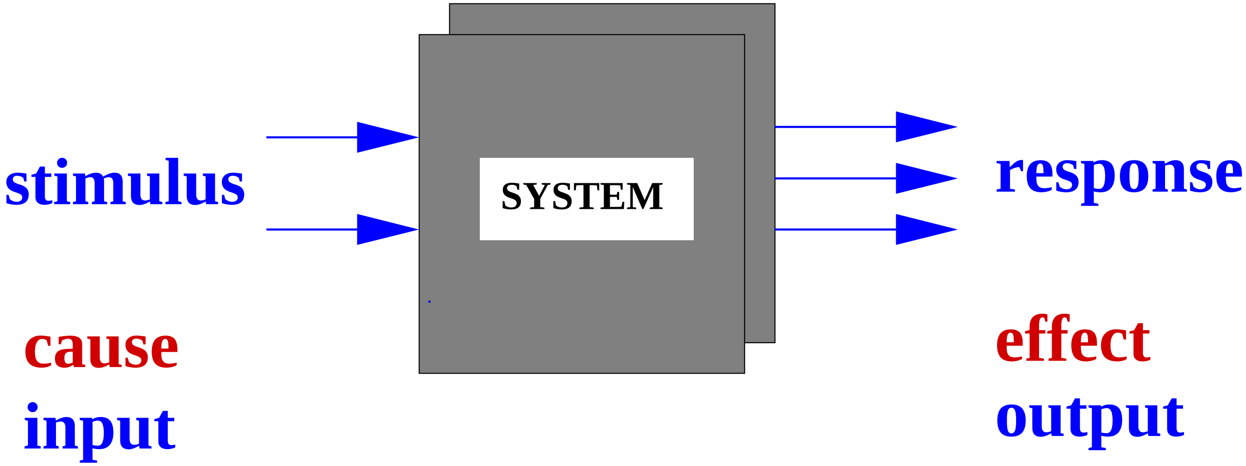
How are open systems formalized ?

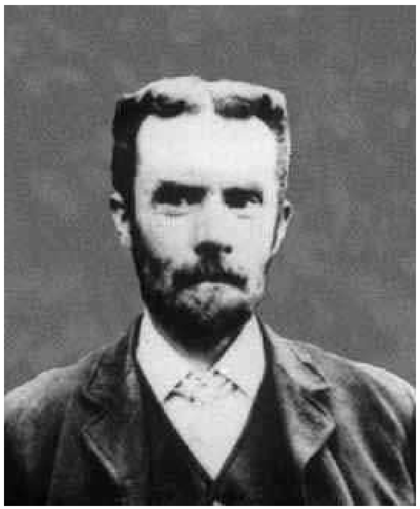
Classical approach: input/output systems



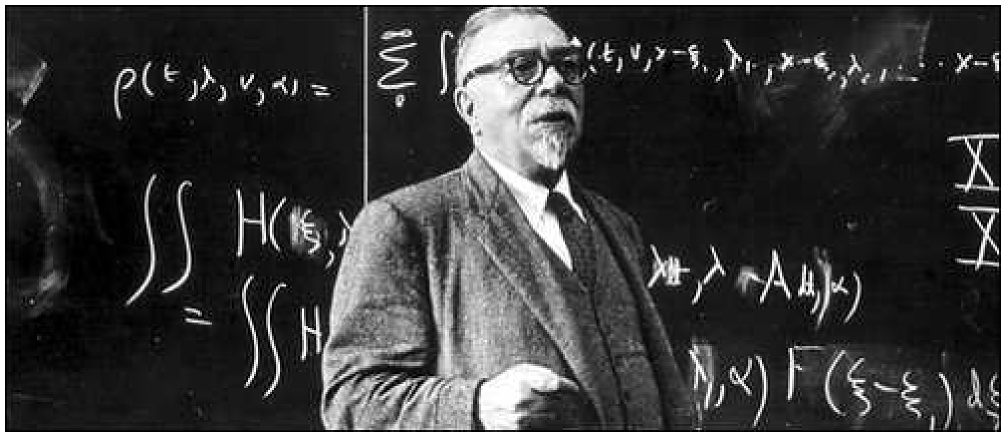
Convolutions, transfer functions, impedances, ...

Classical approach: input/output systems





Oliver Heaviside (1850-1925)



Norbert Wiener (1894-1964)

and many electrical circuit theorists

Input/state/output systems

Around 1960: a paradigm shift to

$$\frac{d}{dt}x = f(x,u),\; y = g(x,u)$$

Rudolf Kalman (1930-)

This framework turned out to be very effective and useful, for example in model order reduction (MOR).

Model order reduction

MOR for linear systems (≅ rational approximation)

$$\frac{d}{dt}x = Ax + Bu, y = Cx, \quad x \in \mathbb{R}^{\mathtt{n}}, u \in \mathbb{R}^{\mathtt{m}}, y \in \mathbb{R}^{\mathtt{p}}$$

Assume stable (*A* Hurwitz: roots in open left half of $\mathbb{C}$).

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$$u \mapsto y \qquad\qquad y(t) = \int_{-\infty}^{t} Ce^{A(t-t')}Bu(t')\,dt'$$

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¡¡ Approximate this system, this map, by another one

$u \mapsto y$ $\qquad$ $y(t) = \int_{-\infty}^{t} C_{\mathtt{red}}e^{A_{\mathtt{red}}(t-t')}B_{\mathtt{red}}u(t')\,dt'$

'simpler': lower state dim. !! $\qquad x_{\mathtt{red}} \in \mathbb{R}^{\mathtt{n}_{\mathtt{red}}}$, $\boxed{\mathtt{n}_{\mathtt{red}} \ll \mathtt{n}}$

$\rightsquigarrow$ **balancing, AAK, Krylov, POD, etc.**

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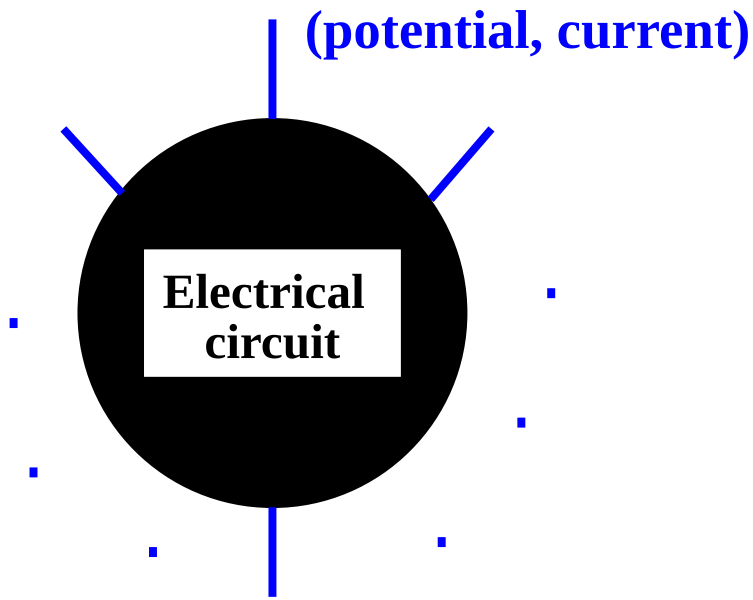
$\exists$ effective methods for MOR for stable LTI i/o systems

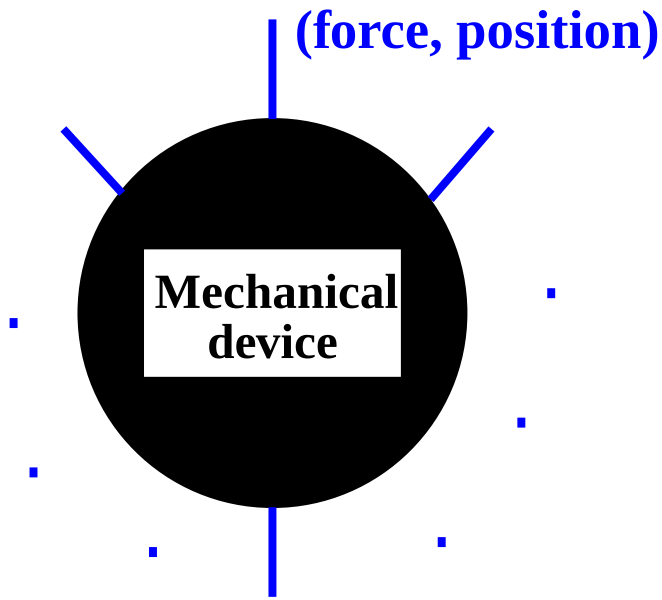
Drawbacks of input/output thinking

Physical systems with terminals



Associated with each terminal there are two variables. Which should be considered input? output?

Physical systems with terminals

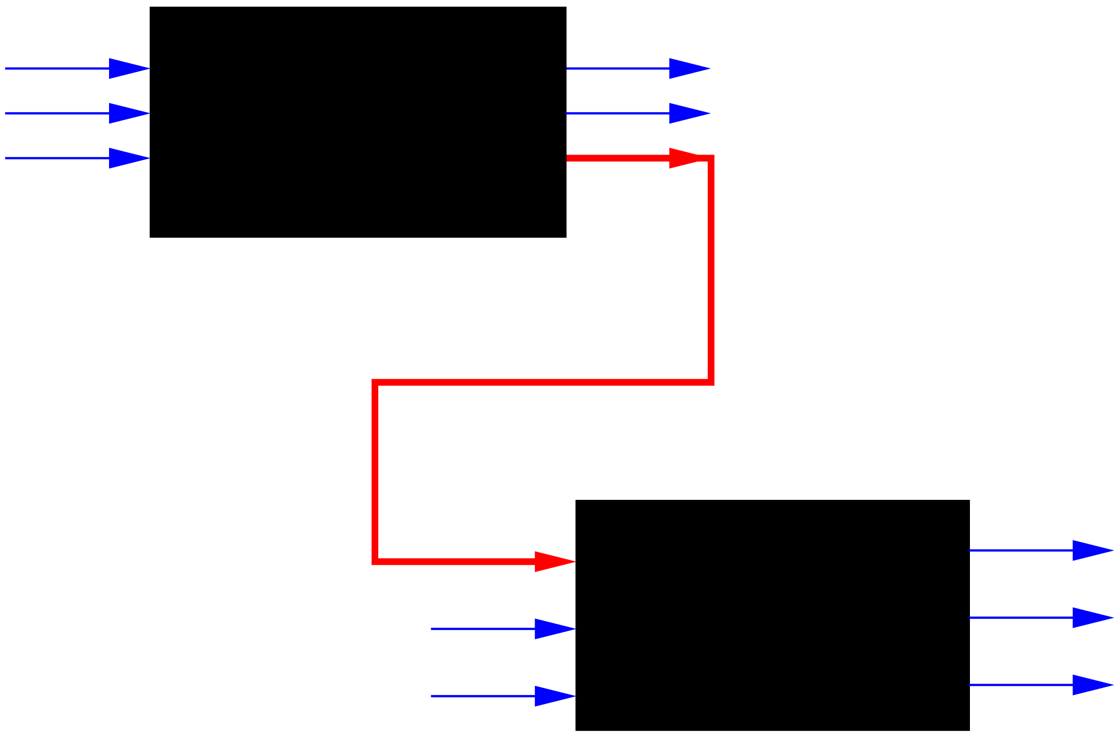


Associated with each terminal there are **two** variables. Which should be considered **input? output?**

- mechanical systems (terminal var's: force & position)
- *et cetera*

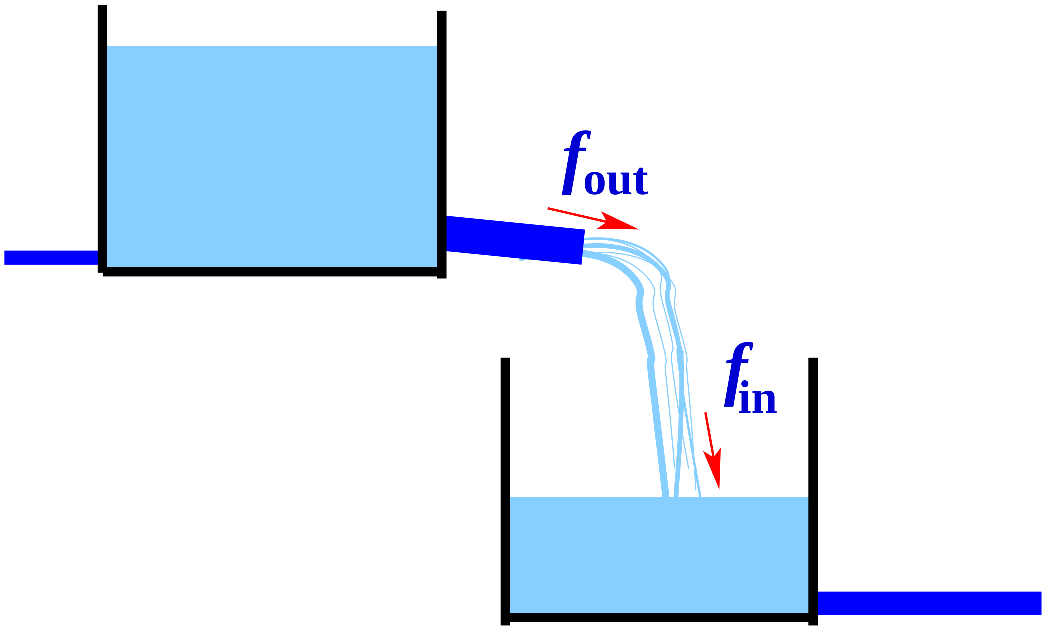
System interconnection as output-to-input assignment

The classical view of system interconnection:



System interconnection as output-to-input assignment

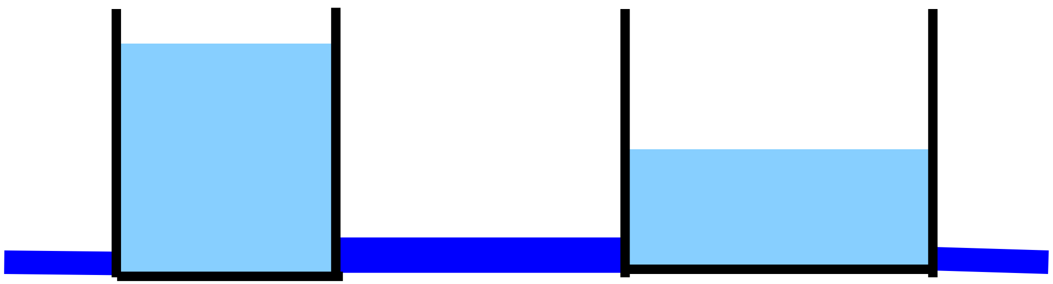
There are many examples where output-to-input connection is eminently natural.



System interconnection as output-to-input assignment

There are many examples where output-to-input connection is eminently natural.

But for other interconnections, i/o is more problematic.



Interconnection = variable sharing, not output-to-input assignment

Ceterum censeo

The input/output approach as the primary and universal view of open systems is a misconception.

Physical systems are not signal processors !

Ceterum censeo

The input/output approach as the primary and universal view of open systems is a misconception.

Physical systems are not signal processors !

How should we formalize open systems, if not as input/output systems?

How does MOR function then?

Linear time-invariant differential systems

LTIDSs

LTIDSs

We consider systems described by linear, constant-coefficient, differential equations

$$R\left(\frac{d}{dt}\right)w = 0$$

with R **a polynomial matrix,** $R \in \mathbb{R}[\xi]^{\bullet \times \mathtt{w}}$

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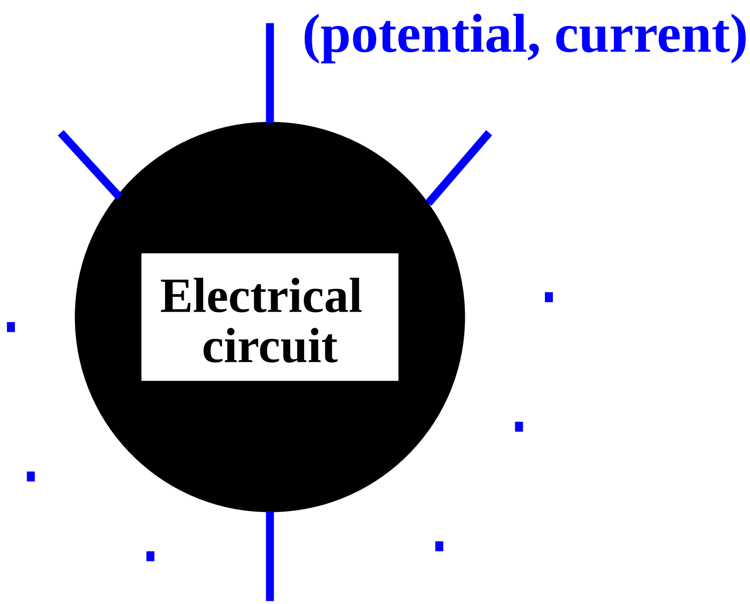

$$w = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_{\mathtt{w}} \end{bmatrix} = \begin{bmatrix} V_1 \\ I_1 \\ V_2 \\ I_2 \\ \vdots \\ V_N \\ I_N \end{bmatrix}$$

All system variables are treated on the same footing.

A model = a relation (rather than a map)

LTIDSs

We consider systems described by linear, constant-coefficient, differential equations

$$R\left(\frac{d}{dt}\right)w = 0$$

Behavior $\mathscr{B}$**:= the set of solutions**

If you so like, assume the solutions in $\mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$**.**

LTIDSs - Rational symbol representation

$$R\left(\frac{d}{dt}\right)w = 0$$

readily generalized to the case where R **is a matrix of** **rational functions** $R \in \mathbb{R}(\xi)^{\bullet\times\mathtt{w}}$

$$R = P^{-1}Q, \quad P, Q \in \mathbb{R}[\xi]^{\bullet\times\bullet},$$ **left coprime**

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$\rightsquigarrow$ **A LTID behavior has many representations.**

A very special representation

Norm-preserving image representations

Let $\mathfrak{B}$ be the behavior of a controllable LTIDS.
Then it allows an observable 'image representation'

$w = M(\frac{d}{dt})\ell$ **with** $M \in \mathbb{R}(\xi)^{\mathtt{w}\times\bullet}$

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i.e.

$$\int_{-\infty}^{+\infty} ||w(t)||^2\, dt = \int_{-\infty}^{+\infty} ||\ell(t)||^2\, dt$$

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M **cannot be polynomial, it must be rational.**
Obviously M **must also be proper.**

Can also make M **stable (meaning: its poles are in the left half of the complex plane).**

Distance between behaviors

Distance between subspaces

A model is a behavior, a set (of trajectories).

Hence the distance between LTIDSs translates into the distance between linear subspaces.

Distance between subspaces

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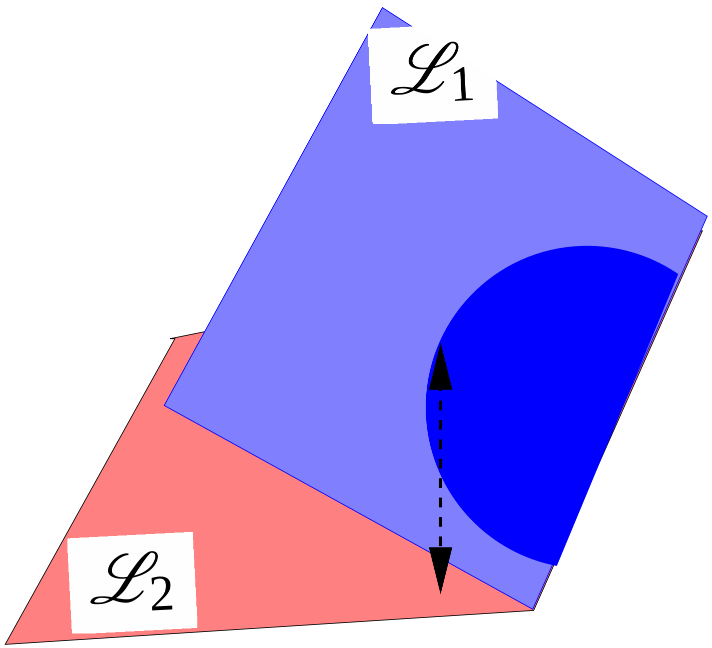
$\mathscr{L}_1, \mathscr{L}_2$ **linear subspaces of a Hilbert space.**

$$\overrightarrow{d}(\mathscr{L}_1, \mathscr{L}_2) := \mathbf{sup}_{x_1 \in \mathscr{L}_1, ||x_1||=1} \ \mathbf{inf}_{x_2 \in \mathscr{L}_2} ||x_1 - x_2||$$



closest point
on unit sphere of $\mathscr{L}_1$
from $\mathscr{L}_2$

Distance between subspaces

$$\mathbf{gap}(\mathscr{L}_1, \mathscr{L}_2) := \mathbf{max}\left\{\overrightarrow{d}(\mathscr{L}_1, \mathscr{L}_2), \overrightarrow{d}(\mathscr{L}_2, \mathscr{L}_1)\right\}$$

$$0 \leq \mathbf{gap}(\mathscr{L}_1, \mathscr{L}_2) \leq 1$$

Association of a subspace of a Hilbert space to a LTID behavior

The behavior $\mathscr{B}$ of a LTIDS is not a subspace of a Hilbert space.

Which subspace of which Hilbert space should we associate with a LTID behavior $\mathscr{B}$?

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$$\mathscr{B} \mapsto \mathscr{B}^{\mathscr{L}_2} := \mathscr{B} \cap \mathscr{L}_2(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$$

Distance between behaviors

Define the distance between two LTID behaviors as

$$d(\mathscr{B}_1, \mathscr{B}_2) := \textbf{gap}(\mathscr{B}_1^{\mathscr{L}_2}, \mathscr{B}_2^{\mathscr{L}_2})$$

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Keep notation $\mathscr{B}$ for $\mathscr{B}^{\mathscr{L}_2} = \mathscr{B} \cap \mathscr{L}_2(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$.

$$\forall w_1 \in \mathscr{B}_1, \exists w_2 \in \mathscr{B}_2 \textbf{ such that } ||w_1 - w_2|| \leq \textbf{gap}(\mathscr{B}_1, \mathscr{B}_2)||w_1||$$

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Small gap $\Rightarrow$ the LTIDSs are ‘close’.

‘Phenomena’ in $\mathscr{B}_1$ are well approximated by ‘phenomena’ in $\mathscr{B}_2$, and vice-versa.

Computation of the gap

Distance between LTID behaviors

- **How to compute the gap?**
- **Model reduce according to the gap!**

Formula for the gap for LTID behaviors

$\mathscr{B}_1, \mathscr{B}_2$ **LTID behaviors.**

Take norm-preserving image representations

$$w = M_1(\frac{d}{dt})\ell_1, \qquad w = M_2(\frac{d}{dt})\ell_2$$

Then

$$\textbf{gap}(\mathscr{B}_1, \mathscr{B}_2) = ||M_1(i\omega)M_1(-i\omega)^\top - M_2(i\omega)M_2(-i\omega)^\top||_{\mathscr{L}_\infty}$$
$$\leq ||M_1(i\omega) - M_2(i\omega)||_{\mathscr{L}_\infty}$$

MOR in the gap

Balanced MOR

Let *M* be the transfer function of a stable input/output system (strictly proper rational function, `n` poles, all in open left half plane).

Associated with *M* there are nonnegative real numbers (the Hankel singular values)

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_{\mathtt{n}_{\mathtt{red}}} \geq \cdots \geq \sigma_{\mathtt{n}} > 0$$

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leading to M_{red} stable input/output reduced system with $\mathtt{n}_{\mathtt{red}}$ poles, in LHP.

Balanced model reduction $\Rightarrow$

$$||M(i\omega) - M_{\text{red}}(i\omega)||_{\mathcal{H}_\infty} \leq 2 \sum_{\textbf{neglected Hankel SVs of } G} \sigma_{\mathtt{k}}$$

Reduction of a stable norm-preserving representation

Start with a LTID behavior $\mathscr{B}$. Represent $\mathscr{B}$ by a norm-preserving, stable image representation

$$\boxed{w = M(\tfrac{d}{dt})\ell} \quad \textbf{with} \quad M \in \mathbb{R}(\xi)^{\mathtt{w}\times\bullet}$$

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Now MOR (in the sense of the state dimension $\cong$ the order of the underlying ODE), in the classical way, viewed as a stable input/output system (input ℓ, output w) using balancing

$$\rightsquigarrow \quad \boxed{w = M_{\mathbf{red}}(\tfrac{d}{dt})\ell}$$

Error bound (classical - 'twice the sum of the tail'):

$$||M(i\omega) - M_{\mathbf{red}}(i\omega)||_{\mathscr{H}_\infty} \leq 2 \sum_{\text{neglected Hankel SVs of } M} \sigma_k$$

Behavior approximation and gap error bound

Start with stable norm-preserving representation of $\mathscr{B}$

$$w = M(\frac{d}{dt})\ell \quad \textbf{with} \quad M \in \mathbb{R}(\xi)^{\mathtt{w}\times\bullet}$$

MOR using balancing $\rightsquigarrow$ $\quad w = M_{\mathtt{red}}(\frac{d}{dt})\ell$.

Call the behavior of the reduced system $\mathscr{B}_{\mathtt{red}}$.

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Error bound

$$\begin{aligned}\mathbf{gap}(\mathscr{B}, \mathscr{B}_{\mathbf{red}}) &= \|M(i\omega)M(-i\omega)^\top - M_{\mathbf{red}}(i\omega)M_{\mathbf{red}}(-i\omega)^\top\|_{\mathscr{L}_\infty} \\ &\leq \|M(i\omega) - M_{\mathbf{red}}(i\omega)\|_{\mathscr{H}_\infty} \\ &\leq 2 \sum_{\text{neglected Hankel SVs of } M} \sigma_k\end{aligned}$$

Gap error bound

$\forall\, w \in \mathscr{B}\ \exists\, w' \in \mathscr{B}_{\mathtt{red}}$ **such that**

$$||w - w'|| \leq \left(2 \sum_{\text{neglected Hankel SVs of } M} \sigma_k \right) ||w||$$

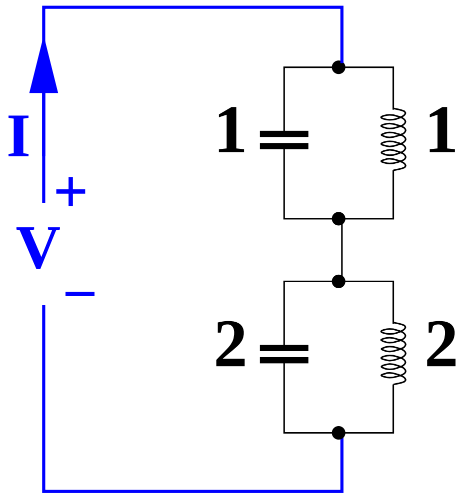
and vice-versa.

$\sum_{\text{neglected Hankel SVs of } M} \sigma_k$ **small**

$\Rightarrow$ **as linear subspaces,**
$\mathscr{B}_{\mathtt{red}}$ **is a good approximation of** $\mathscr{B}$
in the gap metric.

Example

LCLC circuit



system order = 4. Reduce to 2!

kernel $$\left(1+5\frac{d^2}{dt^2}+4\frac{d^4}{dt^4}\right)V=\left(3\frac{d}{dt}+6\frac{d^3}{dt^3}\right)I$$

image $$\begin{bmatrix} I \\ V \end{bmatrix} = \begin{bmatrix} 1+5\frac{d^2}{dt^2}+4\frac{d^4}{dt^4} \\ 3\frac{d}{dt}+6\frac{d^3}{dt^3} \end{bmatrix}\ell$$

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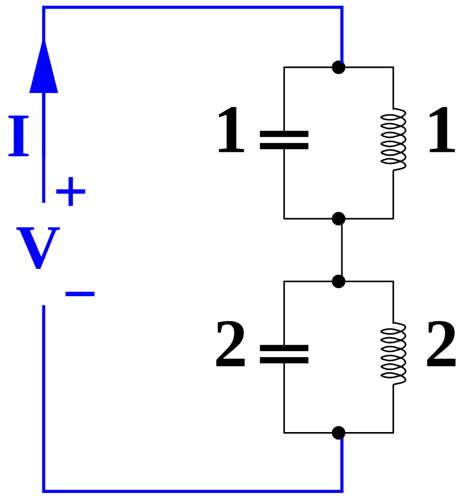

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stable norm-preserving image

$$\begin{bmatrix} I \\ V \end{bmatrix} = \frac{1}{1+3\frac{d}{dt}+5\frac{d^2}{dt^2}+6\frac{d^3}{dt^3}+4\frac{d^4}{dt^4}} \begin{bmatrix} 1+5\frac{d^2}{dt^2}+4\frac{d^4}{dt^4} \\ 3\frac{d}{dt}+6\frac{d^3}{dt^3} \end{bmatrix}\ell$$

Apply balancing algorithm $\rightsquigarrow$

LCLC circuit

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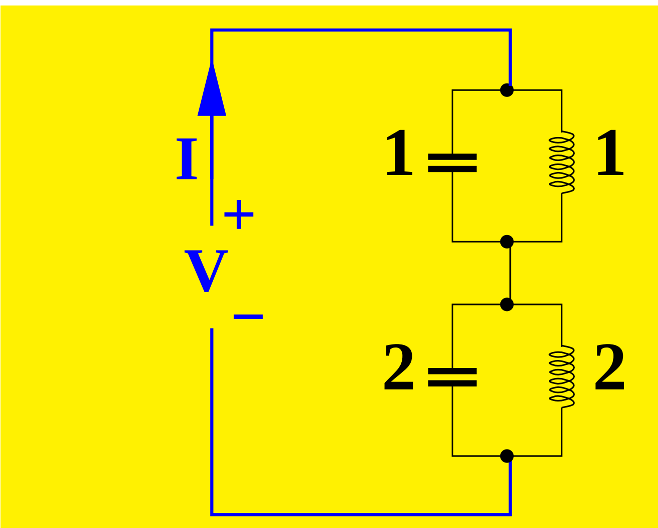
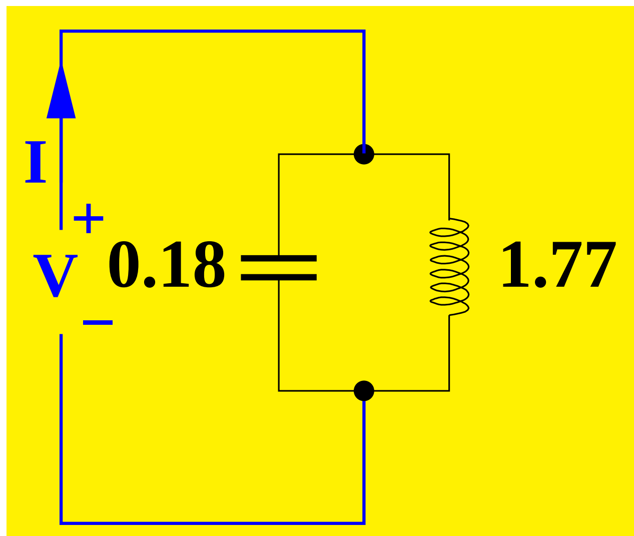
red. order = 2 $$\begin{bmatrix} I \\ V \end{bmatrix} = \frac{1}{\frac{d^2}{dt^2}+0.1861\frac{d}{dt}+0.3298} \begin{bmatrix} \frac{d^2}{dt^2}+0.3298 \\ 0.1861\frac{d}{dt} \end{bmatrix} \ell$$



$\rightsquigarrow$



Recapitulation

- **The gap is a measure of the distance between closed linear subspaces of a Hilbert space.**
- **Through the $\mathscr{L}_2$ behavior, the gap gives a measure of the distance between controllable LTIDSs.**

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- **Through the $\mathscr{L}_2$ behavior, the gap gives a measure of the distance between controllable LTIDSs.**

- **Observable norm-preserving image representations of LTIDSs allow to compute the gap,**
- **and lead to a model reduction algorithm with an error bound in the gap.**

The lecture frames are available from/at

`http://www.esat.kuleuven.be/~jwillems`

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Thank you

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