



OPEN AND CONNECTED

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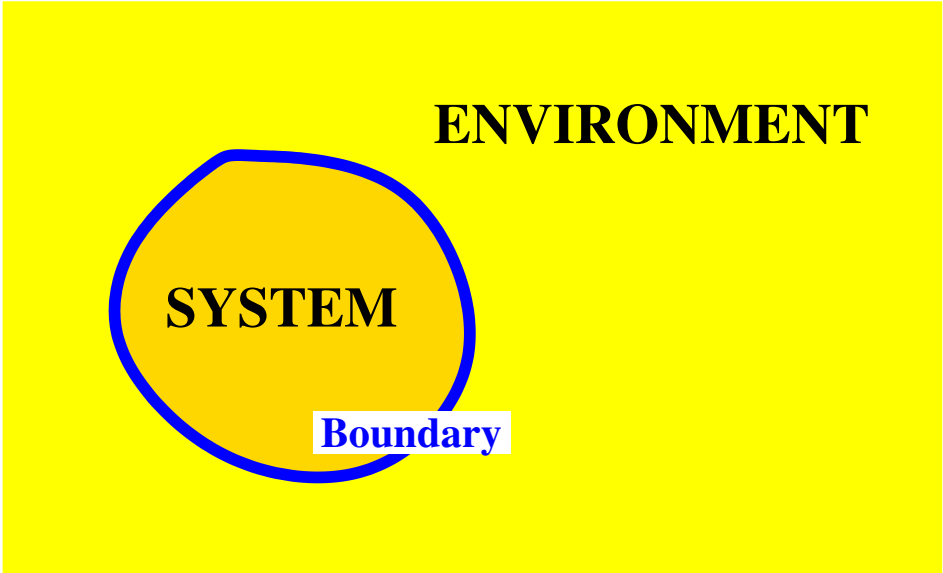
Open and Connected

The central tenets of our field:

Systems are **open** and consist of
interconnected subsystems.

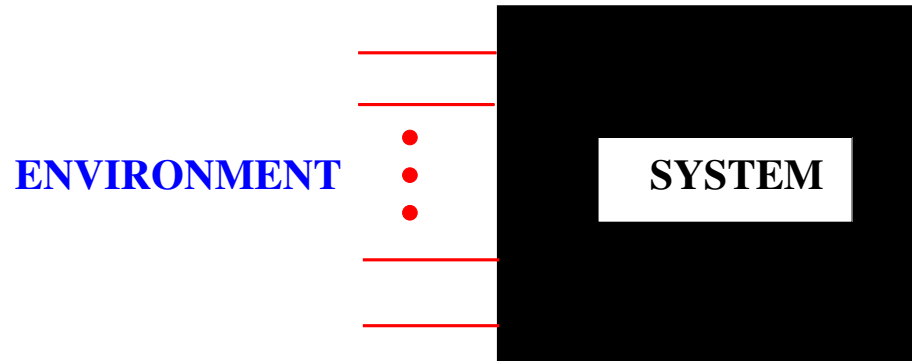
Synthesis of systems consists of
interconnecting subsystems.

Open



Open

In this lecture, we think of this interaction boundary as 'terminals'



electrical components with '**wires**'

mechanical components with '**pins**'

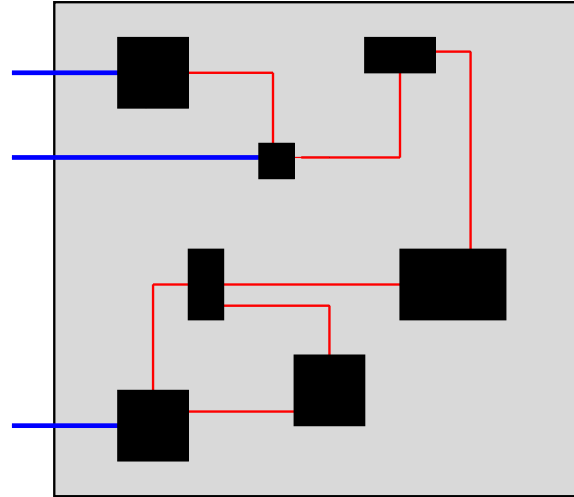
hydraulic components with '**ducts**'

signal processors with **inputs and outputs**

motors with electrical & mech. terminals

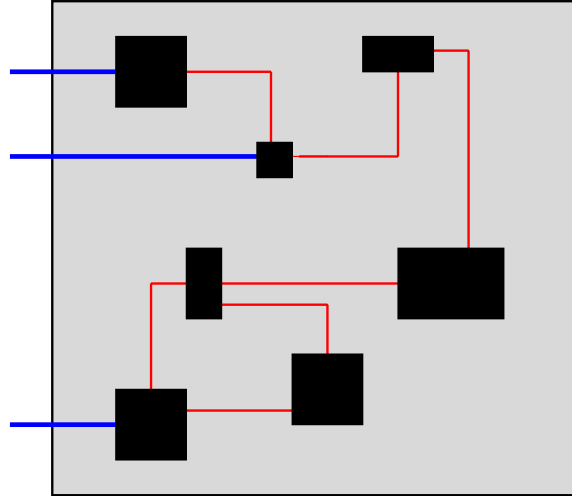
computer terminal, etc., etc., etc.

Connected



An interconnection architecture with subsystems

Connected



Think of:

electrical circuits

mechanical constructions, structures

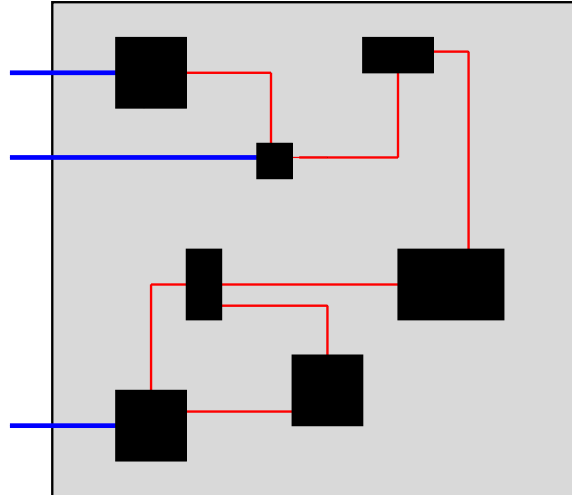
hydraulic systems

networks of **signal processors**

computers

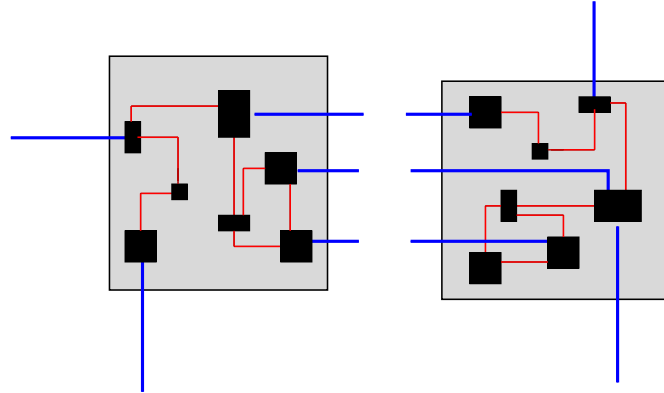
essentially all **engineering systems**

Connected

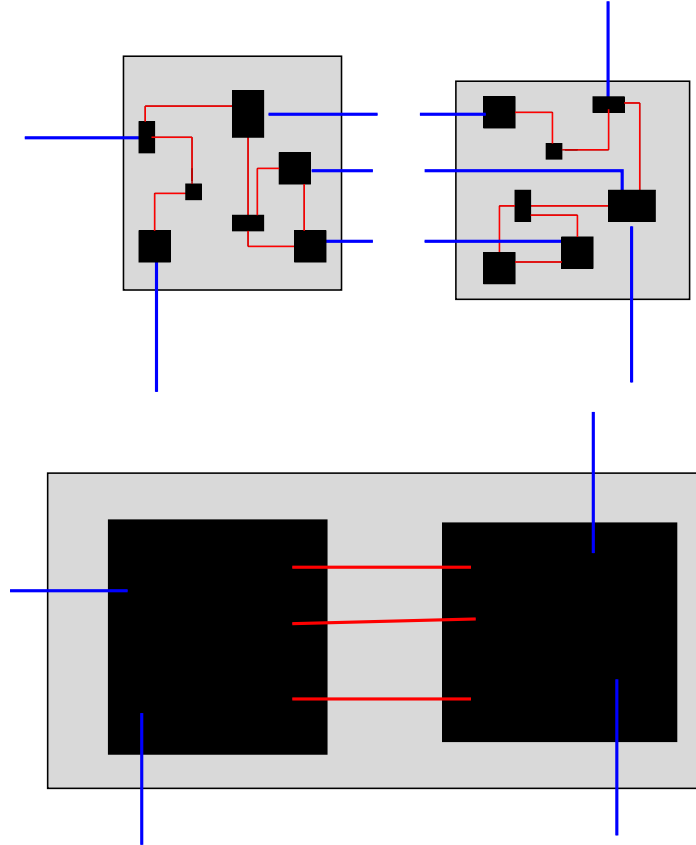


Observe the **hierarchical** nature

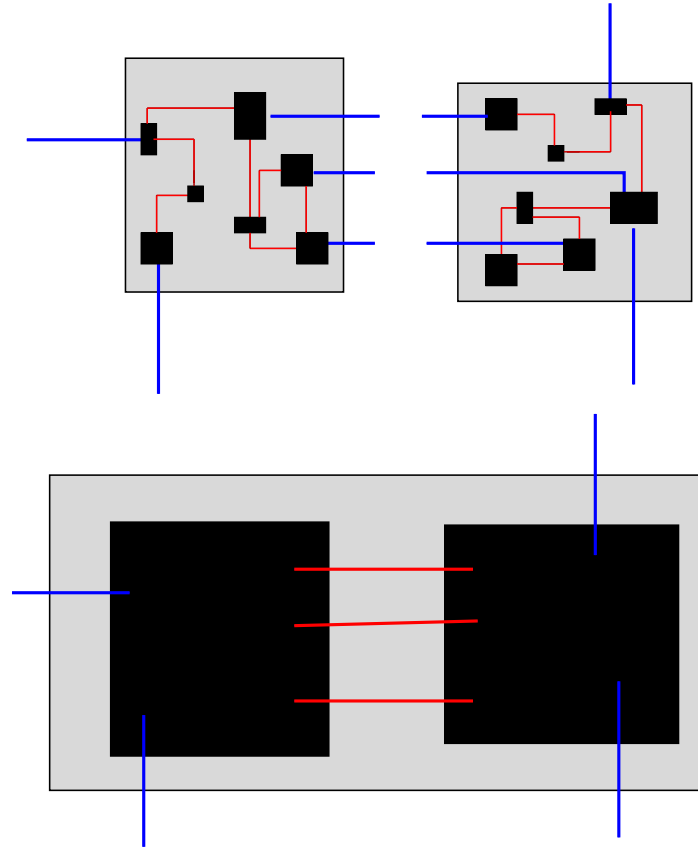
Interconnect



Interconnect



Interconnect



Reverse process: **'tearing'** & **'zooming'** & **'linking'**:

very useful in **modeling**.

Mathematization

What are the appropriate concepts / mathematization?

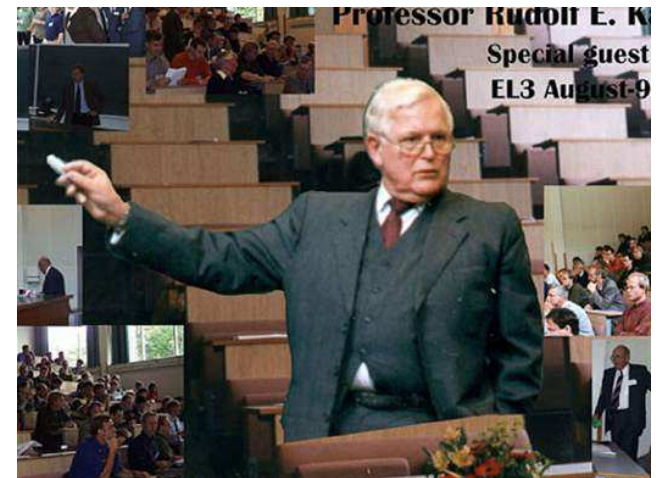
What is an **open** dynamical system?

How do we deal with **interconnections**?

How does **control** fit in?

Mathematization

1. Get the physics right
2. The rest is mathematics



**R.E. Kalman, Opening lecture
IFAC World Congress, Prague, July 4, 2005**

THEMES

1. Open and connected
2. A brief history of systems theory
3. **Control, interconnection, inputs and outputs**
4. Models and behaviors
5. Linear time-invariant differential systems
6. Controllability and stabilizability
7. **Representations of linear differential systems**
8. PDE's

The paradigm of closed systems

Axiomatization

K.1, K.2, & K.3

$$\rightsquigarrow \frac{d^2}{dt^2} w(t) + \frac{1}{|w(t)|^2} = 0$$

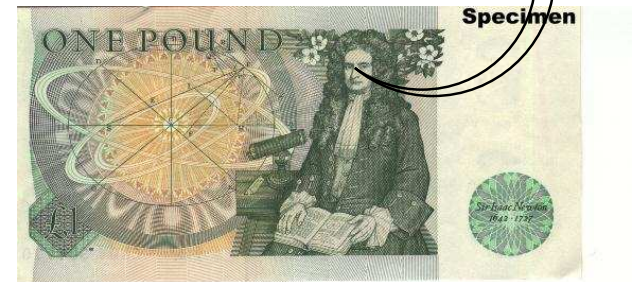
$$\rightsquigarrow \frac{d}{dt} x = f(x)$$

\rightsquigarrow 'dynamical systems', flows

\rightsquigarrow **closed systems** as paradigm of dynamics



Hypotheses
non
fingo



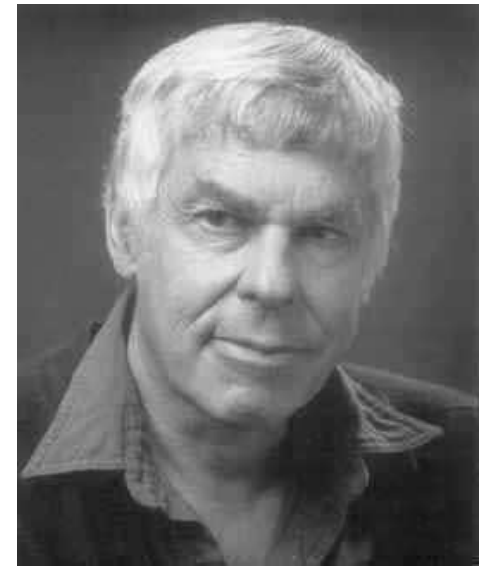
Axiomatization



Henri Poincaré (1854-1912)



George Birkhoff (1884-1944)



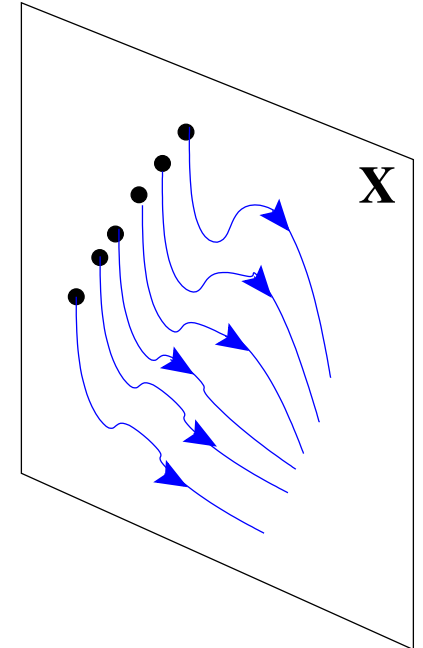
Stephen Smale (1930-)

Axiomatization

A *dynamical system* is defined by
a **state space** X and
a **state transition function**

$\phi : \dots$ such that \dots

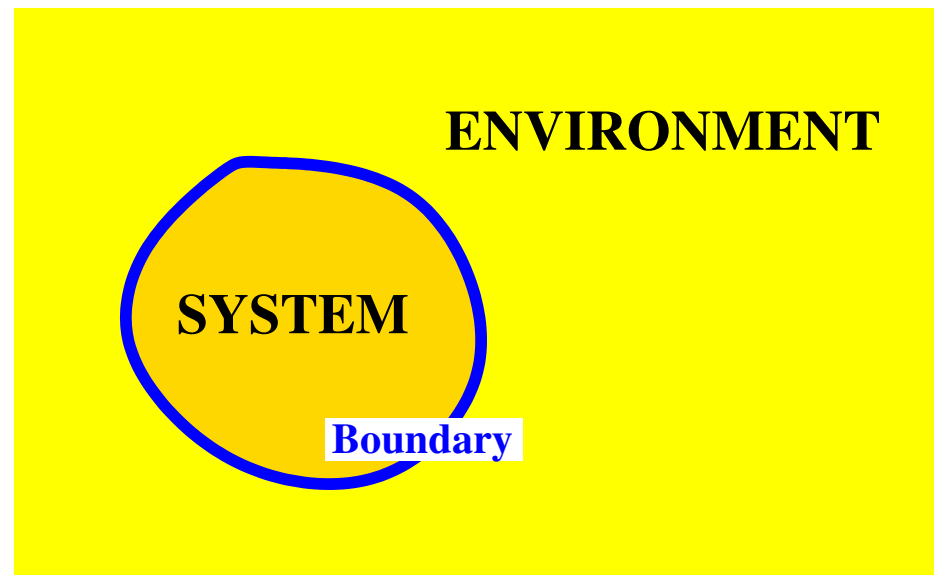
$\phi(t, x)$ = state at time t starting from state x



How could they forget about Newton's second law,
about Maxwell's eq'ns,
about thermodynamics,
about tearing & zooming & linking, ...?

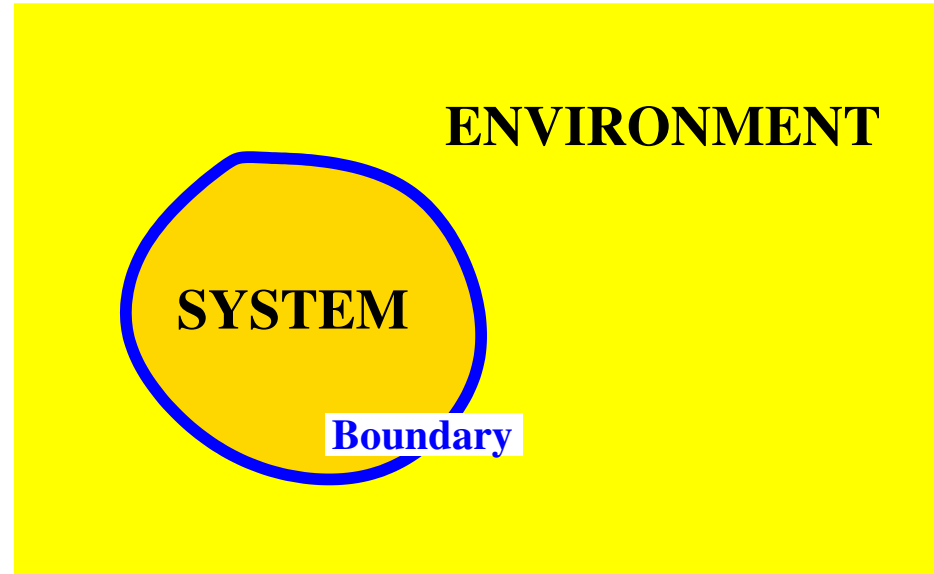
Closed systems

‘Fixed boundary conditions’



~> **an absurd situation: to model a system,
we have to model also the environment!**

Closed systems



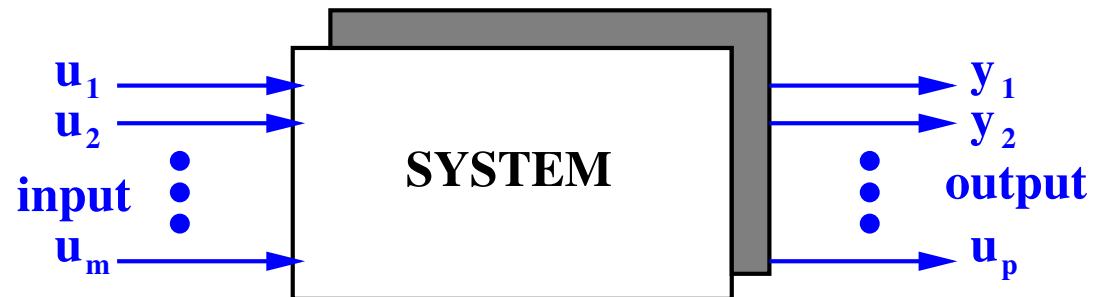
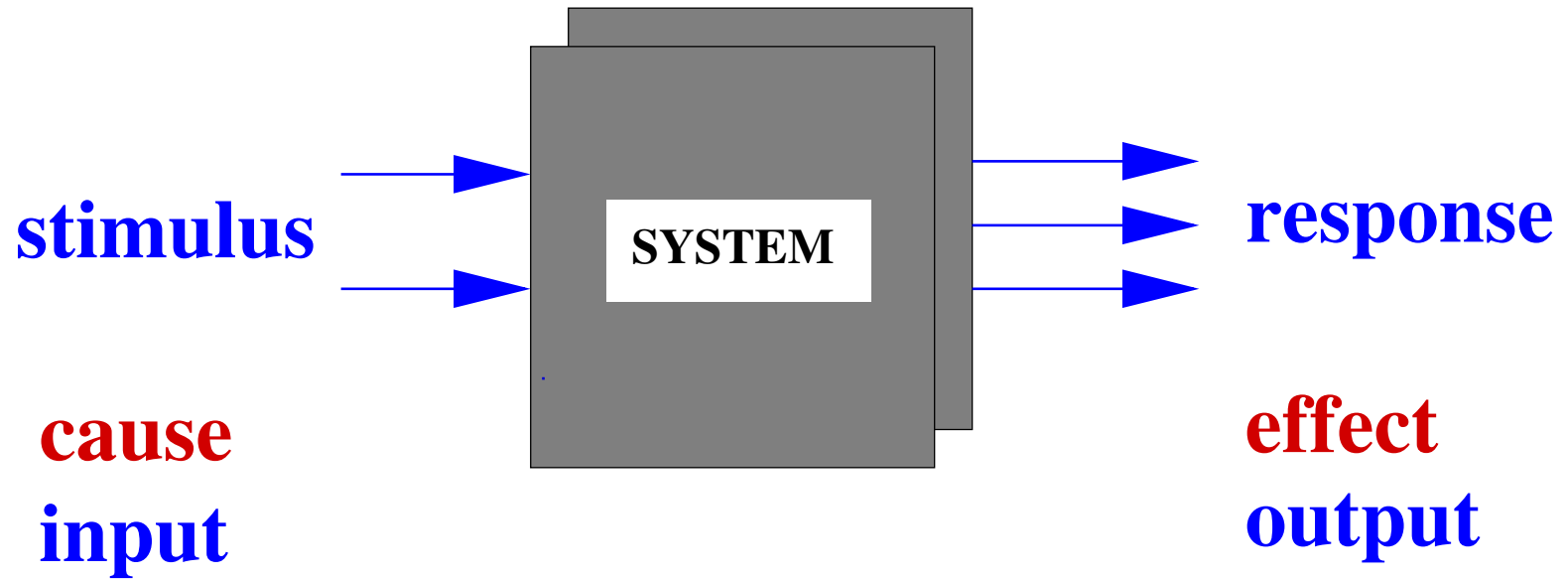
Chaos theory, cellular automata, sync, etc.,

'function' in this framework ...

Meanwhile, in engineering, ...

The paradigm of input/output systems

Input/output systems



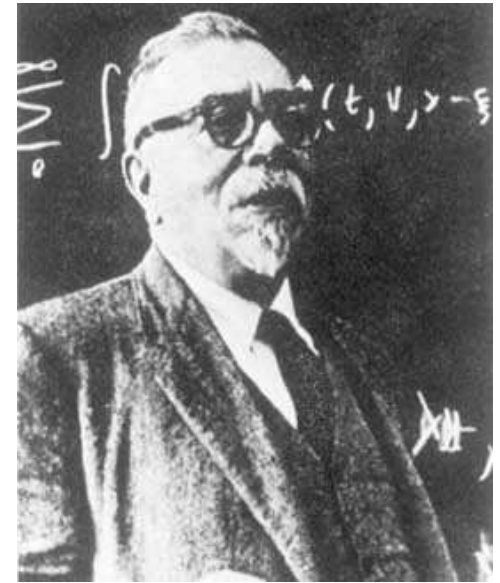
The originators



Lord Rayleigh (1842-1919)



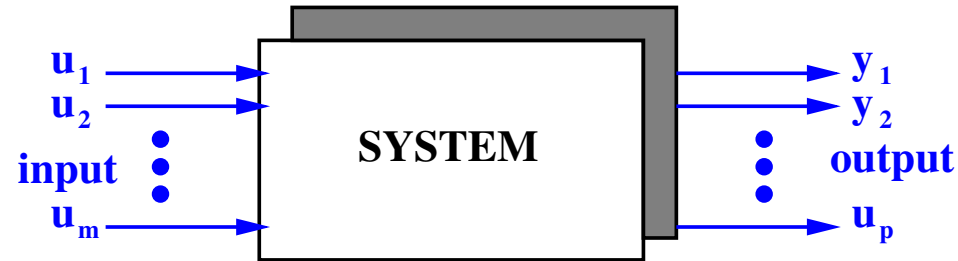
Oliver Heaviside (1850-1925)



Norbert Wiener (1894-1964)

and the many electrical circuit theorists ...

Mathematical description



$$y(t) = \int_{0 \text{ or } -\infty}^t H(t - t') u(t') dt'$$

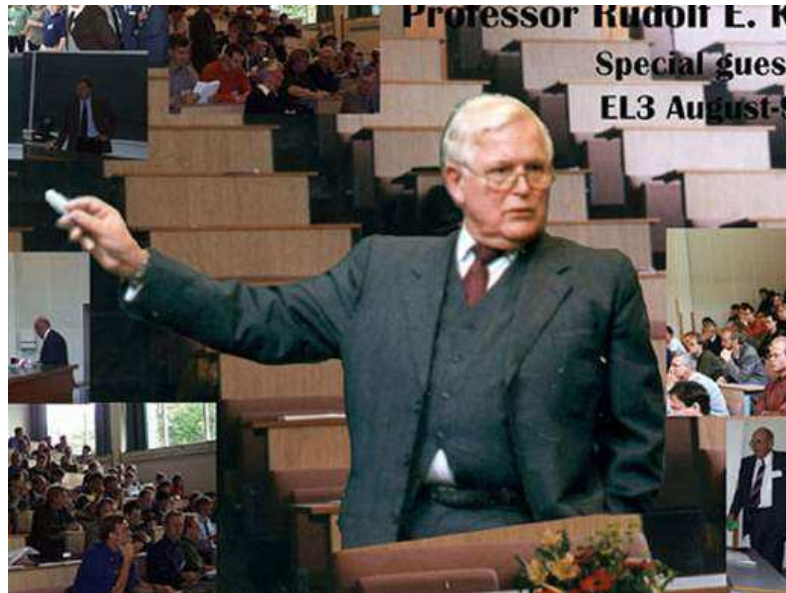
$$y(t) = H_0(t) + \int_{-\infty}^t H_1(t - t') u(t') dt' + \int_{-\infty}^t \int_{-\infty}^{t'} H_2(t - t', t' - t'') u(t') u(t'') dt' dt'' + \dots$$

These models fail to deal with **'initial conditions'**.

A physical system is **SELDOM** an i/o [map](#)

Input/state/output systems

$$\leadsto \frac{d}{dt} \mathbf{x} = f(\mathbf{x}, \mathbf{u}), \quad \mathbf{y} = g(\mathbf{x}, \mathbf{u})$$

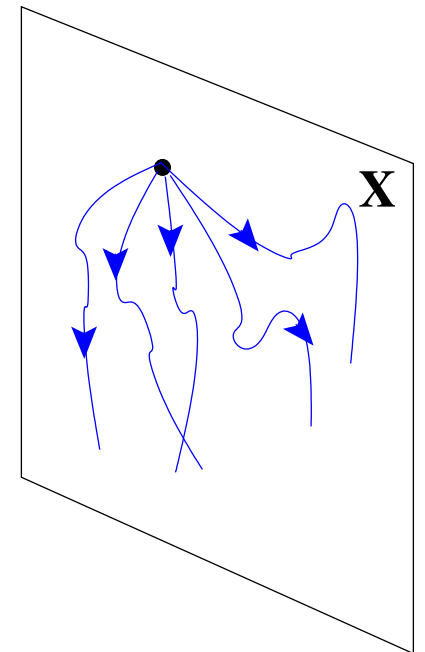


Rudolf Kalman (1930-)

'Axiomatization'

State transition function:

$\phi(t, x, u)$: state reached at time t from x using input u .

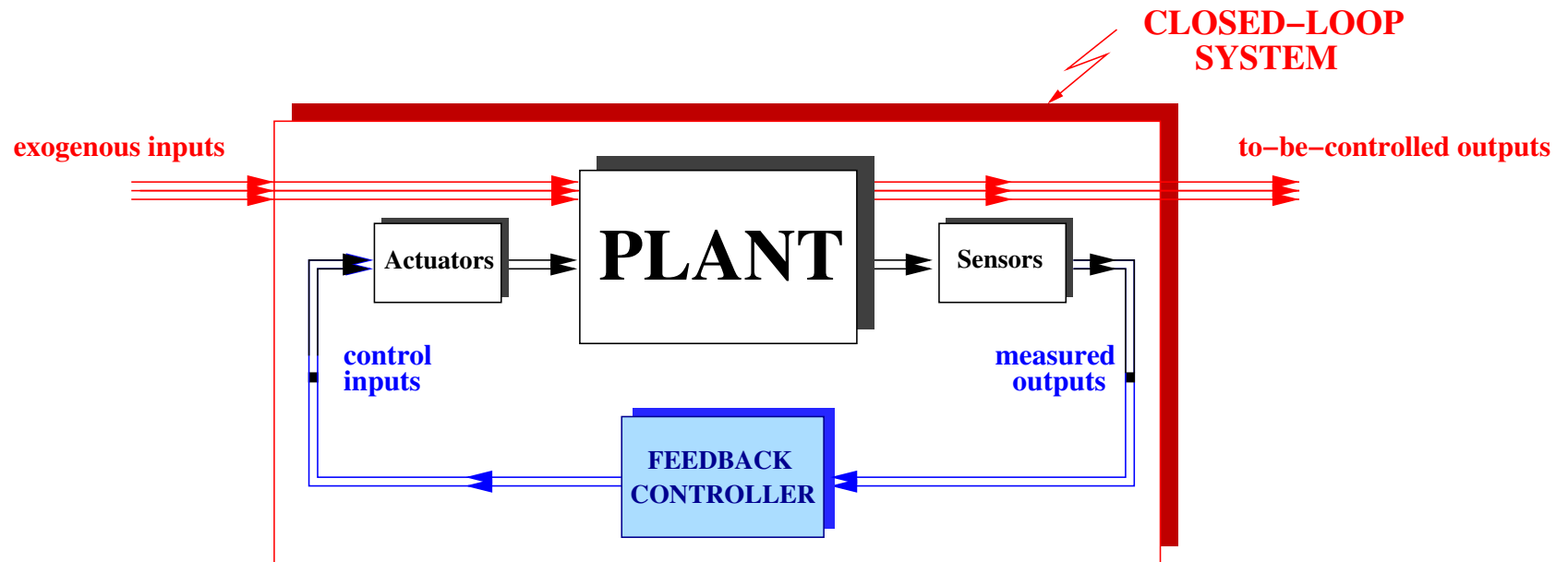


Read-out function:

$g(x, u)$: output value with state x and input value u .

The **input/state/output** view turned out to be
a very effective and fruitful paradigm

- for **control** (stabilization, robustness, ...)



The **input/state/output** view turned out to be
a very effective and fruitful paradigm

- for **control** (stabilization, robustness, ...)
- **prediction** of one signal from another, **filtering**
- understanding **system representations**
(transfer f'n, input/state/output, etc.)
- model simplification, **reduction**
- **system ID:** models from data
- etc., etc., etc.

The need for a new paradigm

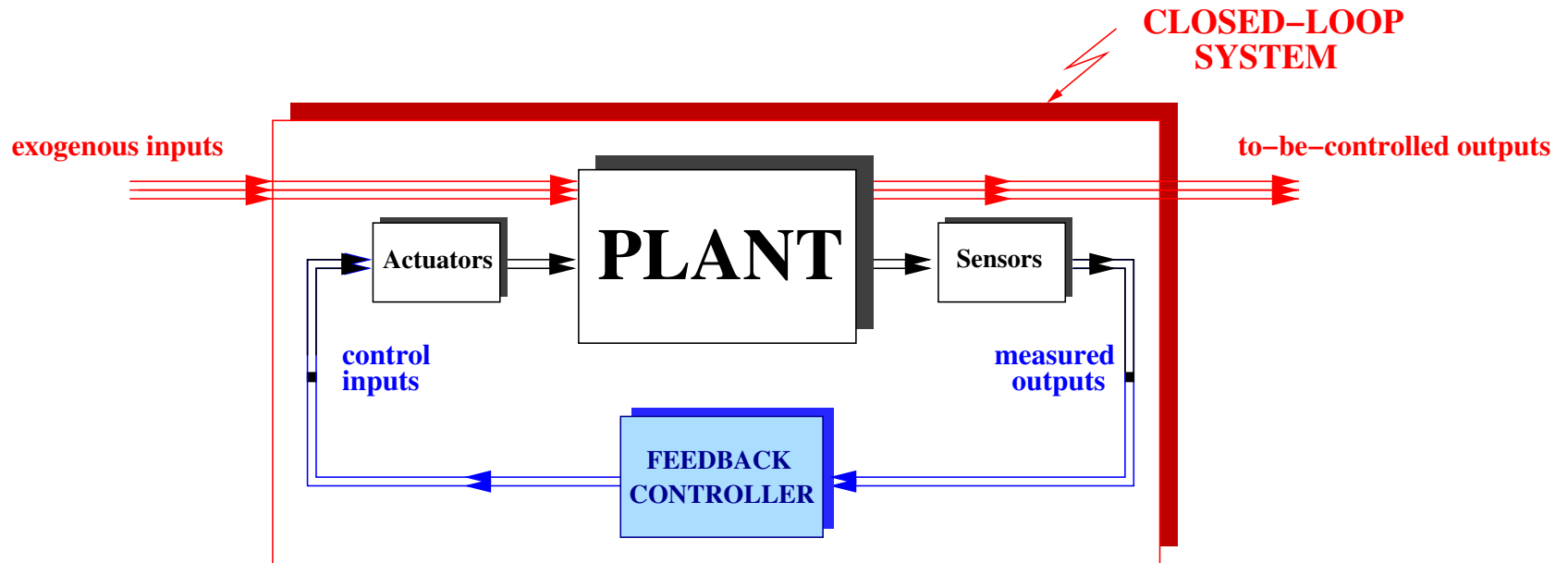
What is a linear system?

Let's take a closer look at the i/o framework ...

in control

Difficulties with i/o

active control



versus **passive control**

Car dampers, heat fins, pressure valves, ...

Controllers without sensors and actuators

Difficulties with i/o

active control versus passive control

Controlling turbulence

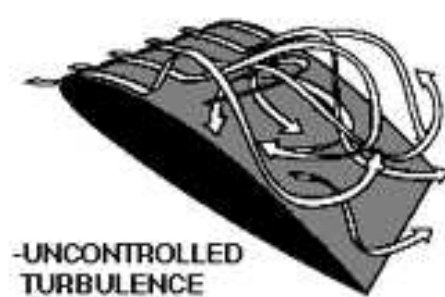
for airplanes, sharks, dolphins, golf balls, bicycling helmets, etc.



Difficulties with i/o

active control versus passive control

Controlling turbulence



Difficulties with i/o

active control versus **passive** control

Controlling turbulence

Nagano 1998



Difficulties with i/o

active control versus **passive** control

Controlling turbulence

Nagano 1998

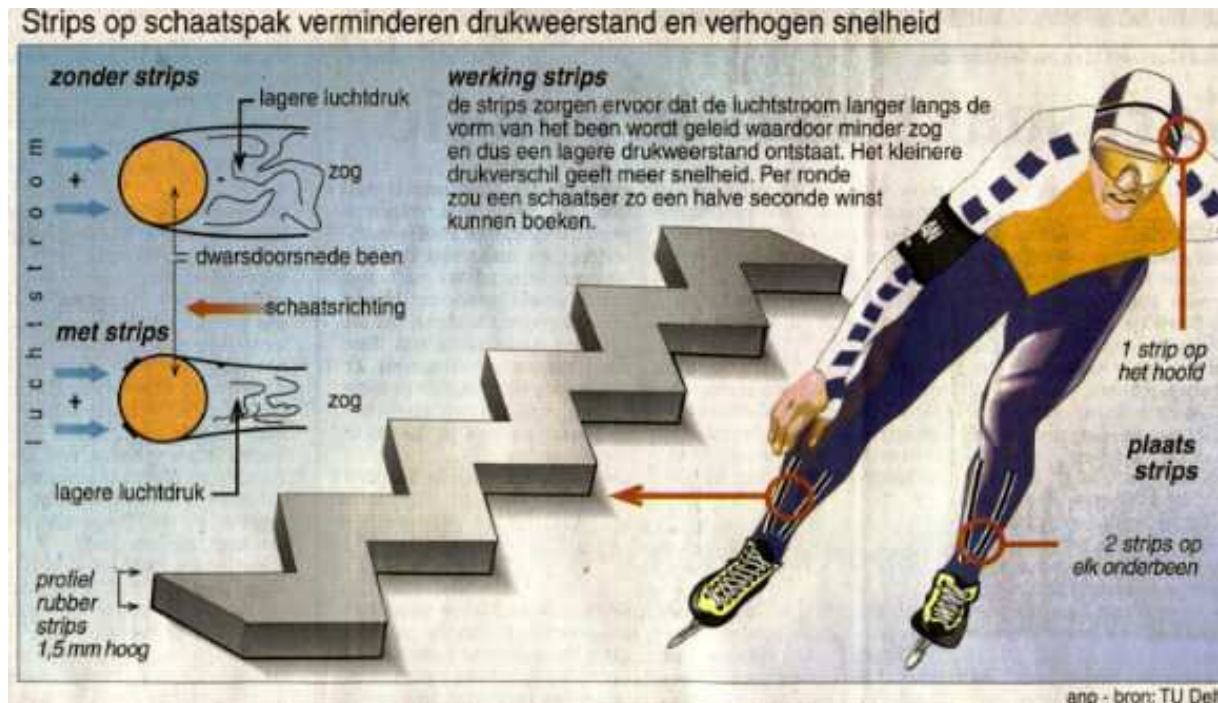


Difficulties with i/o

active control versus passive control

Controlling turbulence

Nagano 1998

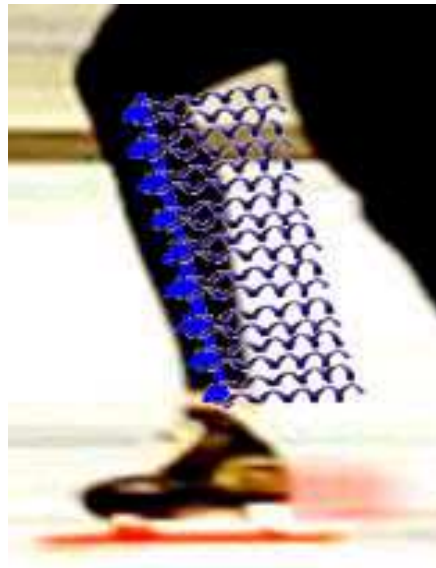


Difficulties with i/o

active control versus passive control

Controlling turbulence

Nagano 1998

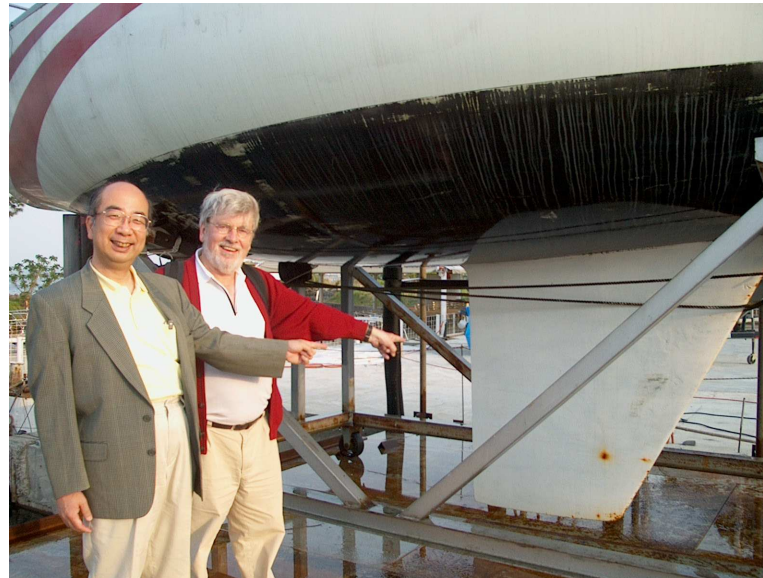


These are beautiful **controllers!** But, the only people not calling this "**control**", are the **control engineers** ...

Difficulties with i/o

active control versus **passive** control

Another example: the stabilizer of a ship

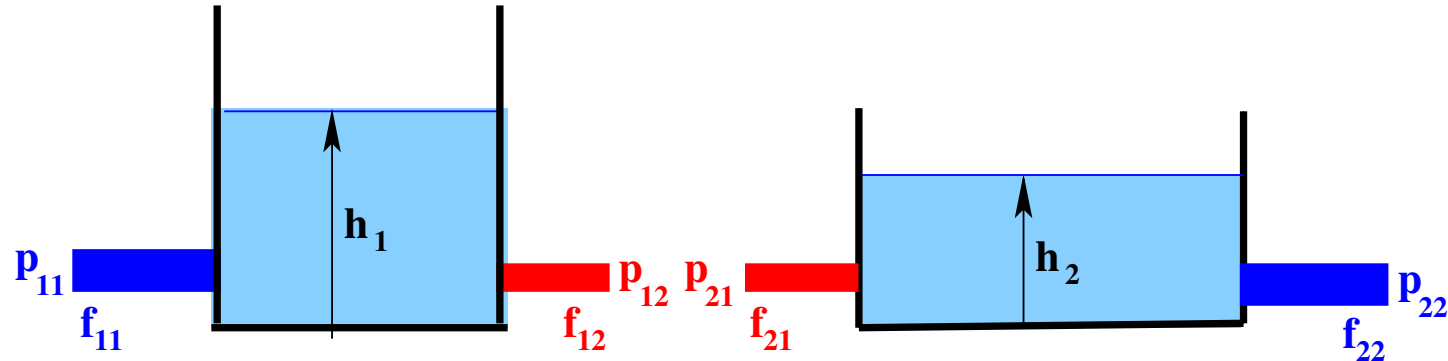


These are beautiful **controllers**! But, the only people not calling this **”stabilization”**, are the **control engineers** ...

Let's take a closer look at the i/o framework ...

for interconnection

i/o and interconnection



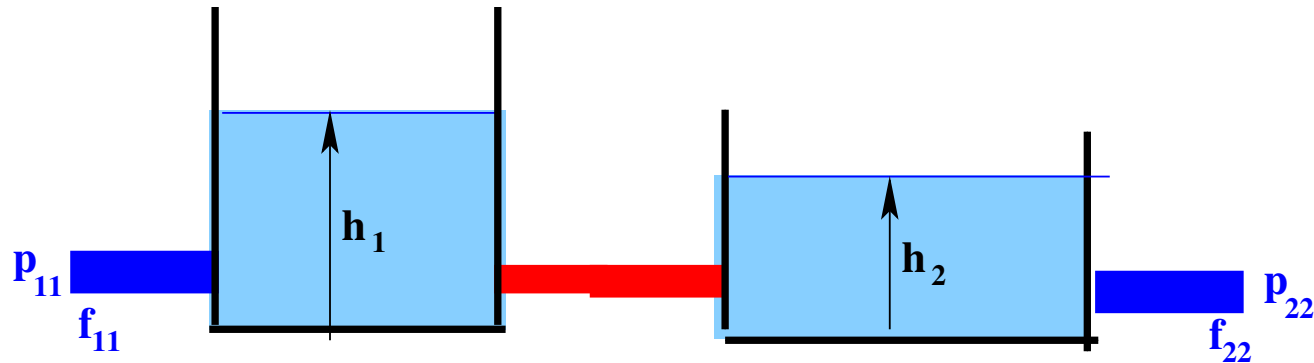
$$\frac{d}{dt}h_1 = F_1(h_1, p_{11}, p_{12}), f_{11} = H_{11}(h_1, p_{11}), f_{12} = H_{12}(h_1, p_{12})$$

$$\frac{d}{dt}h_2 = F_1(h_2, p_{21}, p_{22}), f_{21} = H_{21}(h_2, p_{21}), f_{22} = H_{22}(h_2, p_{22})$$

inputs: the pressures $p_{11}, p_{12}, p_{21}, p_{22}$

outputs: the flows $f_{11}, f_{12}, f_{21}, f_{22}$

i/o and interconnection



$$\frac{d}{dt}h_1 = F_1(h_1, p_{11}, p_{12}), f_{11} = H_{11}(h_1, p_{11}), f_{12} = H_{12}(h_1, p_{12})$$

$$\frac{d}{dt}h_2 = F_2(h_2, p_{21}, p_{22}), f_{21} = H_{21}(h_2, p_{21}), f_{22} = H_{22}(h_2, p_{22})$$

Interconnection:

$$p_{12} = p_{21}, f_{12} + f_{21} = 0$$

This identifies 2 inputs **AND (NOT WITH)** 2 outputs,

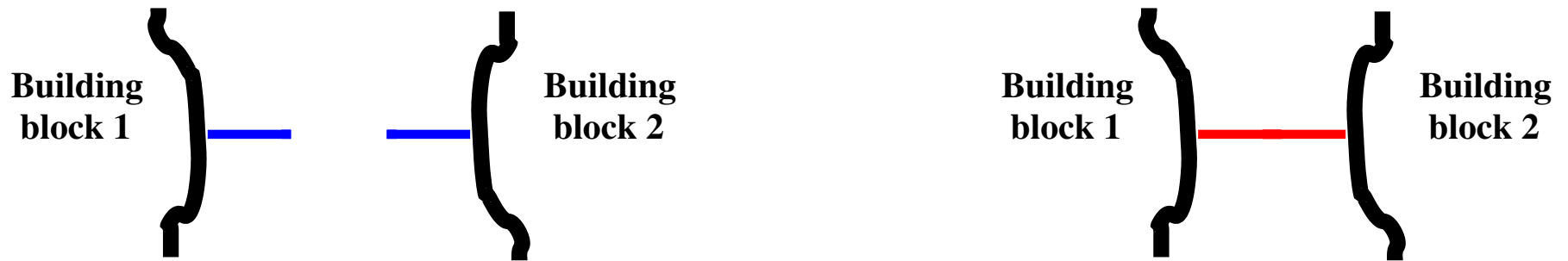
the sort of thing SIMULINK[©] forbids.

This is **the rule, not the exception** (in hydraulics, mechanics,...)

Interconnection is not input-to-output assignment!

Sharing variables, not input-to-output assignment,

is the basic mechanism by which systems interact.



Before interconnection:

the variables on the interconnected terminals are **independent**.

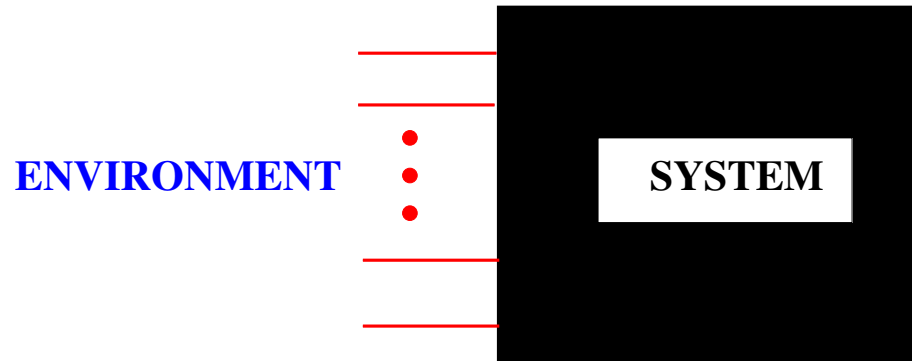
After interconnection: they are set **equal**.

Let's take a closer look at the i/o framework ...

for modeling

i/o in modeling

Physical systems often interact with their environment through **physical** terminals

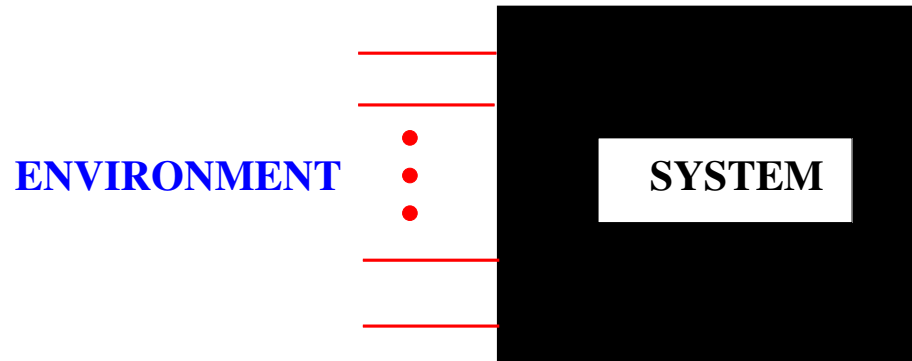


On each of these terminals many variables 'live':

- voltage & current
- position & force
- pressure & flow
- price & demand
- angle & momentum
- etc. & etc.

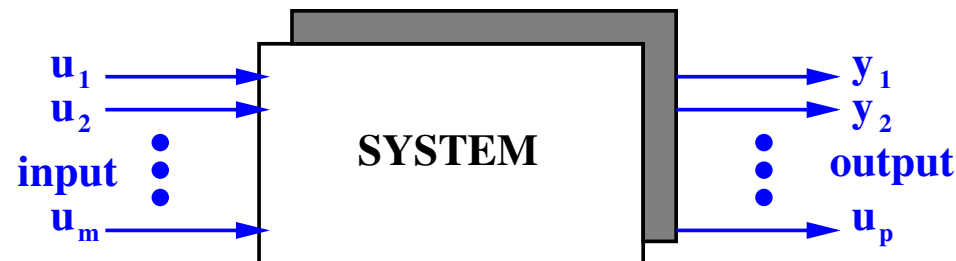
i/o in modeling

Physical systems often interact with their environment through **physical** terminals



Situation is NOT:

on one terminal there is an input, on another there is an output.



This picture is very misleading pedagogically, variables act at same place, indivisibly.

i/o in modeling

Physical systems often interact with their environment through **physical** terminals

The selection of what is an input and what is an output

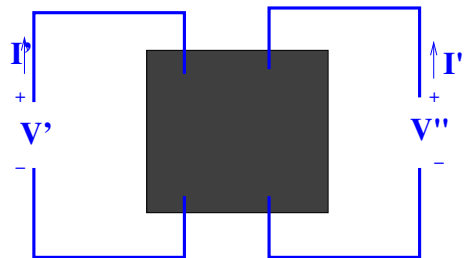
- most often does not need to be made
- if it made, it should be made **after** the modeling is done
- sometimes it **cannot** be made

i/o in modeling

Physical systems often interact with their environment through **physical** terminals

The selection of what is an input and what is an output

- does not need to be made
- if it made, it should be made **after** the modeling is done



voltage controlled?

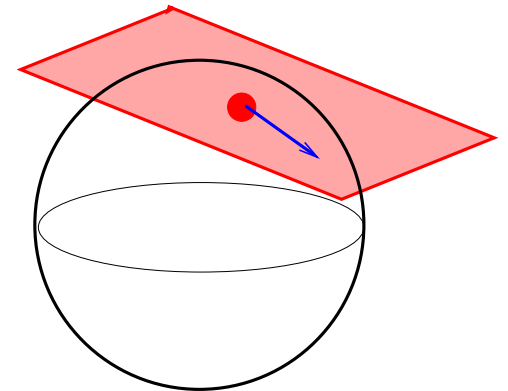
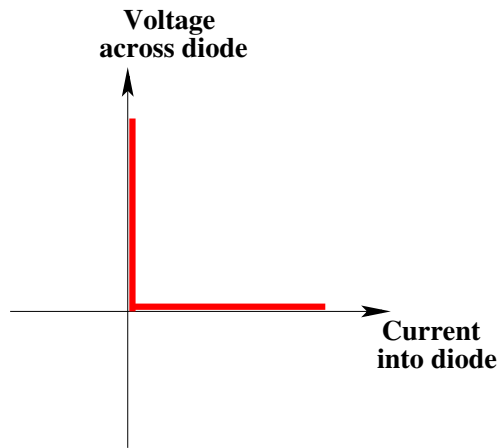
- sometimes it **cannot** be made

i/o in modeling

Physical systems often interact with their environment through **physical** terminals

The selection of what is an input and what is an output

- does not need to be made
- if it made, it should be made **after** the modeling is done
- sometimes it **cannot** be made



variables: (x, v) $\frac{d}{dt}x = v$

tangent bundle of the sphere is not 'trivial'

Conclusion

The inability of the i/o framework to properly deal with

(i) **interconnections**

and

(ii) **passive control**

is lethal.

Just as the state, the input/output partition needs to be **constructed** from first principles models. Contrary to the state, such a partition **may not be useful**, or even possible

We need a better, more flexible, universal, simpler framework that properly deals with

open & connected.

At last, a framework that deals with these difficulties

General formalism

Generalities

What is a model? As a **mathematical** concept.

What is a **dynamical** system?

What is the role of **differential equations** in thinking about dynamical models?

Generalities

Intuition

We have a ‘phenomenon’ that produces ‘outcomes’ (‘events’).
We wish to **model** the outcomes that **can** occur.

Before we model the phenomenon:

the outcomes are in a set, which we call the *universum*.

After we model the phenomenon:

the outcomes are declared (thought, believed)
to belong to the *behavior* of the model,
a subset of this universum.

This subset is what we consider the mathematical model.

Generalities

This way we arrive at the

Definition

A *math. model* is a subset \mathcal{B} of a universum \mathcal{U} of outcomes

$$\mathcal{B} \subseteq \mathcal{U}.$$

\mathcal{B} is called the *behavior* of the model.

For example, **the ideal gas law** states that the temperature T , pressure P , volume V , and quantity (number of moles) N of an ideal gas satisfy

$$\frac{PV}{NT} = R$$

with R a universal constant.

Generalities

So, before Boyle, Charles, and Avogadro got into the act, T , P , V and N may have seemed unrelated, yielding

$$\mathcal{U} = \mathbb{R}_+^4.$$

The ideal gas law restricts the possibilities to

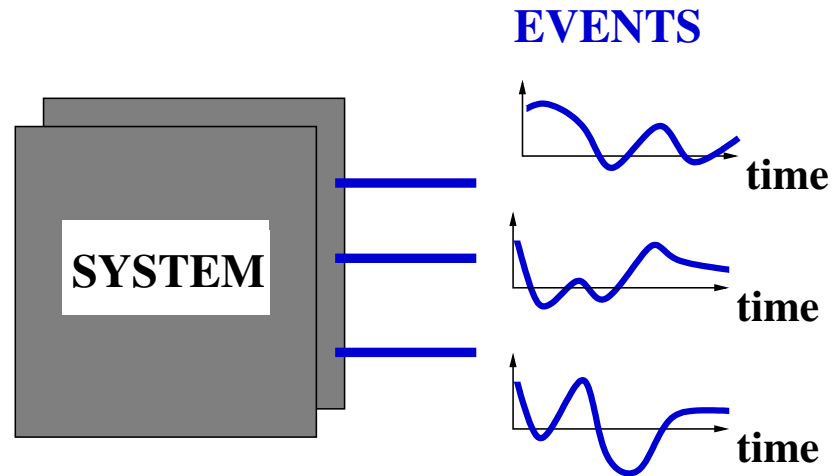
$$\mathcal{B} = \{(T, P, V, N) \in \mathbb{R}_+^4 \mid PV/NT = R\}$$

Features

- **Generality, applicability**
- **shows the role of model equations**
- \rightsquigarrow **notion of equivalent models**
- \rightsquigarrow **notion of more powerful model**
- **Structure, symmetries**
- **...**

Dynamical systems

In dynamics, the outcomes are functions of time \rightsquigarrow



Which event trajectories are possible?

Dynamical systems

Definition

A dynamical system = $\Sigma := (\mathbb{T}, \mathbb{W}, \mathcal{B})$

with $\mathbb{T} \subseteq \mathbb{R}$, the *time-axis* (= the relevant time instances),
 \mathbb{W} , the *signal space*

(= where the variables take on their values),

$\mathcal{B} \subseteq \mathbb{W}^{\mathbb{T}}$ the *behavior* (= the admissible trajectories).

Dynamical systems

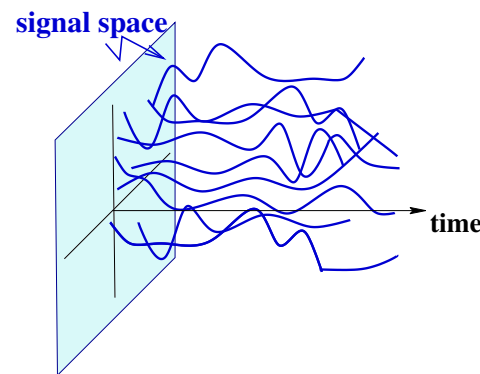
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Totality of 'legal' trajectories =: the behavior

Dynamical systems

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For a trajectory ('an event') $w : \mathbb{T} \rightarrow \mathbb{W}$, we thus have:

$w \in \mathcal{B}$: the model **allows** the trajectory w ,

$w \notin \mathcal{B}$: the model **forbids** the trajectory w .

Dynamical systems

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Usually,

$\mathbb{T} = \mathbb{R}$, or $[0, \infty)$, etc. (in continuous-time systems),
or \mathbb{Z} , or \mathbb{N} , etc. (in discrete-time systems).

Dynamical systems

Definition

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$\mathcal{B} \subseteq \mathbb{W}^{\mathbb{T}}$ the *behavior* (= the admissible trajectories).

Usually,

$\mathbb{W} \subseteq \mathbb{R}^w$ (in lumped systems),

a function space

(in distributed systems, time a distinguished variable),

a finite set (in DES)' etc.

Dynamical systems

Definition

A dynamical system = $\Sigma := (\mathbb{T}, \mathbb{W}, \mathcal{B})$

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(= where the variables take on their values),

$\mathcal{B} \subseteq \mathbb{W}^{\mathbb{T}}$ the *behavior* (= the admissible trajectories).

Emphasis:

$$\mathbb{T} = \mathbb{R},$$

$$\mathbb{W} = \mathbb{R}^w,$$

\mathcal{B} = solution set of system of (linear constant coefficient)
ODE's, or difference eqn's, or PDE's. \rightsquigarrow 'differential systems'.

A series of examples

Examples

Kepler's laws.

K1+K2+K3 obviously define a dynamical system $\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$

$$\mathbb{T} = \mathbb{R}, \quad \mathbb{W} = \mathbb{R}^3,$$

$\mathfrak{B} =$ all $w : \mathbb{R} \rightarrow \mathbb{R}^3$ that satisfy Kepler's 3 laws.

Nice example of a dynamical model 'without equations'.

Examples

Kepler's laws.

K1+K2+K3 obviously define a dynamical system $\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$

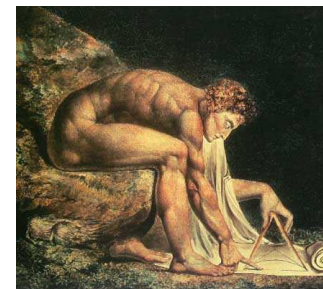
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Nice example of a dynamical model 'without equations'.

Is it a differential system?

This question turned out to be of revolutionary importance...



Examples

Flows: $\frac{d}{dt}x(t) = f(x(t)),$

\mathfrak{B} = all state trajectories.

Observed flows: $\frac{d}{dt}x(t) = f(x(t)); y(t) = h(x(t)),$

\mathfrak{B} = all possible output trajectories.

Examples

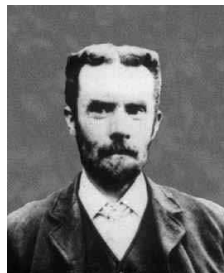
Input / output systems

$$f_1\left(\mathbf{y}(t), \frac{d}{dt}\mathbf{y}(t), \frac{d^2}{dt^2}\mathbf{y}(t), \dots, t\right) \\ = f_2\left(\mathbf{u}(t), \frac{d}{dt}\mathbf{u}(t), \frac{d^2}{dt^2}\mathbf{u}(t), \dots, t\right)$$

$\mathbb{T} = \mathbb{R}$ (time),

$\mathbb{W} = \mathbb{U} \times \mathbb{Y}$ (input \times output signal spaces),

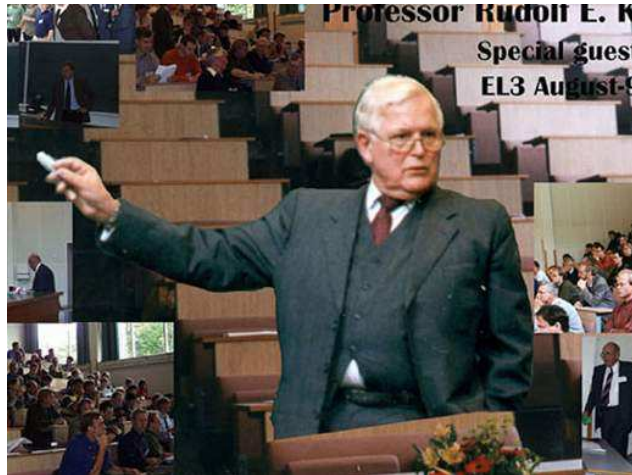
$\mathfrak{B} =$ all input / output pairs.



Examples

Input / state / output systems

$$\frac{d}{dt}\mathbf{x}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t), \quad \mathbf{y}(t) = \mathbf{h}(\mathbf{x}(t), \mathbf{u}(t), t)$$



What do we want to call the behavior?

the $(\mathbf{u}, \mathbf{y}, \mathbf{x})$'s, or the (\mathbf{u}, \mathbf{y}) 's?

Is the (\mathbf{u}, \mathbf{y}) behavior described by a differential eq'n?

Examples

Codes

$\mathcal{C} \subseteq \mathbb{A}^{\mathbb{I}} = \text{the code};$ yields the system $\Sigma = (\mathbb{I}, \mathbb{A}, \mathcal{C})$.

Redundancy structure, error correction possibilities, etc., are visible in the code behavior \mathcal{C} . **It is the central object of study.**

Formal languages

\mathbb{A} = a (finite) alphabet,

$\mathcal{L} \subseteq \mathbb{A}^* = \text{the language} = \text{all 'legal' 'words' } a_1 a_2 \cdots a_k \cdots$

$\mathbb{A}^* = \text{all finite strings with symbols from } \mathbb{A}.$

yields the system $\Sigma = (\mathbb{N}, \mathbb{A}, \mathcal{L})$.

Examples: All words appearing in the *Webster* dictionary

All \LaTeX documents.

Examples

Thermodynamics: a theory of **open** systems, but **NOT** i/o.

\mathcal{L}^\bullet : Linear time-invariant differential systems

More structure

$$\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$$

is said to be **linear**

if \mathbb{W} is a vector space, and \mathfrak{B} a linear subspace of $\mathbb{W}^{\mathbb{T}}$.

More structure

$$\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$$

is said to be **time-invariant**

if $\mathbb{T} = \mathbb{R}, \mathbb{R}_+, \mathbb{Z}$, or \mathbb{Z}_+ and if \mathfrak{B} satisfies

$$\sigma^t \mathfrak{B} \subseteq \mathfrak{B} \text{ for all } t \in \mathbb{T}.$$

σ^t denotes the **shift**, $\sigma^t f(t') := f(t' + t)$.

More structure

$$\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$$

is said to be **differential**

if $\mathbb{T} = \mathbb{R}$, or \mathbb{R}_+ , etc., and if \mathfrak{B} is the solution set of a (system of) ODE's.

a **difference system** if, etc.

or equivalently(!), completeness, or equivalently(!)

\mathfrak{B} is closed - topology of pointwise conv.

\mathcal{L}^w

$$R_0 w + R_1 \frac{d}{dt} w + R_2 \frac{d^2}{dt^2} w + \cdots + R_n \frac{d^n}{dt^n} w = 0.$$

$R \in \mathbb{R}^{\bullet \times w} [\xi]$ $R\left(\frac{d}{dt}\right)w = 0$ defines the

linear, time-invariant, differential system: $\Sigma = (\mathbb{R}, \mathbb{R}^w, \mathfrak{B})$ with

$$\mathfrak{B} = \{w \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^w) \mid R\left(\frac{d}{dt}\right)w = 0\}.$$

\mathcal{L}^w

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NOTATION

\mathcal{L}^\bullet : all such systems (with any - finite - number of variables)

\mathcal{L}^w : with w variables

$\mathfrak{B} \in \mathcal{L}^w$ (no ambiguity regarding \mathbb{T}, \mathbb{W})

\mathcal{L}^w

$$R_0 w + R_1 \frac{d}{dt} w + R_2 \frac{d^2}{dt^2} w + \cdots + R_n \frac{d^n}{dt^n} w = 0.$$

$R \in \mathbb{R}^{\bullet \times w} [\xi]$ $R\left(\frac{d}{dt}\right)w = 0$ defines the

linear, time-invariant, differential system: $\Sigma = (\mathbb{R}, \mathbb{R}^w, \mathfrak{B})$ with

$$\mathfrak{B} = \{w \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^w) \mid R\left(\frac{d}{dt}\right)w = 0\}.$$

NOMENCLATURE

Elements of \mathcal{L}^\bullet : **linear differential systems**

$R\left(\frac{d}{dt}\right)w = 0$: a **kernel representation** of the
corresponding $\Sigma \in \mathcal{L}^\bullet$ or $\mathfrak{B} \in \mathcal{L}^\bullet$

Overview

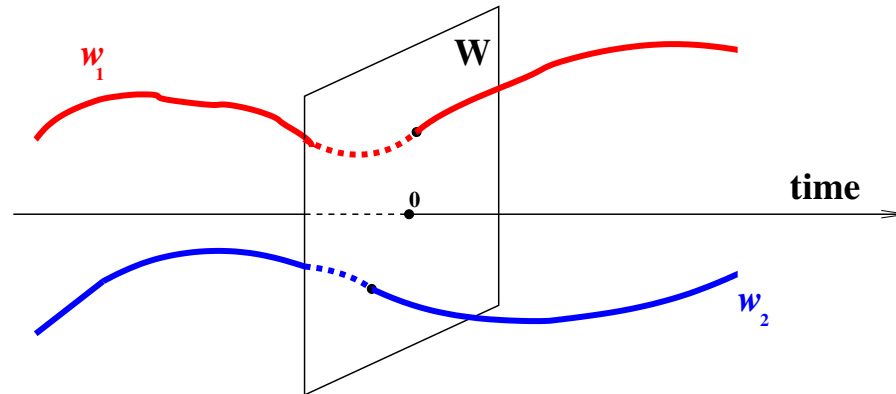
Starting from this vantage point, a rich theory has been developed

1. Modeling by **tearing**, **zooming**, and **linking**
2. **Controllability** and **stabilizability**
3. **Control by interconnection:**
from stabilization to LQ and \mathcal{H}_∞ -control
4. **Observability, observers** and the like
5. **SYSID**, the MPUM, subspace ID
6. **System representations**
7. PDE's
8. etc., etc., ...

Controllability

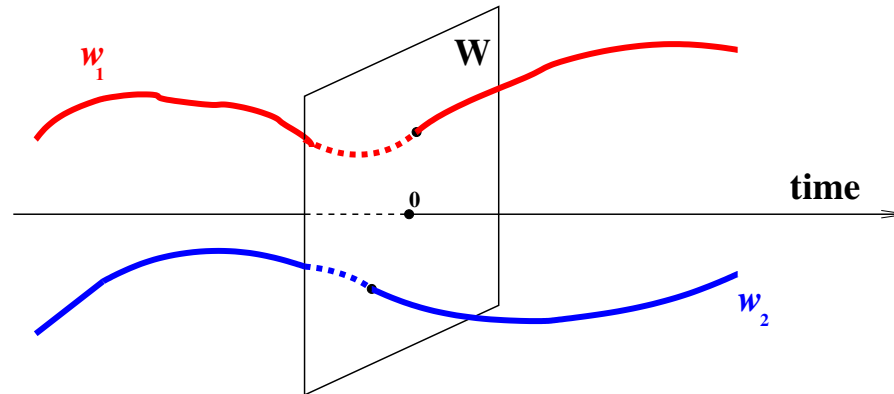
Controllability

Take any two trajectories $w_1, w_2 \in \mathfrak{B}$.

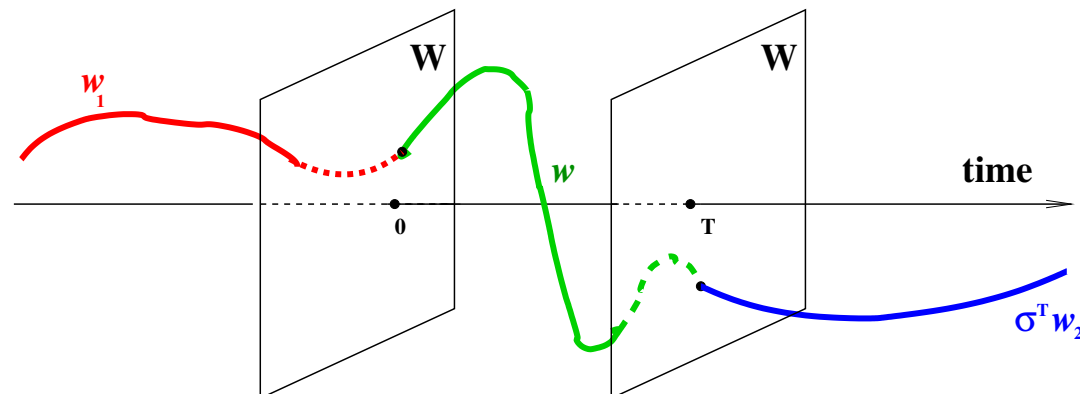


Controllability

Take any two trajectories $w_1, w_2 \in \mathcal{B}$.



Controllability:



Controllability

The time-invariant system $\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$ is said to be

controllable

if for all $w_1, w_2 \in \mathfrak{B}$ there exists $w \in \mathfrak{B}$ and $T \geq 0$ such that

$$w(t) = \begin{cases} w_1(t) & t < 0 \\ w_2(t - T) & t \geq T \end{cases}$$

Controllability $:\Leftrightarrow$

legal trajectories must be **'patch-able', 'concatenable'**.

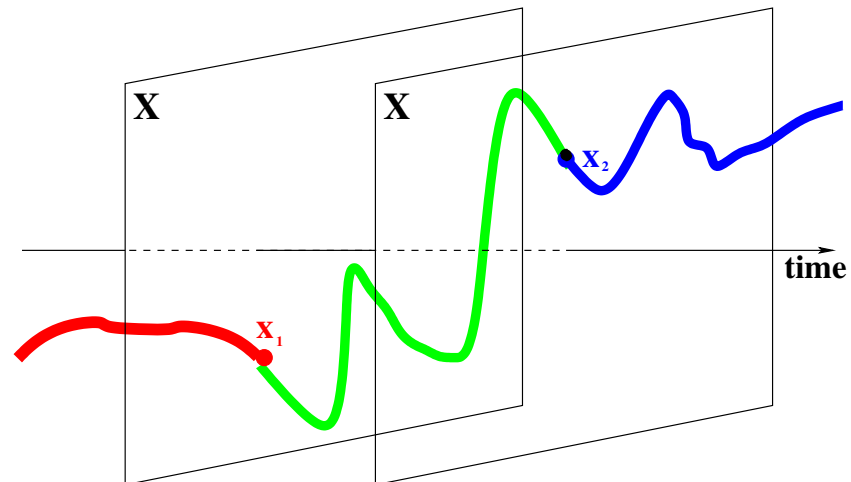
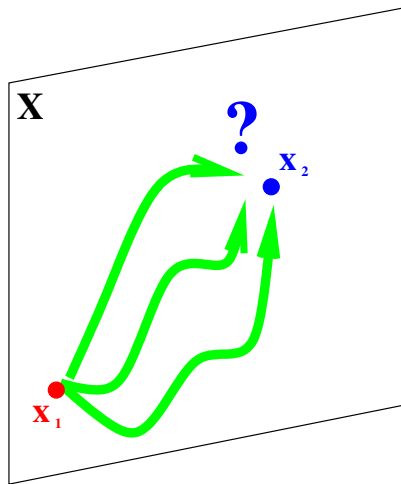
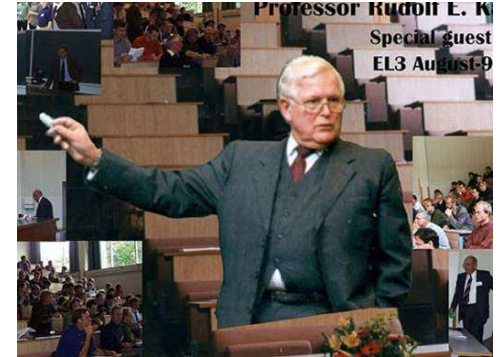
State Controllability

Special case: classical Kalman definitions for

$$\frac{d}{dt}x = f(x, u).$$

controllability: variables = state or (input, state)

This is a **special case** of our controllability:



State Controllability

Special case: classical Kalman definitions for

$$\frac{d}{dt}x = f(x, u).$$

controllability: variables = state or (input, state)

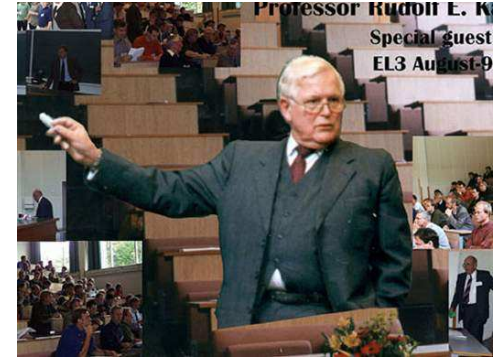
If a system is not (state) controllable, why is it?

Insufficient influence of the control?

Or bad choice of the state?

Or not properly editing the equations?

Kalman's definition addresses a rather special situation.



Tests

Given a system representation, derive algorithms in terms of the parameters for controllability.

Consider the system $\mathfrak{B} \in \mathfrak{L}^\bullet$ defined by

$$R \left(\frac{d}{dt} \right) w = 0.$$

Under what conditions on $R \in \mathbb{R}^{\bullet \times w} [\xi]$ does it define a **controllable system**?

Theorem: $R \left(\frac{d}{dt} \right) w = 0$ defines a controllable system
 \Leftrightarrow
 $\text{rank} (R (\lambda)) = \text{constant over } \lambda \in \mathbb{C}.$

Tests

Notes:

• If $R \left(\frac{d}{dt} \right) w = 0$ has R of full row rank, then

controllability $\Leftrightarrow R(\lambda)$ is of full row rank $\forall \lambda \in \mathbb{C}$.

Equivalently, R is **right-invertible** as a polynomial matrix

(\Leftrightarrow **'left prime'**).

Tests

Notes:

- $\frac{d}{dt}x = Ax + Bu, w = x$ or (x, u) is controllable iff

$$\text{rank}([A - \lambda I \quad B]) = \dim(x) \quad \forall \lambda \in \mathbb{C}.$$

Popov-Belevich-Hautus test for controllability.

Of course,

$$\Leftrightarrow \text{rank} \left(\begin{bmatrix} B & AB & \dots & A^{\dim(x)-1} B \end{bmatrix} \right) = \dim(x).$$

Tests

Notes:

● When is

$$p \left(\frac{d}{dt} \right) w_1 = q \left(\frac{d}{dt} \right) w_2$$

controllable? $p, q \in \mathbb{R}[\xi]$, not both zero.

$$\text{Controllable} \iff \text{rank}([p(\lambda) \quad -q(\lambda)]) = 1 \quad \forall \lambda \in \mathbb{C}.$$

Iff p and q are co-prime. No common factors!

Testable via Sylvester matrix, etc.

Generalizable.

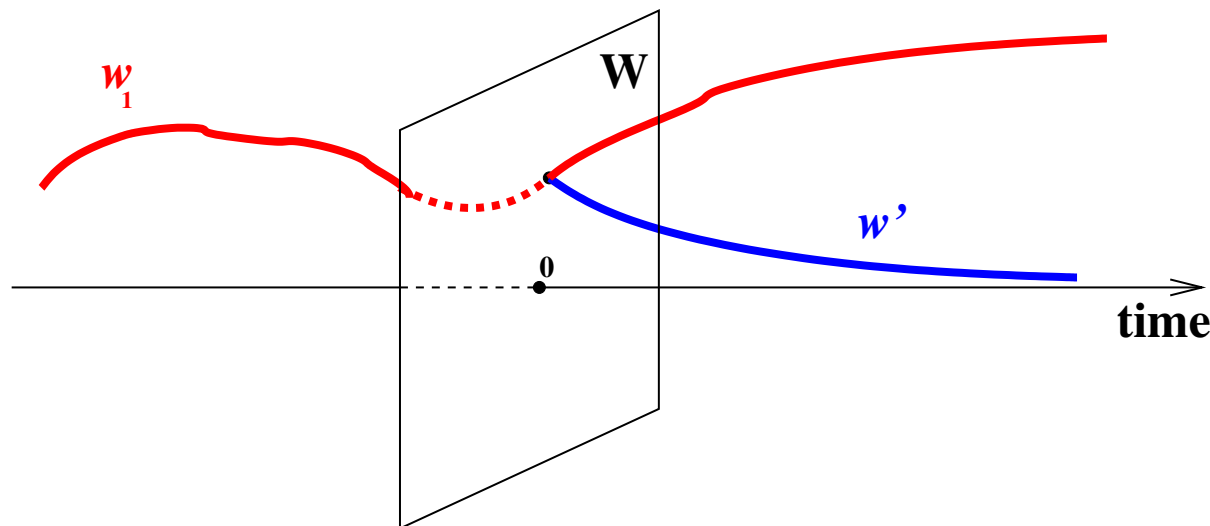
Stabilizability

The system $\Sigma = (\mathbb{T}, \mathbb{R}^w, \mathcal{B})$ is said to be **stabilizable** if, for all $w \in \mathcal{B}$, there exists $w' \in \mathcal{B}$ such that

$$w(t) = w'(t) \text{ for } t < 0 \text{ and } w'(t) \xrightarrow[t \rightarrow \infty]{} 0.$$

Stabilizability $:\Leftrightarrow$

legal trajectories can be steered to a desired point.



Stabilizability

Consider the system defined by

$$R \left(\frac{d}{dt} \right) w = 0.$$

Under which conditions on $R \in \mathbb{R}^{\bullet \times w} [\xi]$ does it define a **stabilizable system**?

Theorem: $R \left(\frac{d}{dt} \right) w = 0$ defines a stabilizable system
 \Leftrightarrow
 $\text{rank} (R (\lambda)) = \text{constant over } \{\lambda \in \mathbb{C} \mid \text{Real} (\lambda) \geq 0\}.$

Image representations

Representations of \mathcal{L}^\bullet : $R \left(\frac{d}{dt} \right) w = 0$

called a *'kernel' representation*. Sol'n set $\in \mathcal{L}^\bullet$, by definition.

$$R \left(\frac{d}{dt} \right) w = M \left(\frac{d}{dt} \right) \ell$$

called a *'latent variable' representation* of the behavior of the w -variables.

'Elimination th'm' $\Rightarrow \in \mathcal{L}^\bullet$.

Image representations

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called a *'latent variable' representation* of the behavior of the w -variables.

'Elimination th'm' $\Rightarrow \in \mathcal{L}^\bullet$.

Missing link:

$$w = M \left(\frac{d}{dt} \right) \ell$$

called an *'image' representation* of $\mathfrak{B} = \text{im} \left(M \left(\frac{d}{dt} \right) \right)$.

Elimination theorem \Rightarrow every image is also a kernel.

?? Which kernels are also images ??

Controllability!

Image representations

Theorem: (Controllability and image representations):

The following are equivalent for $\mathfrak{B} \in \mathcal{L}^\bullet$:

1. \mathfrak{B} is **controllable**
2. \mathfrak{B} admits an **image representation**

$$w = M \left(\frac{d}{dt} \right) \ell$$

3. etc., etc.

Numerical test

- **Image representation leads to an effective numerical test.**
- **∃ similar results & algorithms for time-varying systems.**
- **∃ partial results for nonlinear systems.**

Polynomial representations

Representations with $\mathbb{R}[\xi]$ -matrices of $\mathfrak{B} \in \mathfrak{L}^\bullet$

1. $R \left(\frac{d}{dt} \right) w = 0$ by definition
2. WLOG: R full row rank, in which case uniqueness up to pre-multiplication by unimodular
3. R left prime over $\mathbb{R}[\xi]$ ($\exists S : RS = I$) $\Leftrightarrow \mathfrak{B}$ controllable
4. $w = M \left(\frac{d}{dt} \right) \ell \Leftrightarrow \mathfrak{B}$ controllable
5. if controllable,
WLOG: M right prime over $\mathbb{R}[\xi]$ ($\exists N : NM = I$)
'observable image representation': $\exists N : \ell = N \left(\frac{d}{dt} \right) w$.

Representations with rational symbols

Let $G \in \mathbb{R}^{\bullet \times w}(\xi)$.

What does $G\left(\frac{d}{dt}\right)w = 0$ mean?

Representations with rational symbols

The behavior defined by $G\left(\frac{d}{dt}\right)w = 0$ is defined as that of

$$Q\left(\frac{d}{dt}\right)w = 0$$

with $G = P^{-1}Q$ a left co-prime factorization over $\mathbb{R}[\xi]$ of G .

Equivalently, the output nulling behavior of

$$\frac{d}{dt}x = Ax + Bw, 0 = Cx + D\left(\frac{d}{dt}\right)w$$

(A, B) contr., tf. f'n G .

Representations with rational symbols

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Representations with $\mathbb{R}(\xi)$ -matrices of $\mathfrak{B} \in \mathcal{L}^\bullet$.

1. WLOG, with G (strictly) proper, etc.
2. G left prime over ring of stable rational f'ns $\Leftrightarrow \mathfrak{B}$ stabilizable
3. $w = G\left(\frac{d}{dt}\right)\ell \Leftrightarrow \mathfrak{B}$ controllable
4. if controllable, WLOG: G right prime over stable rational f'ns
'observable im. repr'on': $\exists F$ stable rational : $\ell = F\left(\frac{d}{dt}\right)w$.

PDE's

PDE's

Much of the theory also holds for PDE's.

$T = \mathbb{R}^n$, the set of independent variables, often $n = 4$,

$W = \mathbb{R}^w$, the set of dependent variables,

$\mathcal{B} =$ **sol'ns of a linear constant coefficient system of PDE's.**

PDE's

Much of the theory also holds for PDE's.

$\mathbb{T} = \mathbb{R}^n$, the set of independent variables, often $n = 4$,

$\mathbb{W} = \mathbb{R}^w$, the set of dependent variables,

$\mathcal{B} =$ **sol'ns of a linear constant coefficient system of PDE's.**

Let $R \in \mathbb{R}^{\bullet \times w}[\xi_1, \dots, \xi_n]$, and consider

$$R \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \right) w = 0. \quad (*)$$

Define the associated behavior

$$\mathcal{B} = \{ w \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}^w) \mid (*) \text{ holds} \}.$$

Notation for n -D linear differential systems:

$$(\mathbb{R}^n, \mathbb{R}^w, \mathcal{B}) \in \mathcal{L}_n^w, \quad \text{or } \mathcal{B} \in \mathcal{L}_n^w.$$

Example

Maxwell's eq'ns, diffusion eq'n, wave eq'n, . . .



$$\begin{aligned}\nabla \cdot \vec{E} &= \frac{1}{\epsilon_0} \rho, \\ \nabla \times \vec{E} &= -\frac{\partial}{\partial t} \vec{B}, \\ \nabla \cdot \vec{B} &= 0, \\ c^2 \nabla \times \vec{B} &= \frac{1}{\epsilon_0} \vec{j} + \frac{\partial}{\partial t} \vec{E}.\end{aligned}$$

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$\mathbb{T} = \mathbb{R} \times \mathbb{R}^3$ (time and space) $n = 4$,

$$w = (\vec{E}, \vec{B}, \vec{j}, \rho)$$

(electric field, magnetic field, current density, charge density),

$$\mathbb{W} = \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}, w = 10,$$

\mathcal{B} = set of solutions to these PDE's.

Note: 10 variables, 8 equations! $\Rightarrow \exists$ free variables. 'open' system.

Submodule theorem

$R \in \mathbb{R}^{\bullet \times \bullet}[\xi_1, \dots, \xi_n]$ defines $\mathfrak{B} = \ker \left(R \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \right) \right)$,

but not vice-versa.

?? \exists 'intrinsic' characterization of $\mathfrak{B} \in \mathfrak{L}_n^w$??

Is there a mathematical 'object' that characterizes a $\mathfrak{B} \in \mathfrak{L}_n^w$?

Define the **annihilators** of $\mathfrak{B} \in \mathfrak{L}_n^w$ by

$$\mathfrak{N}_{\mathfrak{B}} := \left\{ n \in \mathbb{R}^w[\xi_1, \dots, \xi_n] \mid n^\top \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \right) \mathfrak{B} = 0 \right\}.$$

Proposition:

$\mathfrak{N}_{\mathfrak{B}}$ is a $\mathbb{R}[\xi_1, \dots, \xi_n]$ sub-module of $\mathbb{R}^w[\xi_1, \dots, \xi_n]$.

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Proposition:

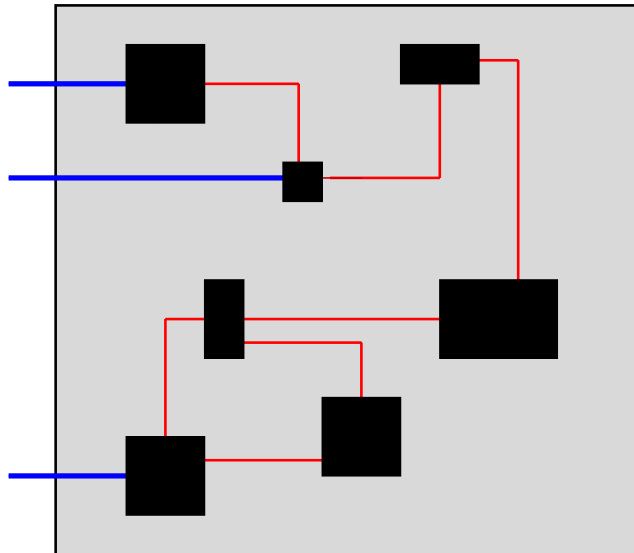
$\mathfrak{N}_{\mathfrak{B}}$ is a $\mathbb{R}[\xi_1, \dots, \xi_n]$ sub-module of $\mathbb{R}^w[\xi_1, \dots, \xi_n]$.

Theorem:

$\mathfrak{L}_n^w \xleftrightarrow{\text{bijective}} \text{submodules of } \mathbb{R}^w[\xi_1, \dots, \xi_n]$

Elimination theorem

Motivation: In many problems, we want to eliminate variables. For example, **first principle modeling**



~> model containing both variables the model aims at (**manifest** variables), and auxiliary variables introduced in the modeling process (**latent** variables).

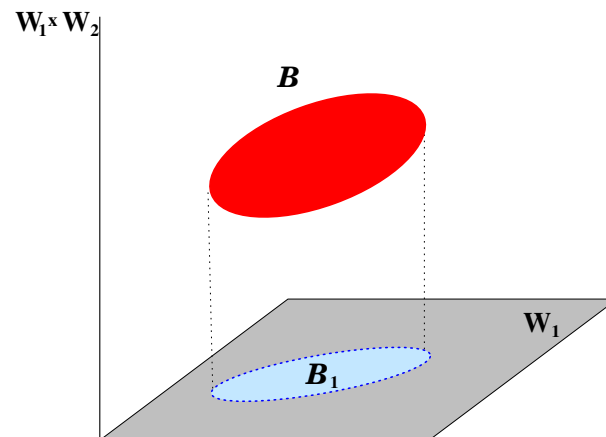
¿ Can these latent variables be eliminated from the equations ?

Elimination theorem

This leads to the following important question, first in polynomial matrix language. Consider

$$R_1\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right)w_1 = R_2\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right)w_2.$$

Obviously, the behavior of the (w_1, w_2) 's is described by a system of PDE's. **¿ Is the behavior of the w_1 's alone also ?**



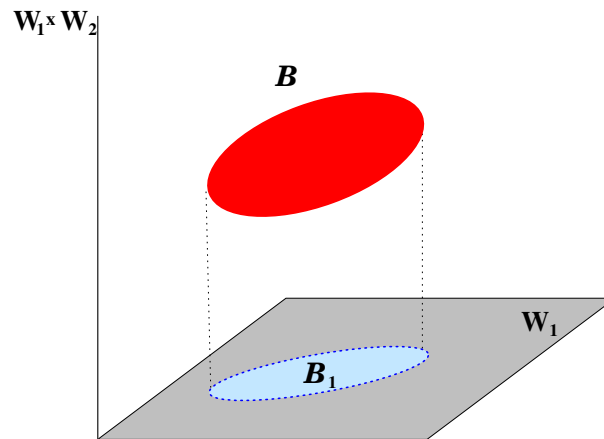
Elimination theorem

In the language of behaviors:

Let $\mathcal{B} \in \mathcal{L}_n^{w_1 + w_2}$. Define

$$\mathcal{B}_1 = \{w_1 \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}^{w_1}) \mid \exists w_2 \text{ such that } (w_1, w_2) \in \mathcal{B}\}.$$

Does this 'projection' \mathcal{B}_1 belong to $\mathcal{L}_n^{w_1}$?



Theorem:

It does!

\mathcal{L}^\bullet is closed under projection !!

The elimination theorem

There exist effective algorithms for $(R_1, R_2) \mapsto R$.

↪ Computer algebra, Gröbner bases.

It follows from all this that \mathcal{L}_n^\bullet has very nice properties. In particular, it is **closed** under:

- **Intersection:** $(\mathcal{B}_1, \mathcal{B}_2 \in \mathcal{L}_n^w) \Rightarrow (\mathcal{B}_1 \cap \mathcal{B}_2 \in \mathcal{L}^w)$
- **Addition:** $(\mathcal{B}_1, \mathcal{B}_2 \in \mathcal{L}_n^w) \Rightarrow (\mathcal{B}_1 + \mathcal{B}_2 \in \mathcal{L}_n^w)$
- **Projection:** $(\mathcal{B} \in \mathcal{L}_n^{w_1+w_2}) \Rightarrow (\Pi_{w_1} \mathcal{B} \in \mathcal{L}_n^{w_1}) \Pi_{w_1}$: projection
- **Action of a linear differential operator:**
 $(\mathcal{B} \in \mathcal{L}_n^{w_1}, P \in \mathbb{R}^{w_2 \times w_1}[\xi_1, \dots, \xi_n]) \Rightarrow (P(\frac{d}{dt})\mathcal{B} \in \mathcal{L}_n^{w_2}).$
- **Inverse image of a linear differential operator:**
 $(\mathcal{B} \in \mathcal{L}_n^{w_2}, P \in \mathbb{R}^{w_2 \times w_1}[\xi_1, \dots, \xi_n]) \Rightarrow (P(\frac{d}{dt}))^{-1}\mathcal{B} \in \mathcal{L}_n^{w_1}.$

Elimination theorem

Which PDE's describe (ρ, \vec{E}, \vec{j}) in Maxwell's equations ?

Eliminate \vec{B} from Maxwell's equations \rightsquigarrow

$$\begin{aligned}\nabla \cdot \vec{E} &= \frac{1}{\epsilon_0} \rho, \\ \epsilon_0 \frac{\partial}{\partial t} \nabla \cdot \vec{E} + \nabla \cdot \vec{j} &= 0, \\ \epsilon_0 \frac{\partial^2}{\partial t^2} \vec{E} + \epsilon_0 c^2 \nabla \times \nabla \times \vec{E} + \frac{\partial}{\partial t} \vec{j} &= 0.\end{aligned}$$

$$R \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \right) w = 0$$

is called a **kernel representation** of the associated $\mathfrak{B} \in \mathfrak{L}_n^W$.

Another representation: **image representation**

$$w = M \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \right) \ell.$$

‘Elimination’ thm $\Rightarrow \text{im} \left(M \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \right) \right) \in \mathfrak{L}_n^W !$

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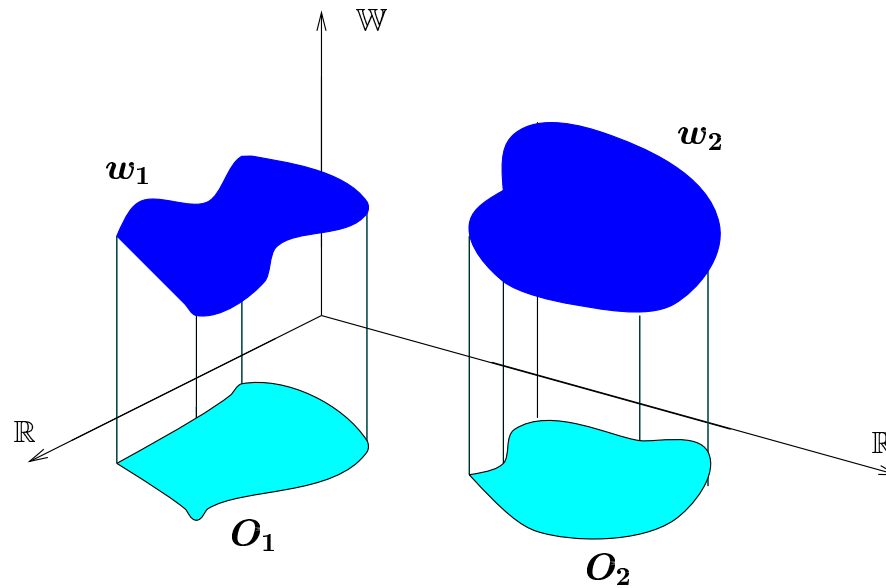
‘Elimination’ thm $\Rightarrow \text{im} \left(M \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \right) \right) \in \mathfrak{L}_n^w !$

Which linear diff. systems admit an image representation???

$\mathfrak{B} \in \mathfrak{L}_n^w$ admits an image representation iff it is **‘controllable’**.

Controllability for PDE's

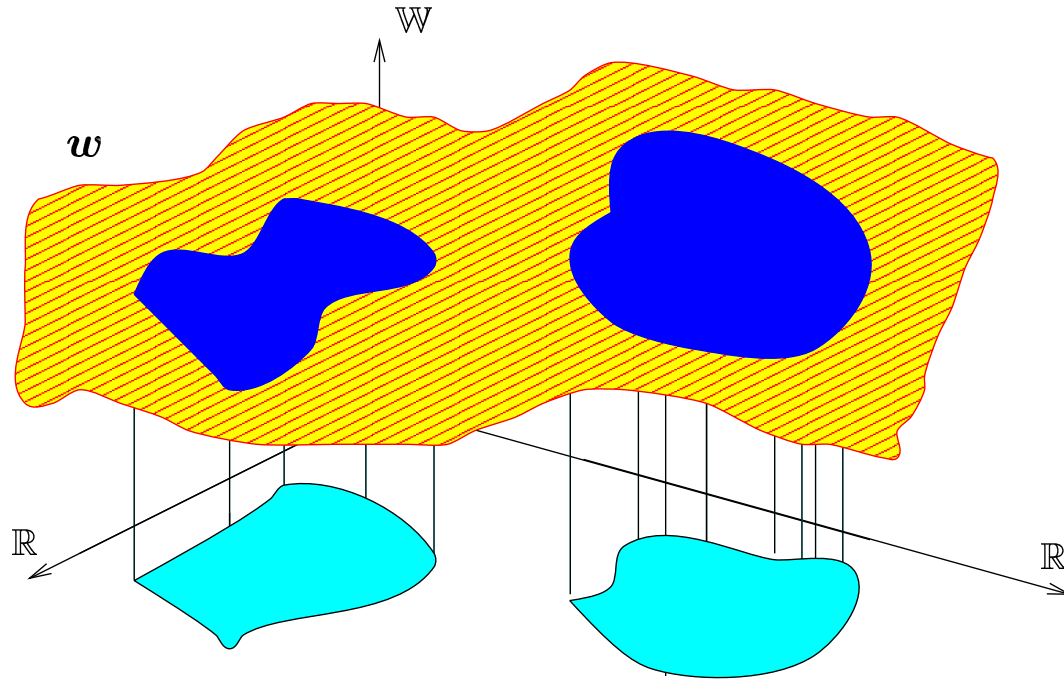
Controllability def'n in pictures:



$$w_1, w_2 \in \mathcal{B}.$$

Controllability for PDE's

$\exists w \in \mathfrak{B}$ 'patches' $w_1, w_2 \in \mathfrak{B}$.



Controllability $:\Leftrightarrow$ 'patch-ability'.

Are Maxwell's equations controllable ?

The following equations in the *scalar potential* $\phi : \mathbb{R} \times \mathbb{R}^3 \rightarrow \mathbb{R}$ and the *vector potential* $\vec{A} : \mathbb{R} \times \mathbb{R}^3 \rightarrow \mathbb{R}^3$, generate exactly the solutions to Maxwell's equations:

$$\vec{E} = -\frac{\partial}{\partial t}\vec{A} - \nabla\phi,$$

$$\vec{B} = \nabla \times \vec{A},$$

$$\vec{j} = \epsilon_0 \frac{\partial^2}{\partial t^2} \vec{A} - \epsilon_0 c^2 \nabla^2 \vec{A} + \epsilon_0 c^2 \nabla (\nabla \cdot \vec{A}) + \epsilon_0 \frac{\partial}{\partial t} \nabla \phi,$$

$$\rho = -\epsilon_0 \frac{\partial}{\partial t} \nabla \cdot \vec{A} - \epsilon_0 \nabla^2 \phi.$$

Proves controllability. Illustrates the interesting connection

controllability $\Leftrightarrow \exists$ potential!

Conclusion

The flexibility and generality of the behavioral approach in modeling, for system representations, for passive control, dealing with PDE's, etc. is evident.

Exemplified by the notion of controllability.

Conclusion

The flexibility and generality of the behavioral approach in modeling, for system representations, for passive control, dealing with PDE's, etc. is evident.

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**Nature and Nature's laws lay hid in night
God said, 'Let Newton be' and all was light**

**Mathematical Systems Theory lay bound by might
Ratio said, 'Let Behaviors be' and all was right**

Details & copies of the lecture frames are available from/at

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