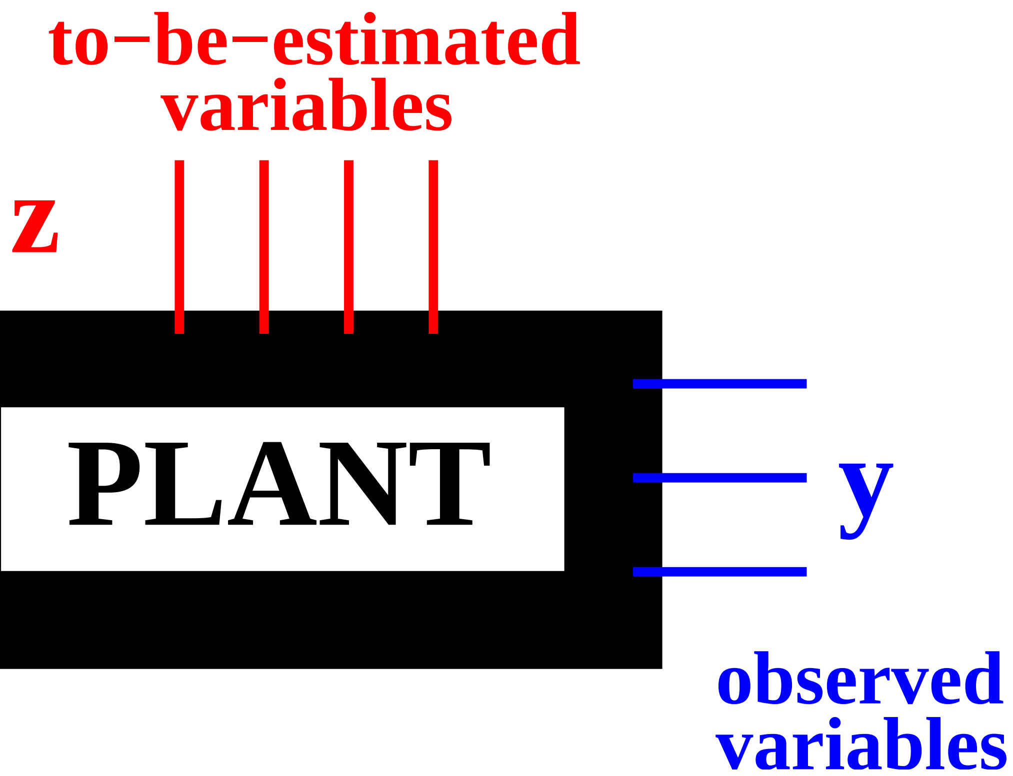
A NEW LOOK AT OBSERVERS

Jan C. Willems, K.U. Leuven, Belgium

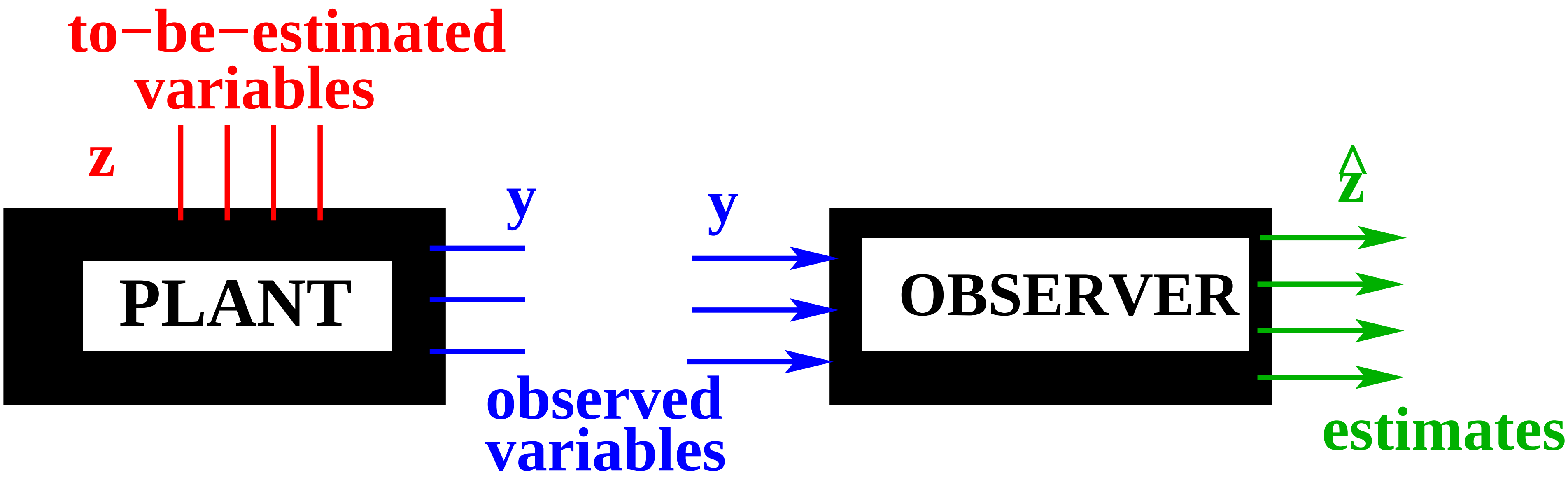
MMAR, Szczecin, Poland **August 27, 2007**

Introduction

THEME

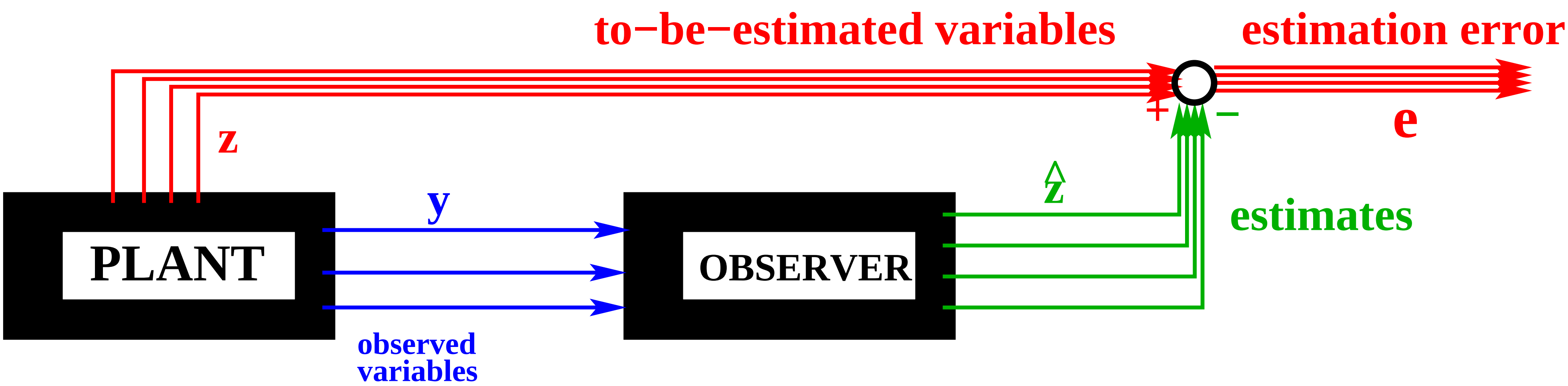


THEME



to−be−estimated
variables
z
PLANT
y
y
observed
variables
OBSERVER
$\hat{z}$
estimates

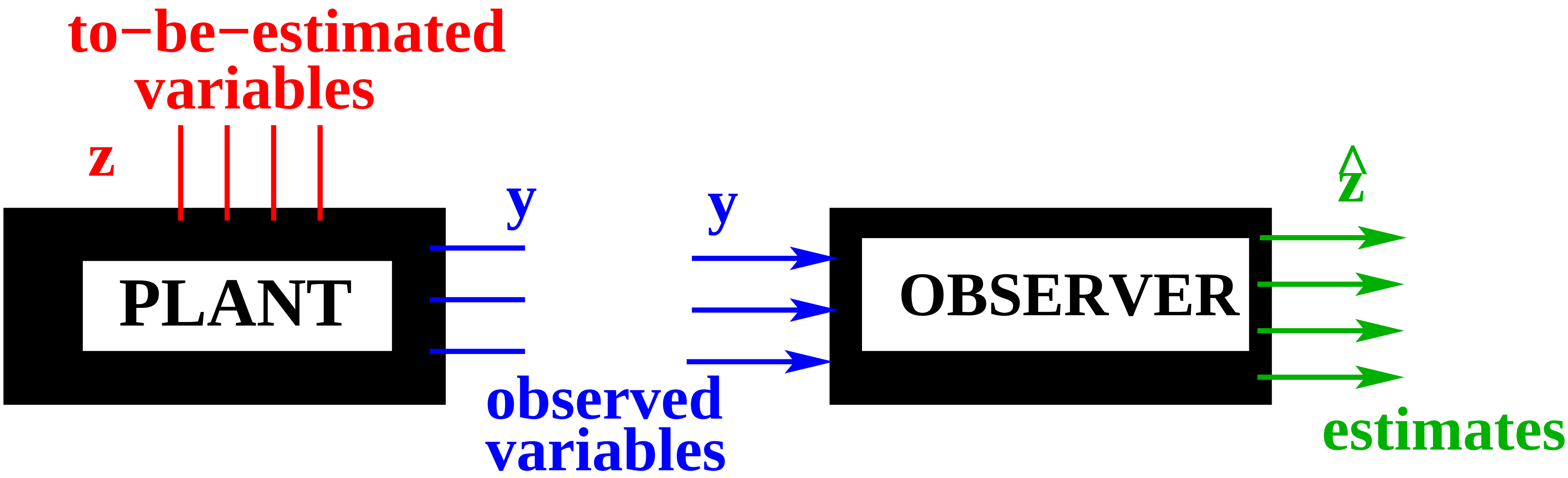
THEME



!! Keep estimation error

small, zero, convergent to zero, etc. !!

THEME



- **How to model the relation between the observed and the to-be-estimated variables ?**
- **Find the observer/filter algorithm !**

Joint work with Jochen Trumpf

Australian National University

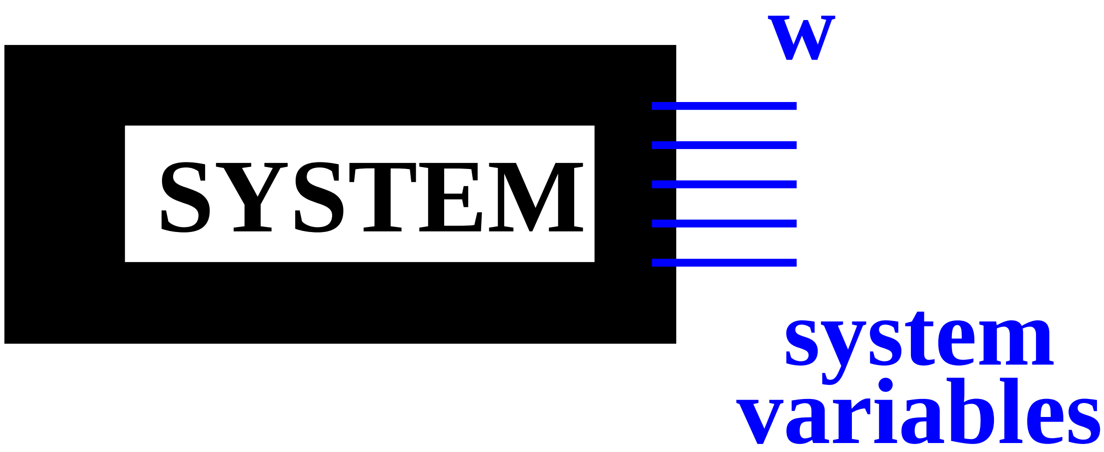
Message

Observers mean more

Controllers mean less

Systems & Their Properties

Behaviors



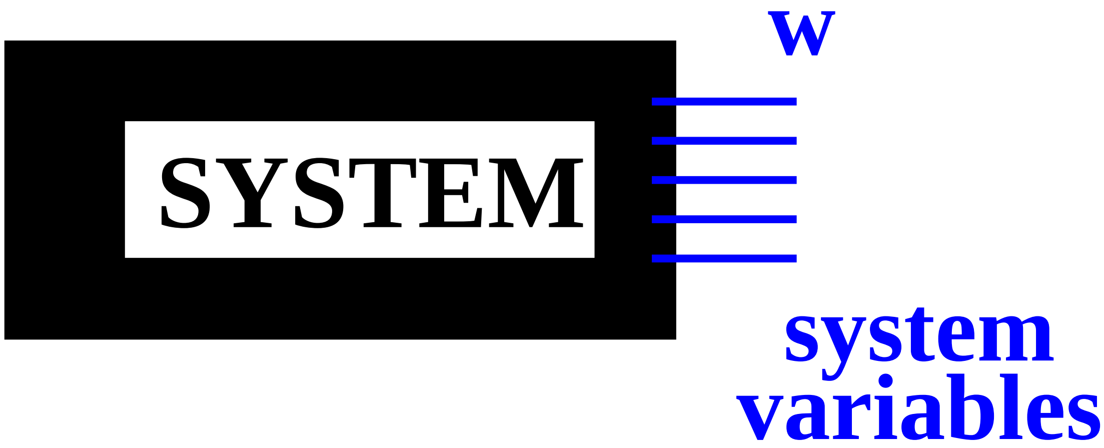
A *dynamical system* is $\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$

$\mathbb{T} \subseteq \mathbb{R}$ **'time-set'**

$\mathbb{W}$ **'signal space'**

$\mathfrak{B} \subseteq \mathbb{W}^{\mathbb{T}}$ **'behavior'**

Behaviors



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Consider $w : \mathbb{T} \longrightarrow \mathbb{W}$

$w \in \mathfrak{B}$ **the model allows the trajectory** w

$w \notin \mathfrak{B}$ **the model forbids the trajectory** w

Behaviors

A *dynamical system* is $\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$

$\mathbb{T} \subseteq \mathbb{R}$	**'time-set'**	**today** $\mathbb{T} = \mathbb{R}$
$\mathbb{W}$	**'signal space'**	**today** $\mathbb{W} = \mathbb{R}^{\mathtt{w}}$
$\mathfrak{B} \subseteq \mathbb{W}^{\mathbb{T}}$	**'behavior'**	**today LTIDS**

Behaviors

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Linear time-invariant differential system (LTIDS):

$\mathfrak{B} =$ **all solutions of**

$$R\left(\frac{d}{dt}\right) w = 0$$

where $R \in \mathbb{R}[\xi]^{\bullet \times \mathtt{w}}$

Behaviors

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The behavior is all there is !

LTIDSs

Let $\mathfrak{L}^\bullet$ denote the set LTIDSs.

$\mathfrak{L}^\bullet$ is closed under projection

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$$R_1\left(\frac{d}{dt}\right)w_1 = R_2\left(\frac{d}{dt}\right)w_2 \qquad (*)$$

$R_1, R_2 \in \mathbb{R}[\xi]^{\bullet\times\bullet}$.

LTIDSs

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$$\mathfrak{B}_1 := \{w_1 \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^\bullet) \mid \exists w_2 \text{ such that } (*)\}$$

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Then $\mathfrak{B}_1 \in \mathfrak{L}^\bullet$ $\qquad \rightsquigarrow R\left(\frac{d}{dt}\right)w = 0$

LTIDSs

Let $\mathfrak{L}^\bullet$ denote the set LTIDSs.

$\mathfrak{L}^\bullet$ is closed under projection

$$F\left(\frac{d}{dt}\right)w = 0, F \in \mathbb{R}[\xi]^{\bullet\times \mathtt{w}}$$

is a consequence of

$$R\left(\frac{d}{dt}\right)w = 0, R \in \mathbb{R}[\xi]^{\bullet\times \mathtt{w}} \text{ if}$$

$$R\left(\frac{d}{dt}\right)w = 0 \;\; \Rightarrow \;\; F\left(\frac{d}{dt}\right)w = 0$$

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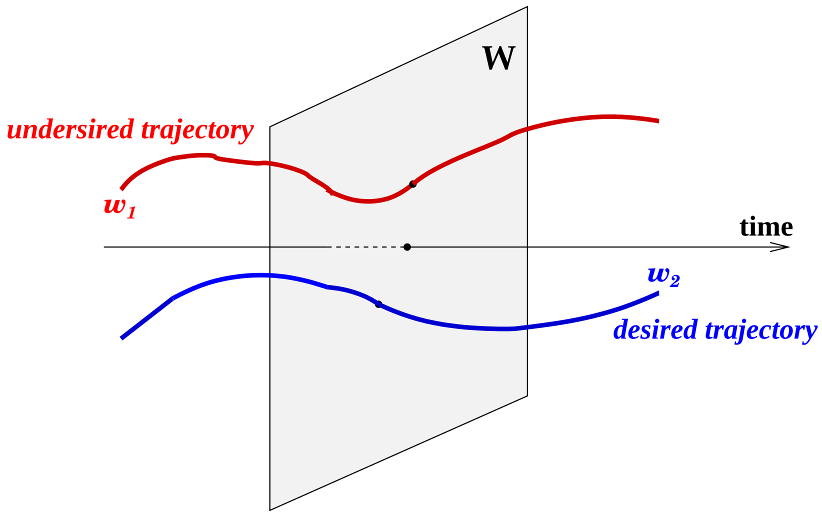
Consequence $\Leftrightarrow$ $F = F'R$

System Properties

- **Controllable**
- **Stabilizable**
- **Autonomous**
- **Stable**

System Properties

$\Sigma = (\mathbb{R}, \mathbb{W}, \mathfrak{B})$ **is** **controllable** $:\Leftrightarrow$



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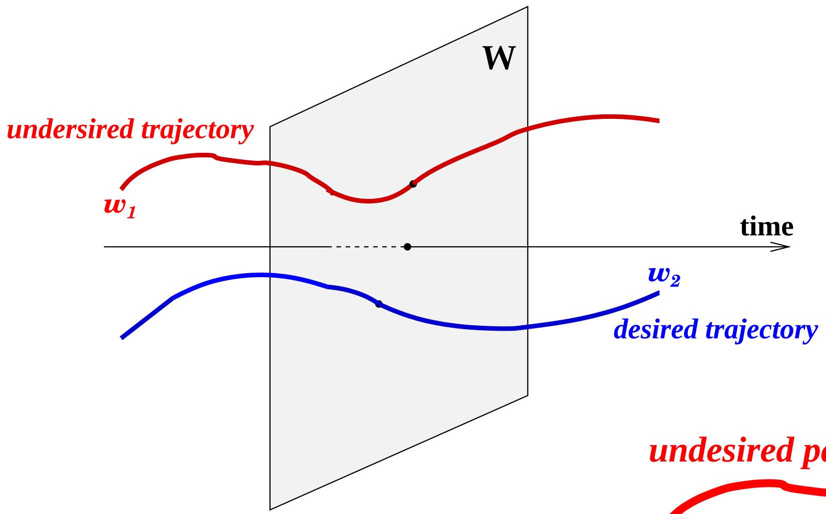
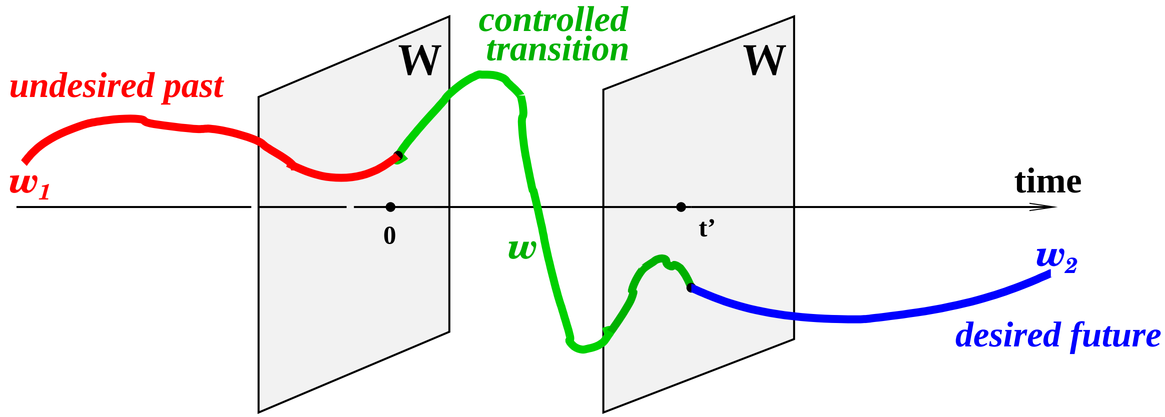
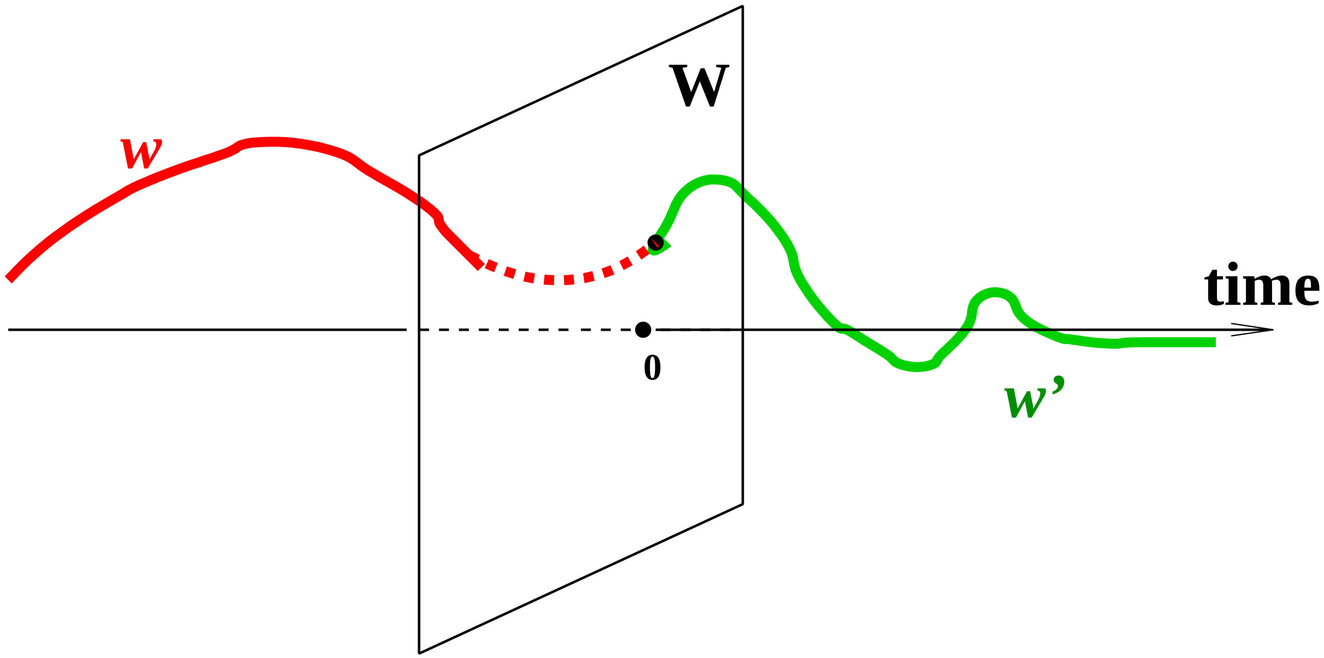




Behavioral controllability of a dynamical system

System Properties

$\Sigma = (\mathbb{R}, \mathbb{W}, \mathfrak{B})$ **is stabilizable :**$\Leftrightarrow$



System Properties

$\Sigma = (\mathbb{R}, \mathbb{W}, \mathfrak{B})$ is **autonomous** $:\Leftrightarrow$

$$w_1, w_2 \in \mathfrak{B} \text{ and } w_1(t) = w_2(t) \text{ for } t < 0$$
$$\Rightarrow w_1(t) = w_2(t) \text{ for } t \geq 0$$

'past implies future'

stable $:\Leftrightarrow \quad w \in \mathfrak{B} \Rightarrow w(t) \to 0 \text{ as } t \to \infty$

System Properties

$$\boldsymbol{R}\left(\frac{d}{dt}\right)\boldsymbol{w} = \mathbf{0}$$

defines a controllable system iff

$\mathrm{rank}\left(\boldsymbol{R}\left(\boldsymbol{\lambda}\right)\right)$ **is the same for all** $\boldsymbol{\lambda} \in \mathbb{C}$

a stabilizable one .... $\lambda \in$ **the closed RHP**

System Properties

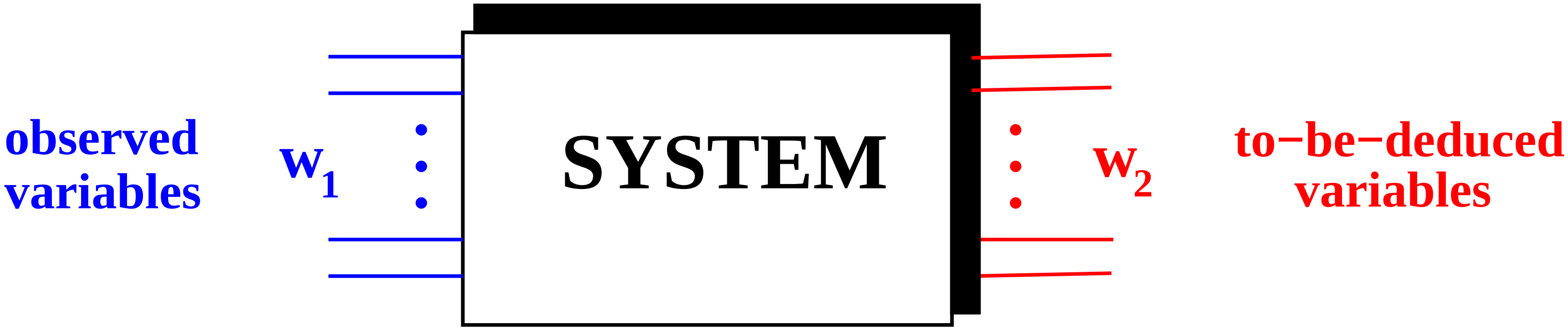
$$\boldsymbol{R}\left(\frac{d}{dt}\right)\boldsymbol{w} = \mathbf{0}$$

defines an autonomous system iff

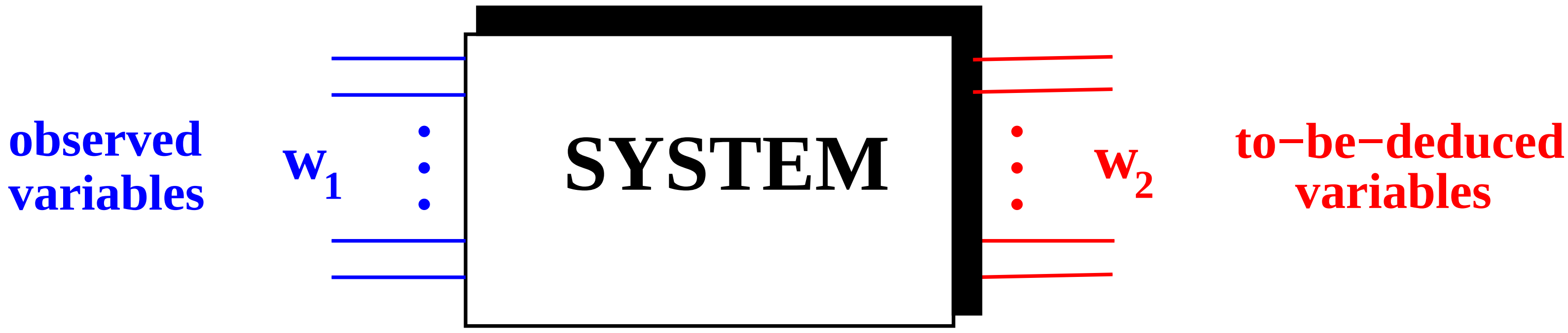
$\boldsymbol{R}(\boldsymbol{\lambda})$ full column rank $\forall$ but finite number $\boldsymbol{\lambda} \in \mathbb{C}$

a stable one ... $\lambda \in$ the closed LHP

Relations Among Variables



Relations Among Variables

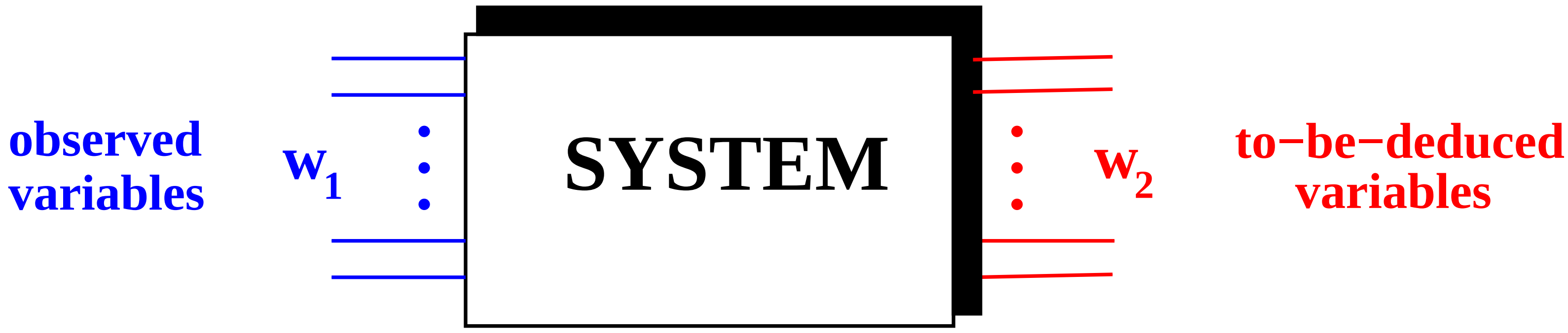


w_1 **is** **observable** **from** w_2 **in** $\Sigma = (\mathbb{T}, \mathbb{W}_1 \times \mathbb{W}_2, \mathfrak{B})$ $:\Leftrightarrow$

$$\left(w_1', w_2'\right), \left(w_1', w_2''\right) \in \mathfrak{B} \text{ and } w_1' = w_1''$$
$$\Rightarrow w_2' = w_2''$$

Observed trajectory implies the to-be-deduced one

Relations Among Variables

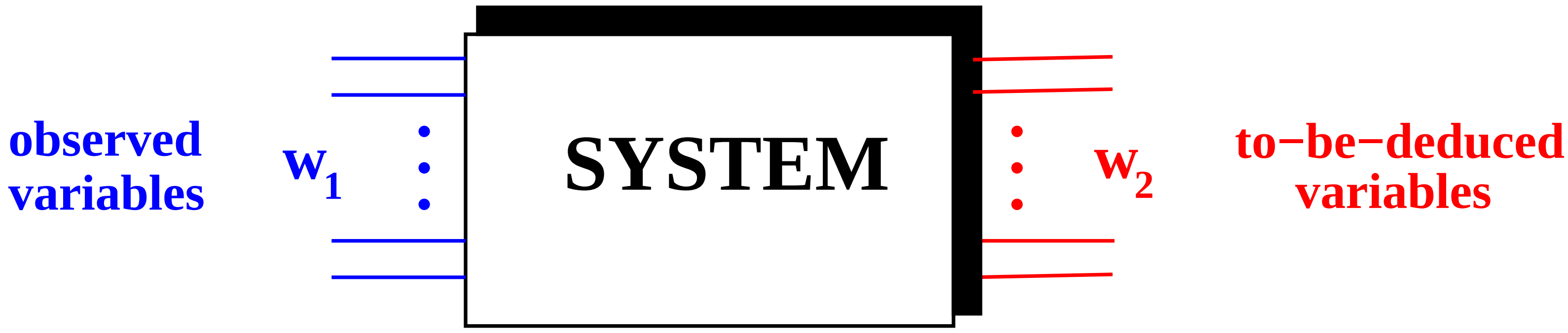


w_1 **is** **detectable** **from** w_2 **in** $\Sigma = (\mathbb{T}, \mathbb{W}_1 \times \mathbb{W}_2, \mathfrak{B})$ $:\Leftrightarrow$

$$\left(w_1', w_2'\right), \left(w_1', w_2''\right) \in \mathfrak{B} \text{ and } w_1' = w_1''$$
$$\Rightarrow w_2'(t) - w_2''(t) \longrightarrow 0 \text{ as } t \longrightarrow \infty$$

Observed trajectory implies the to-be-deduced one asymptotically

Relations Among Variables



$$R_1\left(\frac{d}{dt}\right) w_1 = R_2\left(\frac{d}{dt}\right) w_2$$

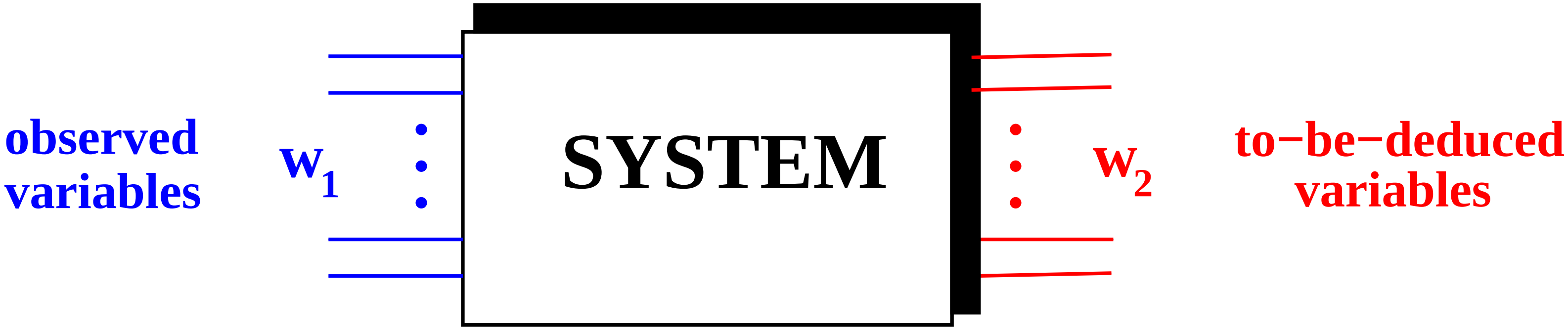
defines an observable system iff

$$R_2(\lambda) \text{ has full column rank } \forall\, \lambda \in \mathbb{C}$$

defines a detectable system iff

...$\forall$ but finite number $\lambda \in$ closed RHP

Relations Among Variables



$$R_1\left(\frac{d}{dt}\right)w_1 = R_2\left(\frac{d}{dt}\right)w_2$$

observable iff there are 'consequences'

$$w_2 = F\left(\frac{d}{dt}\right)w_1$$

detectable iff there are 'consequences'

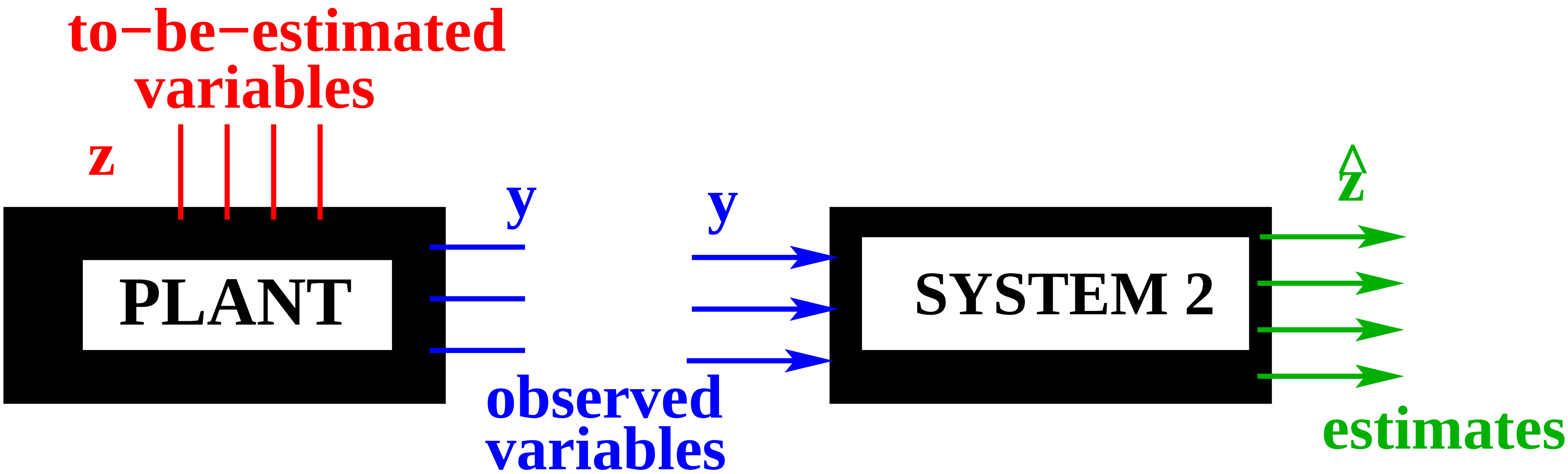
$$H\left(\frac{d}{dt}\right)w_2 = F\left(\frac{d}{dt}\right)w_1, \text{ with } H \text{ 'Hurwitz'}$$

System properties hold beyond the state space setting,

they are representation independent

What is an observer ?

Observers



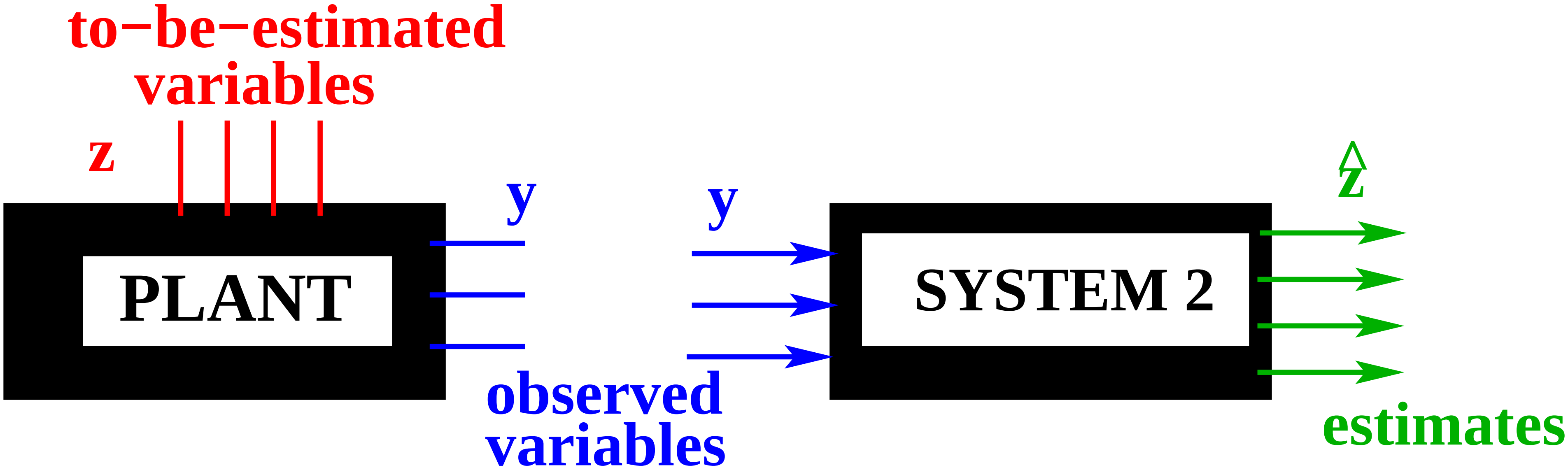
Consider two LTIDS systems.

When is system 2 an observer for the plant?

Denote behaviors by

$$\mathfrak{B}_{\text{plant}} \quad \textbf{and} \quad \hat{\mathfrak{B}}$$

Observers



Condition 1: System 2 simulates the plant, that is

$$\mathfrak{B}_{\text{plant}} \subseteq \hat{\mathfrak{B}}$$

Observers

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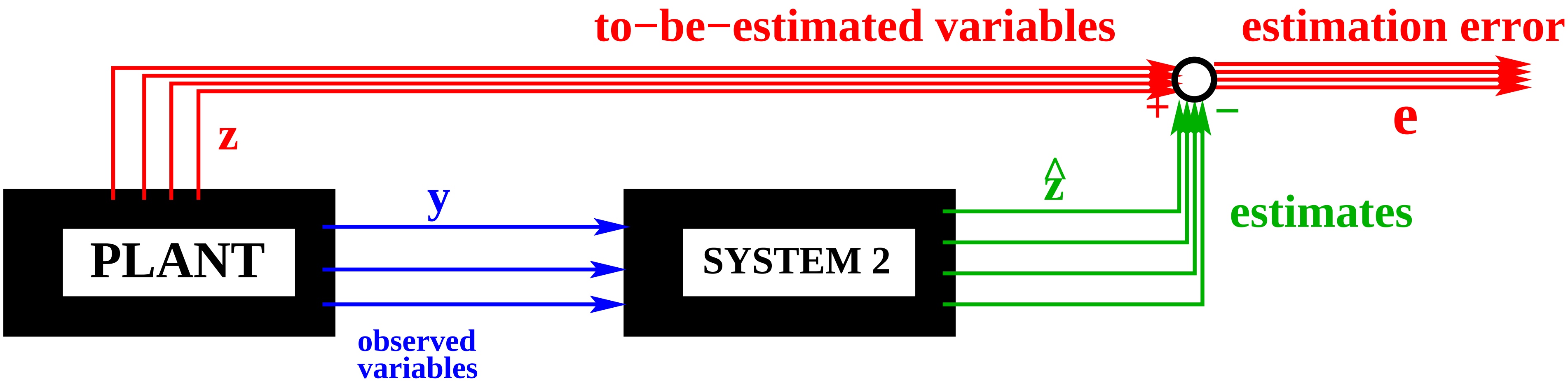

Condition 2: Error behavior, $\mathfrak{B}_{\text{error}}$, is autonomous.

$\mathfrak{B}_{\text{error}} = \{0\}$, **exact observer**

$\mathfrak{B}_{\text{error}}$ **nilpotent, dead-beat observer**

$\mathfrak{B}_{\text{error}}$ **stable, asymptotic observer**

Observers

Condition 1: System 2 simulates the plant, that is

$$\mathfrak{B}_{\text{plant}} \subseteq \hat{\mathfrak{B}}$$

Condition 2: Error behavior, $\mathfrak{B}_{\text{error}}$, is autonomous.

These conditions imply that

1. **it is possible to follow z through y,**
2. **once $z(t') = \hat{z}(t')$ for $t' \in [T - \varepsilon, T], \varepsilon > 0$, there holds $z(t) = \hat{z}(t)$ for $t > T$.**

Observers

Condition 1: System 2 simulates the plant, that is

$$\mathfrak{B}_{\text{plant}} \subseteq \hat{\mathfrak{B}}$$

Condition 2: Error behavior, $\mathfrak{B}_{\text{error}}$, is autonomous.

Condition 3: WLOG, add y is free ('input') in $\hat{\mathfrak{B}}_{\text{plant}}$

Observers

Condition 1: System 2 simulates the plant, that is

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These conditions are not independent.

1 + 3 $\Rightarrow$ 2

controllability of plant + 2 + 3 $\Rightarrow$ 1

Observers

Condition 1: System 2 **simulates** the plant, that is

$$\mathfrak{B}_{\mathrm{plant}} \subseteq \hat{\mathfrak{B}}$$

Condition 2: Error behavior, $\mathfrak{B}_{\mathrm{error}}$, is autonomous.

Condition 3: WLOG, add y is free ('input') in $\hat{\mathfrak{B}}_{\mathrm{plant}}$

Theorem: An observer exists if and only if

$$\{(z, y) \in \mathfrak{B}_{\mathrm{plant}} \mid y = 0\} \text{ is autonomous}$$

Covers

It is easy to find covers. For example, if $\mathfrak{B}$ is given in 'kernel representation'

$$R\left(\frac{d}{dt}\right)w = 0$$

Then $\mathfrak{B}' \supseteq \mathfrak{B}$ iff $\mathfrak{B}'$ has a kernel representation

$$F\left(\frac{d}{dt}\right)R\left(\frac{d}{dt}\right)w = 0$$

for some $F \in \mathbb{R}[\xi]^{\bullet\times\bullet}$.

Covers

Plant:

$$Z\left(\frac{d}{dt}\right) z = Y\left(\frac{d}{dt}\right) y$$

Observer therefore

$$F\left(\frac{d}{dt}\right) Z\left(\frac{d}{dt}\right) \hat{z} = F\left(\frac{d}{dt}\right) Y\left(\frac{d}{dt}\right) y$$

Error dynamics

$$F\left(\frac{d}{dt}\right) Z\left(\frac{d}{dt}\right) e = 0$$

Observer conditions require that FZ is square.

Covers

Given $Z, Y \in \mathbb{R}[\xi]^{\bullet\times\bullet}$, what can be achieved by 'squaring down' Z to FZ?

Achievable error dynamics?

$$F\left(\frac{d}{dt}\right) Z\left(\frac{d}{dt}\right) e = 0$$

Can the observer be made smoothing?

$$F\left(\frac{d}{dt}\right) Z\left(\frac{d}{dt}\right) \hat{z} = F\left(\frac{d}{dt}\right) Y\left(\frac{d}{dt}\right) y$$

$(FZ)^{-1}(FY)$

proper, strictly proper, high-frequency roll-off,

Error Dynamics

Assume that in the plant z is observable from y. Then for any $r \in \mathbb{R}[\xi]$, monic, there exists F such that

$$\det(FZ) = r$$

$r = 1 \rightsquigarrow$ **exact observer**

r **Hurwitz** $\rightsquigarrow$ **asymptotic observer**

$r(\xi) = \xi^{\mathrm{n}} \rightsquigarrow$ **dead-beat observer (discrete-time)**

Combinable with proper, high-frequency roll-off, provided $\mathrm{degree}(r)$ **sufficiently large.**

Error Dynamics

Assume that in the plant z is detectable from y. Then for any $r \in \mathbb{R}[\xi]$, monic, with a given Hurwitz factor (representing the unobservable modes) there exists F such that

$$\det(FZ) = r$$

r **Hurwitz** $\rightsquigarrow$ **asymptotic observer**

Combinable with proper, high-frequency roll-off, provided $\mathrm{degree}(r)$ **sufficiently large.**

Example

Autonomous system, z, y scalar:

$$\begin{bmatrix} Z\left(\frac{d}{dt}\right) & Y\left(\frac{d}{dt}\right) \end{bmatrix} \begin{bmatrix} z \\ y \end{bmatrix} = 0$$

$$\det\left(\begin{bmatrix} Z & Y \end{bmatrix}\right) \neq 0.$$

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$$\det\left(\begin{bmatrix} Z & Y \end{bmatrix}\right) \neq 0.$$

Assume observability $\Rightarrow$ representation

$$Y\left(\frac{d}{dt}\right) y = 0, z = Z\left(\frac{d}{dt}\right) y$$

Example

$$Y\left(\frac{d}{dt}\right)y = 0, z = Z\left(\frac{d}{dt}\right)y$$

Observer:

$$\pi_1\left(\frac{d}{dt}\right)\hat{z} = \left[\pi_1\left(\frac{d}{dt}\right)Z\left(\frac{d}{dt}\right) + \pi_2\left(\frac{d}{dt}\right)Y\left(\frac{d}{dt}\right)\right]y$$

Design with roll-off is simple polynomial algebra.

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Design with roll-off is simple polynomial algebra.

Details & copies of frames are available from/at

`Jan.Willems@esat.kuleuven.be`

`http://www.esat.kuleuven.be/∼jwillems`

Thank you

Thank you

Thank you

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