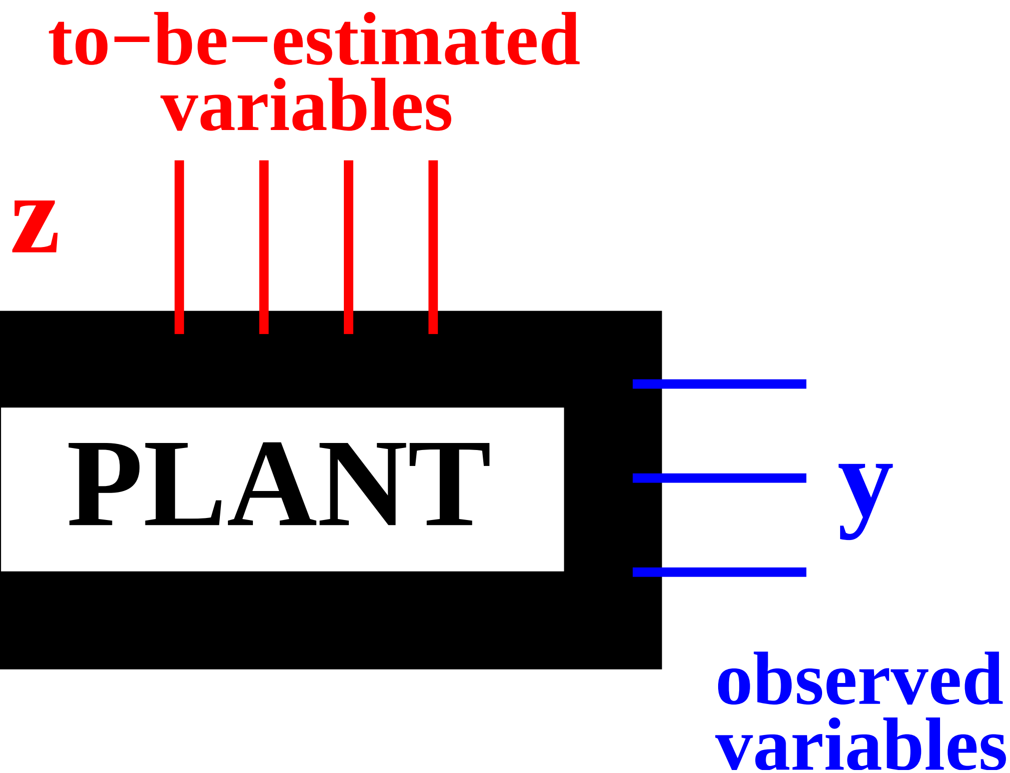
# A NEW LOOK AT OBSERVERS

**Jan C. Willems, K.U. Leuven, Belgium**

**Seminar, Southampton** **November 22, 2007**

# Introduction

## Theme



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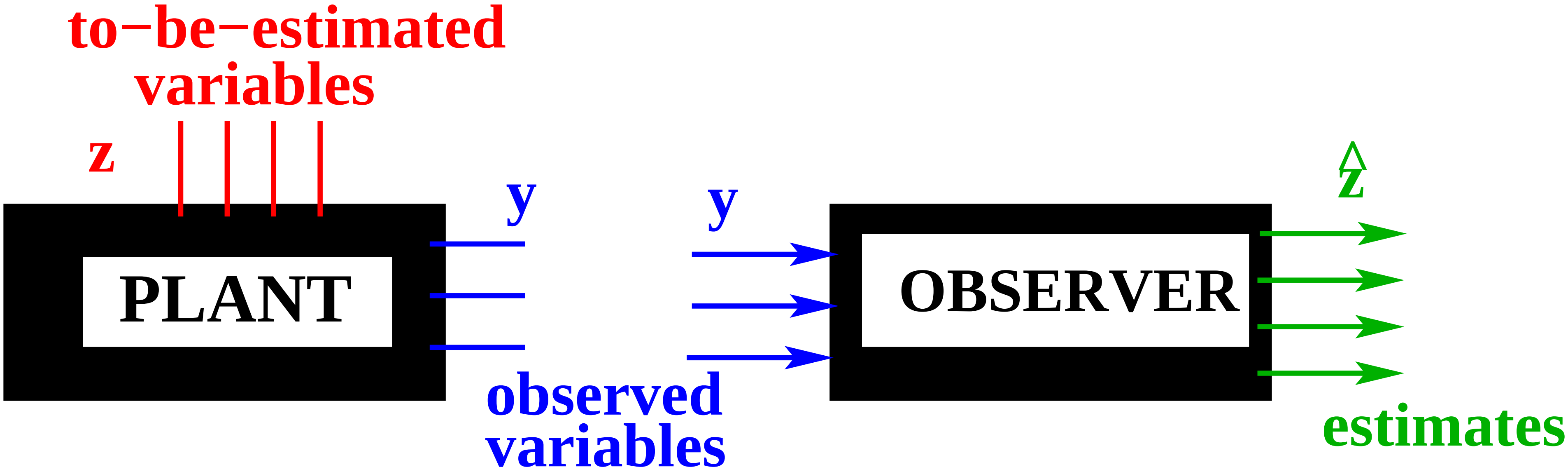
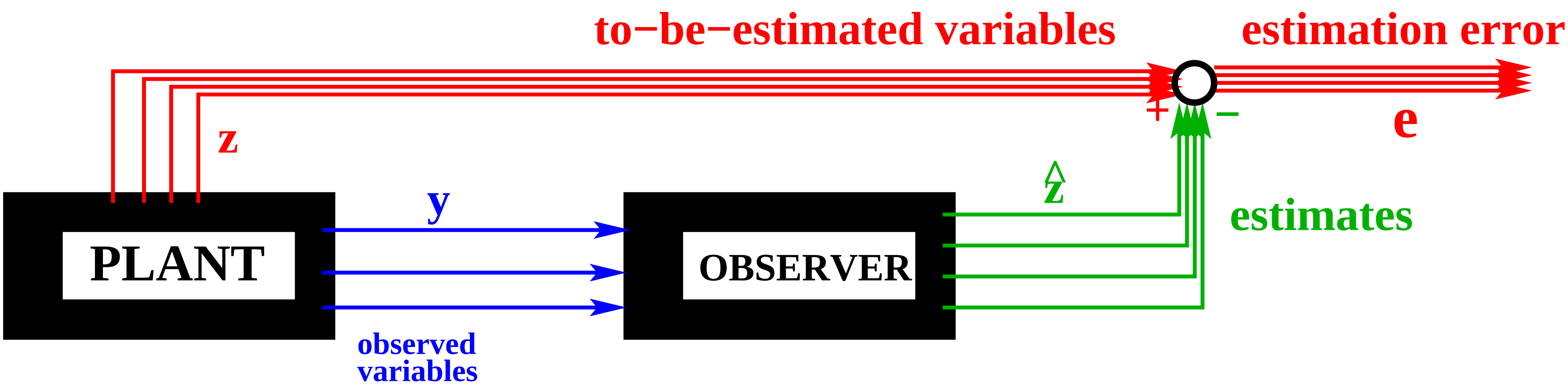


to−be−estimated variables
z
PLANT
y
y
observed variables
OBSERVER
ẑ
estimates

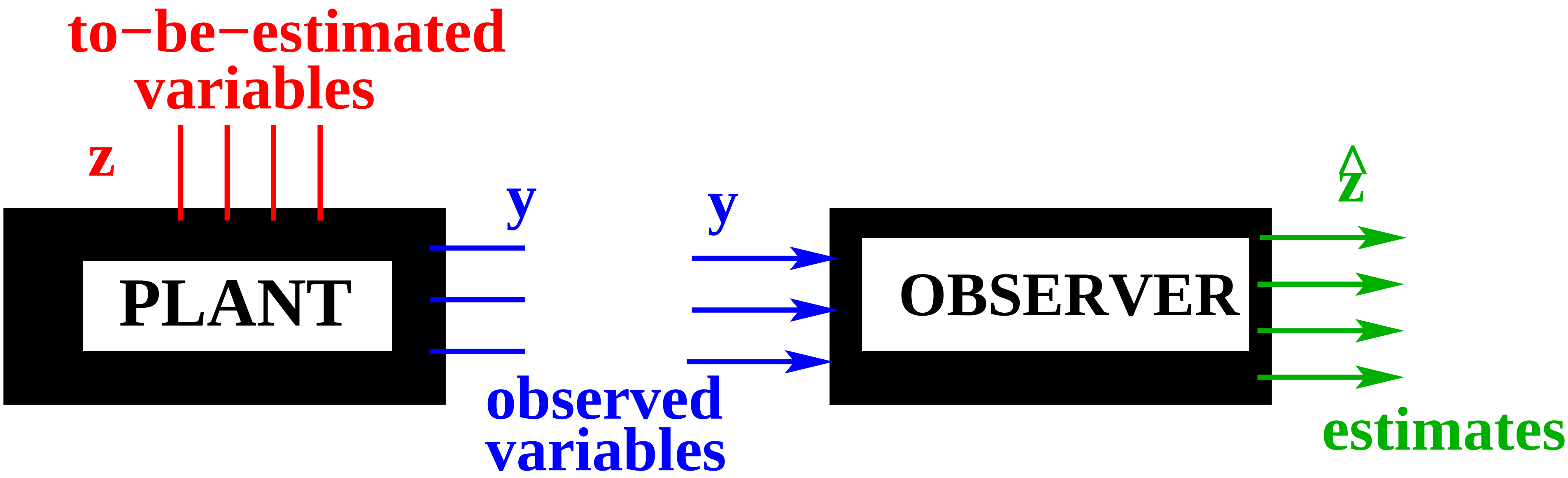
# Theme




**!! Keep estimation error**

**small, zero, convergent to zero, ... !!**

# Theme




- **What is the model that relates the observed with the to-be-estimated variables ?**
- **Find the observer/filter algorithm !**

# Joint Work with

**Maria Elena Valcher**

**Università di Padova**

## Joint Work with

**Jochen Trumpf**

**Australian National University**

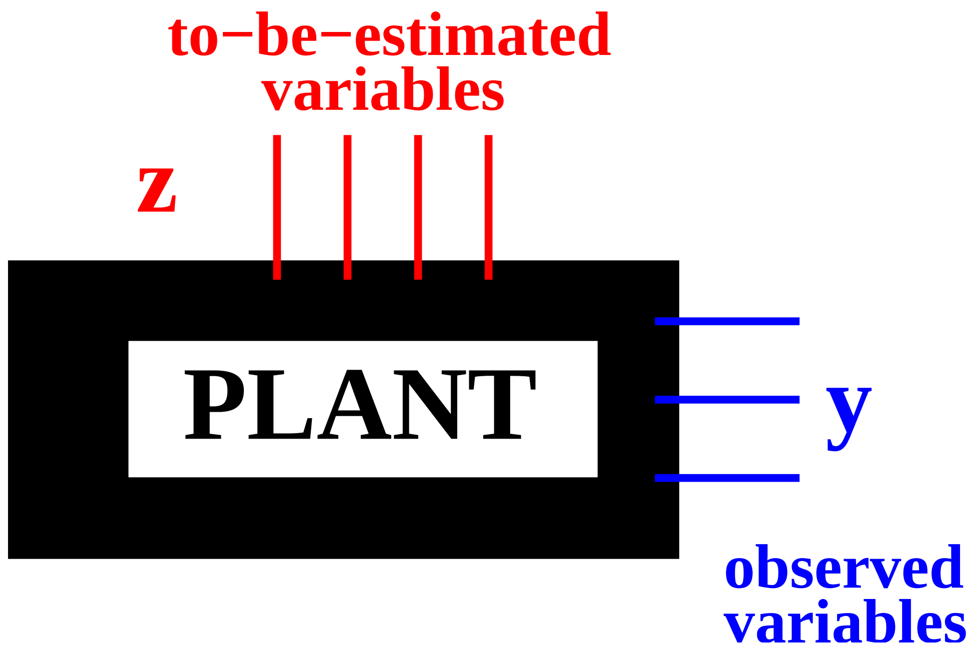
## Message

# Observers mean more

# Controllers mean less

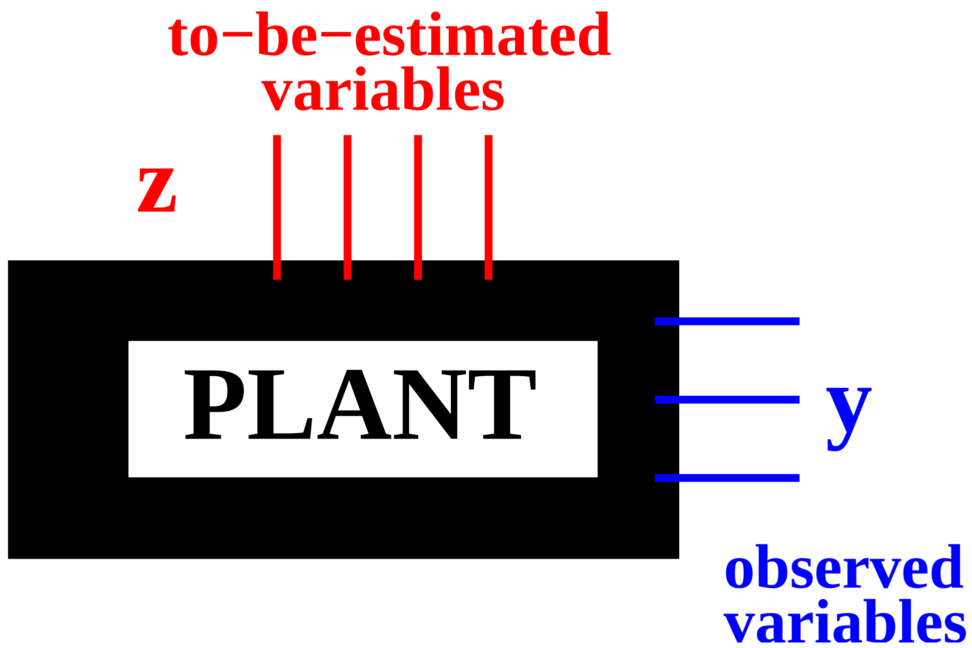
# History

# Wiener Filtering




**Modeling**: Assume $\begin{bmatrix} y \\ z \end{bmatrix}$ is a **stochastic process**

# Wiener Filtering

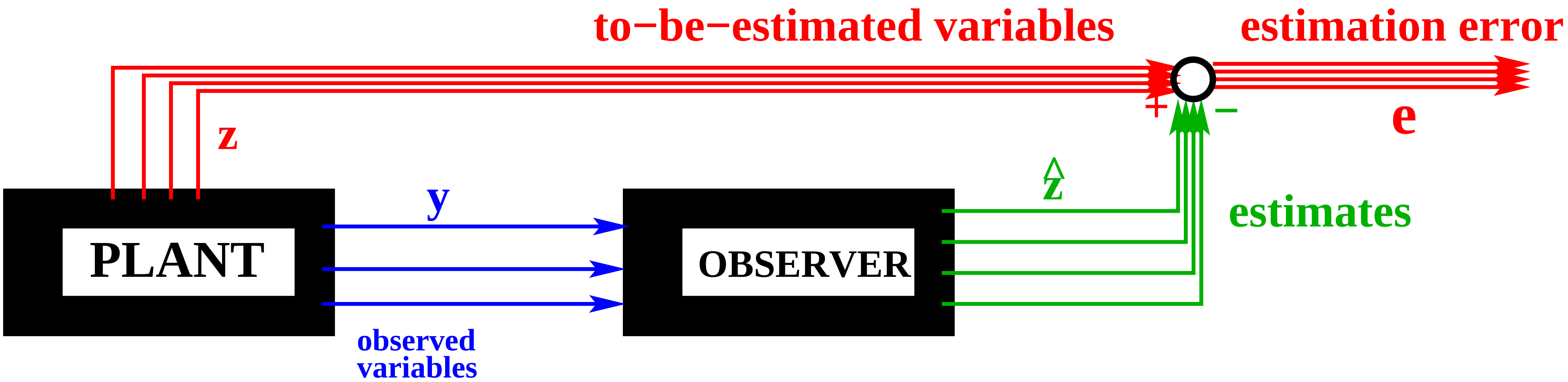



**Modeling**: Assume $\begin{bmatrix} y \\ z \end{bmatrix}$ is a **stochastic process**

**Henceforth time-set $\mathbb{R}$, stationary processes, normal, zero mean**

# Wiener Filtering

**Modeling**: Assume $\begin{bmatrix} y \\ z \end{bmatrix}$ is a **stochastic process**




**Estimation criterion:** $\mathbb{E}\left(|e(t)|^2\right) \rightsquigarrow$ **algorithms**

# Wiener Filtering

**Algorithms are easy to obtain if for the estimate $\hat{z}(t)$, the observations $y(t')$**

**are available for all $t' \in \mathbb{R}$.**

**Much much harder if the observations $y(t')$**

**are available only for $t' \leq t$**

$\leadsto$ **non-anticipating filter**

# Wiener Filtering

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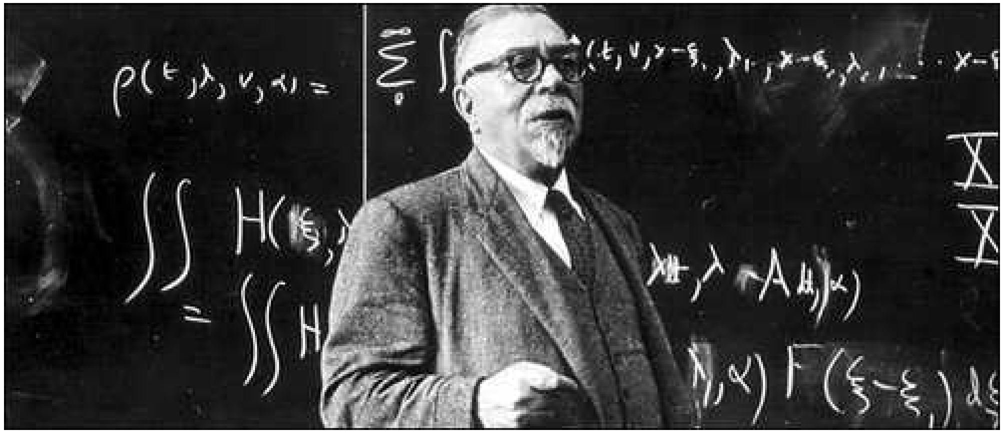
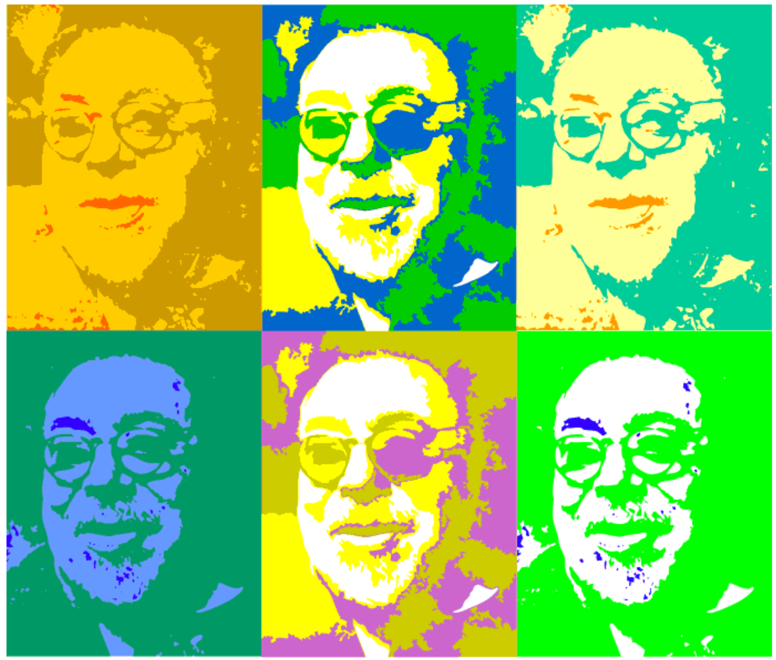
**This is the problem Wiener solved in 1942,**

**in the yellow peril**

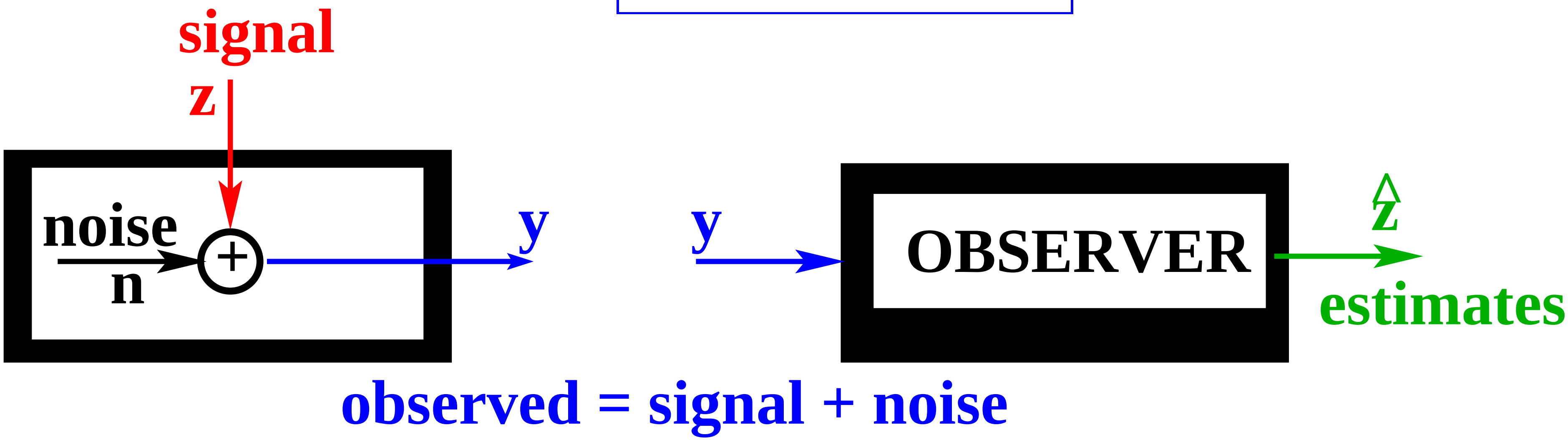
$\leadsto$ **non-anticipating filter , a.k.a. the Wiener filter**

# Wiener Filtering





# The Wiener Filter




**Signal $\perp$ noise**

**Signal spectral density $S_z(s)$**

**Noise white, intensity $\rho_n^2$**

**Filters according to the signal-to-noise ratio.**

# The Wiener Filter

**Knowledge of $y(t)\ \ \forall t \in \mathbb{R} \quad \rightsquigarrow$ tf. fn.**

$$y \mapsto \hat{z} = \boxed{1 - \frac{1}{1 + \frac{S_z(s)}{\rho_n^2}}}\, y$$

**Knowledge of $y(t)$ in past, Wiener filter $\quad \rightsquigarrow \quad$ tf. fn.**

$$y \mapsto \hat{z} = \boxed{1 - \frac{1}{\left[1 + \frac{S_z(s)}{\rho_n^2}\right]_+}}\, y$$

$$1 + \frac{S_z(s)}{\rho_n^2} = \left[1 + \frac{S_z(s)}{\rho_n^2}\right]_+ \left[1 + \frac{S_z(s)}{\rho_n^2}\right]_-$$

**$[\ \ ]_+$ poles & zeros in LHP ‘spectral factorization’**

## The Kalman Filter

**By taking another representation of the stochastic process $\begin{bmatrix} y \\ z \end{bmatrix}$, the optimal non-anticipating filter becomes much easier to compute.**

$$\frac{d}{dt}x = Ax + n_1, y = Cx + n_2, z = Hx, \begin{bmatrix} n_1 \\ n_2 \end{bmatrix} \text{ white}$$

**This is the representation used by Kalman (1960)**

# The Kalman Filter

$$\frac{d}{dt}x = Ax + Bw, \;\; y = Cx + n, \;\; z = Hx$$

$w \perp n$, **both white, intensities** $I$

$$\frac{d}{dt}\hat{x} = A\hat{x} + \Sigma C^\top (y - C\hat{x}), \;\; \hat{z} = H\hat{x}$$

**$\Sigma$ suitable solution of the ARE**

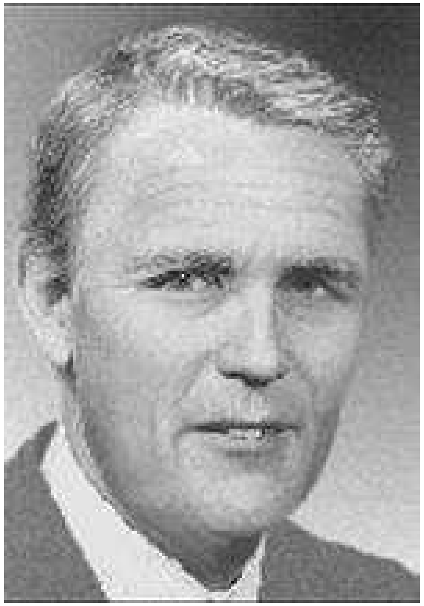
$$A\Sigma + \Sigma A^\top - \Sigma C^\top C\Sigma + BB^\top = 0$$

***Exactly*** **the Wiener filter, but in a form that is**

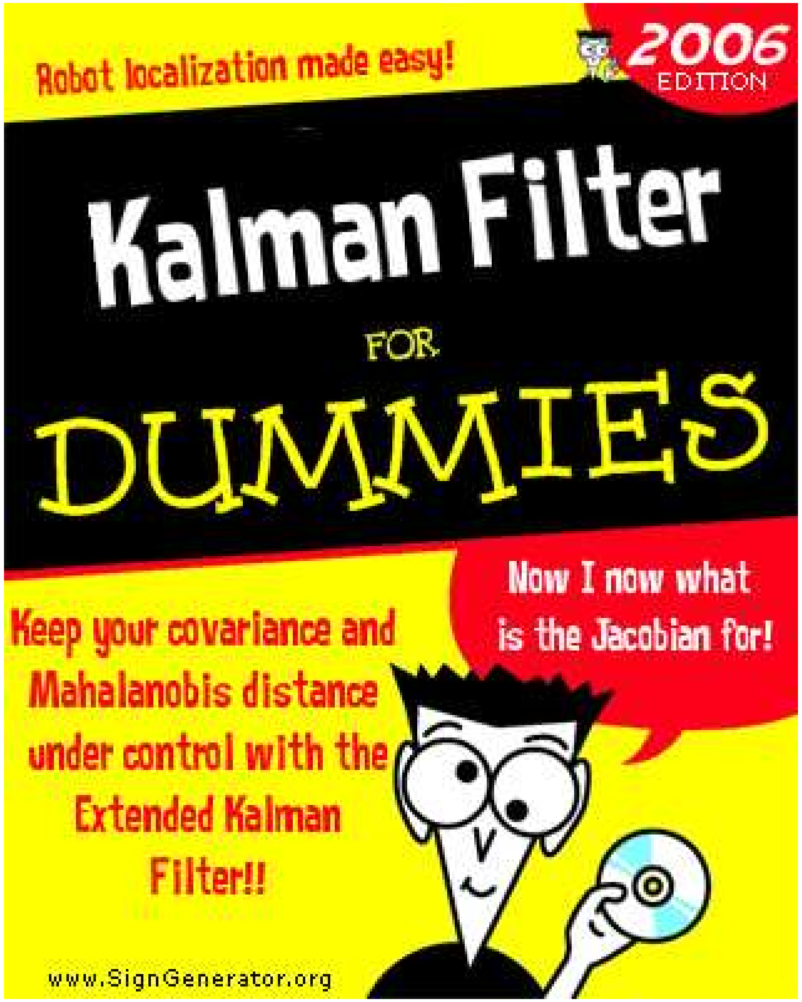
***recursive, algorithmic, generalizable***

***(finite time, time-varying, nonlinear) ...***

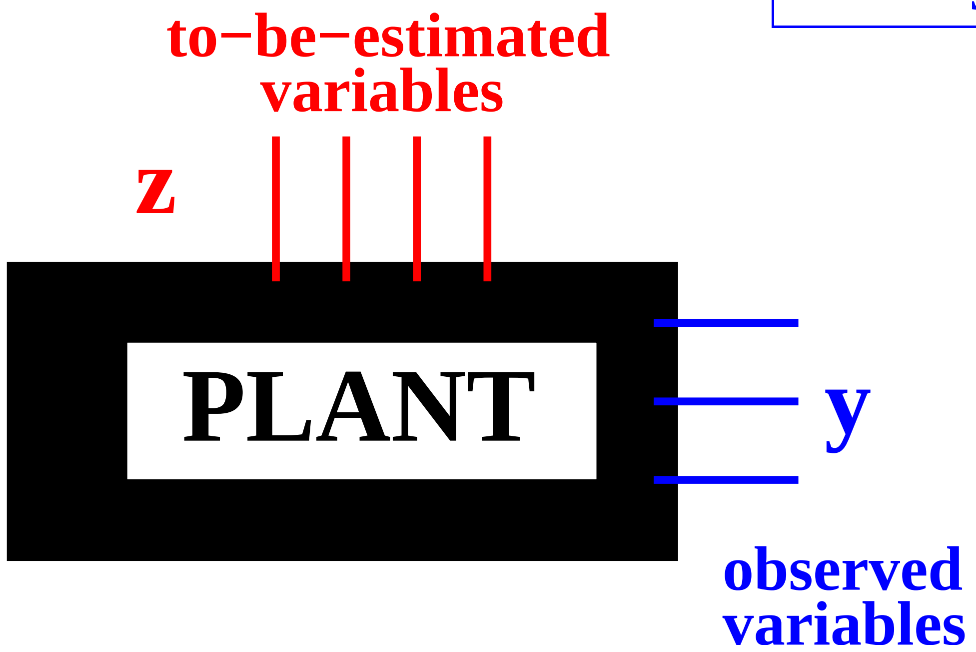
# The Kalman Filter



# The Kalman filter had a tremendous impact !



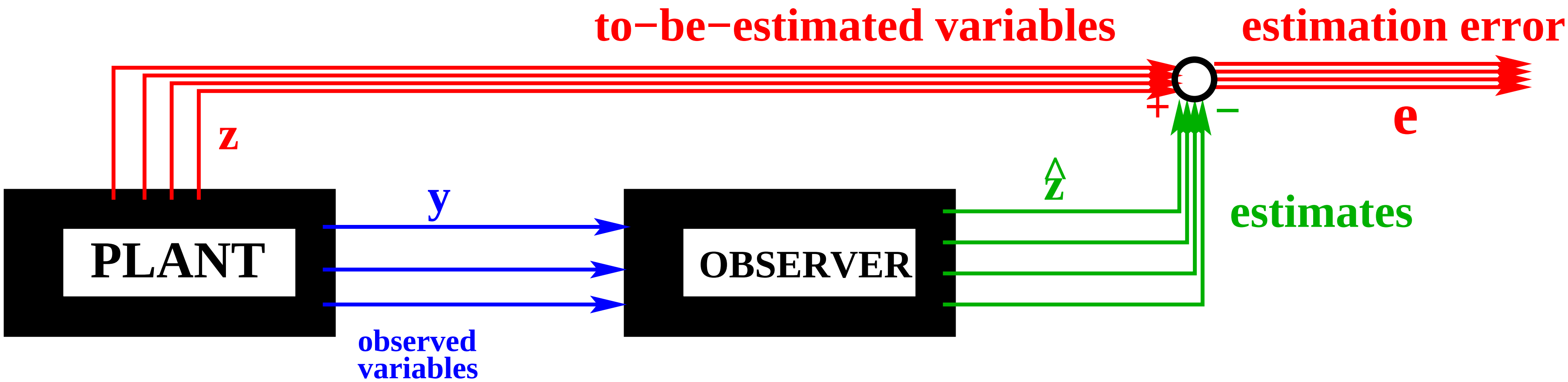
History




**Modeling**: $\begin{bmatrix} y \\ z \end{bmatrix}$ **input/output of a linear system**

$$\frac{d}{dt}x = Ax + Bu, \quad y = \begin{bmatrix} u \\ Cx \end{bmatrix}, \quad z = Hx.$$

# History




**Plant:** $$\frac{d}{dt}x = Ax + Bu, \;\; y = \begin{bmatrix} u \\ Cx \end{bmatrix}, \;\; z = Hx$$

**Observer:** $$\frac{d}{dt}\hat{x} = A\hat{x} + Bu + L\left(Cx - C\hat{x}\right), \;\; \hat{z} = H\hat{x}$$

**Error:** $$\frac{d}{dt}e_x = (A - LC)e_x, \;\; e = He_x$$

# History

*State observer*, **first proposed in 1963 by Luenberger.**

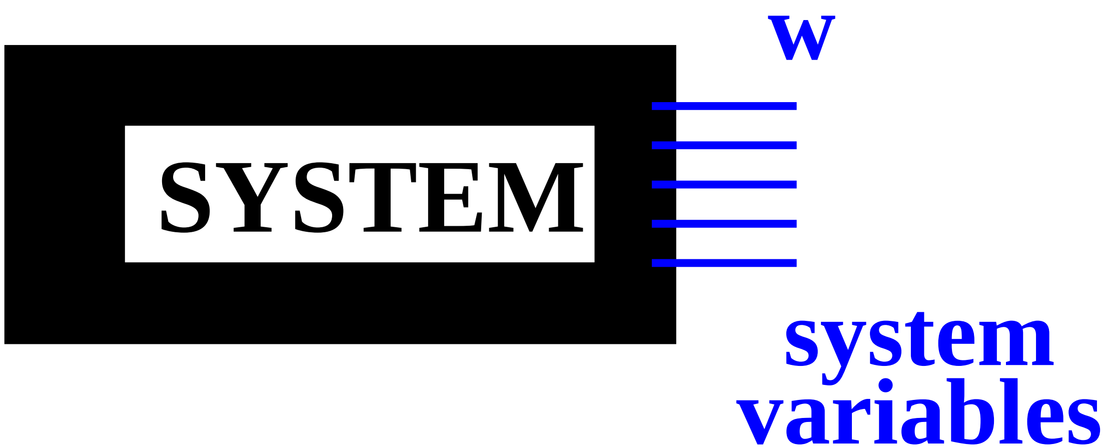
**Many variations (reduced order, dead-beat, ...)**

**Structure inspired by the ‘optimal’ Kalman filter.**

**No stochastic assumptions !**

# Systems & Their Properties

# Behaviors



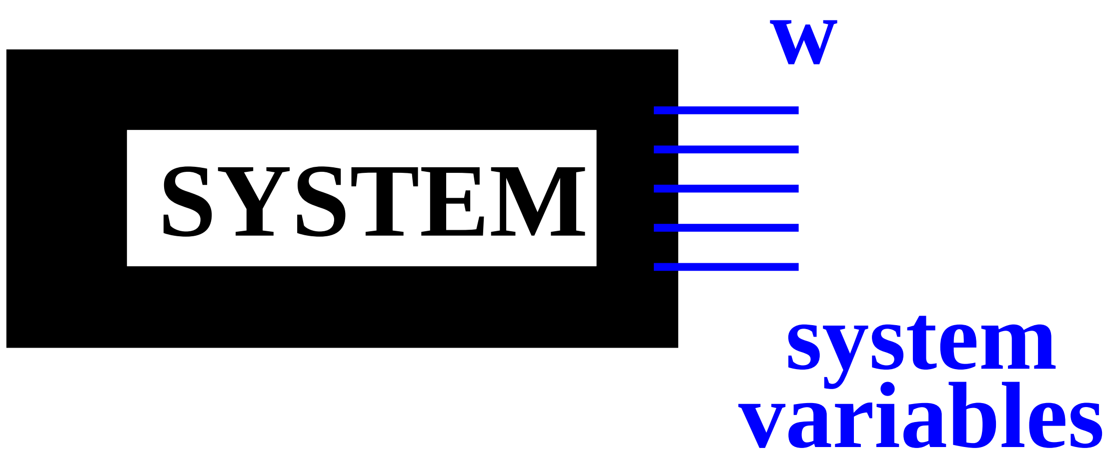

**A** ***dynamical system*** **is** $\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$

$\mathbb{T} \subseteq \mathbb{R}$ **'time-set'**

$\mathbb{W}$ **'signal space'**

$\mathfrak{B} \subseteq \mathbb{W}^{\mathbb{T}}$ **'behavior'**

# Behaviors

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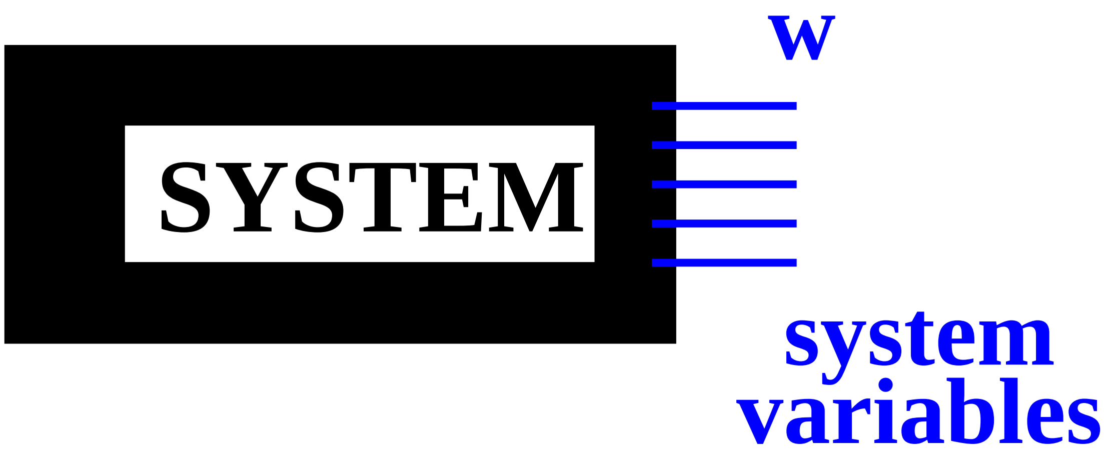
Consider $w : \mathbb{T} \longrightarrow \mathbb{W}$

$w \in \mathfrak{B}$ the model allows the trajectory $w$

$w \notin \mathfrak{B}$ the model forbids the trajectory $w$

# Behaviors

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| | | |
|---|---|---|
| $\mathbb{T} \subseteq \mathbb{R}$ | **'time-set'** | **today,** $\mathbb{T} = \mathbb{R}$ |
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**Linear time-invariant differential system (LTIDS):**

$\mathfrak{B} =$ all solutions of

$$R\left(\frac{d}{dt}\right) w = 0$$

where $R \in \mathbb{R}[\xi]^{\bullet \times \mathtt{w}}$

'kernel representation' (numerous other repr.)

# Behaviors

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$\mathfrak{B}$ **= all** $\mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$**- solutions of**

$$\boldsymbol{R_0 w + R_1 \frac{d}{dt} w + \cdots + R_{\mathrm{n}} \frac{d^{\mathrm{n}}}{dt^{\mathrm{n}}} w = 0}$$

## Behaviors

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**Linear time-invariant differential system (LTIDS):**

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**The behavior is all there is !**

Representations, properties (controllability, observability, symmetries) in terms of behavior

# Behaviors

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**The behavior is all there is !**

**SYSID refers to behavior,**

**control = restricting behavior, ...**

# Behaviors

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**Linear time-invariant differential system (LTIDS):**

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**The behavior is all there is !**

**Physical models specify the behavior !**

# LTIDSs

$\mathfrak{L}^\bullet$: the LTIDSs. $\mathfrak{L}^\bullet$ is closed under projection

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$$R_1\left(\frac{d}{dt}\right)w_1 = R_2\left(\frac{d}{dt}\right)w_2 \qquad (*)$$

$R_1, R_2 \in \mathbb{R}[\xi]^{\bullet\times\bullet}$.

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$$\mathfrak{B}_1 := \{w_1 \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^\bullet) \mid \exists w_2 \text{ such that } (*)\}$$

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**Thm:** $\mathfrak{B}_1 \in \mathfrak{L}^\bullet$ $\quad \leadsto R\left(\frac{d}{dt}\right) w_1 = 0$

$\exists$ algorithms $\quad \boxed{(R_1, R_2) \mapsto R}$

# LTIDSs

$\mathfrak{L}^\bullet$: the LTIDSs. $\mathfrak{L}^\bullet$ is closed under projection

$$F\left(\frac{d}{dt}\right)w = 0, F \in \mathbb{R}[\xi]^{\bullet \times \mathtt{w}}$$

is a consequence of

$$R\left(\frac{d}{dt}\right)w = 0, R \in \mathbb{R}[\xi]^{\bullet \times \mathtt{w}} \quad :\Leftrightarrow$$

$$[\![ R\left(\frac{d}{dt}\right)w = 0 ]\!] \Rightarrow [\![ F\left(\frac{d}{dt}\right)w = 0 ]\!]$$

# LTIDSs

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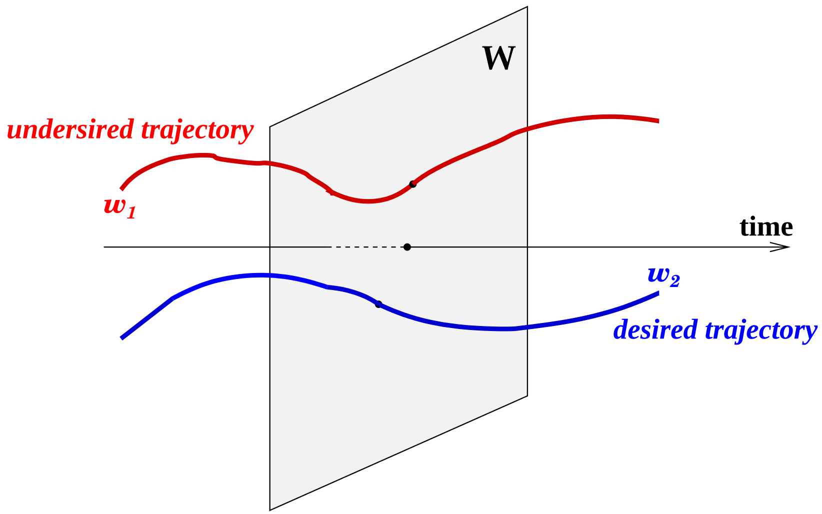
**Thm: Consequence $\Leftrightarrow$ $F = F'R$**

# System Properties

- **Controllable**
- **Stabilizable**
- **Autonomous**
- **Stable**
- **Observable**
- **Detectable**

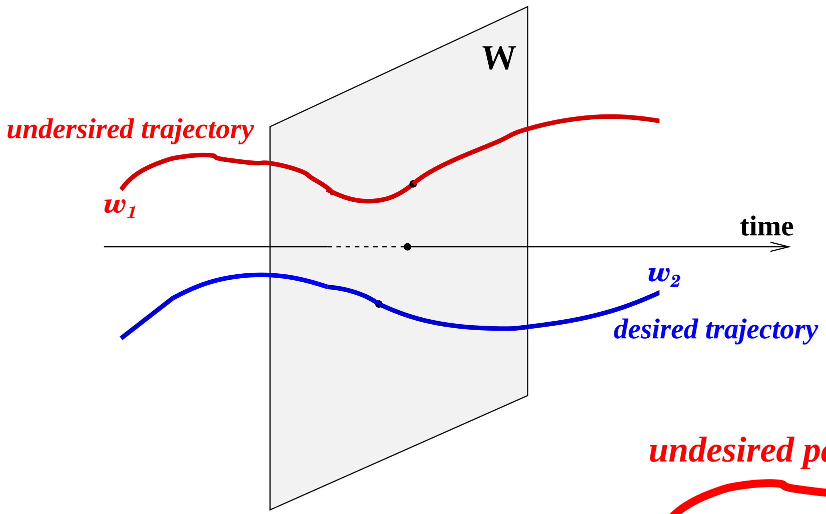
# System Properties

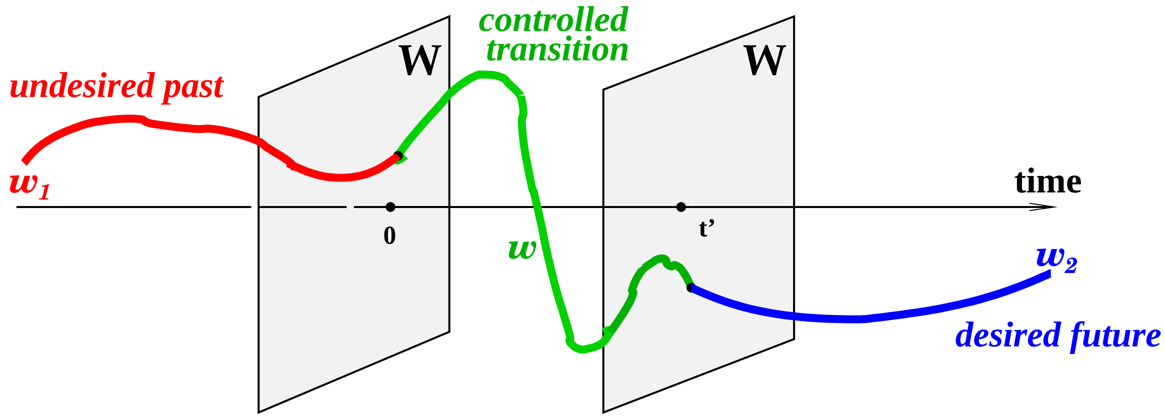
$\Sigma = (\mathbb{R}, \mathbb{W}, \mathfrak{B})$ **is controllable** $:\Leftrightarrow$



# System Properties

$$\Sigma = (\mathbb{R}, \mathbb{W}, \mathfrak{B}) \text{ is } \textbf{controllable} :\Leftrightarrow$$




**Behavioral controllability of a dynamical system**

# System Properties

$\boldsymbol{\Sigma} = (\mathbb{R}, \mathbb{W}, \mathfrak{B})$ **is autonomous** $:\Leftrightarrow$

$$[\![\, w_1, w_2 \in \mathfrak{B} \text{ and } w_1(t) = w_2(t) \text{ for } t < 0 \,]\!] \Rightarrow [\![\, w_1(t) = w_2(t) \text{ for } t \geq 0 \,]\!]$$

**'past implies future'**

**stable** $:\Leftrightarrow \; [\![\, w \in \mathfrak{B} \,]\!] \Rightarrow [\![\, w(t) \to 0 \text{ as } t \to \infty \,]\!]$

# System Properties

$$\boldsymbol{R}\left(\frac{d}{dt}\right)\boldsymbol{w} = \boldsymbol{0}$$

**defines a controllable system iff**

$\operatorname{rank}\left(\boldsymbol{R}\left(\boldsymbol{\lambda}\right)\right)$ **is the same for all** $\boldsymbol{\lambda} \in \mathbb{C}$

**a stabilizable one ....** $\lambda \in$ **the closed RHP**

# System Properties

$$R\left(\frac{d}{dt}\right)w = 0$$

**defines an autonomous system iff**

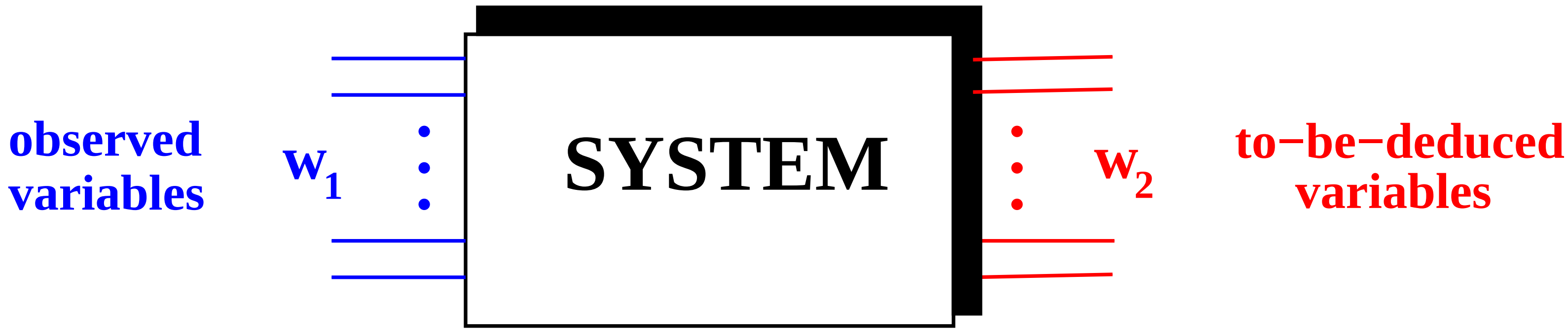
$R(\lambda)$ **full column rank** $\forall$ **but finite number** $\lambda \in \mathbb{C}$

$\exists$ **kernel repr. with** $R$ **square and** $\det(R) \neq 0$**.**

**a stable one ...** $\lambda \in$ **the closed LHP**

$\exists$ $R$ **'Hurwitz'**

# Properties Involving Relations Among Variables



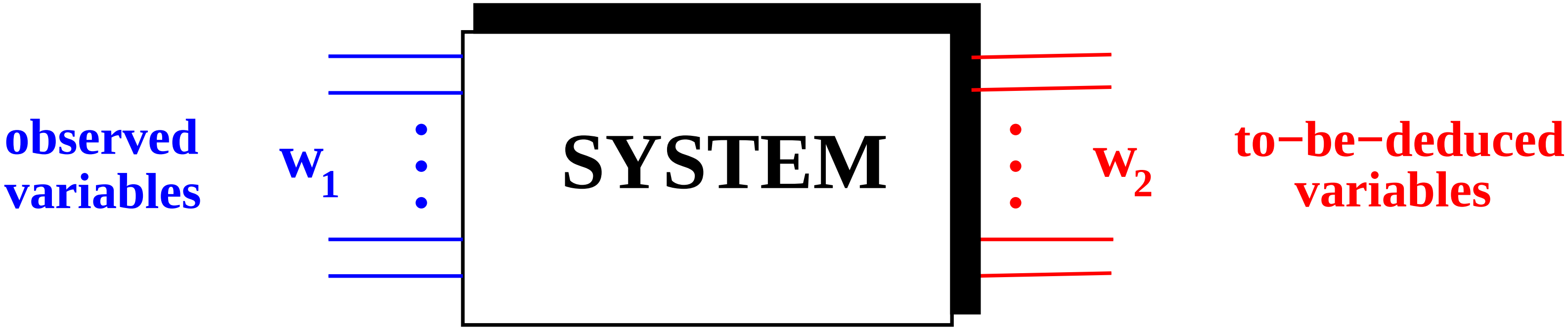
# Properties Involving Relations Among Variables


$w_1$ **is observable from** $w_2$ **in** $\Sigma = (\mathbb{T}, \mathbb{W}_1 \times \mathbb{W}_2, \mathfrak{B})$ $:\Leftrightarrow$

$$[\![ \, (w_1, w_2'), (w_1, w_2'') \in \mathfrak{B} \, ]\!] \Rightarrow [\![ \, w_2' = w_2'' \, ]\!]$$

**Observed trajectory implies the to-be-deduced one**

# Properties Involving Relations Among Variables




$w_1$ is **detectable** from $w_2$ in $\Sigma = (\mathbb{T}, \mathbb{W}_1 \times \mathbb{W}_2, \mathfrak{B})$ :⇔

$$\llbracket \, (w_1, w_2') \, , (w_1, w_2'') \in \mathfrak{B} \, \rrbracket$$

$$\Rightarrow \llbracket \, w_2'(t) - w_2''(t) \longrightarrow 0 \text{ as } t \longrightarrow \infty \, \rrbracket$$

**Observed trajectory implies the to-be-deduced one asymptotically**

# Tests for Observability and Detectability

$$R_1\left(\frac{d}{dt}\right)w_1 = R_2\left(\frac{d}{dt}\right)w_2$$

**defines an observable system iff**

$$R_2(\lambda) \text{ has full column rank } \forall\, \lambda \in \mathbb{C}$$

**defines a detectable system iff**

**...$\forall$ but finite number $\lambda \in$ closed RHP**

# Tests for Observability and Detectability

$$R_1\left(\frac{d}{dt}\right)w_1 = R_2\left(\frac{d}{dt}\right)w_2$$

**observable iff there are 'consequences'**

$$w_2 = F\left(\frac{d}{dt}\right)w_1$$

$$\rightsquigarrow \quad R\left(\frac{d}{dt}\right)w_1 = 0, \;\; w_2 = F\left(\frac{d}{dt}\right)w_1$$

**$\exists$ algorithms ...**

# Tests for Observability and Detectability

$$R_1\left(\frac{d}{dt}\right)w_1 = R_2\left(\frac{d}{dt}\right)w_2$$

**detectable iff there are 'consequences'**

$$H\left(\frac{d}{dt}\right)w_2 = F\left(\frac{d}{dt}\right)w_1, \text{ with } H \text{ 'Hurwitz'}$$

$$\rightsquigarrow \quad R\left(\frac{d}{dt}\right)w_1 = 0, \;\; H\left(\frac{d}{dt}\right)w_2 = F\left(\frac{d}{dt}\right)w_1$$

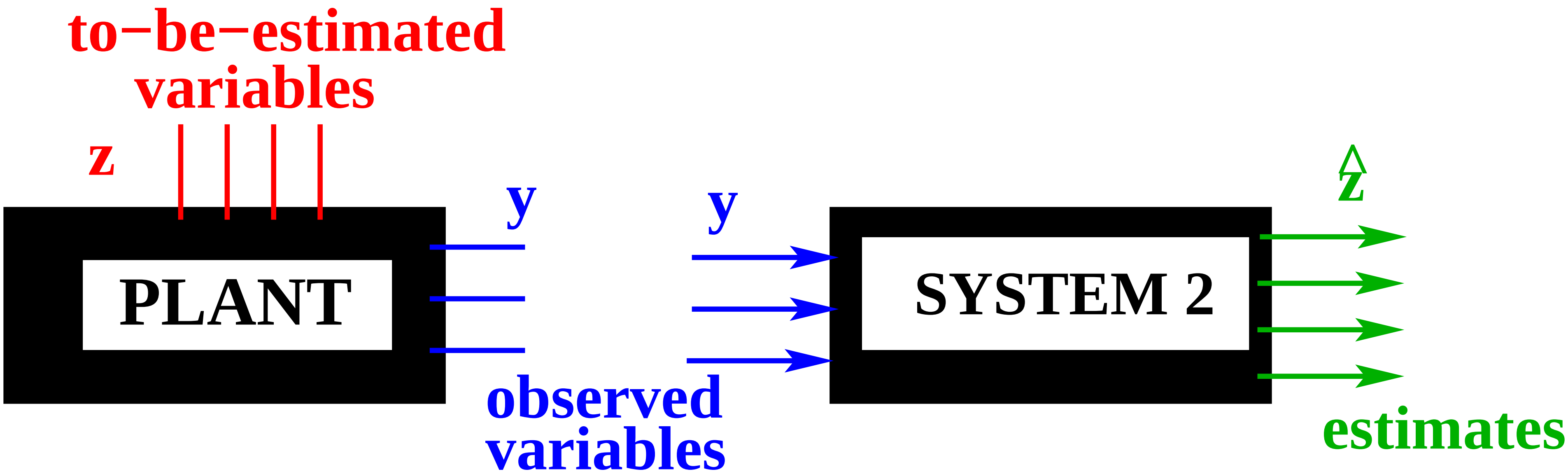
**$\exists$ algorithms ...**

**System properties ought to hold beyond the state space setting,**

**they ought to be representation independent**

# What is an observer ?

# Observers



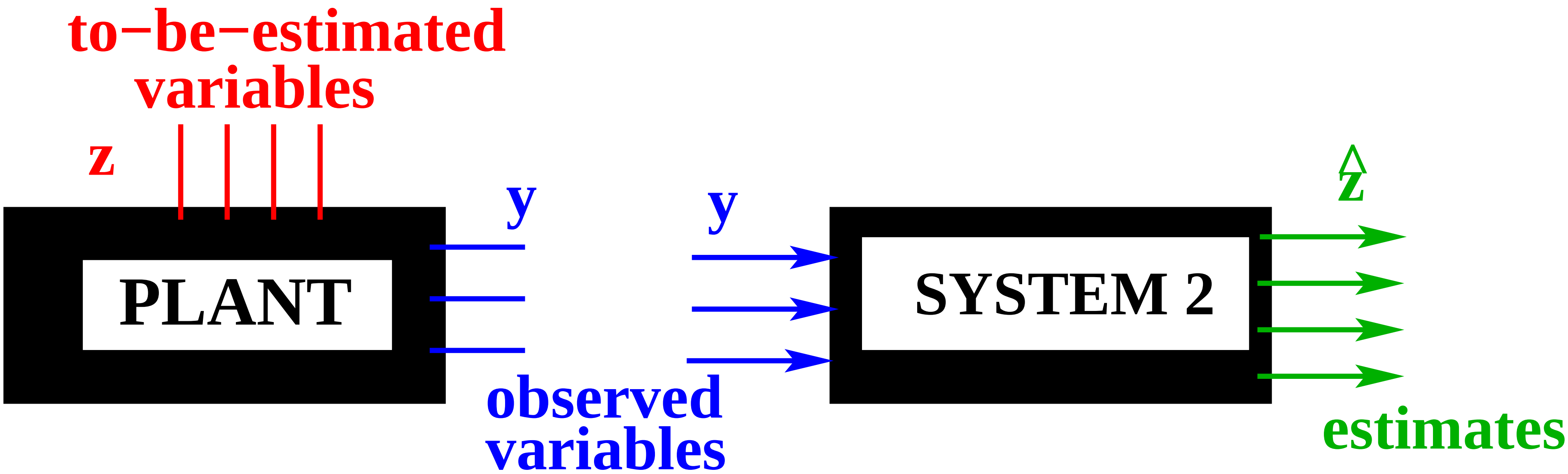

**Consider two LTIDS systems.**

**When is system 2 an observer for the plant?**

**Denote their behavior by**

$$\mathfrak{B}_{\text{plant}} \quad \textbf{and} \quad \hat{\mathfrak{B}}$$

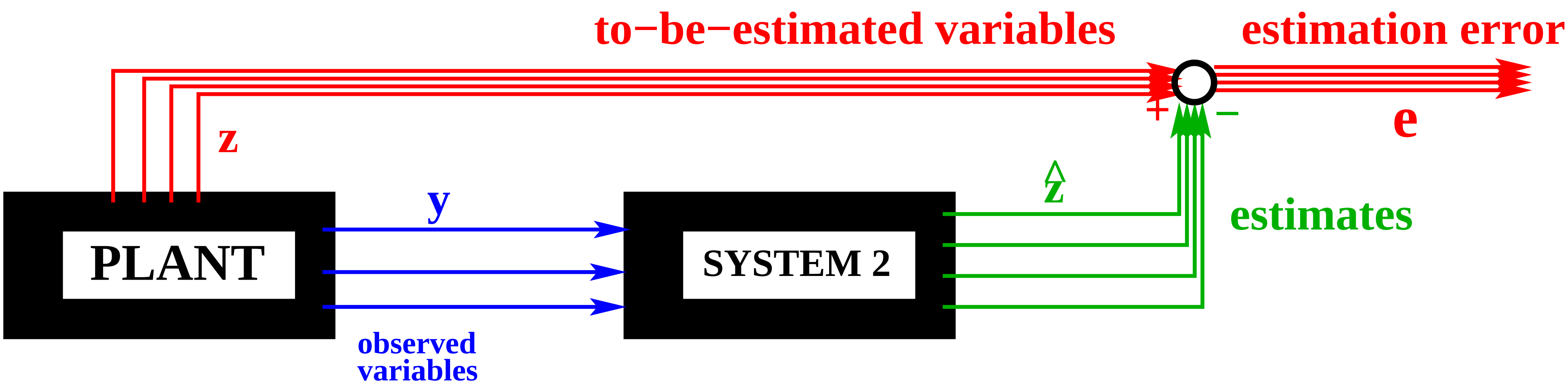
# Observers




**Condition 1**: System 2 simulates the plant, that is

$$\mathfrak{B}_{\text{plant}} \subseteq \hat{\mathfrak{B}}$$

# Observers




**Condition 1:** System 2 simulates the plant, that is

$$\mathfrak{B}_{\text{plant}} \subseteq \hat{\mathfrak{B}}$$

**Condition 2:** Error behavior, $\mathfrak{B}_{\text{error}}$, is autonomous

$\mathfrak{B}_{\text{error}} = \{0\}$, exact observer

$\mathfrak{B}_{\text{error}}$ nilpotent, dead-beat (discr. time)

$\mathfrak{B}_{\text{error}}$ stable, asymptotic observer

# Observers

**Condition 1: System 2 simulates the plant, that is**

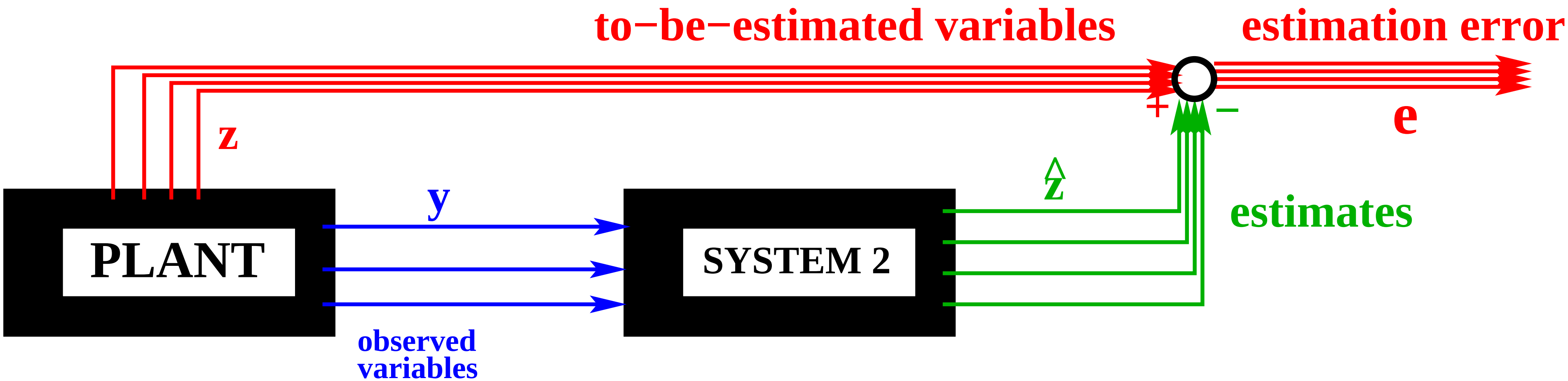
$$\mathfrak{B}_{\text{plant}} \subseteq \hat{\mathfrak{B}}$$

**Condition 2: Error behavior, $\mathfrak{B}_{\text{error}}$, is autonomous**

**These conditions imply that**

1. **it is possible to follow $z$ through $y$,**
2. **once $z(t') = \hat{z}(t')$ for $t' \in [T - \varepsilon, T], \varepsilon > 0$, there holds $z(t) = \hat{z}(t)$ for $t > T$.**

# Observers




**Condition 1**: System 2 **simulates** the plant, that is

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**Condition 2**: Error behavior, $\mathfrak{B}_{\text{error}}$, is **autonomous**

**Condition 3**: WLOG, add $y$ is **free** (‘input’) in $\hat{\mathfrak{B}}$,
$y$ is ‘processed’ in $\hat{\mathfrak{B}}$

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$y$ is 'processed' in $\hat{\mathfrak{B}}$

These conditions are not independent.

1 + 3 ($y$ input) + $\hat{z}$ output $\Rightarrow$ 2

controllability of plant + 2 + 3 $\Rightarrow$ 1

Assume contr. & 3. Then $\mathfrak{B}_{\text{plant}} \subseteq \hat{\mathfrak{B}} \Leftrightarrow$ observer

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**Theorem**: An observer exists if and only if

$\{(z, y) \in \mathfrak{B}_{\text{plant}} \mid y = 0\}$ is autonomous

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**$y$ is 'processed' in $\hat{\mathfrak{B}}$**

**Roughly, observer design $\cong$ finding a cover**

$$\mathfrak{B}_{\text{plant}} \subseteq \hat{\mathfrak{B}}$$

# Observer Design

# Covers

**Essential condition:**

$$\mathfrak{B}_{\text{plant}} \subseteq \hat{\mathfrak{B}}$$

**Easy to find a supsystem, $\mathfrak{B}' \supseteq \mathfrak{B}$, for a given LTIDS $\mathfrak{B}$. For example, from 'kernel representation'**

$$R\left(\frac{d}{dt}\right) w = 0$$

**Then $\mathfrak{B}' \supseteq \mathfrak{B}$ iff $\mathfrak{B}'$ has kernel representation**

$$F\left(\frac{d}{dt}\right) R\left(\frac{d}{dt}\right) w = 0$$

**for some $F \in \mathbb{R}[\xi]^{\bullet\times\bullet}$.**

# Covers

**Plant:**

$$Z\left(\frac{d}{dt}\right) z = Y\left(\frac{d}{dt}\right) y$$

**Observer therefore**

$$F\left(\frac{d}{dt}\right) Z\left(\frac{d}{dt}\right) \hat{z} = F\left(\frac{d}{dt}\right) Y\left(\frac{d}{dt}\right) y$$

**Error dynamics**

$$F\left(\frac{d}{dt}\right) Z\left(\frac{d}{dt}\right) e = 0$$

**Observer conditions: $FZ$ square and non-singular.**

# Covers

Given $Z, Y \in \mathbb{R}[\xi]^{\bullet \times \bullet}$, what can be achieved by $F \in \mathbb{R}[\xi]^{\bullet \times \bullet}$ $(Z, Y) \mapsto (FZ, FY)$ ?

Achievable error dynamics

$$F\left(\frac{d}{dt}\right) Z\left(\frac{d}{dt}\right) e = 0$$

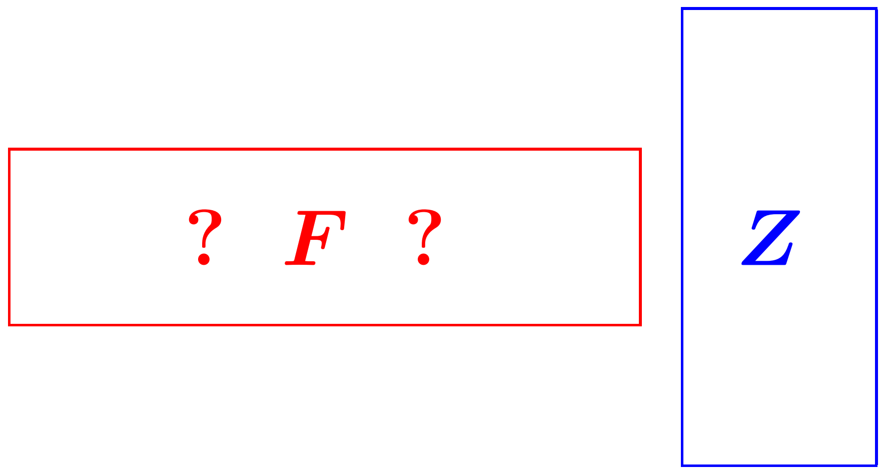
Can the observer be made smoothing?

$$F\left(\frac{d}{dt}\right) Z\left(\frac{d}{dt}\right) \hat{z} = F\left(\frac{d}{dt}\right) Y\left(\frac{d}{dt}\right) y$$

transfer function $(FZ)^{-1}(FY)$

proper, strictly proper, high-frequency roll-off, ...

# Covers




**taking into consideration roll-off of** $(FZ)^{-1}(FY)$

# Error Dynamics

**Assume that in the plant $z$ is observable from $y$.**
**Then $\forall\, r \in \mathbb{R}[\xi]$, monic, $\exists\, F$ such that**

$$\det(FZ) = r$$

$r = 1$ $\rightsquigarrow$ **exact observer**

$r$ **Hurwitz** $\rightsquigarrow$ **asymptotic observer**

$r(\xi) = \xi^d$ $\rightsquigarrow$ **dead-beat observer (discr.-time)**

**Combinable with proper, high-frequency roll-off,**
**provided $\mathrm{degree}(r)$ sufficiently large.**

# Error Dynamics

**Assume $z$ is detectable from $y$. Then for any $r \in \mathbb{R}[\xi]$, monic, with a given Hurwitz factor (representing the unobservable modes) there exists $F$ such that**

$$\det(FZ) = r$$

$r$ **Hurwitz** $\rightsquigarrow$ **asymptotic observer**

**Combinable with proper, high-frequency roll-off, provided $\mathrm{degree}(r)$ sufficiently large.**

## Example

**Autonomous system, $z$, $y$ scalar:**

$$\boldsymbol{R}\left(\frac{d}{dt}\right)\begin{bmatrix} z \\ y \end{bmatrix} = 0$$

$\det(\boldsymbol{R}) \neq 0$.

# Example

**Autonomous system, $z$, $y$ scalar:**

$$R\left(\frac{d}{dt}\right)\begin{bmatrix} z \\ y \end{bmatrix} = 0$$

$\det(R) \neq 0$. **Observability $\Rightarrow$ representation**

$$Y\left(\frac{d}{dt}\right) y = 0, z = Z\left(\frac{d}{dt}\right) y$$

$Y, Z \in \mathbb{R}[\xi]$

# Example

$$Y\left(\frac{d}{dt}\right)y = 0, z = Z\left(\frac{d}{dt}\right)y$$

**Observer:**

$$\pi_1\left(\frac{d}{dt}\right)\hat{z} = \left[\pi_1\left(\frac{d}{dt}\right)Z\left(\frac{d}{dt}\right) + \pi_2\left(\frac{d}{dt}\right)Y\left(\frac{d}{dt}\right)\right]y$$

$\pi_1$ **given, sufficiently high degree, roots arbitrary**

**arbitrary high roll-off by chosing** $\pi_2$

$\rightsquigarrow$ **simple polynomial algebra.**

# Example

$$Y\left(\frac{d}{dt}\right)y = 0, z = Z\left(\frac{d}{dt}\right)y$$

**Observer:**

$$\pi_1\left(\frac{d}{dt}\right)\hat{z} = \left[\pi_1\left(\frac{d}{dt}\right)Z\left(\frac{d}{dt}\right) + \pi_2\left(\frac{d}{dt}\right)Y\left(\frac{d}{dt}\right)\right]y$$

**$\pi_1$ given, sufficiently high degree, roots arbitrary**

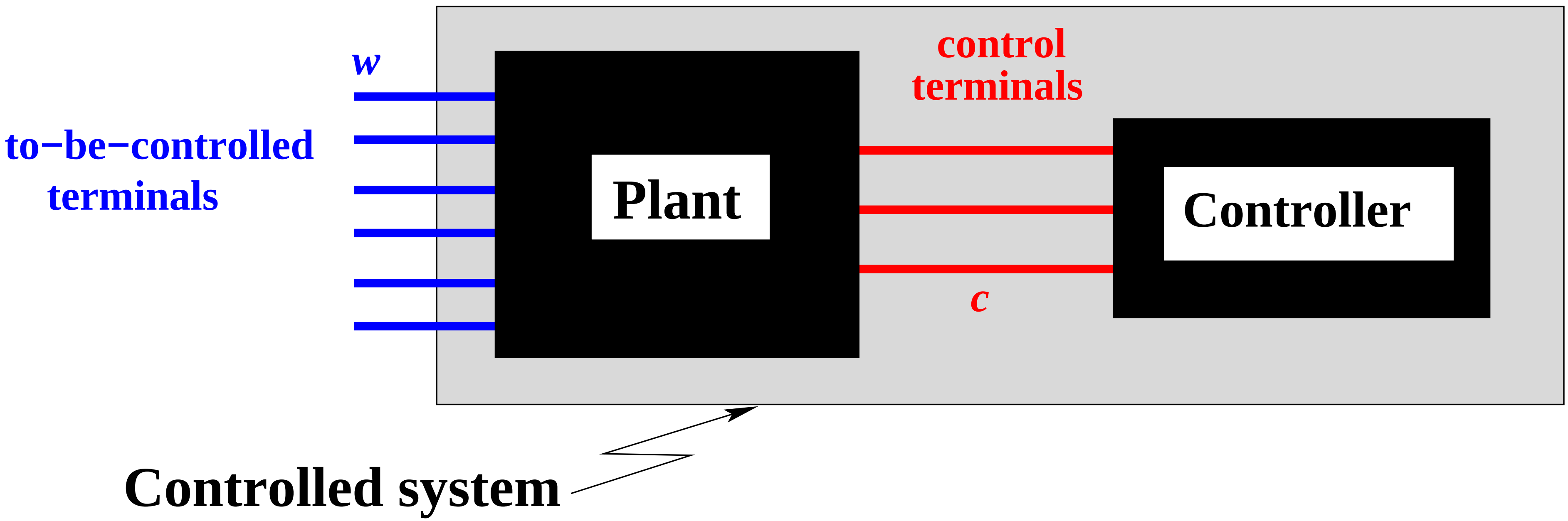
**arbitrary high roll-off by chosing $\pi_2$**

**$\rightsquigarrow$ simple polynomial algebra.**

**When plant is autonomous, the pole placement combinable with arbitrary roll-off**

# Duality with Control

# Control in a Behavioral Setting



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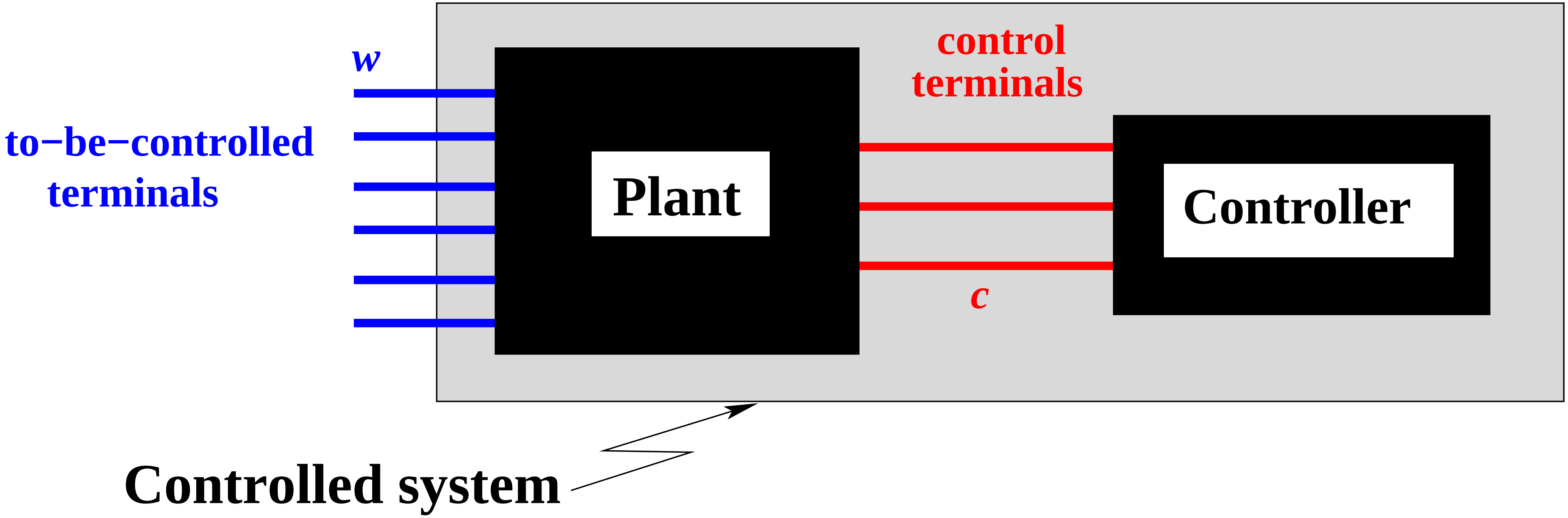


**Behavior of to-be-controlled variables, before controller is applied: $\mathfrak{B}_{\text{plant}}$, after: $\mathfrak{B}_{\text{controlled}}$**

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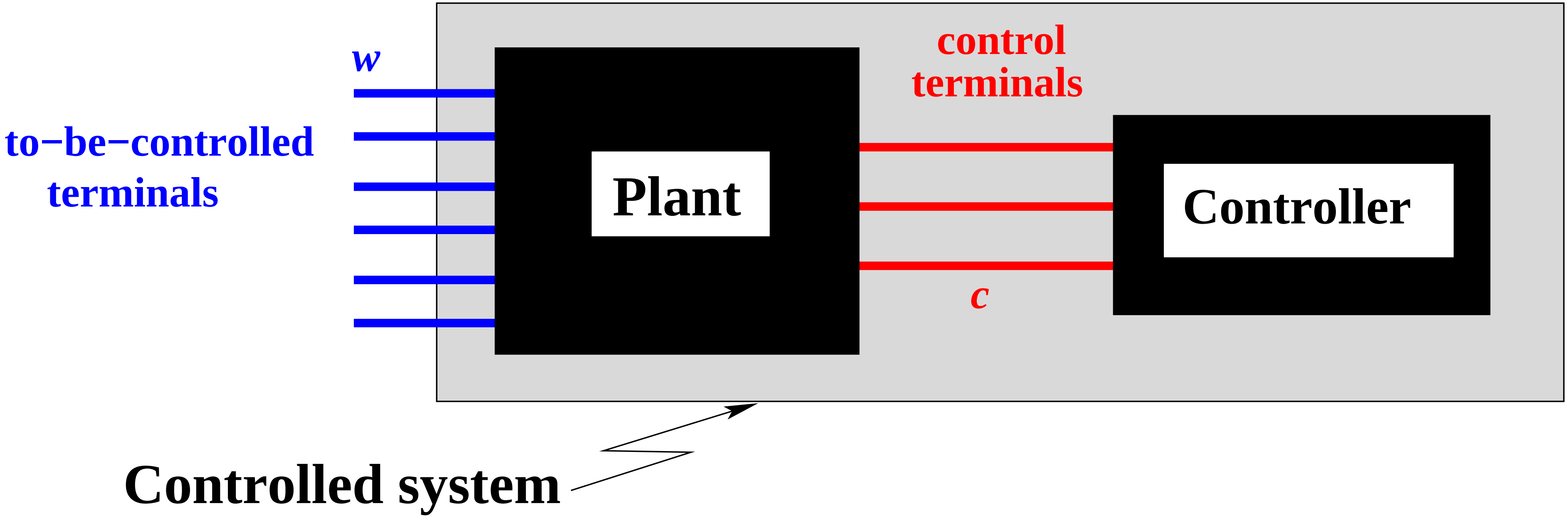
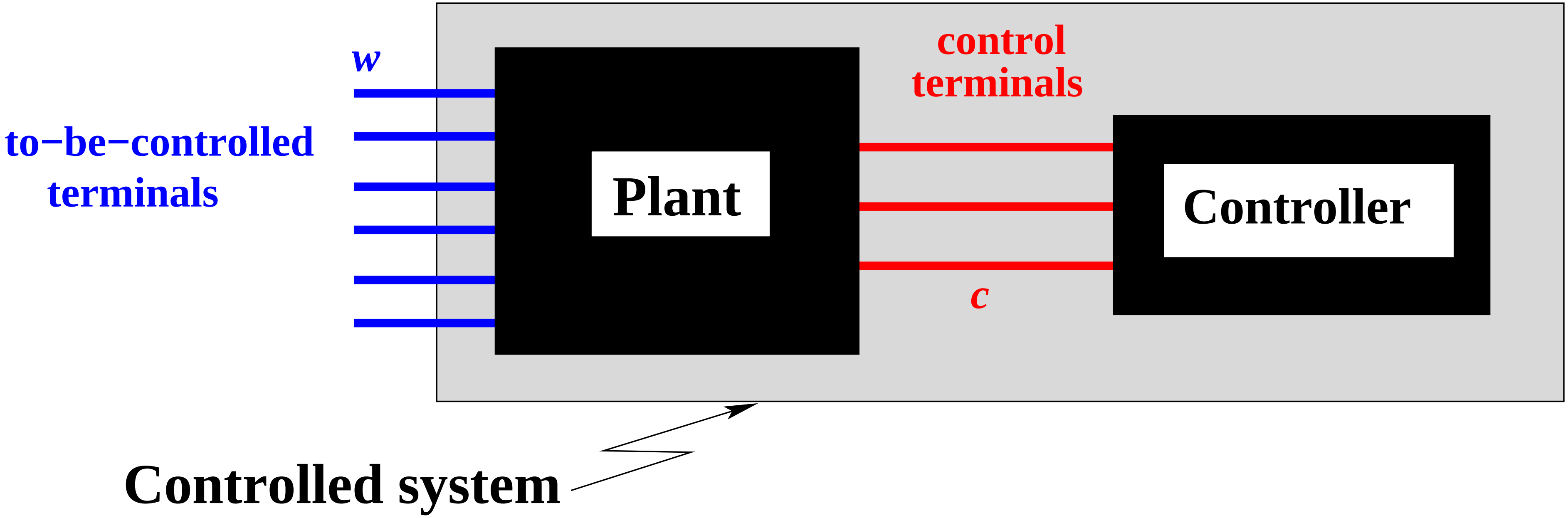



**Behavior of to-be-controlled variables, before controller is applied: $\mathfrak{B}_{\text{plant}}$, after: $\mathfrak{B}_{\text{controlled}}$**

**Obviously,** $\mathfrak{B}_{\text{controlled}} \subseteq \mathfrak{B}_{\text{plant}}$

# Control in a Behavioral Setting




**Behavior of to-be-controlled variables, before controller is applied: $\mathfrak{B}_{\text{plant}}$, after: $\mathfrak{B}_{\text{controlled}}$**

**Obviously,** $\mathfrak{B}_{\text{controlled}} \subseteq \mathfrak{B}_{\text{plant}}$

**If $w$ is observable from $c$ in the plant, then every such $\mathfrak{B}_{\text{controlled}}$ is implementable.**

# Duality

## Given $\mathfrak{B}_{\mathrm{plant}}$, LTIDS

# Duality

**Given $\mathfrak{B}_{\rm plant}$, LTIDS**

**Control $\leadsto$ find a subsystem**

$$\mathfrak{B} \subseteq \mathfrak{B}_{\rm plant}$$

**that meets controller specs.**

**Given**

$$R\left(\frac{d}{dt}\right) w = 0$$

$$C\left(\frac{d}{dt}\right) w = 0$$

**'Squaring up' $R$ to $\begin{bmatrix} R \\ C \end{bmatrix}$**

# Duality

**Given $\mathfrak{B}_{\text{plant}}$, LTIDS**

**Control $\rightsquigarrow$ find a subsystem**

$$\mathfrak{B} \subseteq \mathfrak{B}_{\text{plant}}$$

**that meets controller specs.**

**Observer $\rightsquigarrow$ find a supsystem**

$$\mathfrak{B} \supseteq \mathfrak{B}_{\text{plant}}$$

**that meets observer specs.**

# Duality

**Given $\mathfrak{B}_{\text{plant}}$, LTIDS**

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$$\mathfrak{B} \supseteq \mathfrak{B}_{\text{plant}}$$

**that meets observer specs.**

**Controllers mean less, Observers mean more**

# Extensions

- **Systems defined by rational (rather than polynomial) 'symbols'**
- **Least squares, $\mathcal{H}_\infty$, ...**
- **nD systems, PDEs**

**Details & copies of frames are available from/at**

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`http://www.esat.kuleuven.be/∼jwillems`

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