Models and Behaviors

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K.U. Leuven, Flanders, Belgium

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Where do I come from?


BELGIUM
Zeebrugge
Knokke
Blankenberge
Ostend
Bruges
Veurne
Diksmuide
Ghent
Ypres
Kortrijk
Mouscron
Tournai
Ath
Antwerp
Turnhout
Herentals
Lier
Mechelen
Aarschot
Brussels
Leuven
Hasselt
Tienen
Tongeren
Liège
Verviers
Nivelles
Namur
Charleroi
Hautes Fagnes NP
Chimay
The Ardennes
Bastogne
Bouillon
Arlon
NETHERLANDS
GERMANY
FRANCE
LUXEMBOURG
0 40 km
0 24 miles


BELGIUM
Zeebrugge
Knokke
Blankenberge
Ostend
Bruges
Veurne
Diksmuide
Ghent
Ypres
Kortrijk
Mouscron
Tournai
Ath
Brussels
Mechelen
Antwerp
Lier
Herentals
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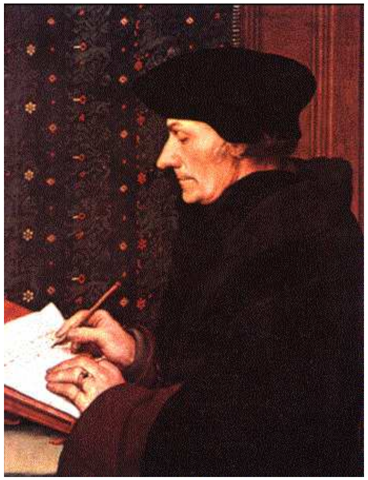
Aarschot
Leuven
Tienen
Hasselt
Tongeren
Liège
Verviers
Nivelles
Namur
Charleroi
Hautes
Fagnes NP
Chimay
The
Ardennes
Bastogne
Bouillon
Arlon
NETHERLANDS
GERMANY
FRANCE
LUXEMBOURG
0
40 km
0
24 miles

Adrianus VI
1459–1523



Erasmus
1469–1536

de la Valleé Poussin
1866–1962

Lemaître
1894–1966

Lecture

Outline

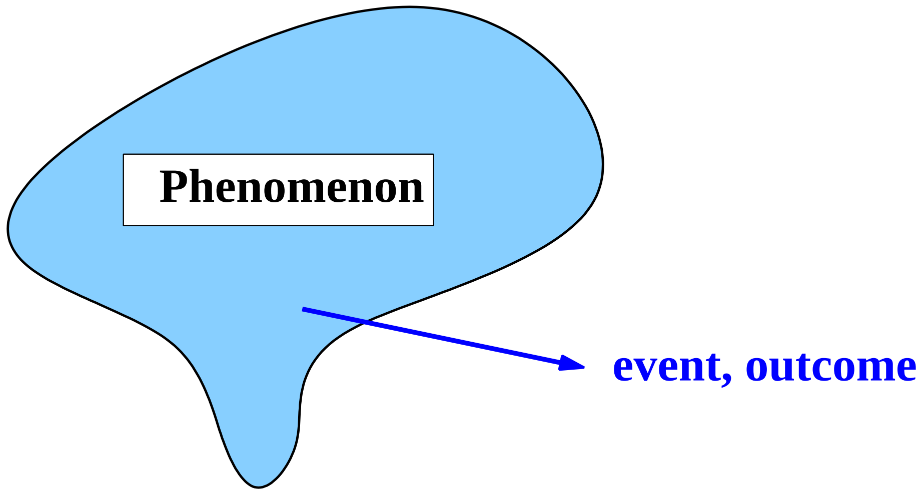
- Mathematical models
- The behavior
- Dynamical systems
- A bit of history
- Linear time-invariant systems
- Kernel representations
- Latent variables
- The elimination theorem

Mathematical models

A bit of mathematics & philosophy

Mathematical models

Assume that we have a 'real' phenomenon that produces *'events'*, *'outcomes'*.



Mathematical models

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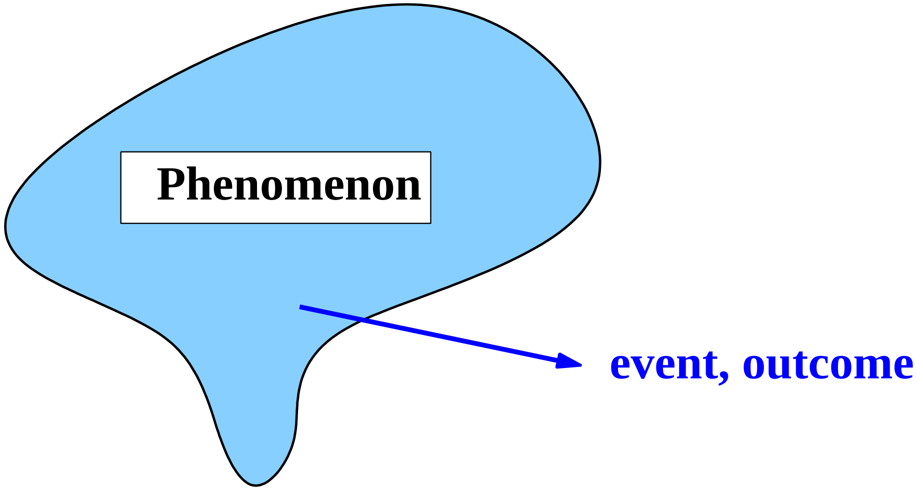

We view a deterministic mathematical model for a phenomenon as a prescription of which events can occur, and which events cannot occur.

Aim of this lecture

- **In the first part of this lecture, we develop this point of view into a mathematical formalism.**

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- **In the second part, we apply this formalism to dynamical systems, and zoom in on linear time-invariant differential systems.**

The universum

Mathematatization

The outcomes can be described in the language of mathematics, as mathematical objects, by answering:

To which universum do the events (before modeling) belong?

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The outcomes can be described in the language of mathematics, as mathematical objects, by answering:

To which universum do the events (before modeling) belong?

- **Do the events belong to a discrete set?**
  $\rightsquigarrow$ ***discrete event phenomena.***
- **Are the events real numbers, or vectors of real numbers?**
  $\rightsquigarrow$ ***continuous phenomena.***
- **Are the events functions of time?**
  $\rightsquigarrow$ ***dynamical phenomena.***
- **Are the events functions of space, or time & space?**
  $\rightsquigarrow$ ***distributed phenomena.***

Mathematization

The outcomes can be described in the language of mathematics, as mathematical objects, by answering:

To which universum do the events (before modeling) belong?

The set where the events belong to is called the universum, denoted by $\mathscr{U}$.

Discrete event phenomena

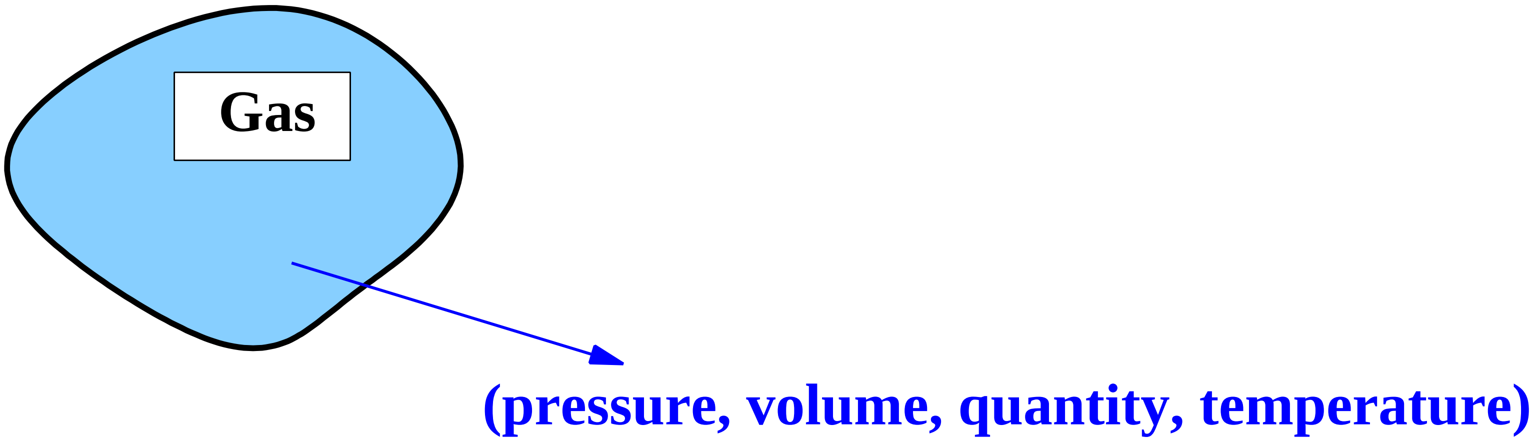
Examples:

- **Words in a natural language**
  $\mathscr{U} \cong \{a, b, c, \ldots, x, y, z\}^{\mathtt{n}}$
  with $\mathtt{n} =$ **the number of letters in the longest word**

Continuous phenomena

Examples:

- **The pressure, volume, quantity, and temperature of a gas in a vessel**

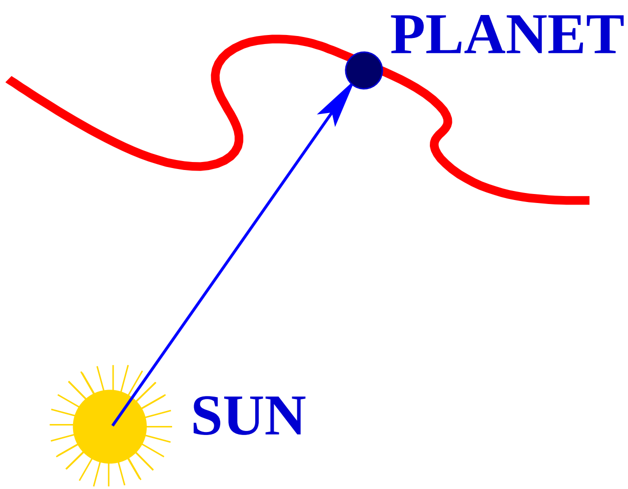


$$\rightsquigarrow \quad \mathscr{U} = (0,\infty) \times (0,\infty) \times (0,\infty) \times (0,\infty)$$

Dynamical phenomena

Examples:

- **Planetary motion**



The events are maps from $\mathbb{R}$ **to** $\mathbb{R}^3$

$$\rightsquigarrow \qquad \mathscr{U} = \{w : \mathbb{R} \to \mathbb{R}^3\}$$

Dynamical phenomena

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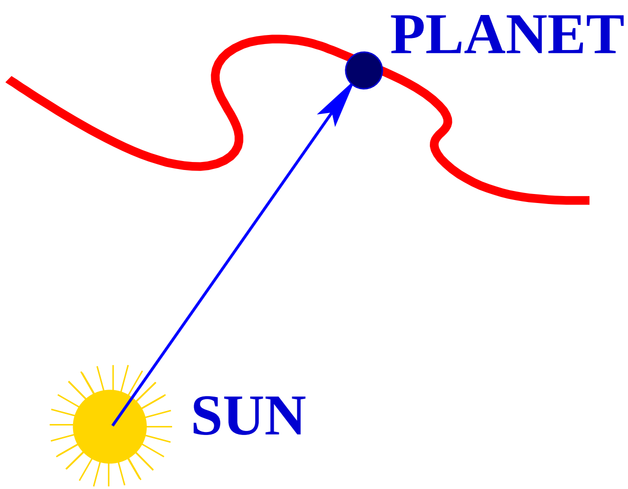

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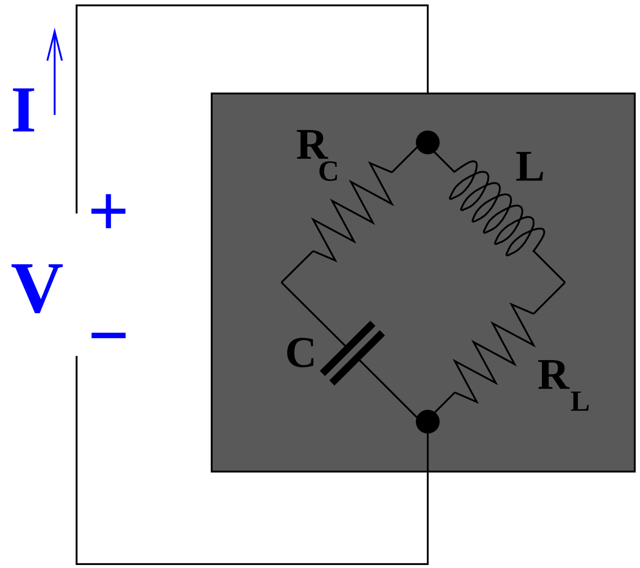
$$\rightsquigarrow \qquad \mathscr{U} = \{w : \mathbb{R} \to \mathbb{R}^3\} = \left(\mathbb{R}^3\right)^{\mathbb{R}}$$

Notation

$A^B :=$ the set of maps from B to A i.e. $A^B := \{f : B \to A\}$

Dynamical phenomena

- **The voltage across and the current into an electrical port with 'dynamics'**



The events are maps from $\mathbb{R}$ to $\mathbb{R}^2$

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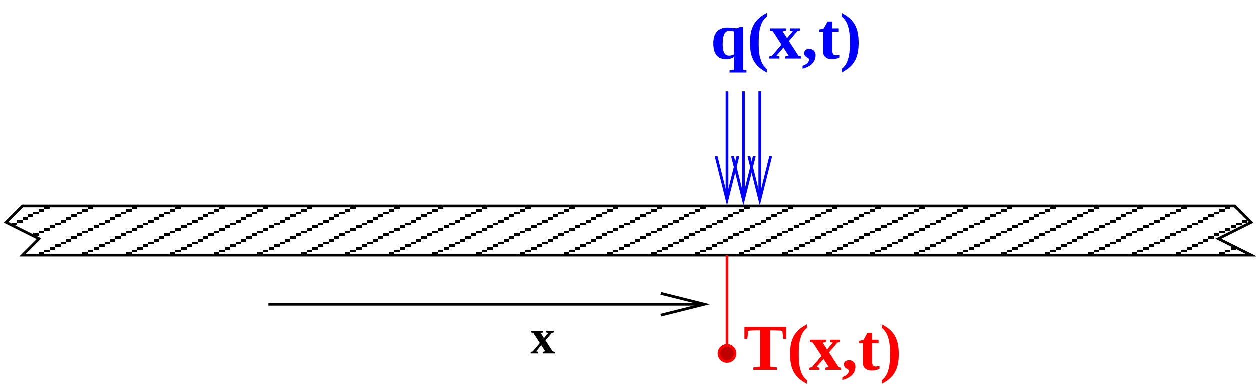
Distributed phenomena

- **Temperature profile of, and heat absorbed by, a rod**



Events: maps from $\mathbb{R} \times \mathbb{R}$ **to** $[0, \infty) \times \mathbb{R}$

$$\rightsquigarrow \qquad \mathscr{U} = \{(T, q) : \mathbb{R}^2 \to [0, \infty) \times \mathbb{R}\} = (\mathbb{R}^2)^{\mathbb{R}^2}$$

A model is a subset: the ‘behavior’

The behavior

Given is a phenomenon with universum $\mathscr{U}$.
Without further scrutiny, every event in $\mathscr{U}$ can occur.

After studying the situation, the conclusion is reached that the events are constrained, that some laws are in force.

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Modeling means that certain events are declared to be impossible, that they cannot occur.

The possibilities that remain constitute what we call the ‘behavior’ of the model.

The behavior

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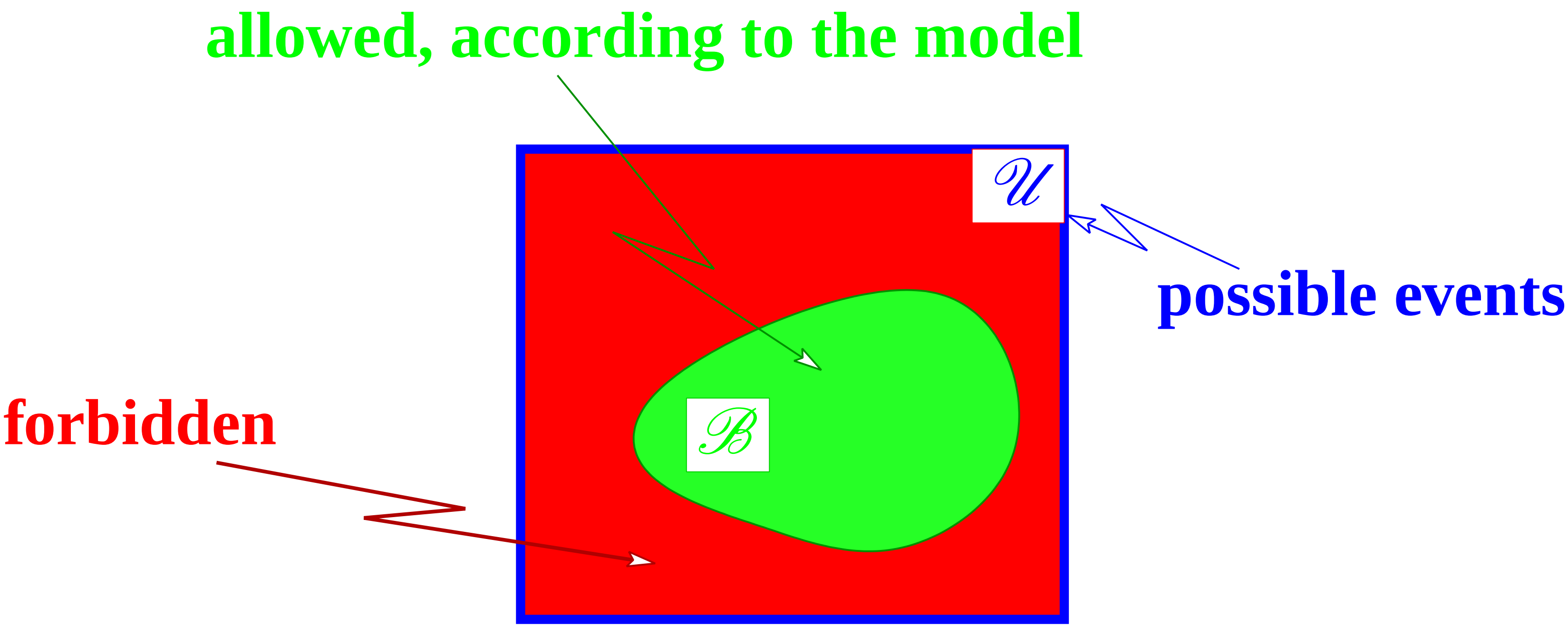
Without further scrutiny, every event in $\mathscr{U}$ can occur.

After studying the situation, the conclusion is reached that the events are constrained, that some laws are in force.

A model is a subset $\mathscr{B}$ of $\mathscr{U}$

$\mathscr{B}$ is called *the behavior* of the model

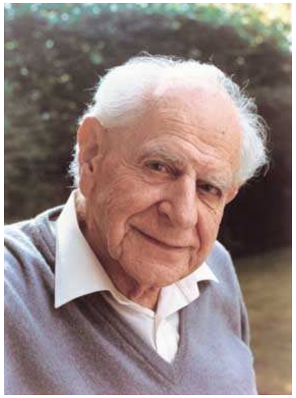
The behavior



The behavior & scientific theory

Every "good" scientific theory is prohibition:
it forbids certain things to happen...
The more a theory forbids, the better it is.

Karl Popper
Conjectures and Refutations:
The Growth of Scientific Knowledge
Routhledge, 1963



Karl Popper
(1902-1994)

Examples

Discrete event phenomena

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  with $\mathtt{n} =$ **the number of letters in the longest word**

  $\mathscr{B}$ **= all words recognized by the spelling checker.**
  For example, SPQR $\notin \mathscr{B}$**.**

  $\mathscr{B}$ **is basically defined by enumeration, by listing its elements.**

Discrete event phenomena

- **32-bit binary strings with a parity check.**

$$\mathscr{U} = \{0,1\}^{32}$$

$$\mathscr{B} = \left\{ a_1 a_2 \cdots a_{31} a_{32} \mid a_k \in \{0,1\} \textbf{ and } a_{32} \overset{(\text{mod } 2)}{=} \sum_{k=1}^{31} a_k \right\}$$

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$\mathscr{B}$ **can be expressed in many other ways. For example,**

$$\mathscr{B} = \{ a_1 a_2 \cdots a_{31} a_{32} \mid a_k \in \{0,1\} \textbf{ and } \sum_{k=1}^{32} a_k \overset{(\mathrm{mod}\ 2)}{=} 0 \}$$

$$\mathscr{B} = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{31} \\ a_{32} \end{bmatrix} \mid \exists \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_{30} \\ b_{31} \end{bmatrix} \textbf{ s.t. } \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{31} \\ a_{32} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ -1 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & -1 & 1 \\ 0 & 0 & \cdots & 0 & -1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_{30} \\ b_{31} \end{bmatrix} \right\}$$

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input/output representation

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kernel representation

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image representation

Continuous phenomena

Examples:

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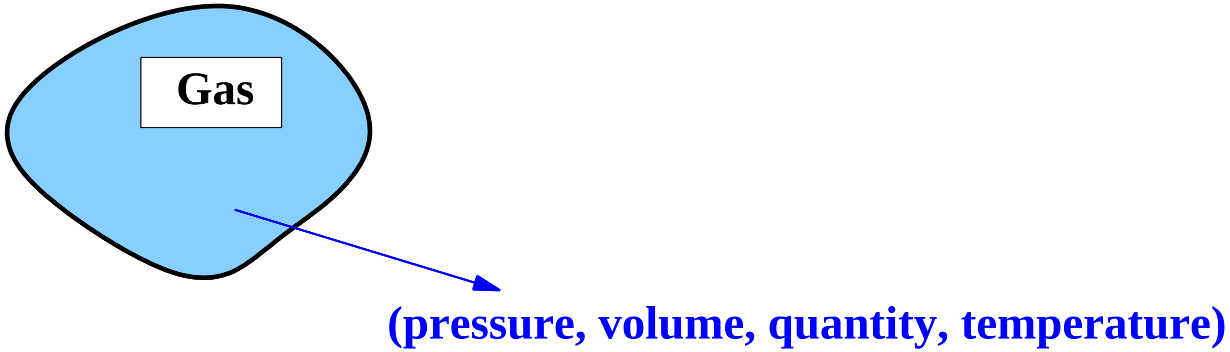
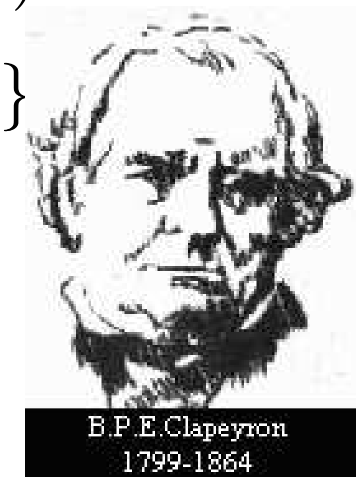


$$\mathscr{U} = (0,\infty) \times (0,\infty) \times (0,\infty) \times (0,\infty)$$

Gas law: $\mathscr{B} = \{(P,V,N,T) \in \mathscr{U} \mid PV = NT\}$



Dynamical phenomena

- **Planetary motion** $\mathscr{U} = \left(\mathbb{R}^3\right)^{\mathbb{R}}$

Kepler's laws $\rightsquigarrow \mathscr{B}$

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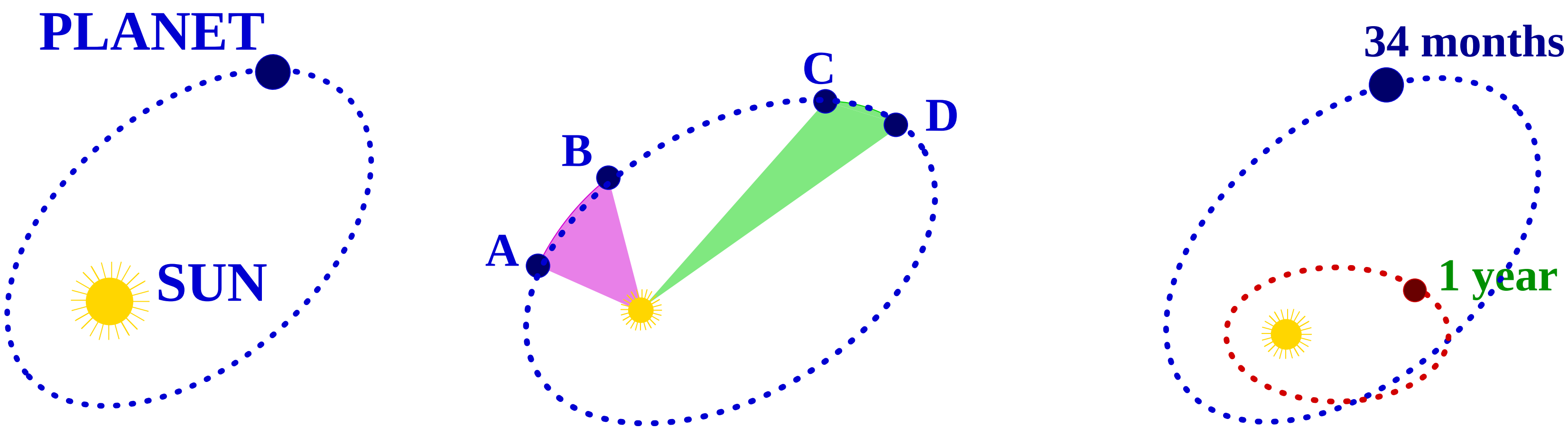


Kepler's laws $\rightsquigarrow \mathscr{B}$ = the orbits $\mathbb{R} \to \mathbb{R}^3$ that satisfy:

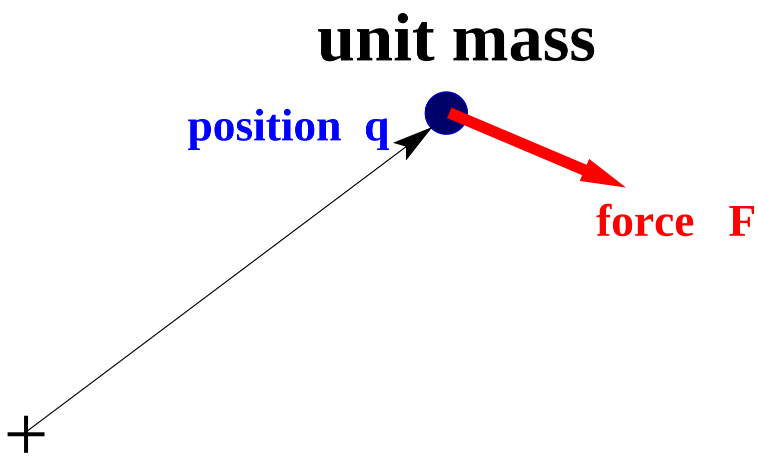
K.1 periodic, ellipses, with the sun in one of the foci;

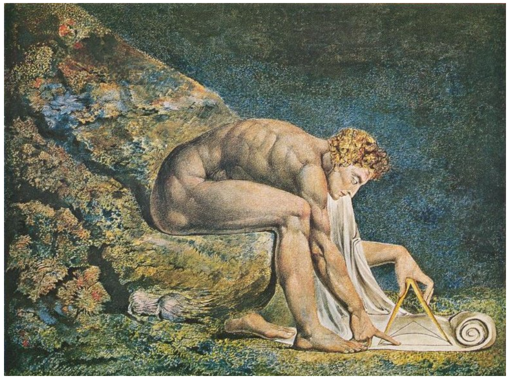
K.2 the vector from sun to planet sweeps out equal areas in equal time;

K.3 the square of the period divided by the third power of the major axis is the same for all the planets

Dynamical phenomena

- **The second law**





Isaac Newton
by William Blake

$$\mathscr{U} = \left(\mathbb{R}^3 \times \mathbb{R}^3\right)^{\mathbb{R}}$$

$$\mathscr{B} = \left\{ (F, q) : \mathbb{R} \to \mathbb{R}^3 \times \mathbb{R}^3 \mid F = \frac{d^2}{dt^2} q \right\}$$

Distributed phenomena

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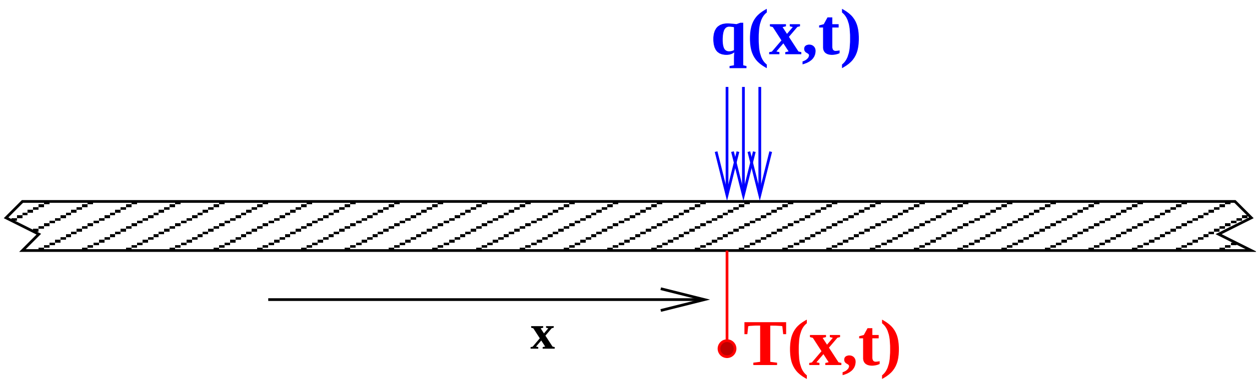

Events: maps from $\mathbb{R} \times \mathbb{R}$ **to** $[0, \infty) \times \mathbb{R}$

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$$\mathscr{B} = \left\{(T, q) : \mathbb{R}^2 \to [0, \infty) \times \mathbb{R} \mid \frac{\partial}{\partial t} T = \frac{\partial^2}{\partial x^2} T + q\right\}$$

Behavioral models

Behavioral models fit the tradition of modeling, but modeling has not been approached in this manner in a deterministic setting.

The behavior captures the essence of what a model articulates.

The behavior is all there is.
Equivalence of models, properties of models,
symmetry, optimality,
system identification (modeling from measured data),
etc., must all refer to the behavior.

Recapitulation

- A model deals with events
- The events belong to a universum, $\mathscr{U}$
- A model is specified by its behavior $\mathscr{B}$, a subset of the event set $\mathscr{U}$
- In dynamical systems, the events are functions of time and the behavior $\mathscr{B}$ is hence a family of time-trajectories.

Dynamical systems

The dynamic behavior

In dynamical systems, ‘events’ are maps, with the time axis as domain, hence functions of time.

It is convenient to distinguish in the notation

the domain of the maps, the **time set**

and their codomain, the **signal space**

the set where the functions take on their values.

The dynamic behavior

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The behavior of a dynamical system is usually described by a system of ordinary differential equations (ODEs) or difference equations.

In contrast to distributed phenomena

$\rightsquigarrow$ partial differential equations (PDEs)

The dynamic behavior

A ***dynamical system*** $:\Leftrightarrow$ $(\mathbb{T}, \mathbb{W}, \mathscr{B})$

$\mathbb{T} \subseteq \mathbb{R}$ **'time set'**

$\mathbb{W}$ **'signal space'**

$\mathscr{B} \subseteq \mathbb{W}^{\mathbb{T}}$ **the 'behavior'**

a family of trajectories $\mathbb{T} \rightarrow \mathbb{W}$

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mostly, $\mathbb{T} = \mathbb{R}, \mathbb{R}_+, \mathbb{Z},$ **or** $\mathbb{N}$ $(\cong \mathbb{Z}_+)$**,**

and, in this lecture, $\mathbb{W} = \mathbb{R}^{\mathtt{w}}$**,**

$\mathscr{B}$ **is a family of**

(finite dimensional) vector-valued time trajectories

The dynamic behavior

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$w : \mathbb{T} \to \mathbb{R}^{\mathtt{w}} \in \mathscr{B} \Leftrightarrow$ **'*w* is compatible with the model'**

$w : \mathbb{T} \to \mathbb{R}^{\mathtt{w}} \notin \mathscr{B} \Leftrightarrow$ **'the model forbids *w*'**

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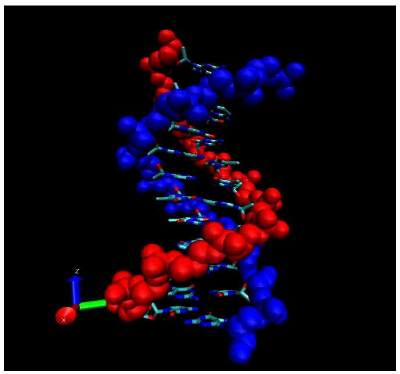
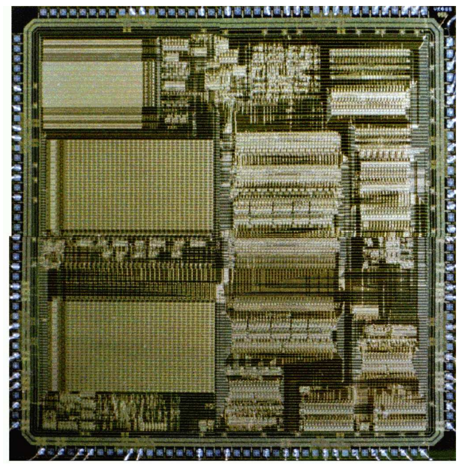
$\mathbb{T} = \mathbb{R}$ **or** $\mathbb{R}_+$ $\rightsquigarrow$ **'continuous-time' systems and ODEs**
$\mathbb{T} = \mathbb{Z}$ **or** $\mathbb{N}$ $\rightsquigarrow$ **'discrete-time' systems and difference eqn's**

We deal with the case $\mathbb{T} = \mathbb{R}$ **only.**

Systems




ZYGOTE



Features

- **open**
- **interconnected**
- **modular**
- **dynamic**

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- **open**
- **interconnected**
- **modular**
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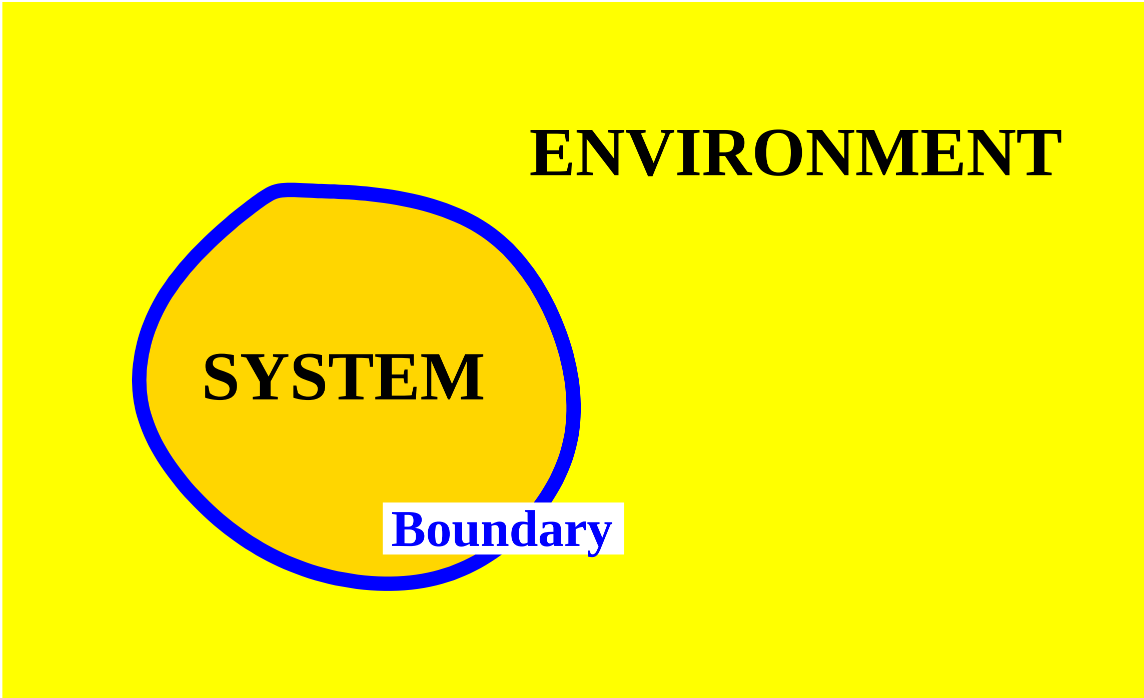
Theme:

develop a suitable mathematical language

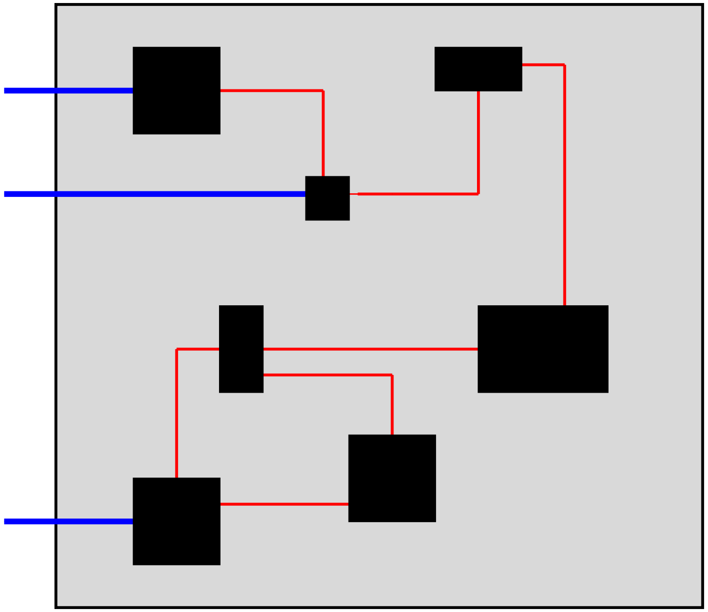
Open, connected, modular, dynamic

Open



Systems interact with their environment

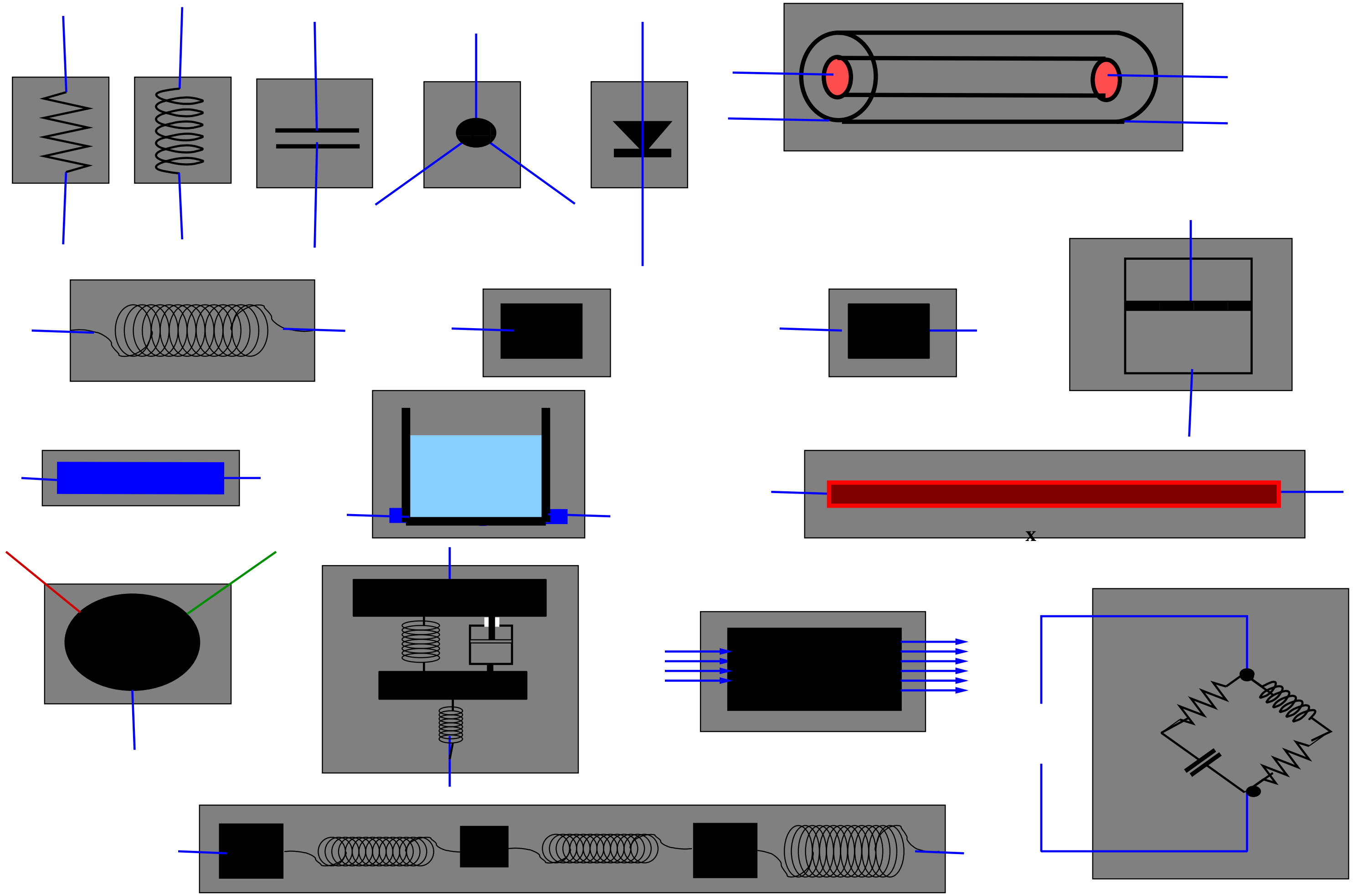
Architecture



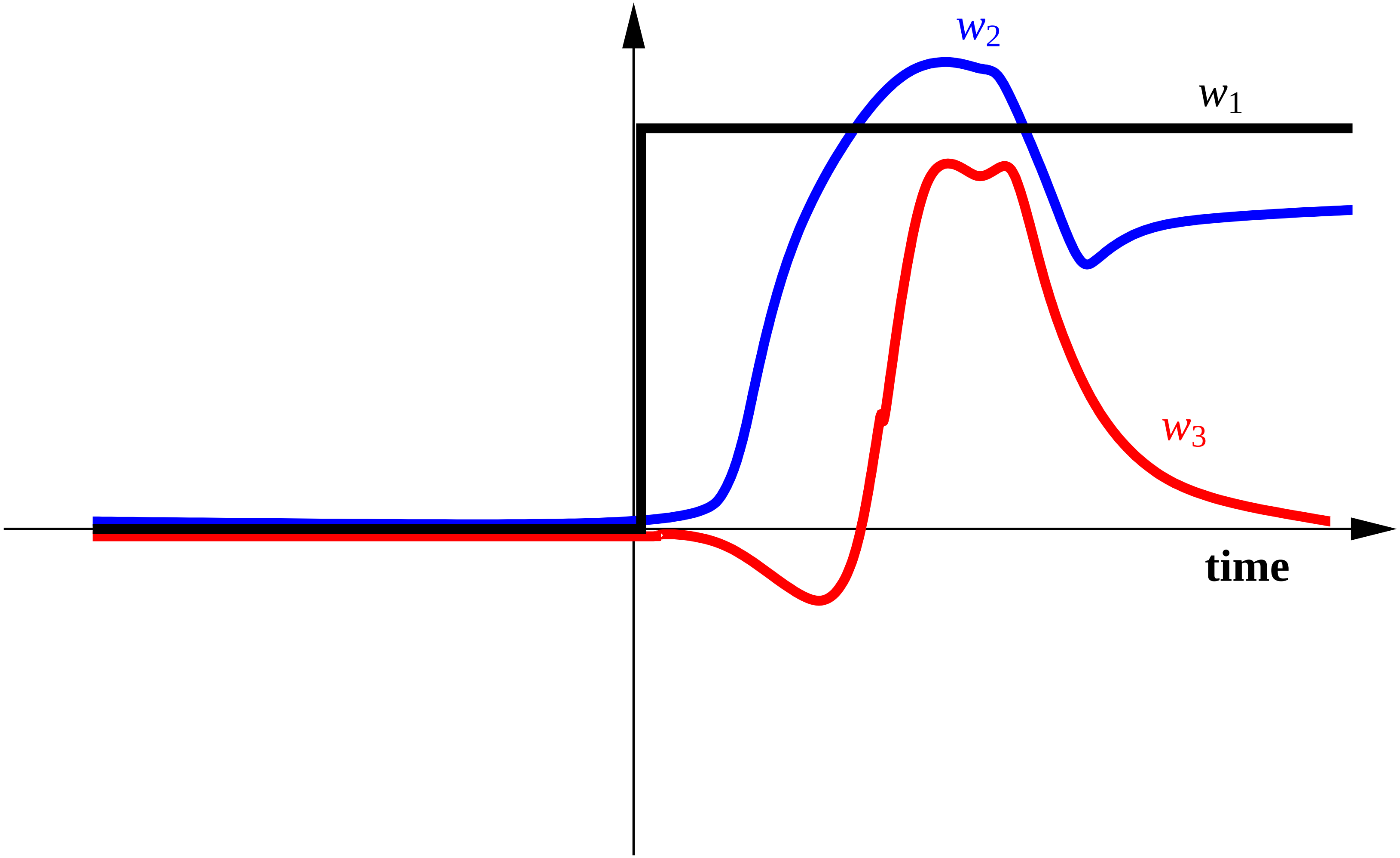
Systems consist of subsystems, interconnected

Modular

Systems consist of an interconnection of ‘building blocks’



Dynamic



There is a delay, an after-effect, memory

The development of the notion

of a dynamical system

a brief causerie

Mathematization

1. Get the physics right
2. The rest is mathematics

R.E. Kalman
Opening lecture
IFAC World Congress
Prague, July 4, 2005

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Prima la fisica, poi la matematica

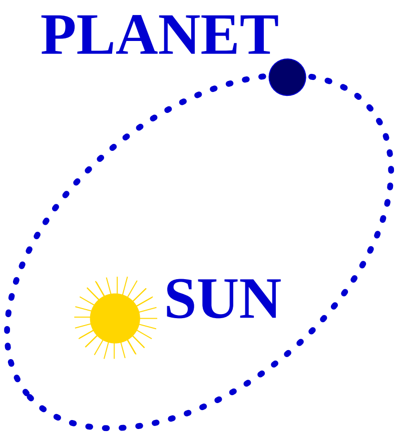
How it all began ...

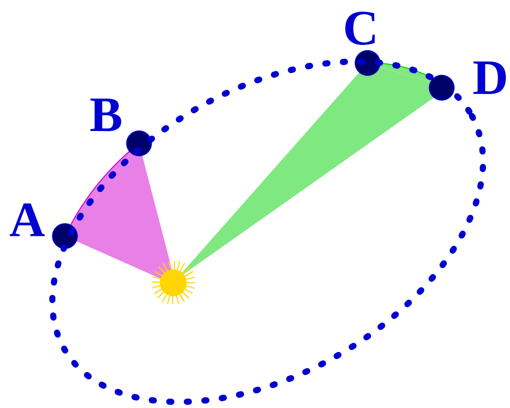
The celestial question

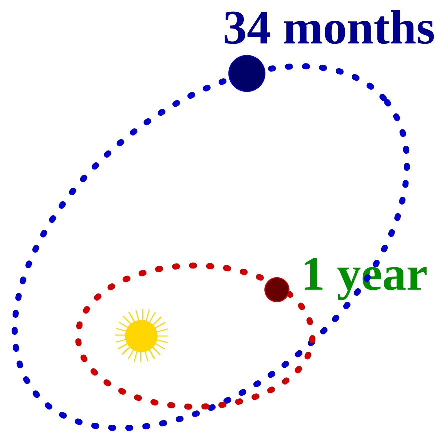
How, for heaven's sake, does it move?

Kepler’s laws

Johannes Kepler
1571-1630







Kepler’s laws:

Ellipse, sun in focus;
= areas in = times;
$(\text{period})^2 \cong (\text{diameter})^3$

The equation of the planet

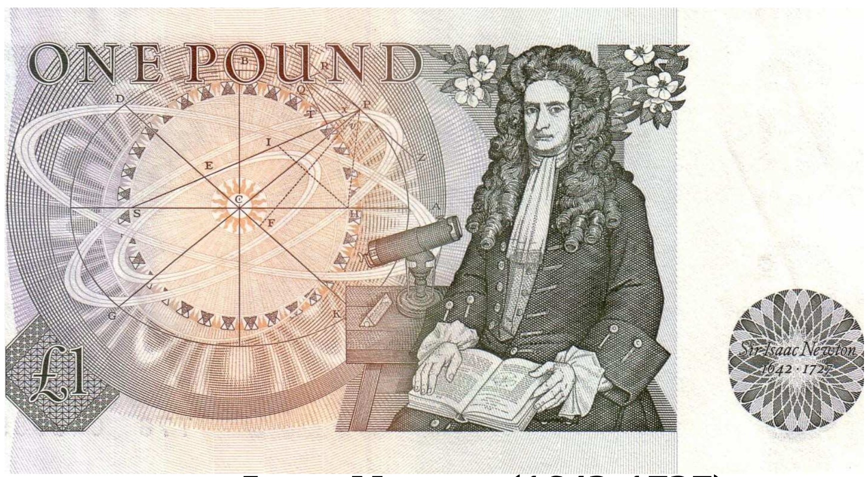
Consequence:

acceleration = function of position and velocity

$$\frac{d^2}{dt^2}w(t) = A(w(t), \frac{d}{dt}w(t))$$

$\rightsquigarrow$ **via calculus and calculation**

$$\frac{d^2}{dt^2}w(t) + \frac{1_{w(t)}}{|w(t)|^2} = 0$$



Isaac Newton (1643-1727)

The equation of the planet

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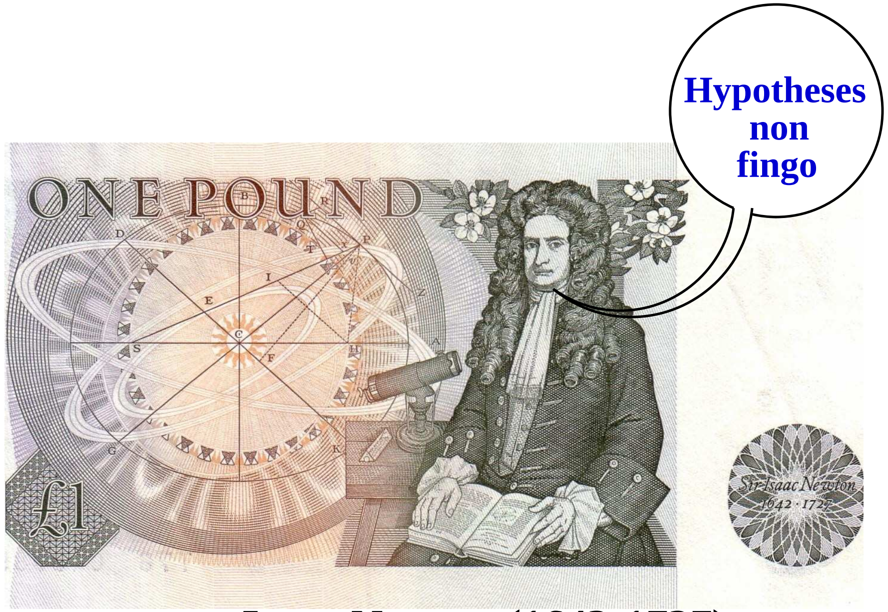
$\leadsto$ **via calculus and calculation**

$$\frac{d^2}{dt^2}w(t) + \frac{1_{w(t)}}{|w(t)|^2} = 0$$

$\cong$ **another representation of K.1, K.2, K.3**



Isaac Newton (1643-1727)

Newton's laws

2-nd law $F'(t) = m\frac{d^2}{dt^2}w(t)$

gravity $F''(t) = m\frac{1_{w(t)}}{|w(t)|^2}$

3-rd law $F'(t) + F''(t) = 0$

$$\Downarrow$$

$$\boxed{\frac{d^2}{dt^2}w(t) + \frac{1_{w(t)}}{|w(t)|^2} = 0}$$

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Isaac Newton by William Blake

$$\Downarrow$$

$$\boxed{\frac{d^2}{dt^2}w(t) + \frac{1_{w(t)}}{|w(t)|^2} = 0}$$

Viewing as interconnection is the key to generalization

The paradigm of *closed* systems

‘Axiomatization’

K.1, K.2, & K.3

$$\rightsquigarrow \quad \frac{d^2}{dt^2}w(t) + \frac{1_{w(t)}}{|\frac{d}{dt}w(t)|^2} = 0$$

$$\rightsquigarrow \ \textbf{with } x = \left(w, \frac{d}{dt}w\right) \quad \frac{d}{dt}x = f(x)$$

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$\rightsquigarrow$ **with** $x = (w, \frac{d}{dt}w)$ $\frac{d}{dt}x = f(x)$

$\rightsquigarrow$ **generalization** $\frac{d}{dt}x = f(x)$

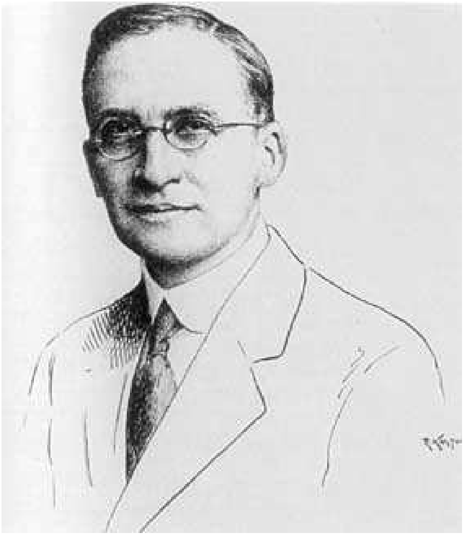
$\rightsquigarrow$ **‘dynamical systems’, flows**

$\rightsquigarrow$ **flows as paradigm of dynamics** $\rightsquigarrow$ **closed systems**

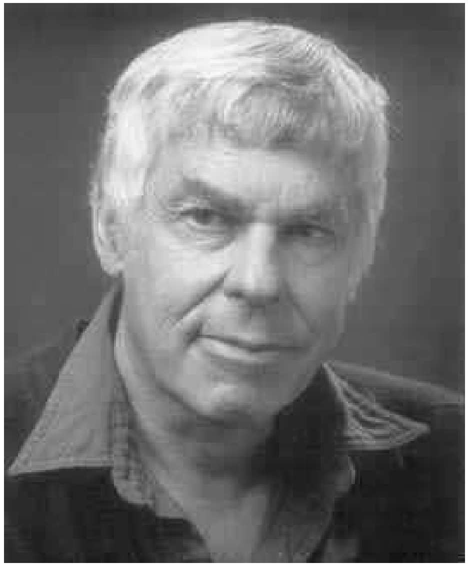
Motion determined by internal initial conditions.

‘Axiomatization’

Henri Poincaré (1854-1912)



George Birkhoff (1884-1944)



Stephen Smale (1930-   )

‘Axiomatization’

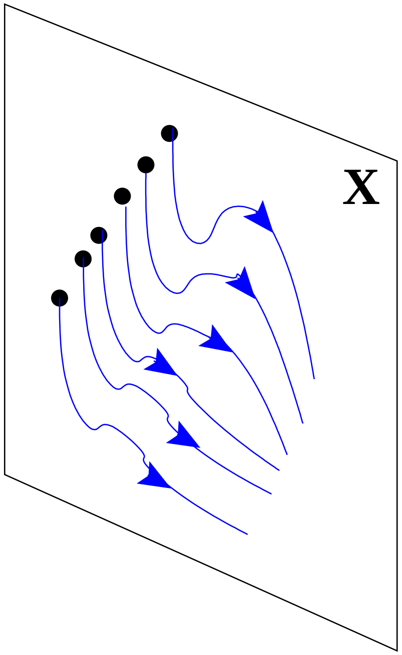
A *dynamical system* is defined by

a state space X and

a state transition function

$\phi :$ $\cdots$ **such that** $\cdots$

$\phi(t, \mathrm{x})$ **= state at time t starting from state** x



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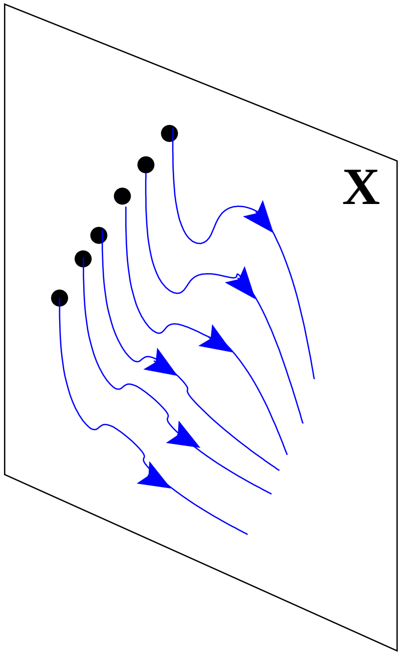

$\phi(t, x)$ **= state at time t starting from state x**

This framework of closed systems is universally used for dynamics in mathematics and physics

‘Axiomatization’

How could they forget Newton’s 2^{nd} law,
about Maxwell’s eq’ns,
about thermodynamics,
about tearing & zooming & linking, ...?

‘Axiomatization’

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Reply: assume ‘fixed boundary conditions’

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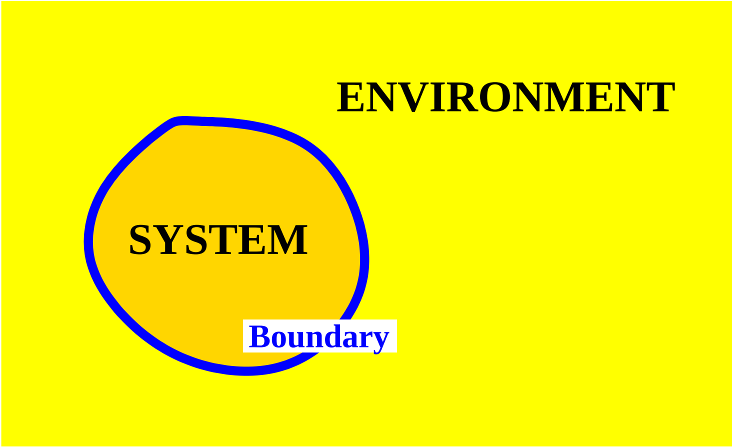

$\leadsto$ to model a system, we have to model also the environment!

‘Axiomatization’

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about tearing & zooming & linking, ...?

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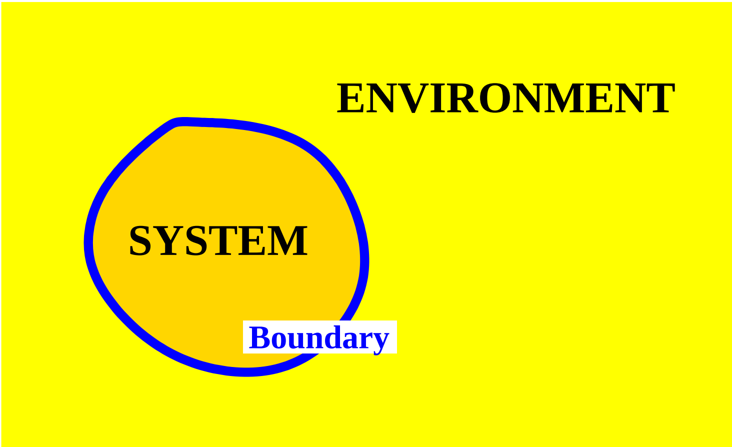


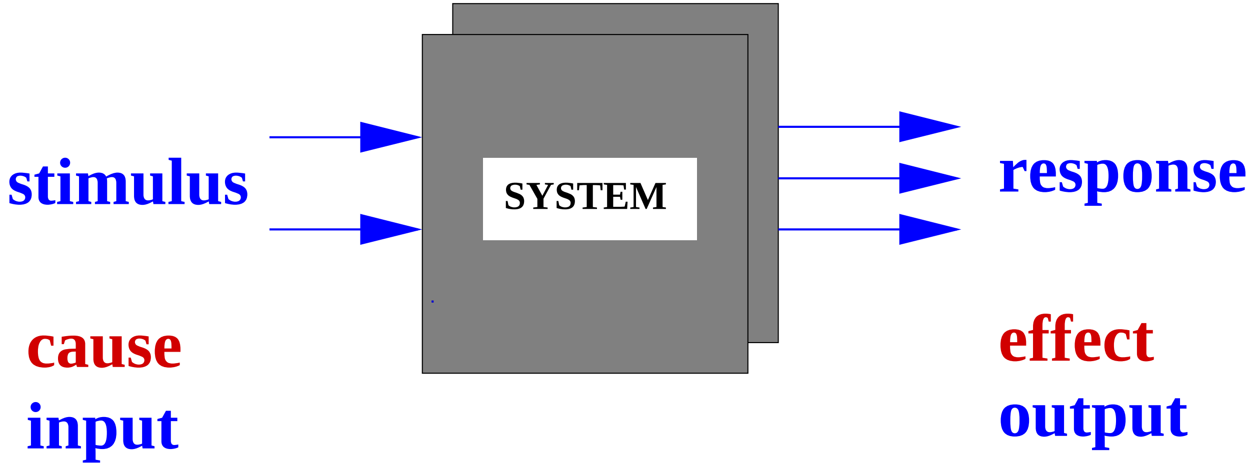
$\rightsquigarrow$ **to model a system, we have to model also the environment!**

Chaos theory, cellular automata, sync, etc., function in this framework ...

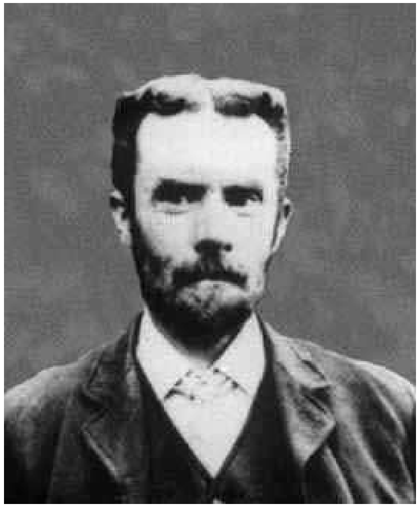
Inputs and outputs

meanwhile, in engineering...

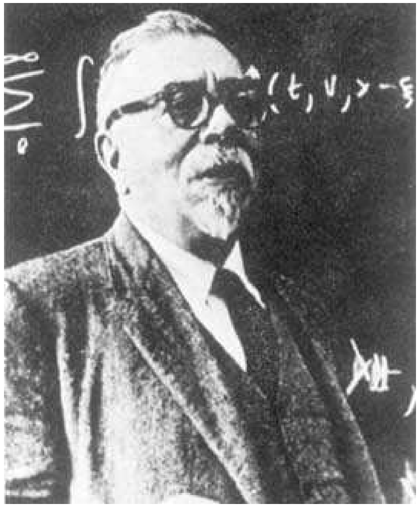
Input/output systems



The originators



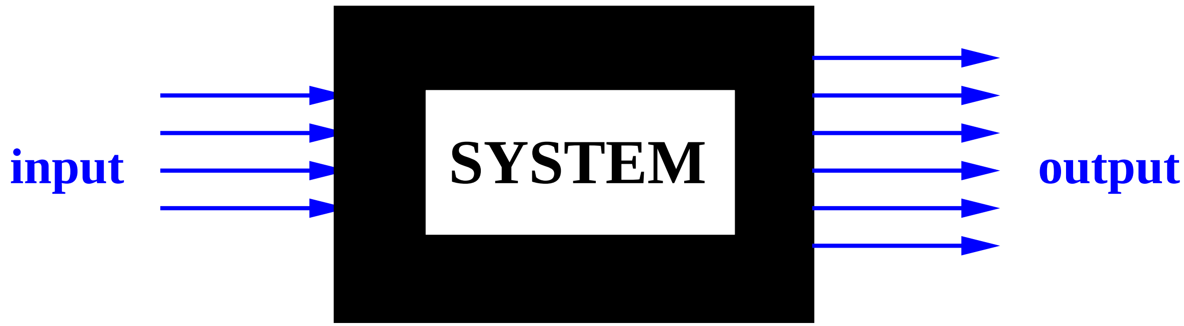
Oliver Heaviside (1850-1925)



Norbert Wiener (1894-1964)

and the many electrical circuit theorists ...

Mathematical description

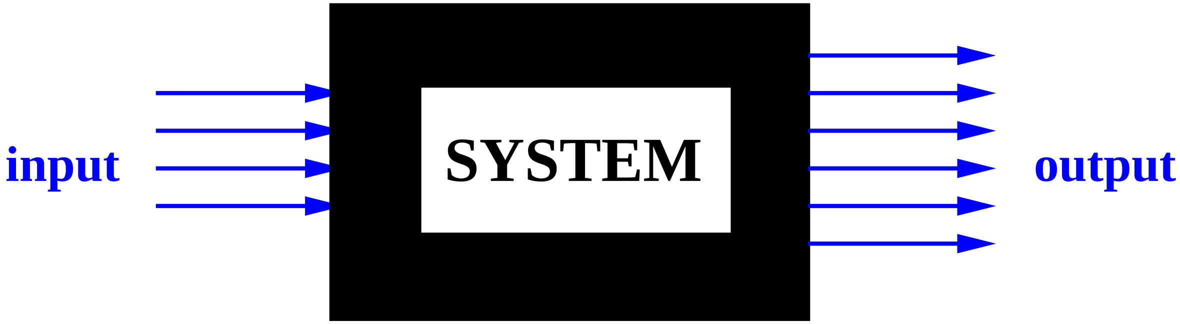


u: **input,** y: **output,**

SISO, LTI case $\rightsquigarrow$ $G(s) = \frac{q(s)}{p(s)}$ **transfer functions, impedances, admittances.**

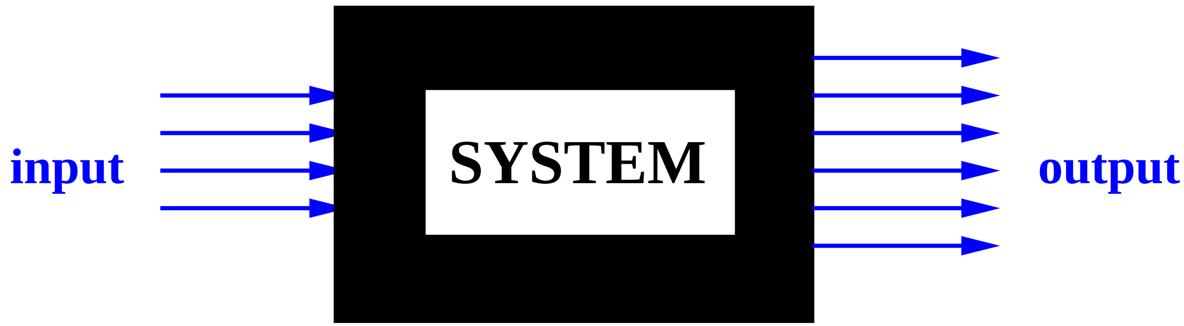
Circuit analysis and synthesis
Classical control
Bode, Nyquist, root-locus.

Mathematical description



$$y(t) = \int_{0 \textbf{ or } -\infty}^{t} H(t-t')u(t')\, dt'$$

Mathematical description



$$y(t) = \int_{0 \textbf{ or } -\infty}^{t} H(t-t')u(t')\,dt'$$

$$y(t) = H_0(t) + \int_{-\infty}^{t} H_1(t-t')u(t')\,dt' + \int_{-\infty}^{t}\int_{-\infty}^{t'} H_2(t-t', t'-t'')u(t')u(t'')\,dt'dt'' + \cdots$$

Awkward nonlinear — far from the physics

Fail to deal with ‘initial conditions’.

Input/state/output systems

Around 1960: a paradigm shift to

$$\frac{d}{dt}x = f(x,u),\ y = g(x,u)$$

Rudolf Kalman (1930-)

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$$\frac{d}{dt}x = f(x,u),\ y = g(x,u)$$

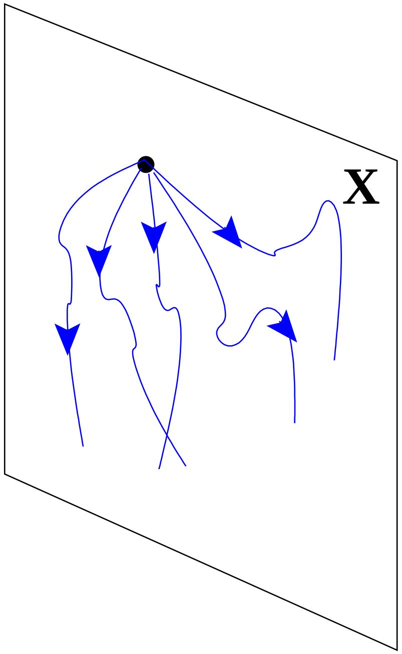
Rudolf Kalman (1930-)

- **open**
- **deals with initial conditions**
- **incorporates nonlinearities, time-variation**
- **models many physical phenomena**
- $\cdots$

'Axiomatization'

State transition function:

$\phi(t, \mathrm{x}, u)$: **state reached at time t from x using input u.**



$$\frac{d}{dt}x = f(x, u), \; y = g(x, u)$$

Read-out function:

$g(\mathrm{x}, \mathrm{u})$: **output value with state x and input value u.**

The input/state/output paradigm

The input/state/output view turned out to be very effective and fruitful

The input/state/output paradigm

The input/state/output view turned out to be very effective and fruitful

- **for modeling**
- **for control (stabilization, robustness, ...)**
- **prediction of one signal from another, filtering**
- **understanding system representations (transfer f'n, input/state/output repr., etc.)**
- **model simplification, reduction**
- **system ID: models from data**
- **etc., etc., etc.**

Linear time-invariant differential systems

LTIDSs

LTIDSs

The dynamical system $(\mathbb{R}, \mathbb{R}^{\mathtt{w}}, \mathscr{B}) \quad \rightsquigarrow \quad \mathscr{B}$ **is said to be**

$$[\![\, \textbf{linear} \,]\!] :\Leftrightarrow [\![\, [\![w_1, w_2 \in \mathscr{B}, \alpha \in \mathbb{R}]\!] \Rightarrow [\![\alpha w_1 + w_2 \in \mathscr{B}]\!] \,]\!]$$

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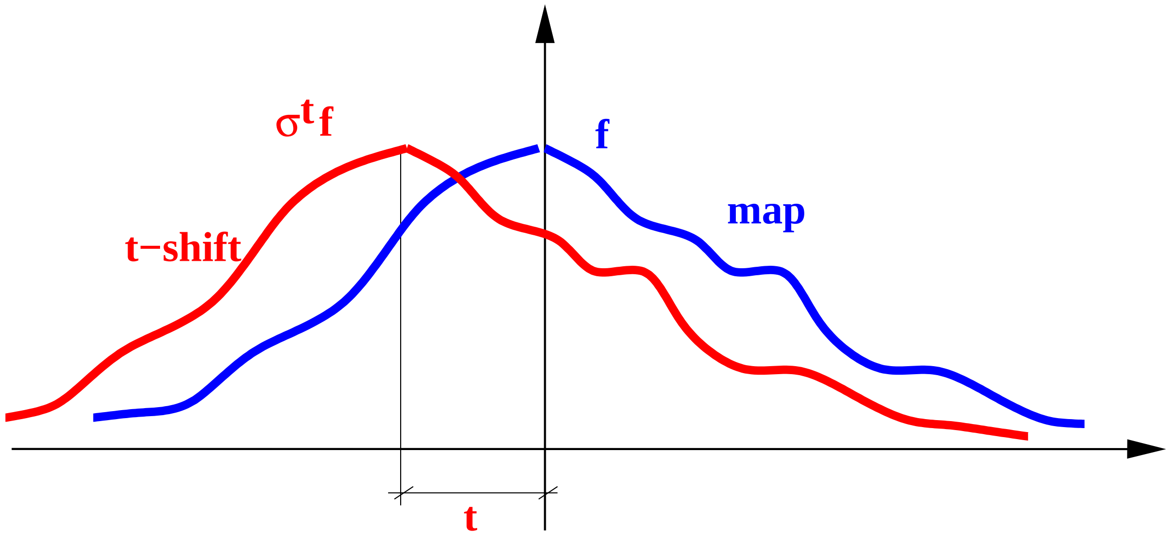
$[\![$ **time-invariant** $]\!] :\Leftrightarrow [\![[\![w \in \mathscr{B}, \sigma^t$ **the** t**-shift**$]\!] \Rightarrow [\![\sigma^t w \in \mathscr{B}\ \forall t \in \mathbb{R}]\!]]\!]$

$$(\sigma^t f)(t') := f(t' + t)$$



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$[\![$ **differential** $]\!] :\Leftrightarrow [\![\mathscr{B}$ **is 'described' by an ODE**$]\!]$**.**

Linearity

This definition of linearity has as a special case

$$u \mapsto y = L(u) \quad L \text{ a linear map}$$

$u \in$ **a space of inputs,** $y \in$ **a space of outputs,** $\qquad w = \begin{bmatrix} u \\ y \end{bmatrix}$**.**

$$\mathscr{B} = \{w = \begin{bmatrix} u \\ y \end{bmatrix} \mid y = L(u)\} = \text{ the ‘graph’ of } L$$

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But, a dynamical system, even an input/output system,

is seldom an input/output map !

Response depends on initial condition, as well as on driving input.

LTIDSs

The dynamical system $(\mathbb{R}, \mathbb{R}^{\mathtt{w}}, \mathscr{B})$ **is**

a linear time-invariant differential system (LTIDS) $:\Leftrightarrow$ **the behavior consists of the set of solutions of a system of linear, constant coefficient, ODEs**

$$R_0 w + R_1 \frac{d}{dt} w + \cdots + R_{\mathtt{n}} \frac{d^{\mathtt{n}}}{dt^{\mathtt{n}}} w = 0.$$

$R_0, R_1, \cdots, R_{\mathtt{n}} \in \mathbb{R}^{\bullet \times \mathtt{w}}$ **real matrices that parametrize the system, and** $w : \mathbb{R} \to \mathbb{R}^{\mathtt{w}}$**.**

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$$\rightsquigarrow \qquad R\left(\tfrac{d}{dt}\right) w = 0$$

with $R(\xi) = R_0 + R_1 \xi + \cdots + R_{\mathtt{n}} \xi^{\mathtt{n}} \in \mathbb{R}[\xi]^{\bullet \times \mathtt{w}}$

a polynomial matrix, usually 'wide' $\boxed{}$ **or square.**

LTIDS

We should define what we mean by a solution of

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For ease of exposition, we take $\mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$ solutions. Hence the behavior defined is

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$\mathscr{B} = \mathtt{kernel}\left(R\left(\frac{d}{dt}\right)\right)$ **‘*kernel representation*’ of this $\mathscr{B}$.**

Notation:

$\mathscr{B} \in \mathscr{L}^{\mathtt{w}}$ $\mathscr{L}^{\mathtt{w}}$ **= the LTIDSs with $\mathtt{w}$ variables**

$\mathscr{B} \in \mathscr{L}^\bullet$, $\mathscr{L}^\bullet$ **= the LTIDSs.**

Representations of LTIDSs

There are numerous representations of LTIDSs

- **As the solutions of** $R\left(\frac{d}{dt}\right)w = 0$ $R \in \mathbb{R}[\xi]^{\bullet \times \mathtt{w}}$ **(our def.)**

  $R\left(\frac{d}{dt}\right) : \mathscr{C}^\infty\left(\mathbb{R}, \mathbb{R}^{\mathtt{coldim}(R)}\right) \to \mathscr{C}^\infty\left(\mathbb{R}, \mathbb{R}^{\mathtt{rowdim}(R)}\right)$ **'kernel repr'n'**

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- **With input/output partition**

  $P\left(\frac{d}{dt}\right)y = Q\left(\frac{d}{dt}\right)u$ $w \cong \begin{bmatrix} u \\ y \end{bmatrix}$ $\det(P) \neq 0, P^{-1}Q$ **proper**

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- **Input/state/output representation**
  in terms of matrices A, B, C, D **such that**
  $\mathscr{B}$ **consists of all** $w's$ **generated by**

  $$\frac{d}{dt}x = Ax + Bu, \; y = Cx + Du \quad w \cong \begin{bmatrix} u \\ y \end{bmatrix}$$

Rudolf E. Kalman
born 1930

Representations of LTIDSs

- $w = M\left(\frac{d}{dt}\right)\ell$ **with** $M \in \mathbb{R}[\xi]^{\mathtt{w}\times\bullet}$
  $M\left(\frac{d}{dt}\right) : \mathscr{C}^\infty\left(\mathbb{R}, \mathbb{R}^{\mathtt{coldim}(M)}\right) \to \mathscr{C}^\infty\left(\mathbb{R}, \mathbb{R}^{\mathtt{rowdim}(M)}\right)$ **'image repr'n'** $\mathscr{B} = \mathtt{image}\left(M\left(\frac{d}{dt}\right)\right)$

Representations of LTIDSs

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- **First principles models often contain 'latent variables' (see later)**
  $\rightsquigarrow \quad R\left(\frac{d}{dt}\right)w = M\left(\frac{d}{dt}\right)\ell$ **'latent variable repr'n'**

$$\mathscr{B} = \{w \mid \exists\, \ell \text{ such that } \ldots\}$$

Representations of LTIDSs

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$$\mathscr{B} = \{w \mid \exists\, \ell \text{ such that } \ldots\}$$

- **Special case:** $\frac{d}{dt}Fx = Ax + Bw$ **DAEs**

$$\mathscr{B} = \{w \mid \exists\, x \text{ such that } \ldots\}$$

Representations of LTIDSs

- **representations with rational symbols**
  $R\left(\frac{d}{dt}\right) w = 0$, $w = M\left(\frac{d}{dt}\right) \ell$, **etc.**
  with $R, M \in \mathbb{R}(\xi)^{\bullet \times \bullet}$**, or proper stable rational, etc.**

Representations of LTIDSs

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  with $R, M \in \mathbb{R}(\xi)^{\bullet\times\bullet}$, **or proper stable rational, etc.**
- **and then, there are the convolution representations**

$$\int_{-\infty}^{+\infty} H(t') w(t-t')\, dt' = 0$$

with the kernel, input/output, image versions

$$y(t) = \int_{-\infty}^{+\infty} H(t') u(t-t')\, dt', \quad w = \begin{bmatrix} u \\ y \end{bmatrix}$$

Representations of LTIDSs

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- **Rich ... but confusing!**

Recapitulation

- **Dynamical systems** $\rightsquigarrow \Sigma = (\mathbb{T}, \mathbb{W}, \mathscr{B})$ **with behavior** $\mathscr{B} \subseteq (\mathbb{W})^{\mathbb{T}}$ **a family of time trajectories**
- **Closed systems: awkward special case**
- **Input/output systems: successful special case**
- **LTIDSs:** $\mathscr{B}$ **is the sol'n set of a system of linear constant coefficient ODEs**

Latent variables

Kernels, images, and projections

A model $\mathscr{B}$ is a subset of $\mathscr{U}$. There are many ways to specify a subset. For example,

- **as the solution set of equations**
- **as an image of a map**
- **as a projection**

Kernels, images, and projections

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$$f : \mathscr{U} \to \bullet; \qquad \mathscr{B} = \{w \mid f(w) = 0\}$$

- **as an image of a map**

$$f : \bullet \to \mathscr{U}; \qquad \mathscr{B} = \{w \mid \exists\, \ell \text{ such that } w = f(\ell)\}$$

- **as a projection**

$$\mathscr{B}_{\text{extended}} \subseteq \mathscr{U} \times \mathscr{L}; \qquad \mathscr{B} = \{w \mid \exists\, \ell \text{ such that } (w, \ell) \in \mathscr{B}_{\text{extended}}\}$$

Kernels, images, and projections

A model $\mathscr{B}$ is a subset of $\mathscr{U}$. There are many ways to specify a subset. For example,

- **as the solution set of equations** **'kernel representation'**

$$f : \mathscr{U} \to \bullet; \qquad \mathscr{B} = \{w \mid f(w) = 0\}$$

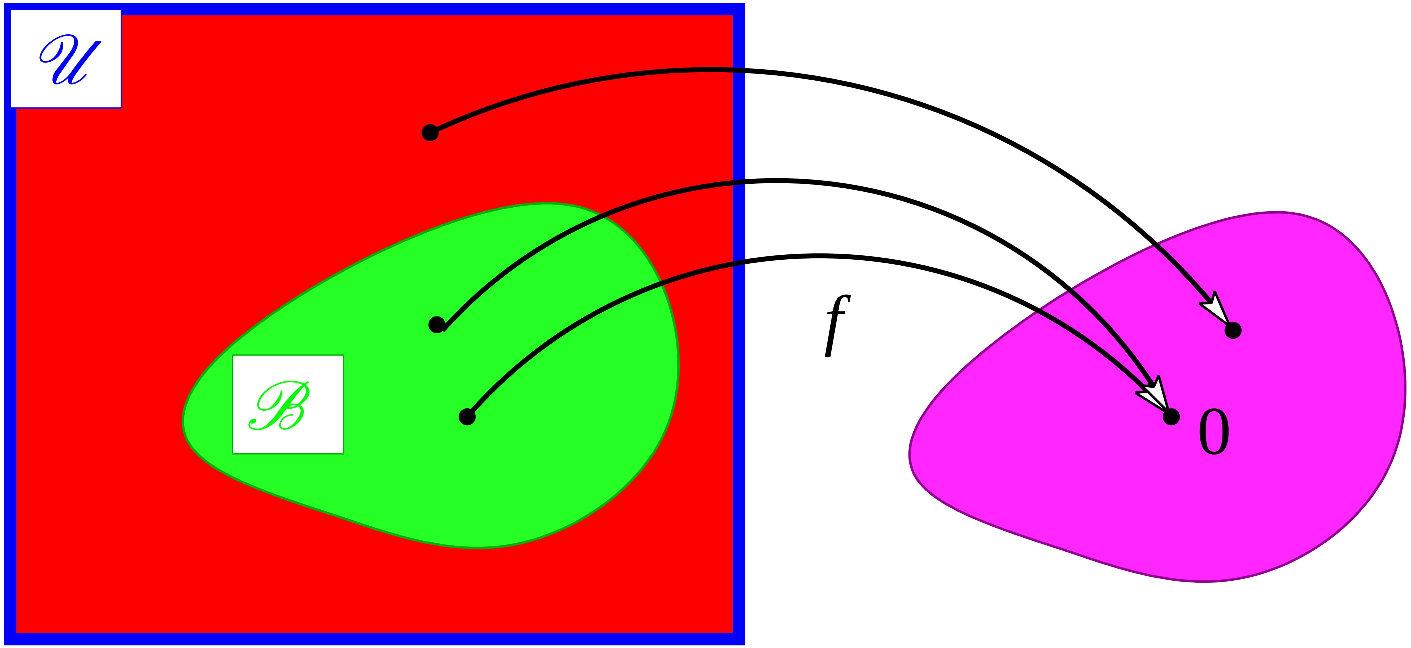
- **as an image of a map** **'image representation'**

$$f : \bullet \to \mathscr{U}; \qquad \mathscr{B} = \{w \mid \exists\, \ell \text{ such that } w = f(\ell)\}$$

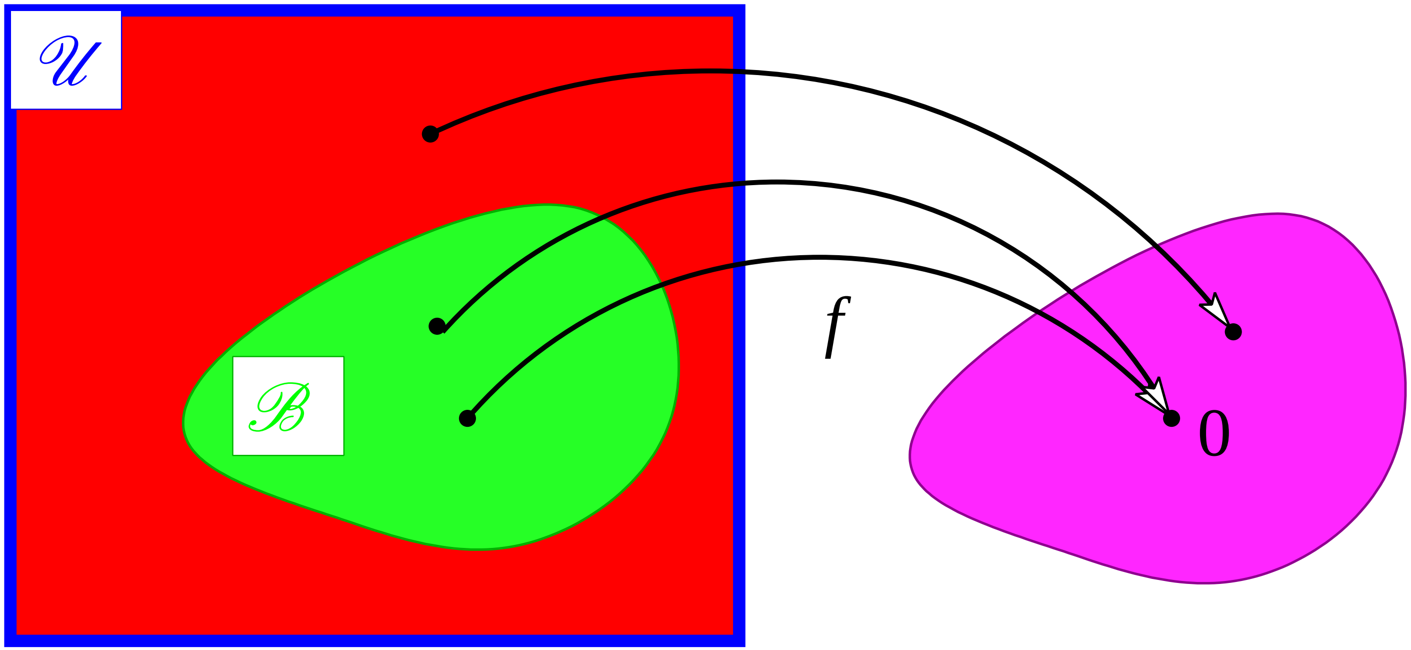
- **as a projection** **'latent variable representation'**

$$\mathscr{B}_{\text{extended}} \subseteq \mathscr{U} \times \mathscr{L}; \qquad \mathscr{B} = \{w \mid \exists\, \ell \text{ such that } (w, \ell) \in \mathscr{B}_{\text{extended}}\}$$

Kernel



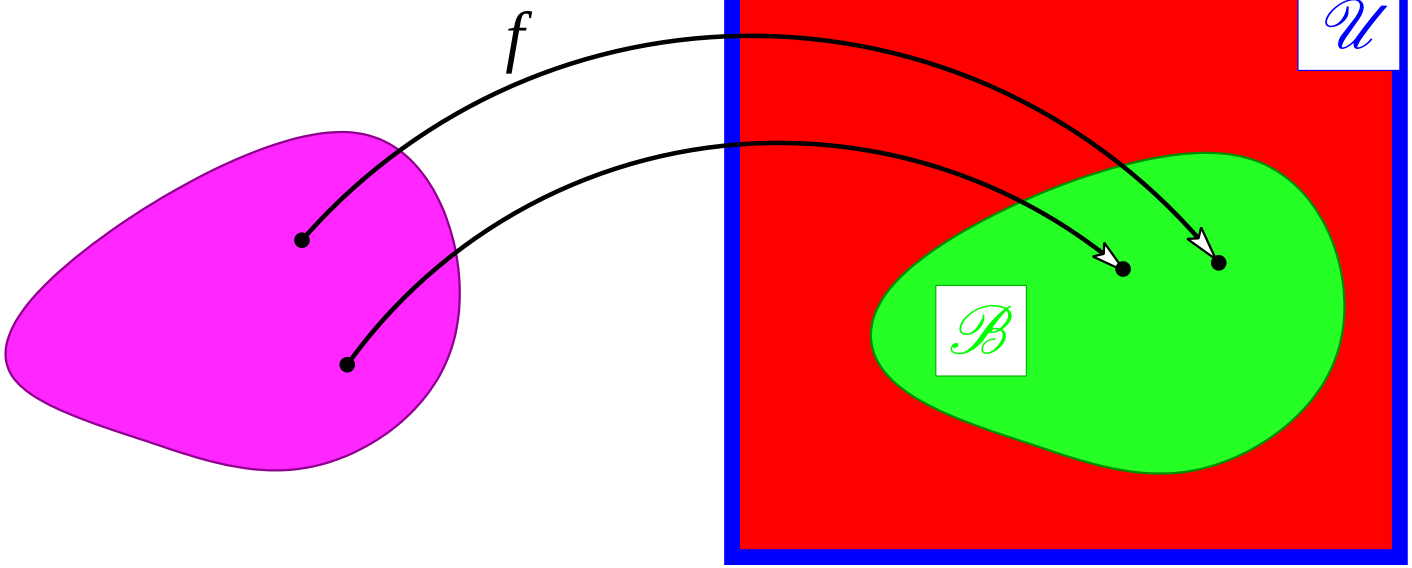
Kernel



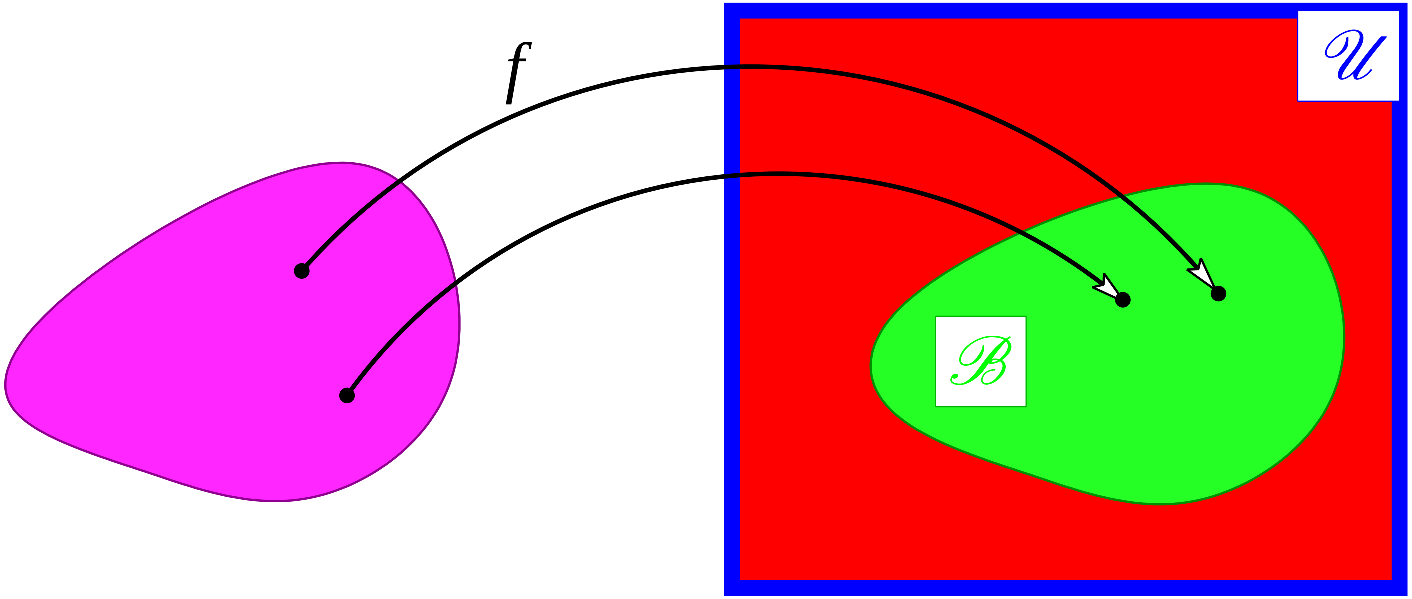
For example, $$p_0 y + p_1 \frac{d}{dt} y + \cdots + p_{\mathtt{n}} \frac{d^{\mathtt{n}}}{dt^{\mathtt{n}}} y$$

$$= q_0 u + q_1 \frac{d}{dt} u + \cdots + q_{\mathtt{n}} \frac{d^{\mathtt{n}}}{dt^{\mathtt{n}}} u, \quad w = \begin{bmatrix} u \\ y \end{bmatrix}$$

Image



Image

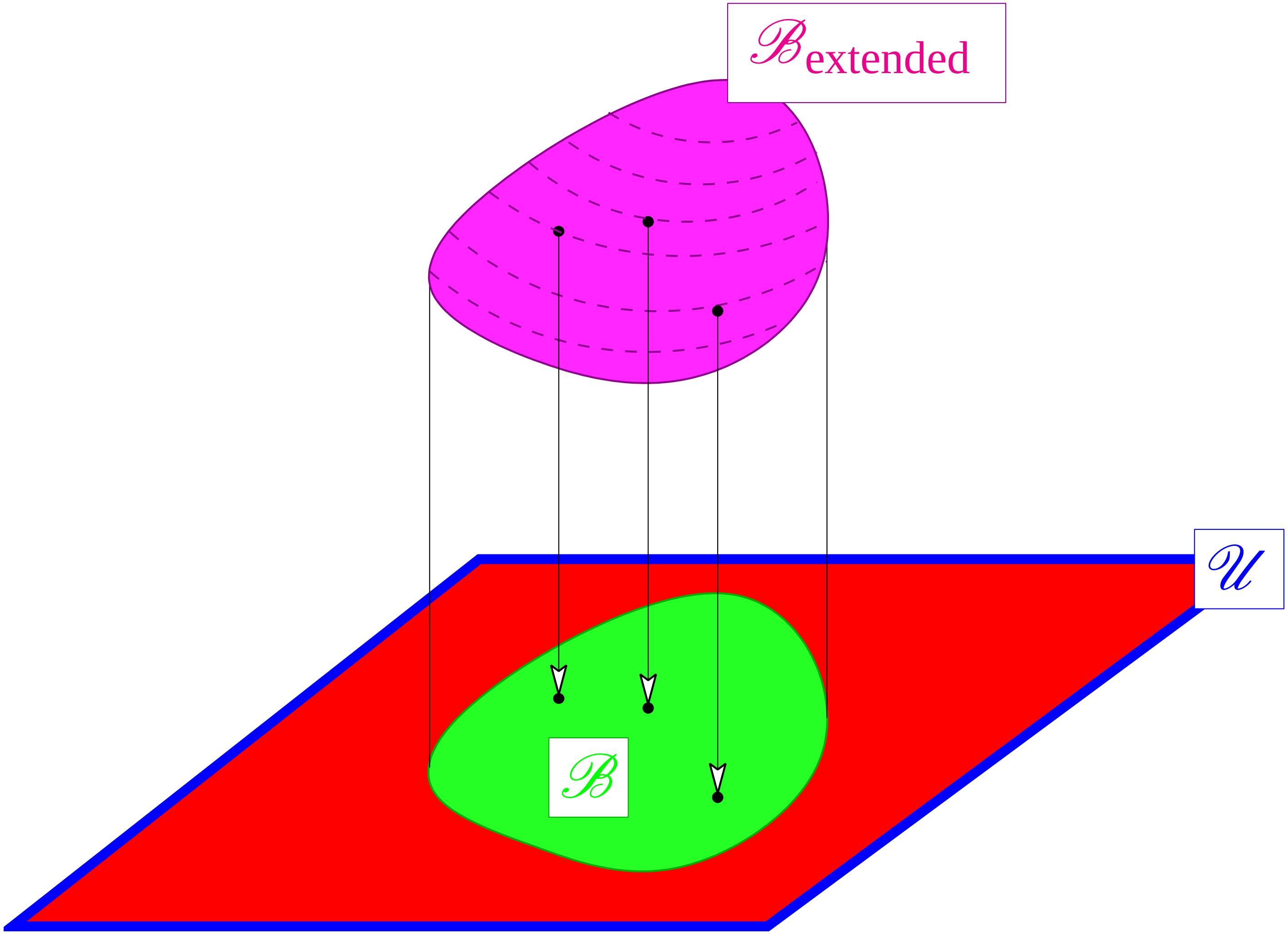


For example,

$$u = p_0\ell + p_1\frac{d}{dt}\ell + \cdots + p_{\mathtt{n}}\frac{d^{\mathtt{n}}}{dt^{\mathtt{n}}}\ell,$$

$$y = q_0\ell + q_1\frac{d}{dt}\ell + \cdots + q_{\mathtt{n}}\frac{d^{\mathtt{n}}}{dt^{\mathtt{n}}}\ell, \quad w = \begin{bmatrix} u \\ y \end{bmatrix}$$

Projection



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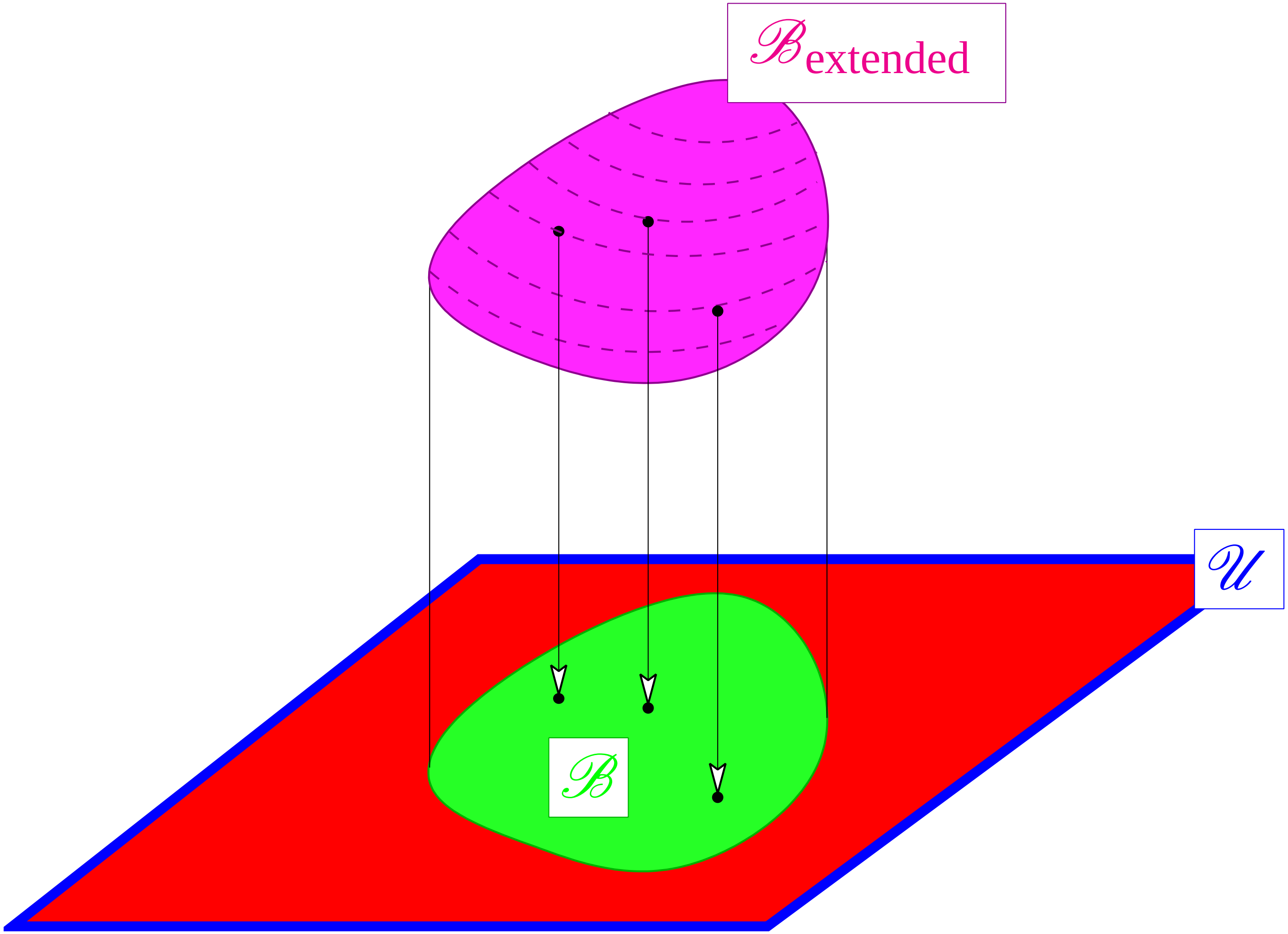

For example, $\frac{d}{dt}x = Ax + Bu, y = Cx + Du, \quad w = \begin{bmatrix} u \\ y \end{bmatrix}$

Latent variable representations

Combining equations with latent variables $\rightsquigarrow$

$\mathscr{B}_{\text{extended}}$ **specified by**

$$\mathscr{B}_{\text{extended}} = \{(w, \ell) \mid f(w, \ell) = 0 = 0\}$$

$$\mathscr{B} = \{w \mid \exists\, \ell \text{ such that } f(w, \ell) = 0\}$$

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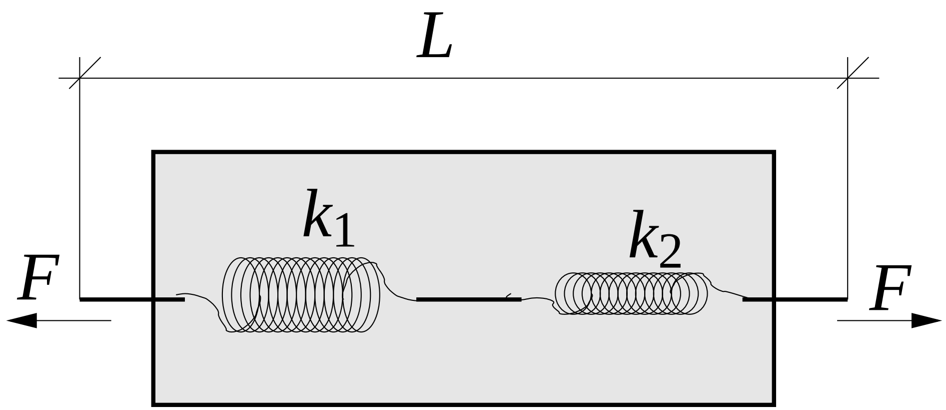
First principles models usually come in this form.
Latent variables naturally emerge from interconnections.

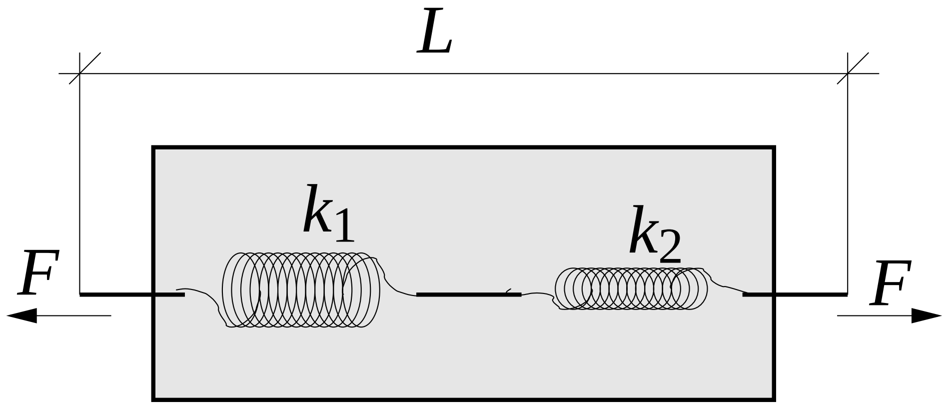
Example

Two springs interconnected in series



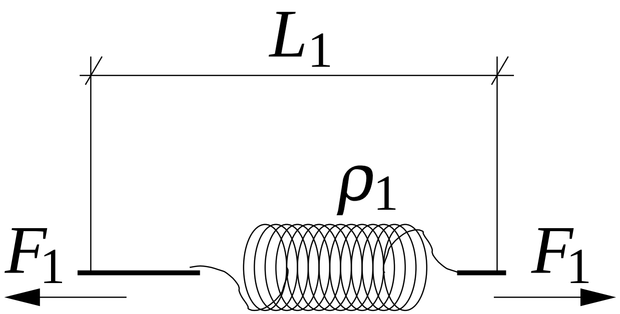
¡¡ Model relation between L and F !!

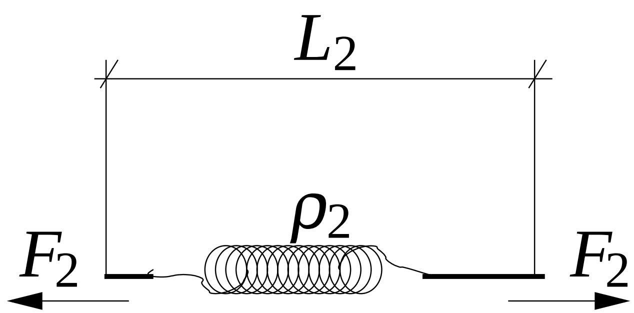
Two springs interconnected in series



¡¡ Model relation between L and F !!

View as interconnection of two springs





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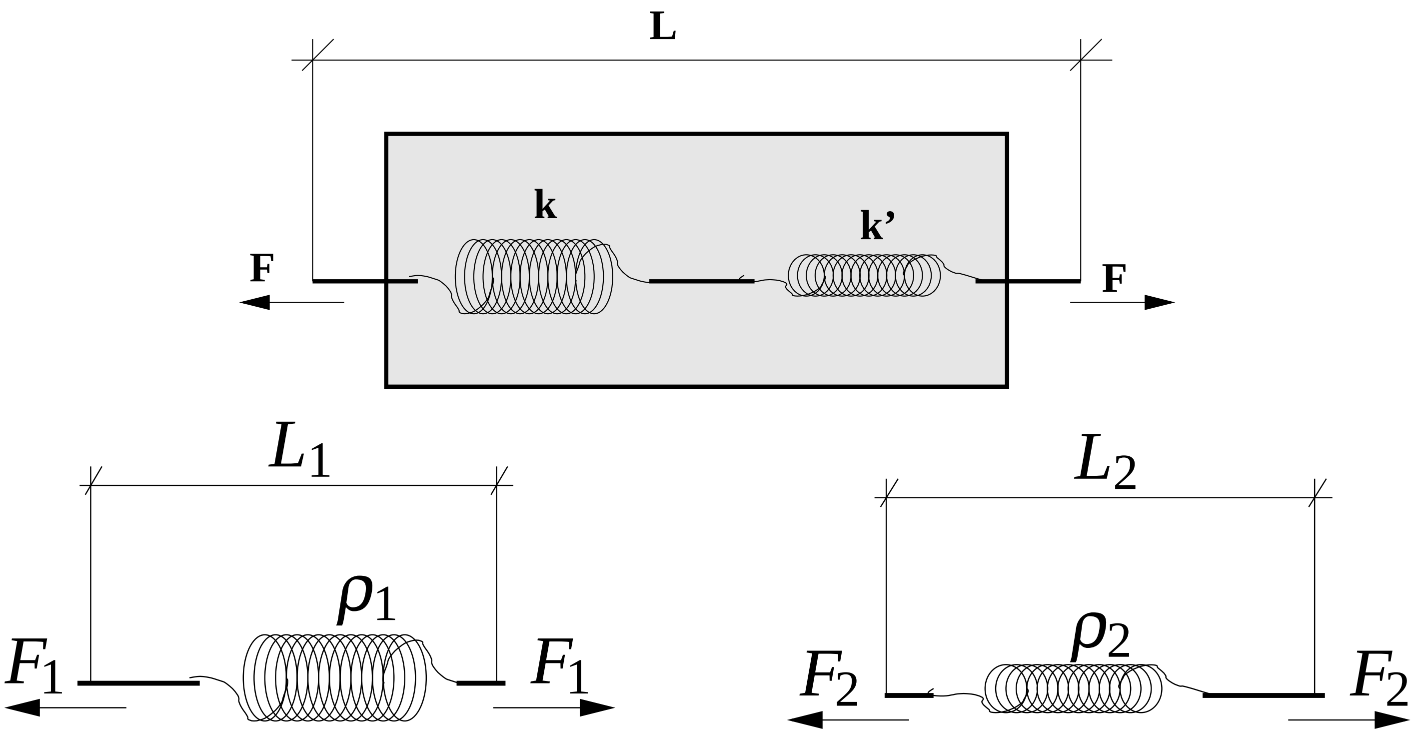

Model for (L,F) (assume that for the individual springs the length is a function of the force exerted).

$$L_1 = \rho_1(F_1) \qquad L_2 = \rho_1(F_2)$$

$$F = F_1 = F_2 \qquad L = L_1 + L_2$$

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L,F: **'manifest variables'** L_1,F_1,L_2,F_2: **'latent variables'**

$\rightsquigarrow$ $$\boxed{L = \rho_1(F) + \rho_2(F)}$$

Latent variables are easily eliminated, for this example.

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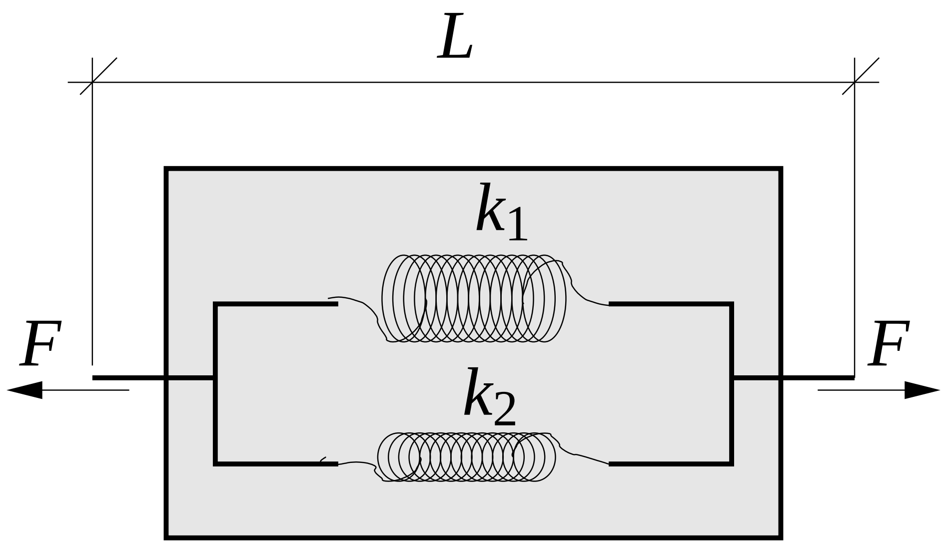
$\rightsquigarrow$ $\boxed{L = \rho_1(F) + \rho_2(F)}$

Latent variables are easily eliminated, for this example.

In the linear case: $L_1 = L_1^* + \rho_1 F_1 \qquad L_2 = L_2^* + \rho_2 F_2$

After elimination $\rightsquigarrow L = L_1^* + L_2^* + (\rho_1 + \rho_2)F$

Two springs interconnected in parallel



’!’! Model relation between L and F !!

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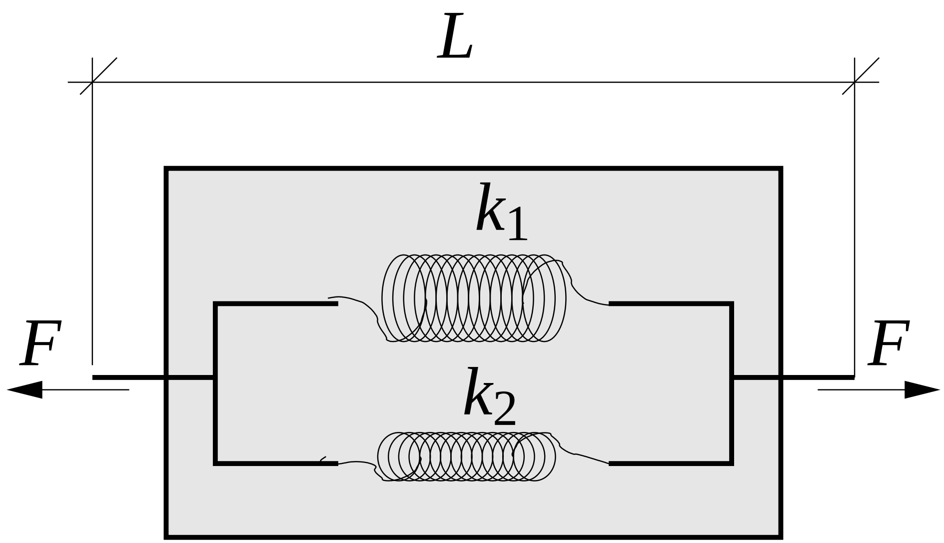
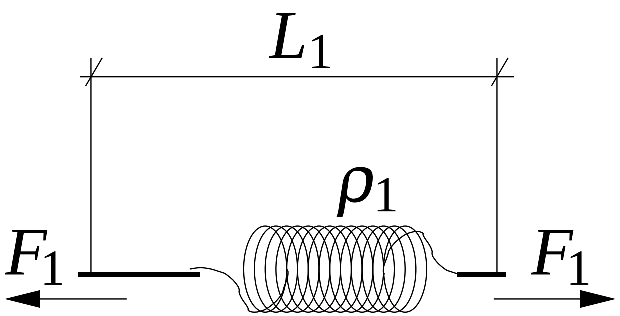
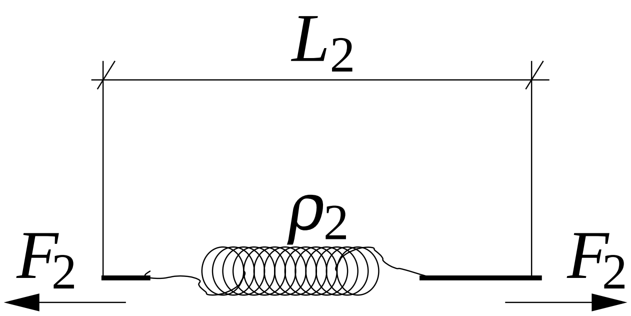


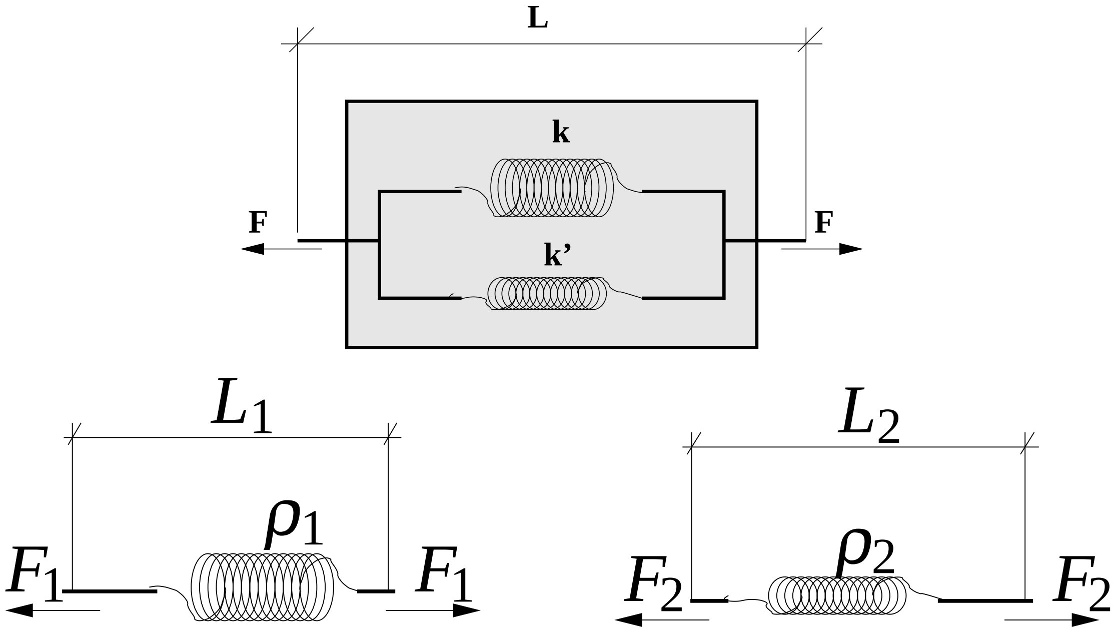
’!’! Model relation between *L* and *F* !!

View as interconnection of two springs





Two springs interconnected in parallel



Model for (L, F) (assume that for the individual springs the length is a function of the force exerted, and neglect the dimensions of the interconnecting mechanism).

$$L_1 = \rho_1(F_1) \qquad L_2 = \rho_1(F_2)$$

$$F = F_1 + F_2 \qquad L = L_1 = L_2$$

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$$
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F & = & F_1 + F_2 & \qquad & L & = & L_1 = L_2
\end{array}
$$

L,F: **'manifest variables'** L_1,F_1,L_2,F_2: **'latent variables'**

$\rightsquigarrow$ $$\mathscr{B} = \{(L,F) \mid \exists\, \alpha : L = \rho_1(\alpha), \quad \rho_1(\alpha) = \rho_2(F-\alpha)\}$$

Latent variables are not easily eliminated, for this example,

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Latent variables are not easily eliminated, for this example, unless we are in the linear case: $L_1 = L_1^* + \rho_1 F_1, L_2 = L_2^* + \rho_2 F_2$

After elimination $\rightsquigarrow L = \frac{\rho_2}{\rho_1+\rho_2} L_1^* + \frac{\rho_1}{\rho_1+\rho_2} L_2^* + \frac{\rho_1\rho_2}{\rho_1+\rho_2} F$

A dynamic example

Elimination problem

First principles models invariably contain (many) auxiliary variables in addition to the variables whose behavior we wish to model.

Elimination problem

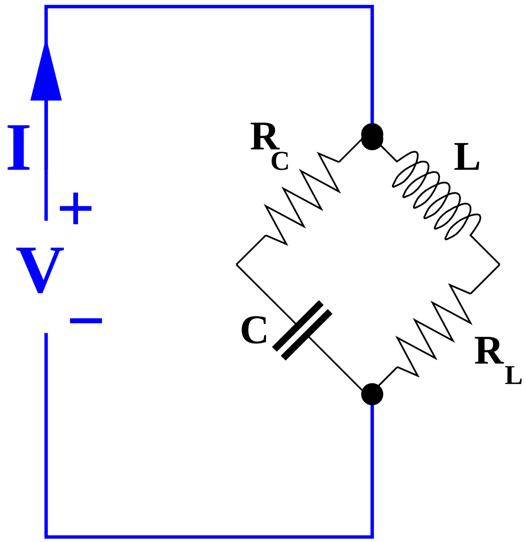
First principles models invariably contain (many) auxiliary variables in addition to the variables whose behavior we wish to model.

Can these latent variables be eliminated?

We illustrate the emergence of latent variables and the elimination question by means of an extensive example in the dynamic systems case.

RLC circuit

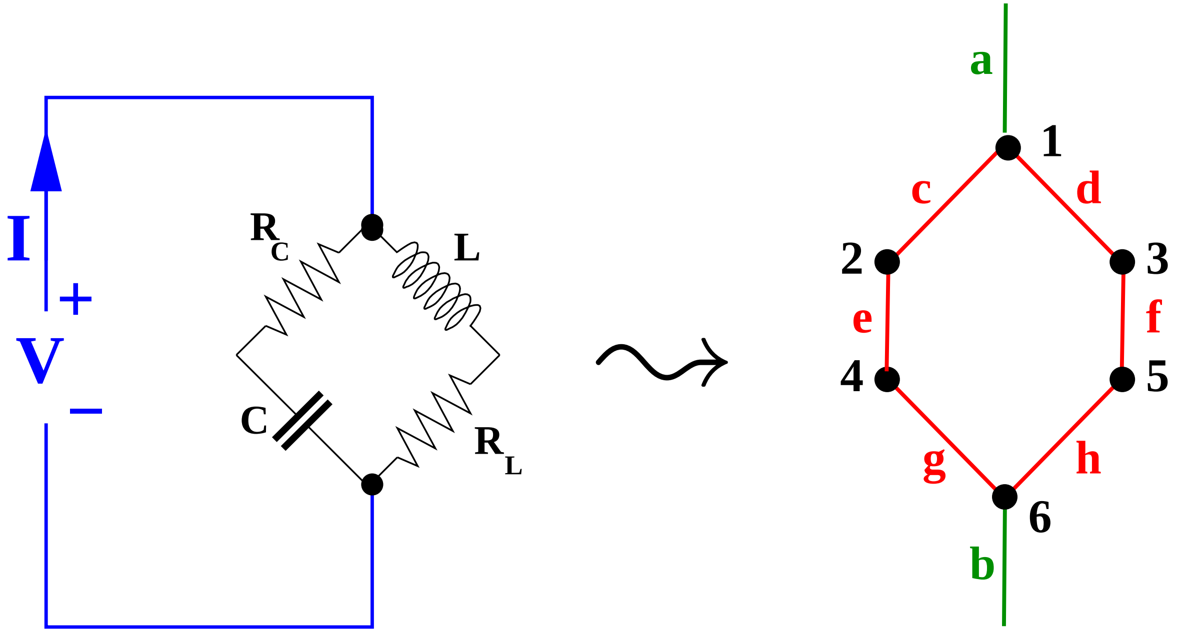
Model the port behavior of



by tearing, zooming, and linking.

RLC circuit

Model the port behavior of

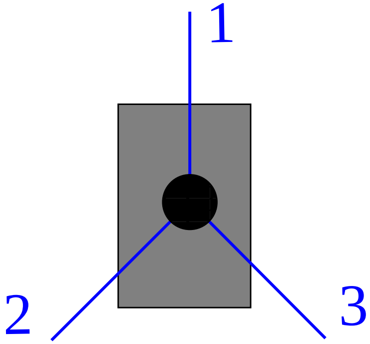


by tearing, zooming, and linking.

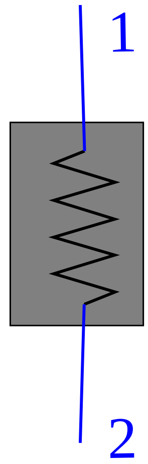
In each vertex there is an element $\leadsto$ module equations involving 2 variables (potential, current) for each terminal,

In each edge a connection $\leadsto$ interconnection equations

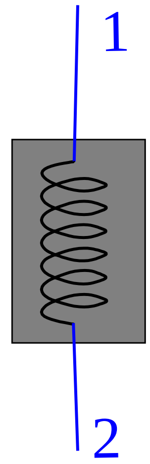
Modules



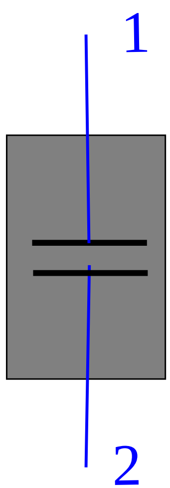
connector 1



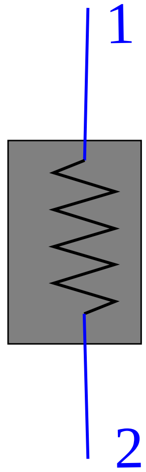
resistor R_C



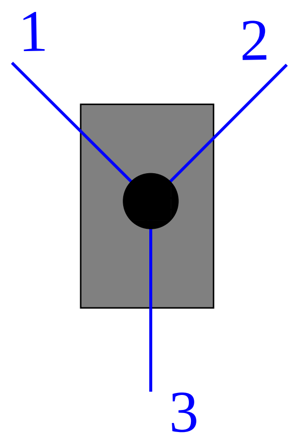
inductor L



capacitor C



resistor R_L



connector 2

Module equations

vertex 1 : $V_{\text{connector}_1,1} = V_{\text{connector}_1,2} = V_{\text{connector}_1,3}$

$I_{\text{connector}_1,1} + I_{\text{connector}_1,2} + I_{\text{connector}_1,3} = 0$

vertex 2 : $V_{R_C,1} - V_{R_C,2} = R_C I_{R_C,1},\ I_{R_C,1} + I_{R_C,2} = 0$

vertex 3 : $L\frac{d}{dt} I_{L,1} = V_{L,1} - V_{L,2},\ I_{L,1} + I_{L,2} = 0$

vertex 4 : $C\frac{d}{dt}\left(V_{C,1} - V_{C,2}\right) = I_{C,1},\ I_{C,1} + I_{C,2} = 0$

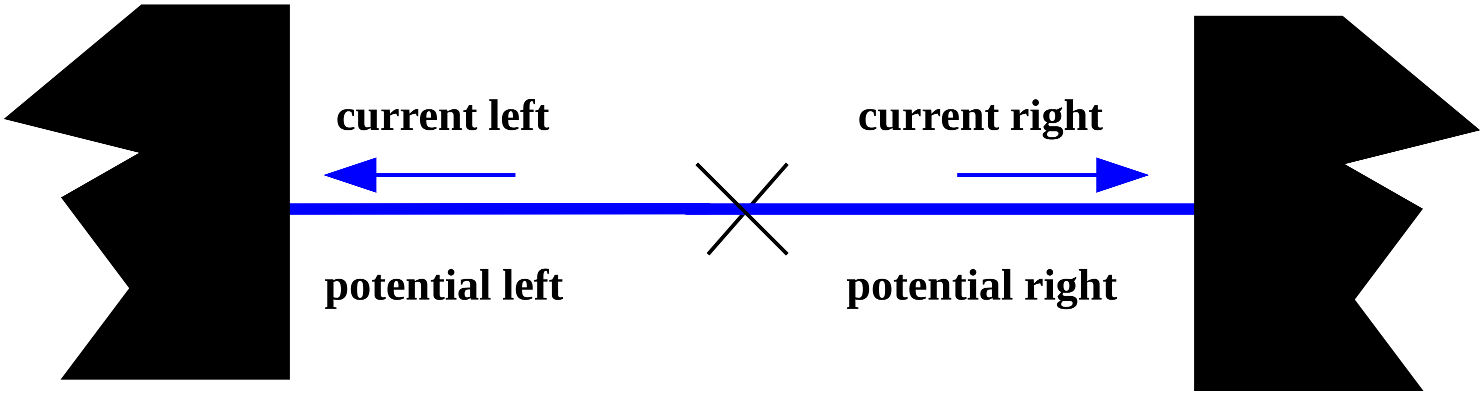
vertex 5 : $V_{R_L,1} - V_{R_L,2} = R_L I_{R_L,1}$

$I_{R_L,1} + I_{R_L,2} = 0$

vertex 6 : $V_{\text{connector}_2,1} = V_{\text{connector}_2,2} = V_{\text{connector}_2,3}$

$I_{\text{connector}_2,1} + I_{\text{connector}_2,2} + I_{\text{connector}_2,3} = 0$

Interconnection



Interconnection of two electrical terminals

Interconnection equations:

$$\textbf{potential left} = \textbf{potential right}$$

$$\textbf{current left} + \textbf{current right} = 0$$

Interconnection equations

$$
\begin{array}{lllllll}
\textbf{edge c}: & V_{R_{C,1}} & = & V_{\text{connector1}_2} & I_{R_{C,1}} + I_{\text{connector1},2} & = & 0 \\
\textbf{edge d}: & V_{L_1} & = & V_{\text{connector1}_3} & I_{L_1} + I_{\text{connector1}_3} & = & 0 \\
\textbf{edge e}: & V_{R_{C,2}} & = & V_{C_1} & I_{R_{C,2}} + I_{C_1} & = & 0 \\
\textbf{edge f}: & V_{L_2} & = & V_{R_{C,1}} & I_{L_2} + I_{R_{L,1}} & = & 0 \\
\textbf{edge g}: & V_{C_2} & = & V_{\text{connector2}_1} & I_{C_2} + I_{\text{connector2}_1} & = & 0 \\
\textbf{edge h}: & V_{R_{L,2}} & = & V_{\text{connector2}_2} & I_{R_{L,2}} + I_{\text{connector2}_2} & = & 0
\end{array}
$$

Manifest variable assignment

$$V_{\text{externalport}} = V_{\text{connector}_1,1} - V_{\text{connector}_2,3}$$

$$I_{\text{externalport}} = I_{\text{connector1}_1}$$

Tableau

vertex 1 : $V_{\text{connector}_1,1} = V_{\text{connector}_1,2} = V_{\text{connector}_1,3}$

$I_{\text{connector}_1,1} + I_{\text{connector}_1,2} + I_{\text{connector}_1,3} = 0$

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$V_{\text{externalport}} = V_{\text{connector}_1,1} - V_{\text{connector}_2,3}$ $I_{\text{externalport}} = I_{\text{connector1}_1}$

Variables and equations

In total 28 latent variables $V_{\text{connector}_1,1}, \ldots, V_{R_{C,1}}, I_{R_{C,1}}, \ldots, I_{\text{connector}_2,3}$
2 manifest variables, $\left(V_{\text{externalport}}, I_{\text{externalport}}\right)$
24 equations.

Which equation(s) govern(s) $\left(V_{\text{externalport}}, I_{\text{externalport}}\right)$

A constant-coefficient linear differential equation that does not contain the branch variables?

Does the fact that all the equations before elimination of the latent (auxiliary) variables are constant-coefficient linear differential equations imply the same after elimination?

The port equation

The port defines the system $\Sigma = (\mathbb{R}, \mathbb{R}^2, \mathscr{B})$ with behavior $\mathscr{B}$ specified by:

<u>Case 1</u>: $CR_C \neq \frac{L}{R_L}$

$$\left(\frac{R_C}{R_L} + \left(1 + \frac{R_C}{R_L}\right) CR_C \frac{d}{dt} + CR_C \frac{L}{R_L} \frac{d^2}{dt^2}\right) V_{\text{externalport}} = \left(1 + CR_C \frac{d}{dt}\right)\left(1 + \frac{L}{R_L}\frac{d}{dt}\right) R_C I_{\text{externalport}}$$

<u>Case 2</u>: $CR_C = \frac{L}{R_L}$

$$\left(\frac{R_C}{R_L} + CR_C \frac{d}{dt}\right) V_{\text{externalport}} = (1 + CR_C)\frac{d}{dt} R_C I_{\text{externalport}}$$

The elimination theorem

Elimination theorem

Theorem

$\mathscr{L}^\bullet$ is closed under projection

Elimination theorem

Theorem

$\mathscr{L}^\bullet$ is closed under projection

Consider

$$\mathscr{B} = \{(w_1, w_2) : \mathbb{R} \to \mathbb{R}^{\mathtt{w}_1} \times \mathbb{R}^{\mathtt{w}_2} \mid (w_1, w_2) \in \mathscr{B}\}$$

Define the projection

$$\mathscr{B}_1 = \{w_1 : \mathbb{R} \to \mathbb{R}^{\mathtt{w}_1} \mid \exists\, w_2 : \mathbb{R} \to \mathbb{R}^{\mathtt{w}_1} \text{ such that } (w_1, w_2) \in \mathscr{B}\}$$

The theorem states that $\boxed{[\![\mathscr{B} \in \mathscr{L}^{\mathtt{w}_1+\mathtt{w}_2}]\!] \Rightarrow [\![\mathscr{B}_1 \in \mathscr{L}^{\mathtt{w}_1}]\!]}$

This is, as seen, important in modeling.

Applications of the elimination theorem

$$[\![\frac{d}{dt}x = Ax + Bu, y = Cx + Du]\!] \Rightarrow [\![P\left(\frac{d}{dt}\right)y = Q\left(\frac{d}{dt}\right)u]\!]$$

$$[\![E\frac{d}{dt}x = Ax + Bw]\!] \Rightarrow [\![R\left(\frac{d}{dt}\right)w = 0]\!]$$

linear DAE's allow elimination of nuisance variables

$$[\![R\left(\frac{d}{dt}\right)w = M\left(\frac{d}{dt}\right)\ell]\!] \Rightarrow [\![R'\left(\frac{d}{dt}\right)w = 0]\!]$$

elimination of latent variables in LTIDSs is always possible.

$$[\![w = M\left(\frac{d}{dt}\right)\ell]\!] \Rightarrow [\![R'\left(\frac{d}{dt}\right)w = 0]\!]$$

Recapitulation

- Models are usually given as equations
- First principles models invariantly contain (many) latent variables
- In LTIDSs, latent variables can be completely eliminated

Summary

The main points

- **A model is a subset $\mathscr{B}$ of a universum $\mathscr{U}$.**
  $\mathscr{B}$ is the behavior of the model.

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End of the lecture