DISSIPATIVE SYSTEMS

Jan C. Willems, K.U. Leuven, Belgium

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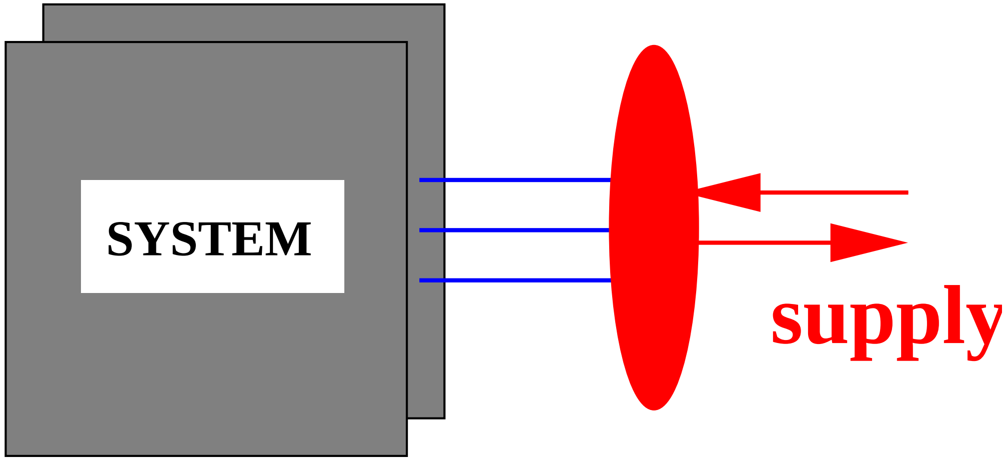
Introduction

Theme



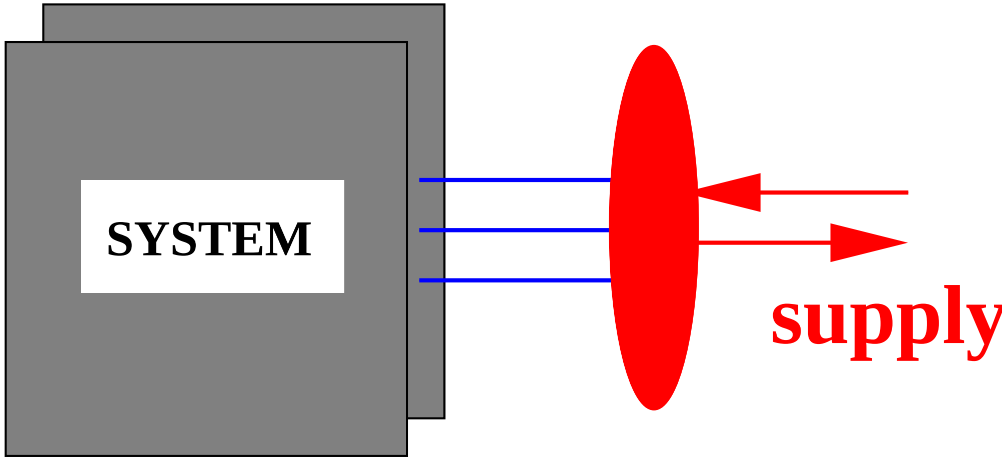
Supply rate:

power, mass-flow rate,

rate of entropy production, information rate,

a quantity used to prove stability, robustness, ...

Theme



A system is **dissipative** if it absorbs supply,

netto = in − out

conservative if netto absorption is zero

Theme

A system is dissipative if it absorbs supply, netto

Dissipative $\cong$

rate of change in storage $\leq$ **supply rate**

Theme

A system is dissipative if it absorbs supply, netto

Dissipative $\cong$

rate of change in storage $\leq$ supply rate

- **Formalize !**
- **Given supply dynamics, what is the storage ?**
- **Does a storage function exist ? Is it unique ?**
- **Characterize set of storage functions !**

Thx !

Dissipative systems

run as a red thread through my scientific life

I owe a lot to many co-workers

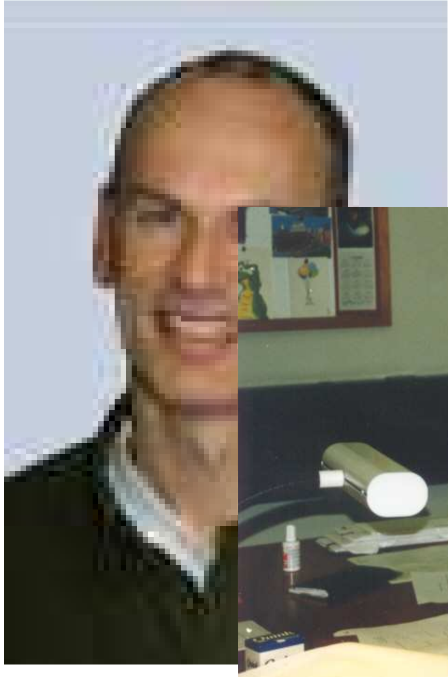
Thx !

Roger Brockett

Arjan van der Schaft

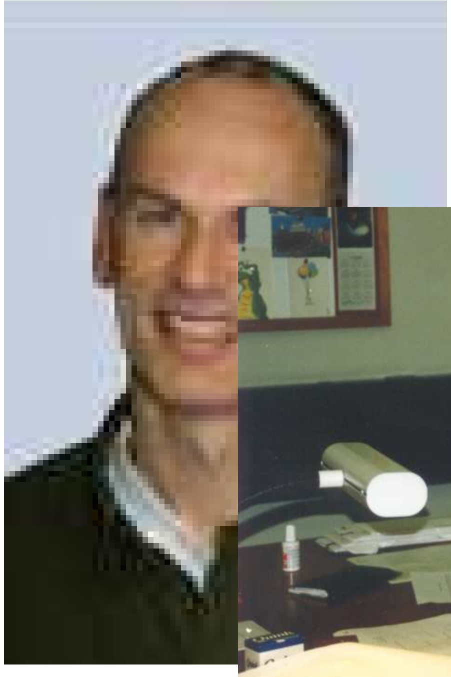
Thx !

Thx !



Siep Weiland

Thx !



Kiyotsugu Takaba

Thx !

Harry Trentelman

Thx !

Paula Rocha

Thx !

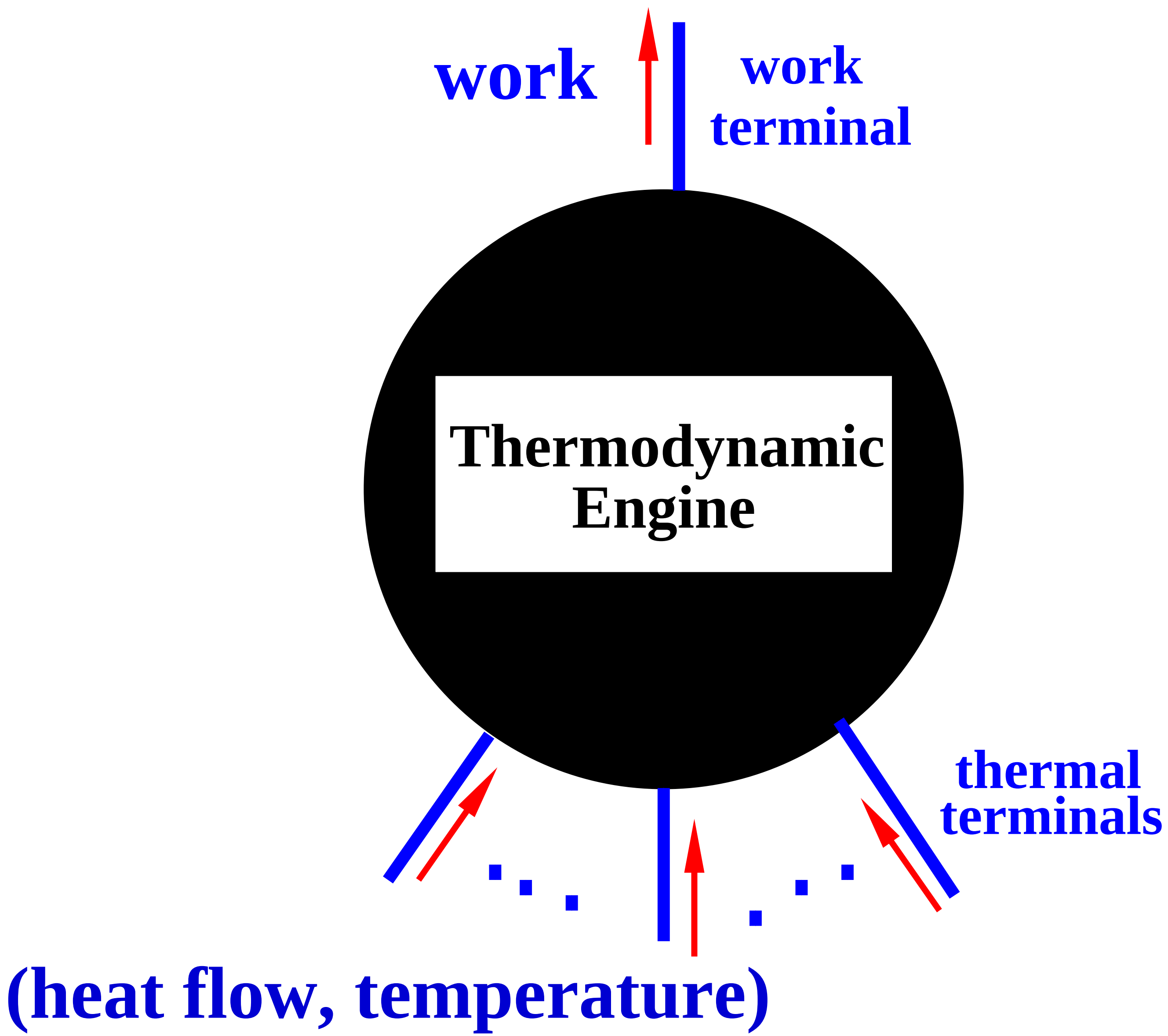
Shiva Shankar

Thx !

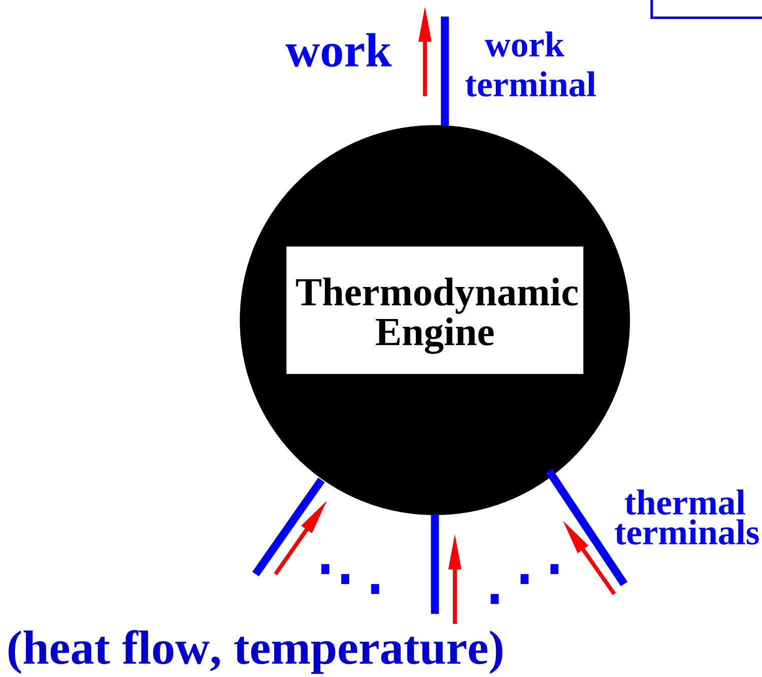
Harish Pillai

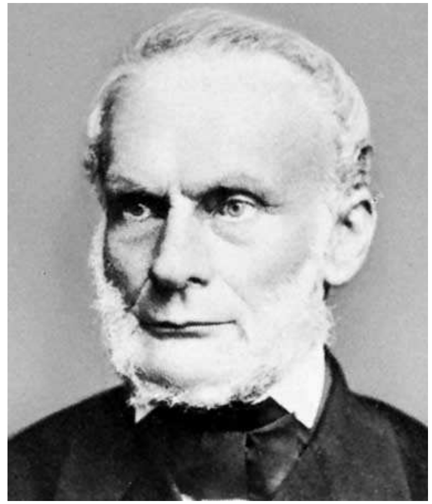
A bit of history

History & Roots



History & Roots



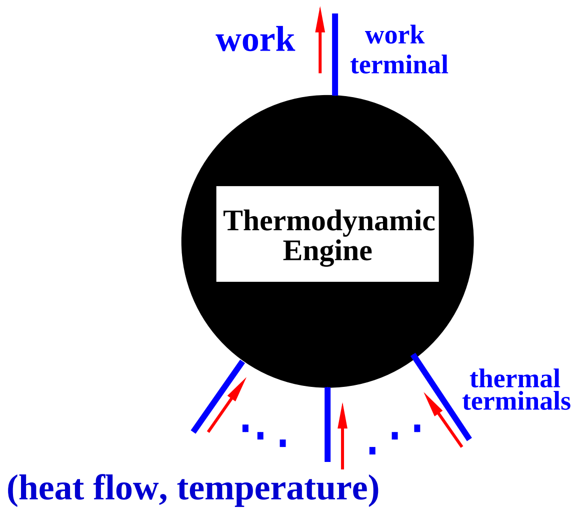


Rudolf Clausius

First and second law of thermodynamics

are statements about dissipativity of open systems

History & Roots



conservative w.r.t. $\left(\sum_{\text{terminals}} \text{heat flows}\right) - \text{work}$

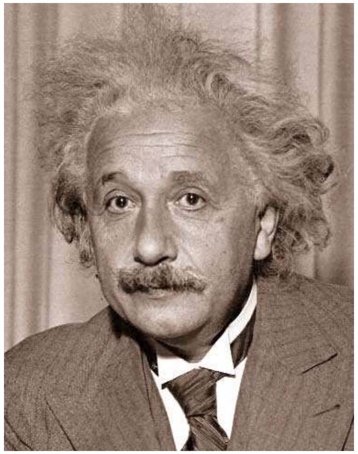
Storage = Internal Energy

dissipative w.r.t. $\sum_{\text{terminals}} -\dfrac{\text{heat flows}}{\text{temperatures}}$

Storage = $-$ Entropy

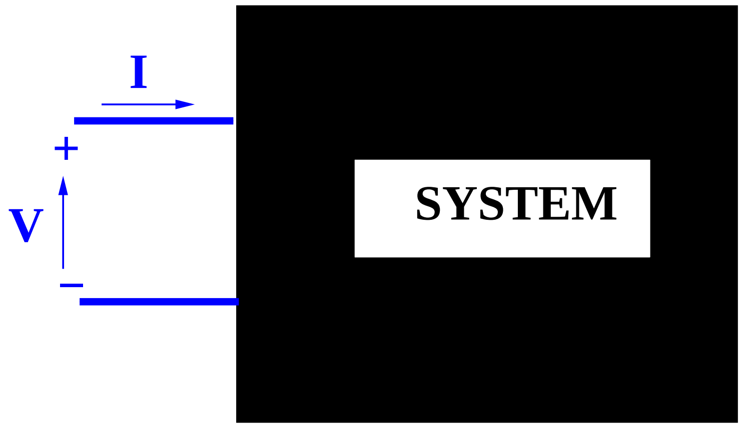
History & Roots

"Thermodynamics is the only physical theory of a universal nature of which I am convinced that it will never be overthrown" **Albert Einstein**



W.H. Haddad, V. Chellaboina, & S. Nersenov, ***Thermodynamics: A Dynamical Systems Approach*, 2006**

History & Roots



Dissipative w.r.t. VI **(= power in)**

$$:\Leftrightarrow \quad \int_{-\infty}^{0} V(t')I(t')\,dt' \geq 0 \quad \textbf{for all } (V, I) \in \mathfrak{B}$$

History & Roots

Dissipative w.r.t. VI **(= power in)**

$$:\Leftrightarrow \quad \int_{-\infty}^{0} V(t')I(t')\,dt' \geq 0 \quad \text{for all } (V, I) \in \mathfrak{B}$$

Linear, time-inv. system, transfer function $G \in \mathbb{R}(\xi)$

$\Leftrightarrow$ G **is positive real**

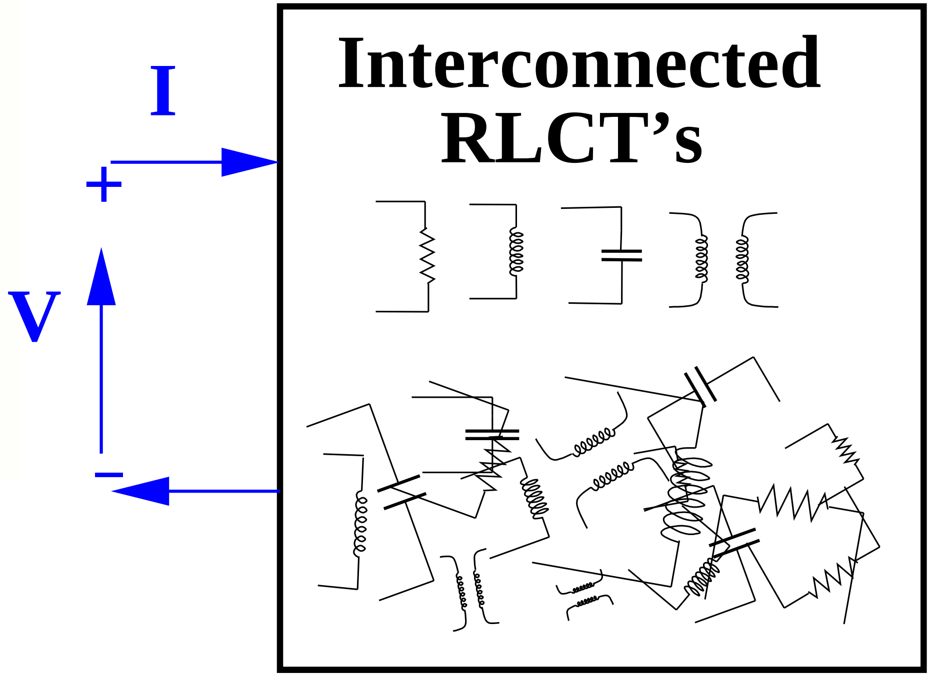
[i.e. $\texttt{Real}\,(G\,(s)) \geq 0$ **for** $\texttt{Real}(s) > 0$**]**

History & Roots

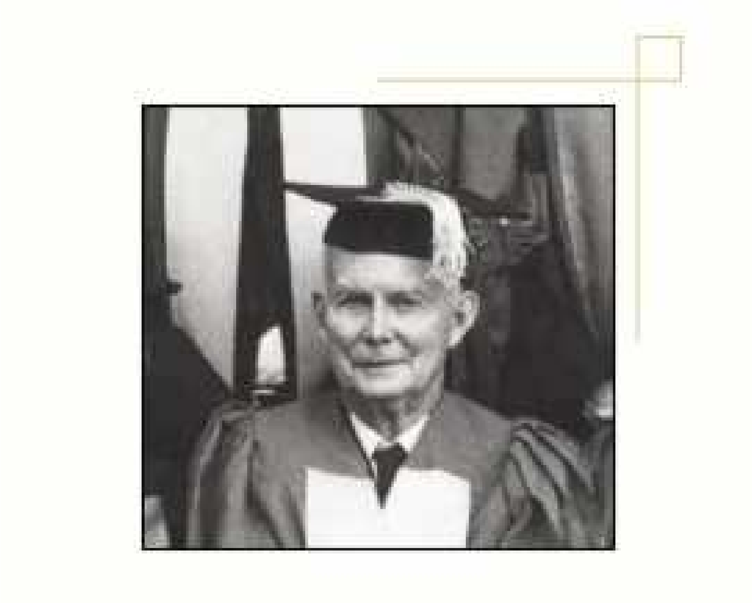
Dissipative $\Leftrightarrow$ G **is positive real**

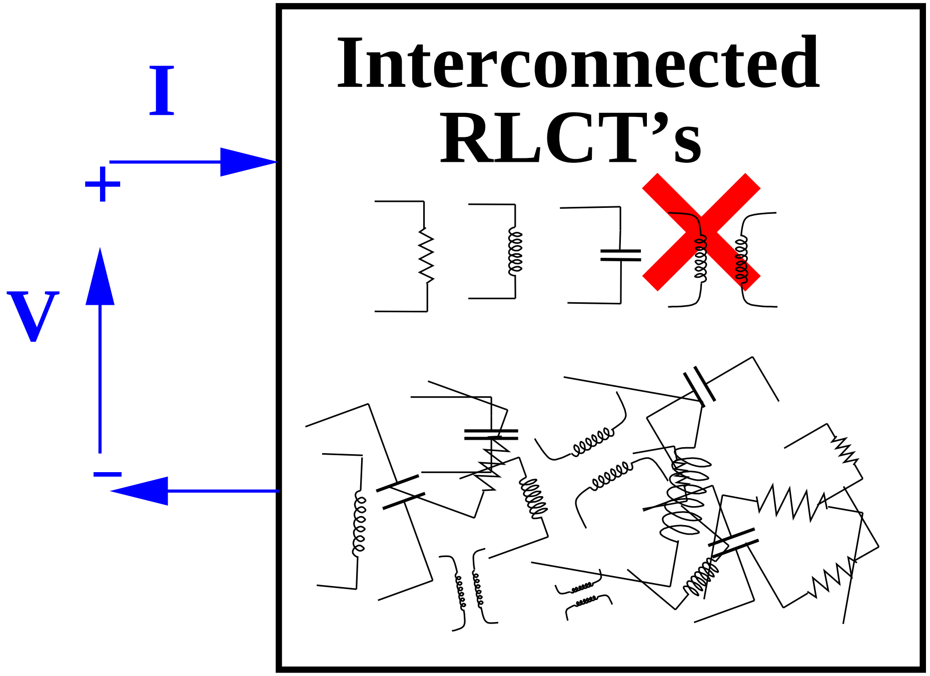
$\Leftrightarrow$ G **is realizable as impedance of a circuit with resistors, inductors, capacitors, and transformers**

Otto Brune, 1931



History & Roots





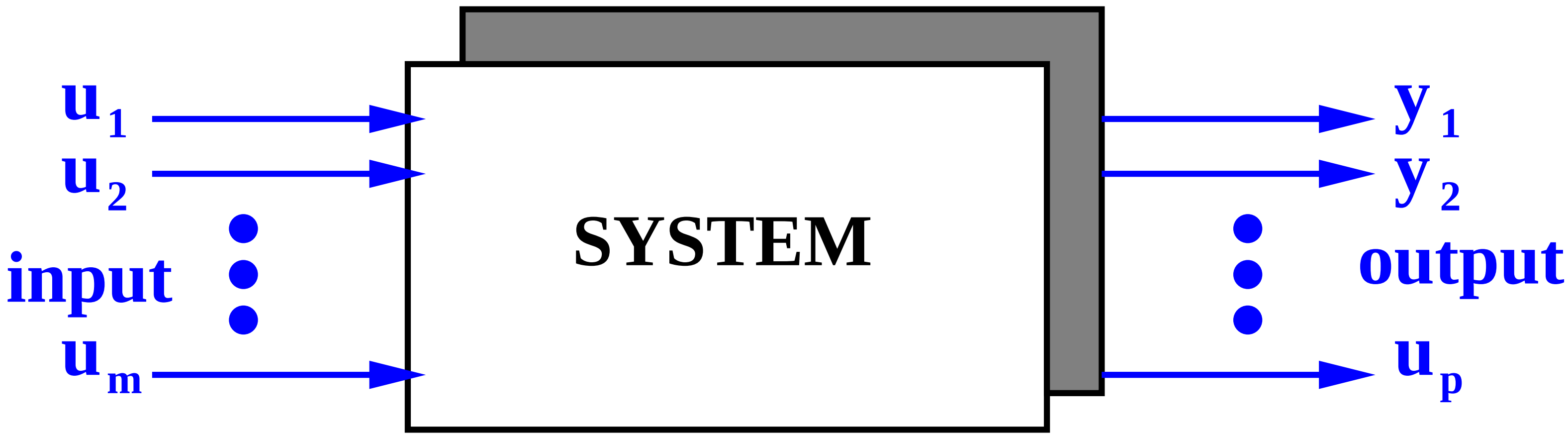
Otto Brune, 1931

Bott & Duffin: transformers not needed (1949)

B.D.O. Anderson & S. Vongpanitlerd, *Network Analysis and Synthesis: A Modern Systems Theory Approach*, 1973

Dissipative input/state/output systems

input/state/output systems



$$\frac{d}{dt}x = f(x,u), \quad y = h(x,u)$$

Behavior $\mathfrak{B} =$ **all sol'ns** $(u,y,x) : \mathbb{R} \to \mathbb{U} \times \mathbb{Y} \times \mathbb{X}$.

input/state/output systems

$$\frac{d}{dt} x = f(x, u), \quad y = h(x, u).$$

Consider

$s : \mathbb{U} \times \mathbb{Y} \longrightarrow \mathbb{R}$ **called the supply rate**

$V : \mathbb{X} \longrightarrow \mathbb{R}$ **called the storage function**

input/state/output systems

$$\frac{d}{dt}x = f(x,u), \quad y = h(x,u).$$

Consider

$s : \mathbb{U} \times \mathbb{Y} \longrightarrow \mathbb{R}$ **called the supply rate**

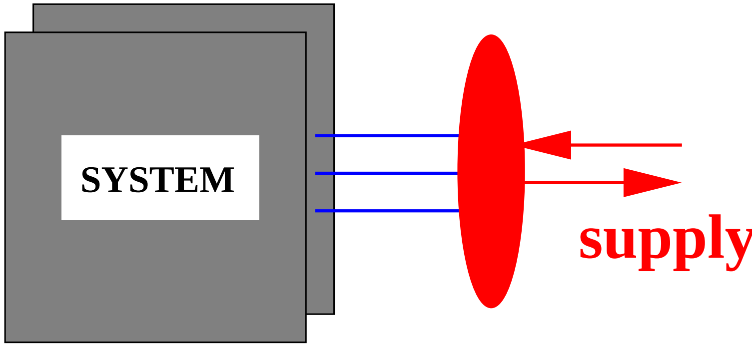
$V : \mathbb{X} \longrightarrow \mathbb{R}$ **called the storage function**

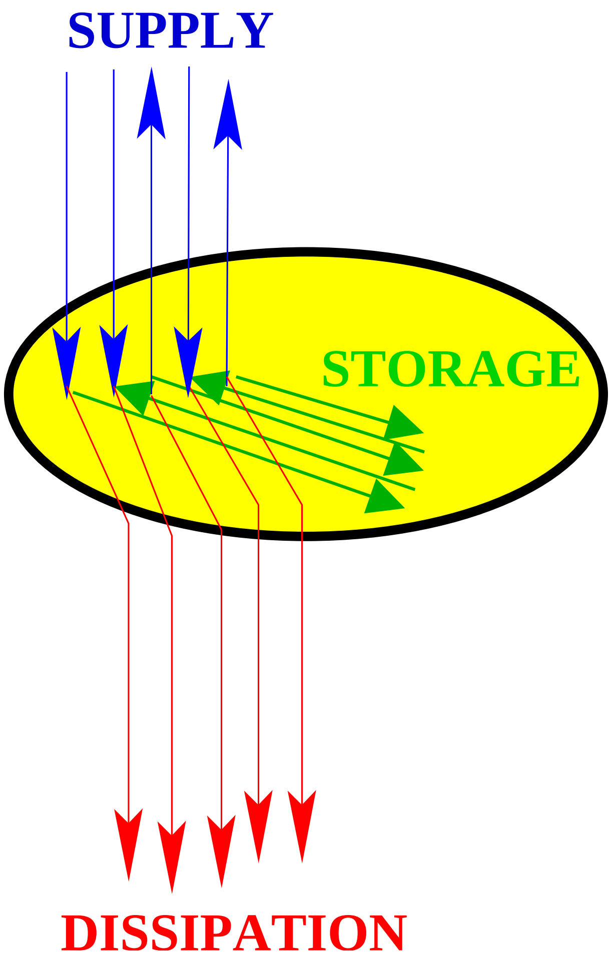
dissipative **w.r.t. supply rate s and with storage V**

$$:\Leftrightarrow \quad \frac{d}{dt}V(x(\cdot)) \leq s(u(\cdot), y(\cdot)) \quad \text{for } (u,y,x) \in \mathfrak{B}$$

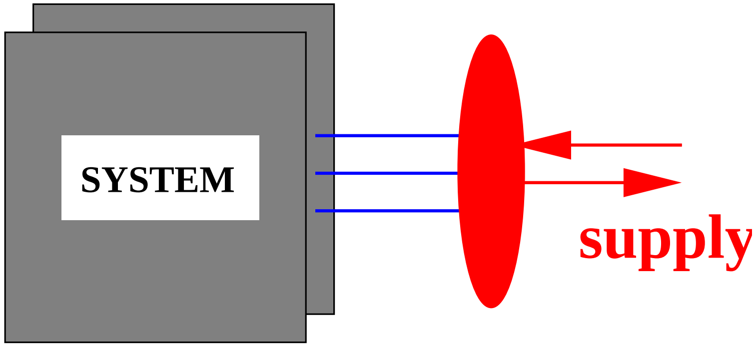
This inequality is called the dissipation inequality

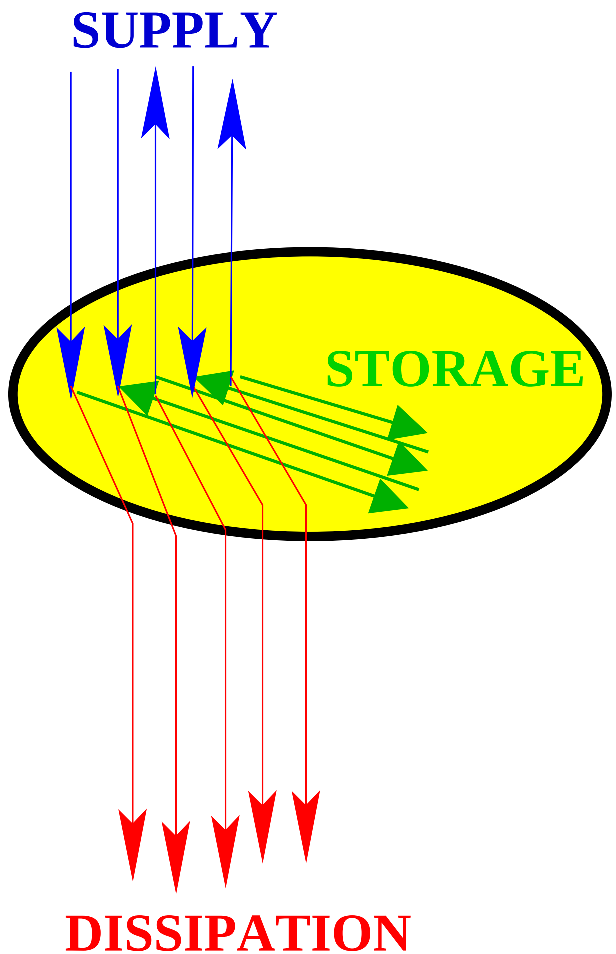
Dissipation inequality





Dissipation inequality





$s(\mathrm{u}, \mathrm{y})$ **models**

something like the power in

$V(\mathrm{x})$ **the stored energy.**

Dissipativity $:\Leftrightarrow$

rate of increase of energy $\leq$ **power delivered**

Lyapunov functions

Special case: isolated systems $\frac{d}{dt}x = f(x) \rightsquigarrow s = 0$

Dissipation inequality $\Leftrightarrow$ $\boxed{\frac{d}{dt}V(x(\cdot)) \leq 0}$

$\rightsquigarrow V$ **is a Lyapunov function**

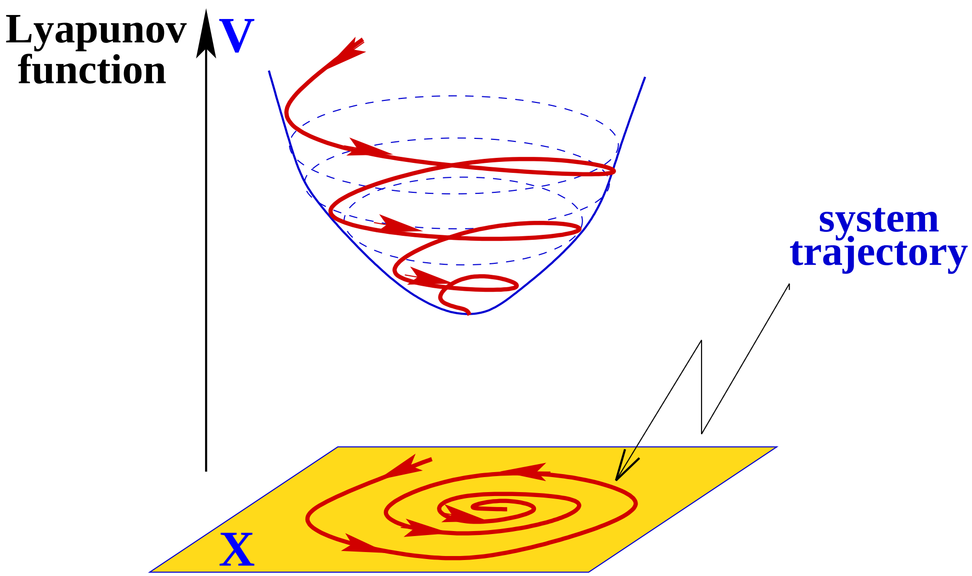
Lyapunov functions

Special case: isolated systems $\frac{d}{dt}x = f(x) \rightsquigarrow s = 0$

Dissipation inequality $\Leftrightarrow$ $\boxed{\frac{d}{dt} V\left(x\left(\cdot\right)\right) \leq 0}$

$\rightsquigarrow V$ **is a Lyapunov function**

Aleksandr
Mikhailovich Lyapunov



Lyapunov functions

Special case: isolated systems $\frac{d}{dt}x = f(x) \rightsquigarrow s = 0$

Dissipation inequality $\Leftrightarrow$ $\boxed{\frac{d}{dt} V(x(\cdot)) \leq 0}$

$\rightsquigarrow$ V **is a Lyapunov function**

Lyapunov f'ns play a remarkably central role.

Dissipative systems:

generalize Lyapunov f'ns to open systems

i/s/o

Rich theory surrounding the construction of storage f'ns, especially in the L inear- Q uadratic case

system: linear; supply rate: quadratic

$\rightsquigarrow$ **LMIs, ARIneq, ARE, KYP,**
robust stability and control,
semi-definite programming, ...

Numerous applications

Dissipative i/s/o systems were covered very well in

"The Continuing Joy of Dissipation Inequalities"

December 14, 2006

Semi-plenary presentation

CDC 2006, San Diego

Frank Allgöwer

ODEs

Dissipative i/s/o systems were covered very well in

"The Continuing Joy of Dissipation Inequalities"

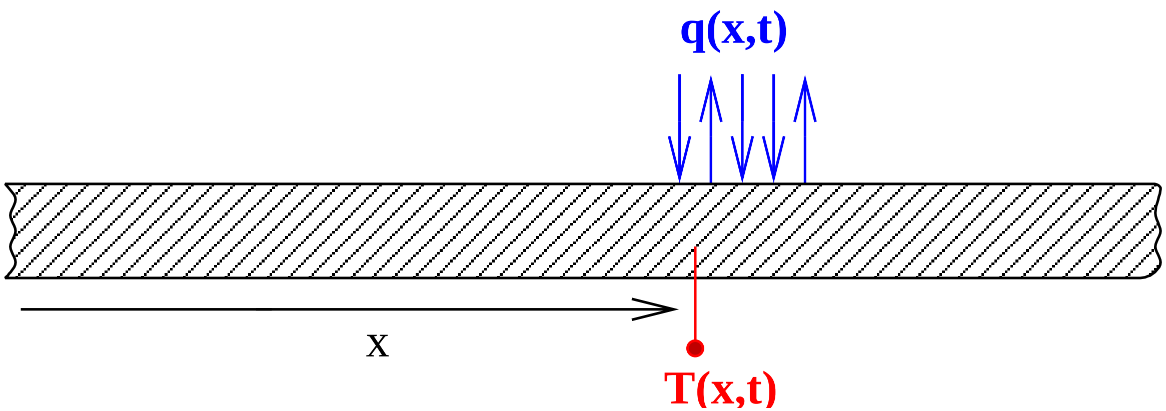
Today, I will concentrate on systems described by PDEs.

Partial differential equations

Results also interesting for ODEs !

PDEs: Examples

Diffusion



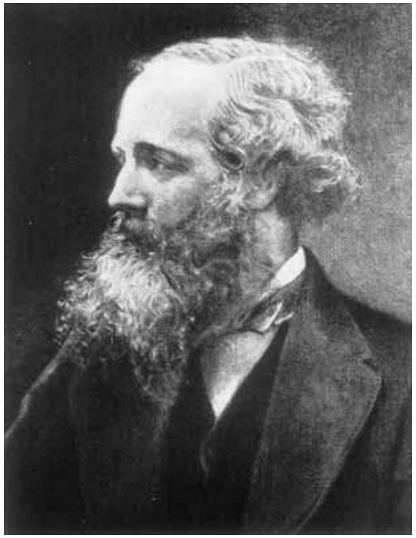
$$\frac{\partial}{\partial t} T = \frac{\partial^2}{\partial x^2} T + q$$

independent variables: (t, x) **time and space**

dependent variables: (T, q) **temperature and heat**

PDEs: Examples

Maxwell's equations for EM fields in free space



$$\nabla \cdot \vec{E} = \frac{1}{\varepsilon_0}\rho\,,$$
$$\nabla \times \vec{E} = -\frac{\partial}{\partial t}\vec{B},$$
$$\nabla \cdot \vec{B} = 0\,,$$
$$c^2\nabla \times \vec{B} = \frac{1}{\varepsilon_0}\vec{j} + \frac{\partial}{\partial t}\vec{E}.$$

independent variables: (t, x, y, z) **time and space**

dependent variables: $(\vec{E}, \vec{B}, \vec{j}, \rho)$

electric field, magnetic field, current density, charge density

PDEs: Notation

$\mathbb{R}[\xi_1, \ldots, \xi_{\mathtt{n}}]$: **polynomials,** $\mathtt{n}$ **indet., real coeff.**

$\mathbb{R}[\xi_1, \ldots, \xi_{\mathtt{n}}]^{\bullet \times \mathtt{w}}$, $\mathbb{R}[\xi_1, \ldots, \xi_{\mathtt{n}}]^{\bullet \times \bullet}$ **matrices of ...**

PDEs: Notation

$$\boldsymbol{R} \in \mathbb{R}\left[\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_{\mathtt{n}}\right]^{\bullet \times \mathtt{w}} \rightsquigarrow \boldsymbol{R}\left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathtt{n}}}\right) \boldsymbol{w} = 0$$

linear constant coefficient PDEs with

$\mathtt{n}$ **independent variables,** $x_1, \ldots, x_{\mathtt{n}}$

$\mathtt{w}$ **dependent variables,** $w_1, \ldots, w_{\mathtt{w}}$

$\mathrm{rowdim}(R)$ **= number of equations**

PDEs: Notation

$$\boldsymbol{R} \in \mathbb{R}\left[\xi_1, \ldots, \xi_{\mathtt{n}}\right]^{\bullet \times \mathtt{w}} \rightsquigarrow \boxed{\boldsymbol{R}\left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathtt{n}}}\right) \boldsymbol{w} = 0}$$

Ex.: Diffusion eq'n $\quad \frac{\partial}{\partial t} T = \frac{\partial^2}{\partial x^2} T + q$

2 indep. variables, (t, x)**,** $\mathtt{w} = 2$**,** $w = (T, q)$**, 1 eq'n.**

$$R(\xi_t, \xi_x) = [\, \xi_t - \xi_x^2 \mid -1 \,]$$

PDEs: Notation

$$\boldsymbol{R} \in \mathbb{R}\,[\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_{\mathtt{n}}]^{\bullet\times\mathtt{w}} \rightsquigarrow \boldsymbol{R}\left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathtt{n}}}\right)\boldsymbol{w} = 0$$

Example: Maxwell's eq'ns

4 independent variables, (t, x, y, z)

$\mathtt{w} = 10$, $w = (\vec{E}, \vec{B}, \vec{j}, \rho)$

8 equations, $R\, 8 \times 10$**, sparse**

PDEs: Notation

$$R \in \mathbb{R}[\xi_1, \ldots, \xi_{\mathtt{n}}]^{\bullet \times \mathtt{w}} \rightsquigarrow \boxed{R\left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathtt{n}}}\right) w = 0}$$

Behavior:

$$\mathfrak{B} = \{ w \in \mathfrak{C}^{\infty}(\mathbb{R}^{\mathtt{n}}, \mathbb{R}^{\mathtt{w}}) \mid R\left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathtt{n}}}\right) w = 0 \}$$

Notation:

$$\mathfrak{B} \in \mathfrak{L}_{\mathtt{n}}^{\mathtt{w}}, \quad \mathfrak{B} = \mathtt{kernel}\left(R\left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathtt{n}}}\right)\right)$$

PDEs: Notation

$$\boldsymbol{R} \in \mathbb{R}[\xi_1, \ldots, \xi_{\mathtt{n}}]^{\bullet \times \mathtt{w}} \rightsquigarrow \boldsymbol{R}\left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathtt{n}}}\right) w = 0$$

We cover only linear constant coefficient PDEs

$\mathfrak{C}^\infty$-solutions

infinite domain, no boundary conditions

'everything' valid for convex, open domain $\Omega \subseteq \mathbb{R}^{\mathtt{n}}$

Basic facts about $\mathfrak{L}^{\mathtt{w}}_{\mathtt{n}}$

Fact 1:

$\mathfrak{L}^{\mathtt{w}}_{\mathtt{n}}$ $\leftrightarrow$ **the submodules of** $\mathbb{R}\left[\xi_{1},\ldots,\xi_{\mathtt{n}}\right]^{\mathtt{w}}$

Basic facts about $\mathfrak{L}_{\mathtt{n}}^{\mathtt{w}}$

Fact 1:

$$\mathfrak{L}_{\mathtt{n}}^{\mathtt{w}} \quad \leftrightarrow \quad \text{the submodules of } \mathbb{R}\left[\xi_1, \ldots, \xi_{\mathtt{n}}\right]^{\mathtt{w}}$$

Fact 2: Elimination theorem

$$\mathfrak{L}_{\mathtt{n}}^{\mathtt{w}} \text{ is closed under projection}$$

$\mathfrak{L}^{\mathtt{w}}_{\mathtt{n}}$: the basics

$\mathfrak{L}^{\mathtt{w}}_{\mathtt{n}}$

Describe $(\rho, \vec{E}, \vec{j})$ in Maxwell's equations

Eliminate $\vec{B}$ from Maxwell's equations $\rightsquigarrow$

$$\begin{aligned}
\nabla \cdot \vec{E} &= \frac{1}{\varepsilon_0}\rho\,, \\
\varepsilon_0 \frac{\partial}{\partial t}\nabla \cdot \vec{E} + \nabla \cdot \vec{j} &= 0, \\
\varepsilon_0 \frac{\partial^2}{\partial t^2}\vec{E} + \varepsilon_0 c^2 \nabla \times \nabla \times \vec{E} + \frac{\partial}{\partial t}\vec{j} &= 0.
\end{aligned}$$

$\mathfrak{L}_{\mathtt{n}}^{\mathtt{w}}$

Fact 1:

$\mathfrak{L}_{\mathtt{n}}^{\mathtt{w}} \quad \leftrightarrow \quad$ **the submodules of** $\mathbb{R}[\xi_1, \ldots, \xi_{\mathtt{n}}]^{\mathtt{w}}$

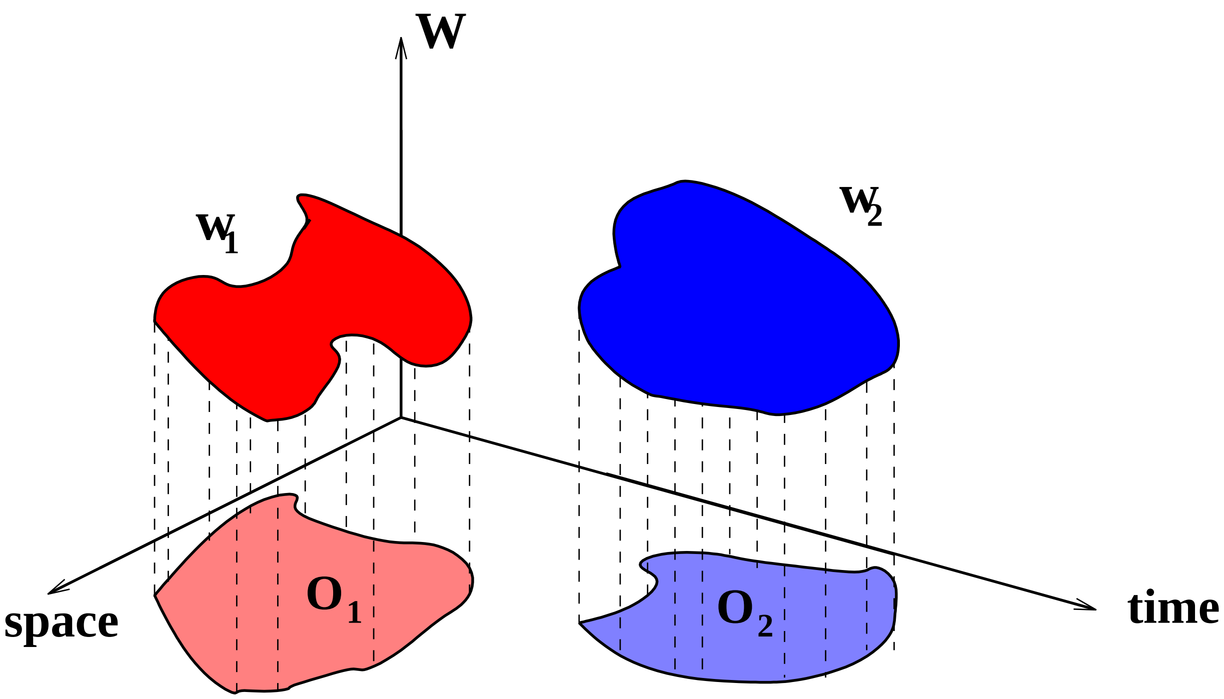
Fact 2: Elimination thm

$\mathfrak{L}_{\mathtt{n}}^{\mathtt{w}}$ **is closed under projection**

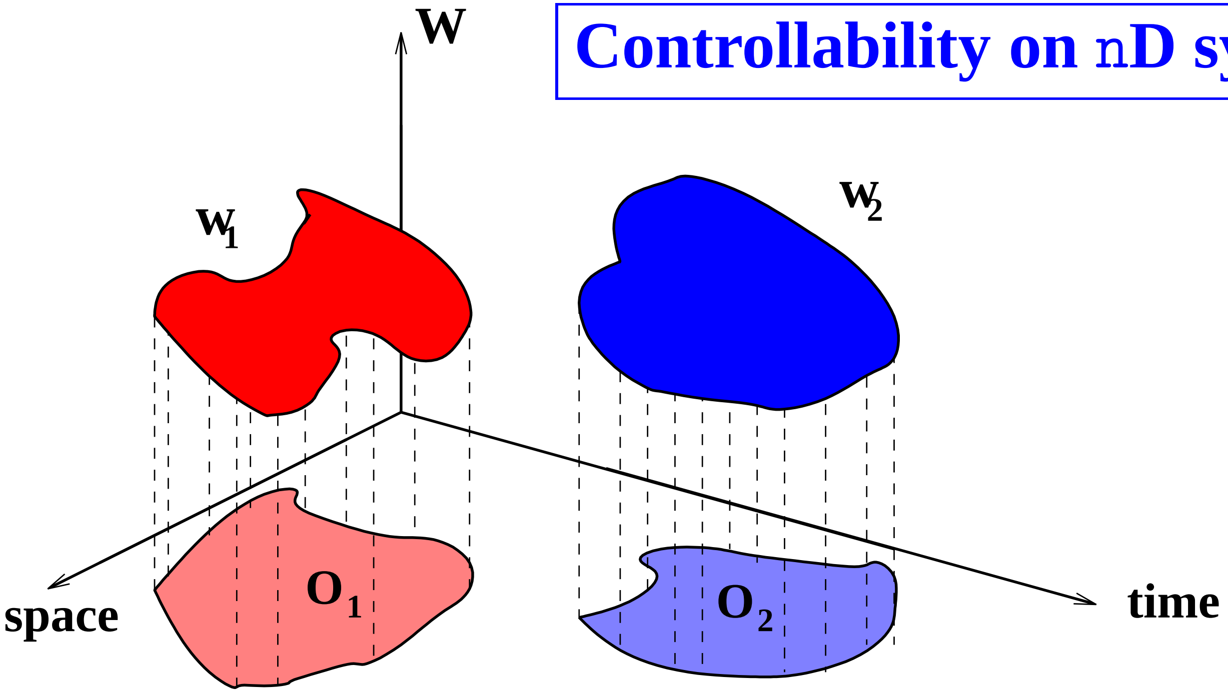
Fact 3:

$\mathfrak{B} \in \mathfrak{L}_{\mathtt{n}}^{\mathtt{w}}$ **is controllable** $\Leftrightarrow$ $\mathfrak{B}$ **is an image**

Controllability on nD systems

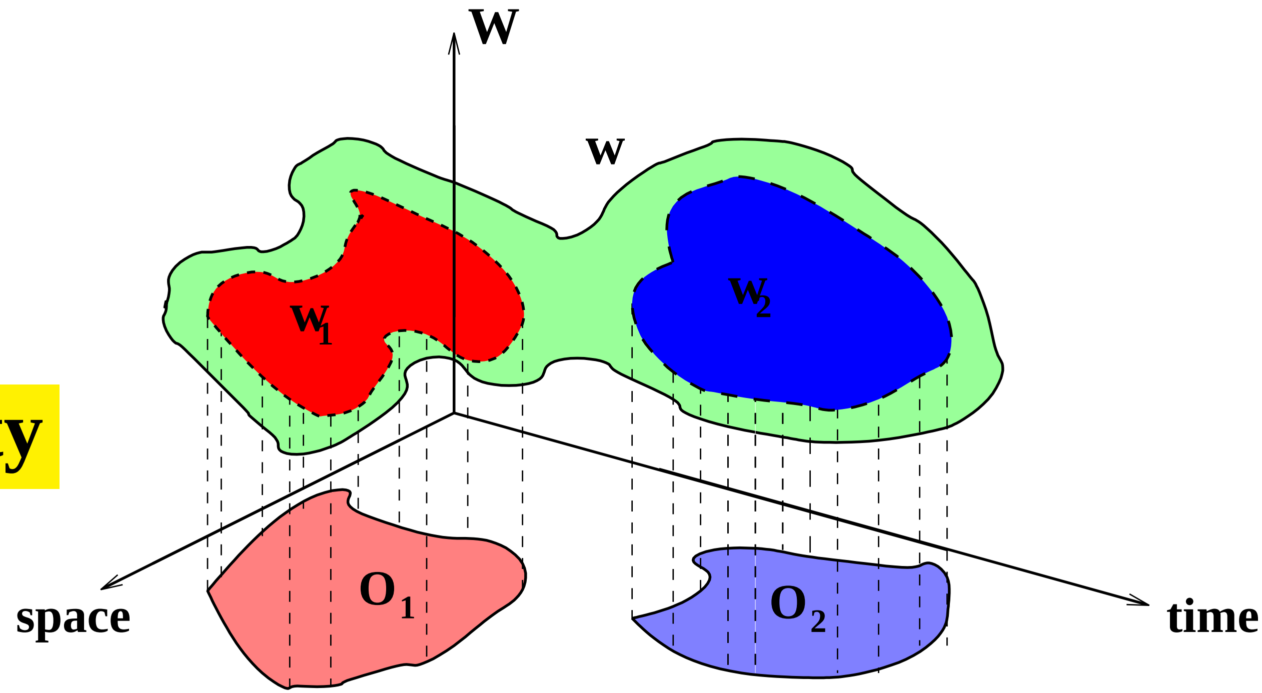


Controllability on nD systems



Controllability

:= Patchability



Controllability on $\mathtt{n}$D systems

$\mathfrak{B} \in \mathfrak{L}_{\mathtt{n}}^{\mathtt{w}}$ **controllable if and only if it has a repr.**

$$w = M\left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathtt{n}}}\right)\ell$$

$$\mathfrak{B} = \mathtt{image}\left(M\left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathtt{n}}}\right)\right)$$

Is an image a kernel ? Always ! $\Leftarrow$ Elimination th'm

Is a kernel an image ? Iff the kernel is controllable !

Controllability on $\mathtt{n}$D systems

$\mathfrak{B} \in \mathfrak{L}_{\mathtt{n}}^{\mathtt{w}}$ controllable if and only if it has a repr.

$$w = M\left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathtt{n}}}\right)\ell$$

$$\mathfrak{B} = \mathtt{image}\left(M\left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathtt{n}}}\right)\right)$$

But, for $\mathtt{n} > 1$, this image representation may not be observable. Images may require hidden variables.

Are EM fields controllable ?

Are EM fields controllable ?

The following eq'ns in the *scalar potential* ϕ :
$\mathbb{R} \times \mathbb{R}^3 \longrightarrow \mathbb{R}$ **and the *vector potential* $\vec{A}$:**
$\mathbb{R} \times \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ **generate exactly the solutions to MEs:**

$$\begin{aligned}
\vec{E} &= -\frac{\partial}{\partial t}\vec{A} - \nabla\phi, \\
\vec{B} &= \nabla \times \vec{A}, \\
\vec{j} &= \varepsilon_0 \frac{\partial^2}{\partial t^2}\vec{A} - \varepsilon_0 c^2 \nabla^2 \vec{A} + \varepsilon_0 c^2 \nabla\left(\nabla \cdot \vec{A}\right) + \varepsilon_0 \frac{\partial}{\partial t}\nabla\phi, \\
\rho &= -\varepsilon_0 \frac{\partial}{\partial t}\nabla \cdot \vec{A} - \varepsilon_0 \nabla^2 \phi.
\end{aligned}$$

Are EM fields controllable ?

$$\vec{E} = -\frac{\partial}{\partial t}\vec{A} - \nabla\phi,$$
$$\vec{B} = \nabla \times \vec{A},$$
$$\vec{j} = \varepsilon_0 \frac{\partial^2}{\partial t^2}\vec{A} - \varepsilon_0 c^2 \nabla^2 \vec{A} + \varepsilon_0 c^2 \nabla\left(\nabla \cdot \vec{A}\right) + \varepsilon_0 \frac{\partial}{\partial t}\nabla\phi,$$
$$\rho = -\varepsilon_0 \frac{\partial}{\partial t}\nabla \cdot \vec{A} - \varepsilon_0 \nabla^2 \phi.$$

Proves controllability. Not observable, cannot be !

controllability $\Leftrightarrow \exists$ potential!

Dissipative distributed systems

Notation

For simplicity of notation & concreteness, $\mathtt{n} = 4$,

independent var., t, time, and x, y, z, space.

$$\nabla\cdot := \left[\ \frac{\partial}{\partial x} \mid \frac{\partial}{\partial y} \mid \frac{\partial}{\partial z}\ \right]$$ **'divergence'**

We henceforth consider only

controllable linear differential systems $\in \mathfrak{L}_4^{\mathtt{w}}$

Dissipative distributed systems

Supply rate $s = w^\top S w$ $S = S^\top \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$

supply rate:

$$s(t,x,y,z) = w(t,x,y,z)^\top \, S \, w(t,x,y,z)$$

Dissipative distributed systems

Definition: $\mathfrak{B} \in \mathfrak{L}_4^{\mathtt{w}}$, **controllable, is said to be**

dissipative **with respect to the supply rate** $w^\top S w$ **if**

$$\int_{\mathbb{R}} \left[\int_{\mathbb{R}^3} w^\top S w \;\; dxdydz \right] \; dt \geq 0$$

for $w \in \mathfrak{B}$ **of compact support, i.e.** $w \in \mathfrak{B} \cap \mathfrak{D}$**.**

$\mathfrak{D} := \mathfrak{C}^\infty$ **and 'compact support'.**

Dissipative distributed systems

Idea: $\left(w^\top S w\right)(x, y, z,\ t)\ \ dxdydz\ \ dt =$

'energy' supplied in the space-cube

$[x, x + dx] \times [y, y + dy] \times [z, z + dz]$

during the time-interval $[t, t + dt]$.

Dissipativity $:\Leftrightarrow$

$$\int_{\mathbb{R}} \left[\int_{\mathbb{R}^3} \left(w^\top S w\right)(x, y, z,\ t)\ dxdydz\right]\ dt \geq 0$$

A dissipative system absorbs net energy in compact support realizations.

Example: EM fields

Maxwell's eq'ns define a dissipative (in fact, a conservative) system w.r.t. $-\vec{E} \cdot \vec{j}$

Indeed, if $\vec{E}, \vec{j}$ **are of compact support and**

$$\varepsilon_0 \frac{\partial}{\partial t} \nabla \cdot \vec{E} + \nabla \cdot \vec{j} = 0,$$

$$\varepsilon_0 \frac{\partial^2}{\partial t^2} \vec{E} + \varepsilon_0 c^2 \nabla \times \nabla \times \vec{E} + \frac{\partial}{\partial t} \vec{j} = 0,$$

$$\int_{\mathbb{R}} \left[\int_{\mathbb{R}^3} -\vec{E} \cdot \vec{j} \; dxdydz \right] dt = 0.$$

The storage and the flux

Local dissipation law

Dissipativity $:\Leftrightarrow$

$$\int_{\mathbb{R}} \left[\int_{\mathbb{R}^3} w^\top S w \, dx dy dz \right] \, dt \geq 0 \qquad \text{for } w \in \mathfrak{B} \cap \mathfrak{D}.$$

Local dissipation law

Dissipativity $:\Leftrightarrow$

$$\int_{\mathbb{R}} \left[\int_{\mathbb{R}^3} w^\top S w \, dxdydz\right] dt \geq 0 \qquad \text{for } w \in \mathfrak{B} \cap \mathfrak{D}.$$

Can this be reinterpreted as:

As the system evolves over time and space,

some of the *supply*, applied locally in time and space

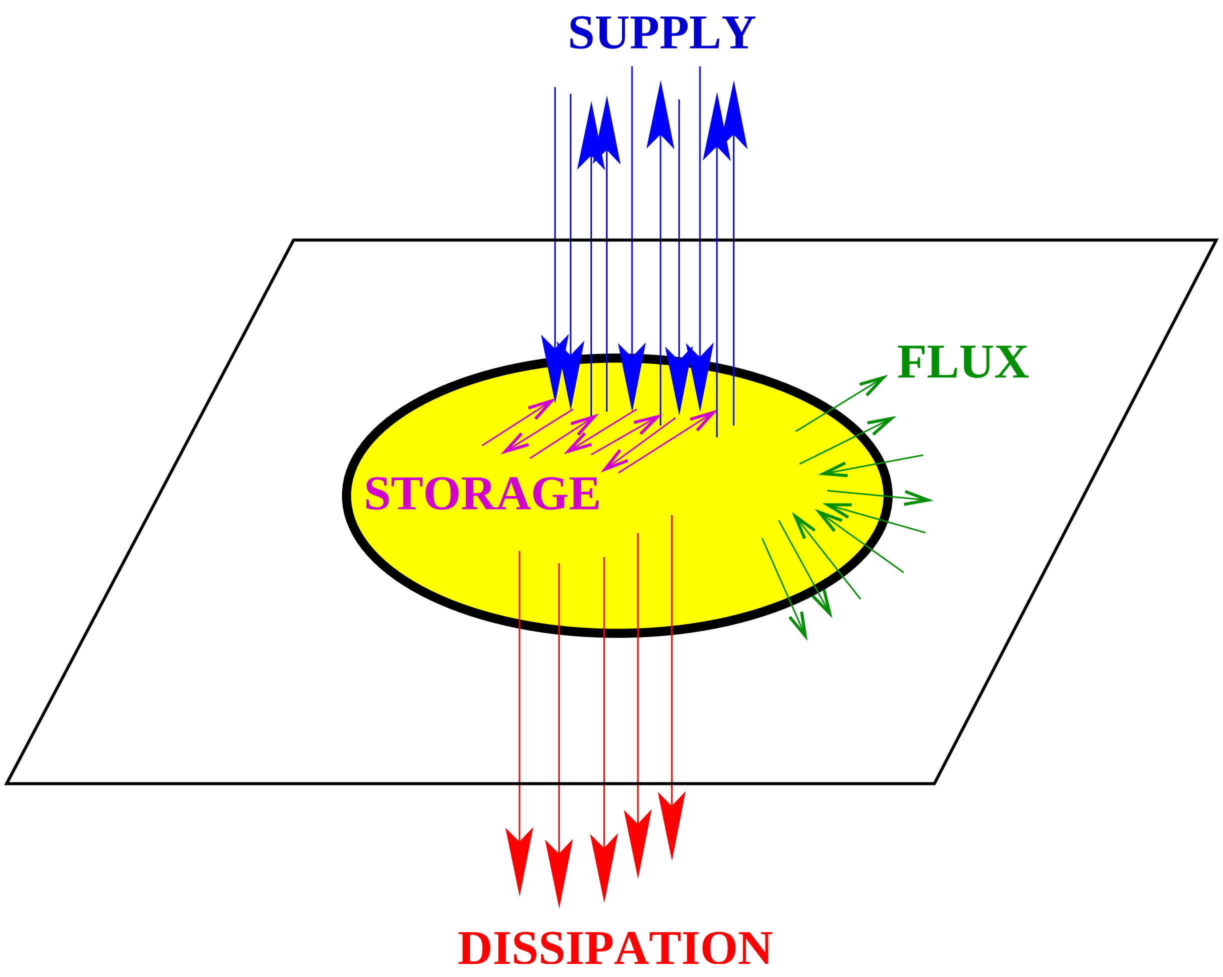
is some locally stored,

some redistributed over space,

some locally dissipated ?

Local dissipation law

!! Invent storage and flux, locally defined in time and space, such that in every spatial domain there holds:



Local dissipation law

!! Invent storage and flux, locally defined in time and space, such that in every spatial domain there holds:

$$\frac{d}{dt}\ \textbf{Storage} + \textbf{Spatial flux} \leq \textbf{Supply.}$$

Supply = stored + radiated + dissipated.

MAIN RESULT (stated for $\mathtt{n} = 4$)

Thm: $\mathfrak{B} \in \mathfrak{L}_4^{\mathtt{w}}$, **controllable.**

Then $\int_{\mathbb{R}} \left[\int_{\mathbb{R}^3} \boldsymbol{w}^\top \boldsymbol{S} \boldsymbol{w} \; \boldsymbol{dxdydz} \right] \; \boldsymbol{dt} \geq \mathbf{0} \; \forall \, \boldsymbol{w} \in \mathfrak{B} \cap \mathfrak{D}$

$$\Updownarrow$$

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$\exists$ image repr. $w = M\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \ell$ **of $\mathfrak{B}$,**
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and functions (QDFs): a real valued S **the** ***storage,*** **and a vector valued** F **the** ***flux,*** **such that the** ***local dissipation law***

$$\frac{\partial}{\partial t} S(\ell) + \nabla \cdot F \leq w^\top S w$$

holds for (w, ℓ) **s.t.** $w = M\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\ell$**.**

Hidden variables

The local law involves

possibly unobservable, - i.e., hidden!

latent variables (the ℓ's).

This gives physical notions as stored energy, entropy, etc., an enigmatic physical flavor.

Energy stored in EM fields

MEs are dissipative (in fact, conservative) with respect to $-\vec{E}\cdot\vec{j},$ **the rate of energy supplied.**

Energy stored in EM fields

Introduce the *stored energy density, S,* and *energy flux density (Poynting vector)*, $\vec{F}$,

$$S\left(\vec{E},\vec{B}\right) := \frac{\varepsilon_0}{2}\vec{E}\cdot\vec{E} + \frac{\varepsilon_0 c^2}{2}\vec{B}\cdot\vec{B},$$

$$\vec{F}\left(\vec{E},\vec{B}\right) := \varepsilon_0 c^2 \vec{E}\times\vec{B}.$$

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Local conservation law for Maxwell's equations:

$$\boxed{\frac{\partial}{\partial t}S\left(\vec{E},\vec{B}\right) + \nabla\cdot\vec{F}\left(\vec{E},\vec{B}\right) = -\vec{E}\cdot\vec{j}.}$$

Energy stored in EM fields

Local conservation law for Maxwell's equations:

$$\frac{\partial}{\partial t} S\left(\vec{E}, \vec{B}\right) + \nabla \cdot \vec{F}\left(\vec{E}, \vec{B}\right) = -\vec{E} \cdot \vec{j}.$$

The storage and flux involve $\vec{B}$,

unobservable from $\vec{E}$ **and** $\vec{j}$.

The proof

The crux of the proof

Solve the ‘factorization equation’

$$\boldsymbol{X}^\top(-\xi_1,\ldots,-\xi_n)\,\boldsymbol{X}(\xi_1,\ldots,\xi_n) = \boldsymbol{Y}(\xi_1,\ldots,\xi_n)$$

with $\boldsymbol{Y} \in \mathbb{R}^{\bullet\times\bullet}[\xi_1,\ldots,\xi_n]$ **given**

$\boldsymbol{X}$ **the unknown** ***Solvable??***

SOS

The factorization equation can be reduced to the following scalar problem

Express $p(\xi_1, ., \xi_{\mathrm{n}}) \in \mathbb{R}[\xi_1, ., \xi_{\mathrm{n}}]$ **as a sum of squares**

$$p = x_1^2 + x_2^2 + \cdots + x_{\mathrm{m}}^2$$

Necessary: $p(t_1, \ldots, t_{\mathrm{n}}) \geq 0,$ **for** t_{k}**'s** $\in \mathbb{R}$**.**
Also sufficient?

SOS

Express $p(\xi_1, ., \xi_n) \in \mathbb{R}[\xi_1, ., \xi_n]$ **as a sum of squares**

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Also sufficient?

This is Hilbert's 17-th problem !

Not solvable over polynomials,

but solvable over rational functions.

Observe

The need to introduce rational functions
in the factorization equation and
an image representation of $\mathfrak{B}$
are the causes of the unavoidable presence
of (possibly unobservable, ‘hidden’) latent variables
in the local dissipation law.

Observe

The stored energy for a spatially distributed is NOT a function of the phenomenological variables w and their partial derivatives, but it is a function of underlying unobservable variables !

Summary

Hightlights

- **The theory of dissipative systems centers around the construction of the storage function**

Hightlights

- **global dissipation $\Leftrightarrow$ local dissipation**

  time-wise and space-wise

Hightlights

- **For $\mathtt{n} > 1$ involves, possibly, hidden variables (similar to $\vec{B}$ in Maxwell's eq'ns)**

  Also relevant for Lyapunov functions for spatially distributed systems

Hightlights

- **The proof** $\cong$ **Hilbert's 17-th problem**

Hightlights

- **Neither controllability nor observability are good assumptions for physical models**

Hightlights

- **Finite dimensional linear system theory:**
  well developed, in systems and control.
  Linear constant coefficient PDEs:
  well developed, in mathematics
  Very relevant physically. Fruitful problem area.

Details & copies of frames are available from/at

`Jan.Willems@esat.kuleuven.be`

`http://www.esat.kuleuven.be/∼jwillems`

Thank you

Thank you

Thank you

Thank you

Thank you

Thank you

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