DISSIPATIVE DYNAMICAL SYSTEMS

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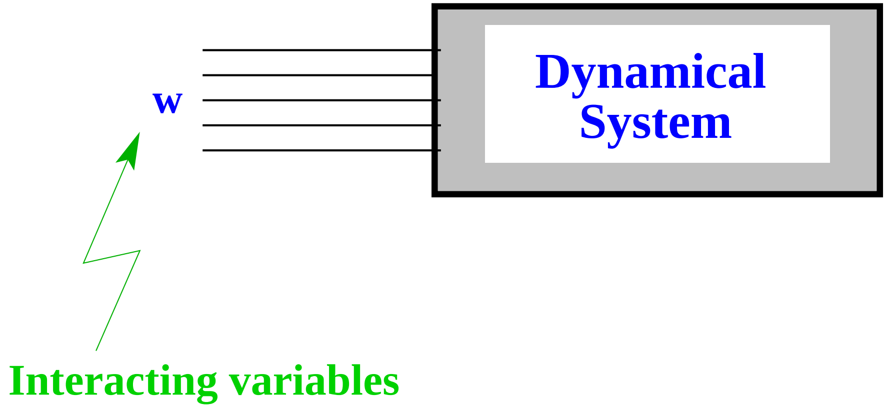
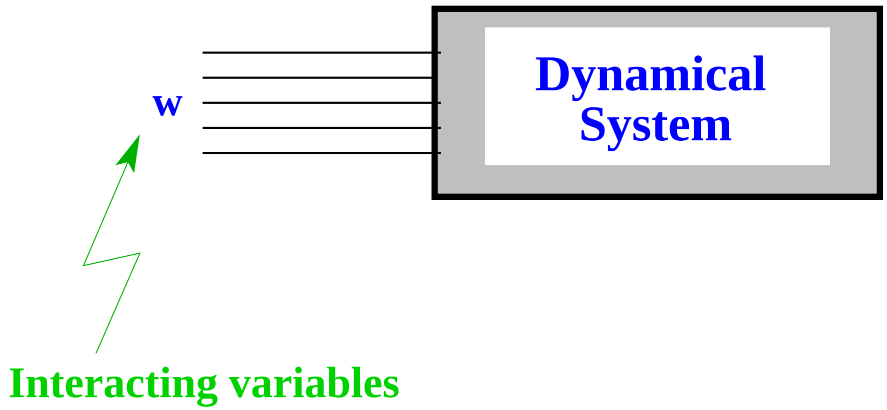
Theme

Dissipativity



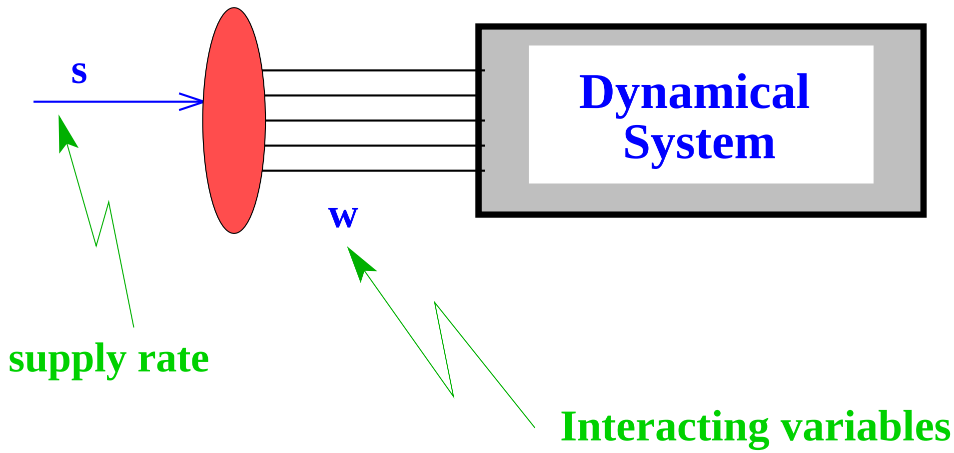
Dissipativity



Assume the variables have a physical meaning, and that a function of the variables means *'supply'*. For example,

- **mass flow**
- **currents** I **and voltages** V $\rightsquigarrow V^\top I =$ **electrical 'power'**
  rate of electrical energy supplied
- **positions** q**, forces** F $\rightsquigarrow F^\top \frac{d}{dt}q =$ **mechanical 'power'**
- **temperature** T **and heat flow** Q $\rightsquigarrow = \frac{Q}{T}$ **'entropy flow'**

Dissipativity



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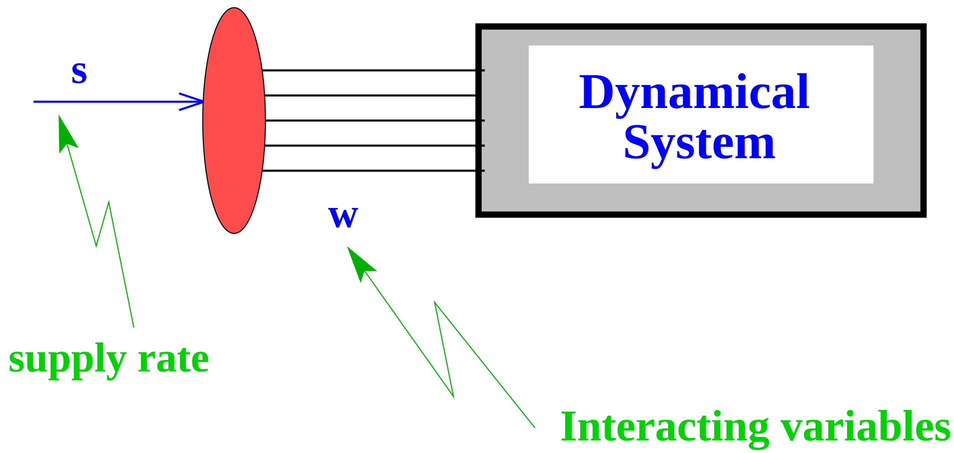
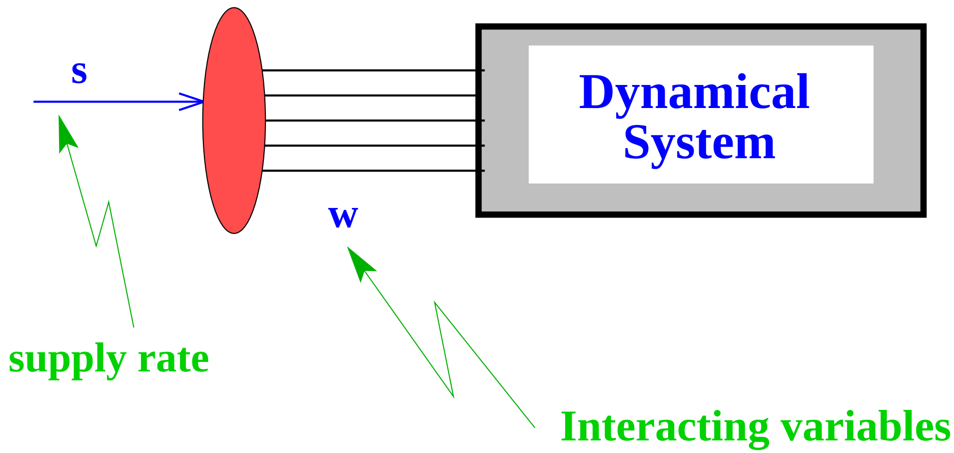


There is a *'supply'*; dissipative: part of the supply is *'lost'*.

- **mass flow** — **loss: leakage**
- **currents I and voltages V $\leadsto V^\top I =$ 'electrical power'**
  loss: heat in resistors
- **positions p, forces F $\leadsto F^\top \frac{d}{dt} q =$ 'mechanical power'**
  heat loss due to friction
- **temperature T and heat flow Q $\leadsto = \frac{Q}{T}$ 'entropy flow'**
  irreversibility

Dissipativity



How do we formalize this concept?

Relevance:

- System theory for physical systems
- Mechanisms for stability
  by interconnecting dissipative systems
- Robustness
- Mechanisms for stabilization by adding friction

Lyapunov functions

Lyapunov functions

Consider the classical dynamical system, the ***'flow'***

$$\Sigma : \frac{d}{dt}x = f(x)$$

with $x \in \mathbb{X} = \mathbb{R}^{\mathtt{n}}$ **the** ***state*** **and** $f : \mathbb{X} \to \mathbb{X}$ **the** ***vectorfield*****.**

Denote the set of solutions $x : \mathbb{R} \to \mathbb{X}$ **by** $\mathfrak{B}$**, the** ***'behavior'.***

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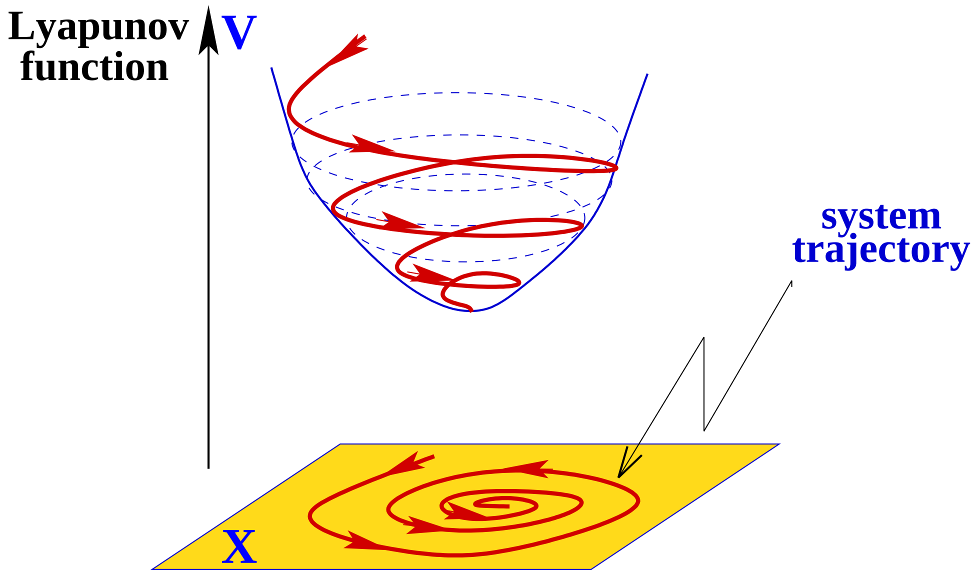
$$V : \mathbb{X} \to \mathbb{R}$$

is said to be a ***Lyapunov function*** **for** Σ **if along** $x \in \mathfrak{B}$

$$\frac{d}{dt} V(x(\cdot)) \leq 0$$

Equivalently, if $\overset{\bullet}{V}^{\Sigma} := \nabla V \cdot f \leq 0.$

Lyapunov type ‘theorem’



$$V(x) > 0 \textbf{ and } \overset{\bullet}{V}^{\Sigma}(x) < 0 \textbf{ for } 0 \neq x \in \mathbb{X}$$

$$\Rightarrow$$

$\forall \ x \in \mathfrak{B},$ **there holds** $x(t) \to 0$ **for** $t \to \infty$ **‘global stability’**

Lyapunov

Lyapunov f'ns play a remarkably central role in the field.

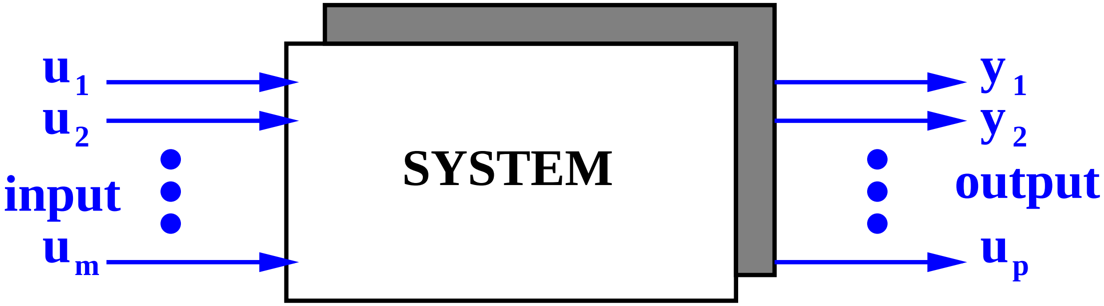
Aleksandr Mikhailovich Lyapunov (1857-1918)

Introduced Lyapunov's 'second method' in his thesis (1899).

The classical notion of a dissipative system

Open systems

'Open' systems are a much more appropriate starting point for the study of dynamics than 'flows'. For example,



$\leadsto$ **the dynamical system**

$$\Sigma: \quad \frac{d}{dt}x = f(x,u), \quad y = h(x,u).$$

$u \in \mathbb{U} = \mathbb{R}^{\mathtt{m}}, y \in \mathbb{Y} = \mathbb{R}^{\mathtt{p}}, x \in \mathbb{X} = \mathbb{R}^{\mathtt{n}}$**: input, output, state.**

Behavior $\mathfrak{B} =$ **all sol'ns** $(u,y,x) : \mathbb{R} \to \mathbb{U} \times \mathbb{Y} \times \mathbb{X}$.

Dissipative dynamical systems

Now consider

$s : \mathbb{U} \times \mathbb{Y} \longrightarrow \mathbb{R}$ **called the** ***supply rate*****,**

$V : \mathbb{X} \longrightarrow \mathbb{R}$ **called the** ***storage functon*****.**

Σ is said to be

dissipative **w.r.t. the supply rate s and with storage V**

$:\Leftrightarrow$

$$\frac{d}{dt} V(x(\cdot)) \leq s(u(\cdot), y(\cdot))$$

for all $(u, y, x) \in \mathfrak{B}$**.**

Dissipation inequality

$$\frac{d}{dt} V(x(\cdot)) \leq s(u(\cdot), y(\cdot))$$

for all $(u, y, x) \in \mathfrak{B}$**.**

This inequality is called the ***dissipation inequality.***

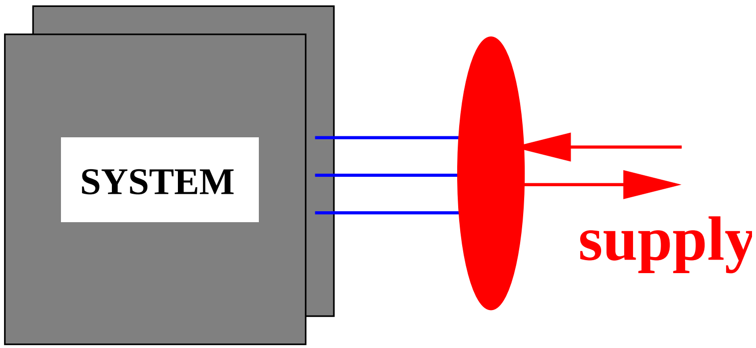
Equivalent to

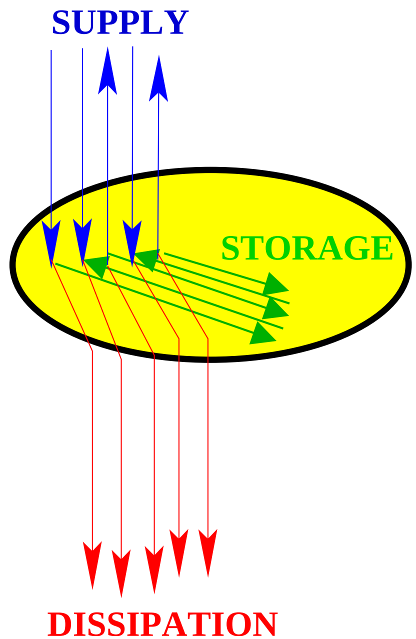
$$\overset{\bullet}{V}^{\Sigma}(\mathrm{x}, \mathrm{u}) := \nabla V(\mathrm{x}) \cdot f(\mathrm{x}, \mathrm{u}) \leq s(\mathrm{x}, h(\mathrm{x}, \mathrm{u}))$$

for all $(\mathrm{u}, \mathrm{x}) \in \mathbb{U} \times \mathbb{X}$**.**

If equality holds: 'conservative' system.

Dissipation inequality





$s(\mathrm{u}, \mathrm{y})$ models something like the power delivered to the system when the input value is u and output value is y.

$V(\mathrm{x})$ then models the internally stored energy.

Dissipativity $:\Leftrightarrow$

rate of increase of internal energy $\leq$ power delivered

The construction of storage functions

Basic question:

Given (a representation of) Σ, the dynamics,
and given s, the supply rate,
is the system dissipative w.r.t. s, i.e.
does there exist a storage function V such that
the dissipation inequality holds?

Analog question of construction of Lyapunov f'n for stable systems.

The construction of storage functions

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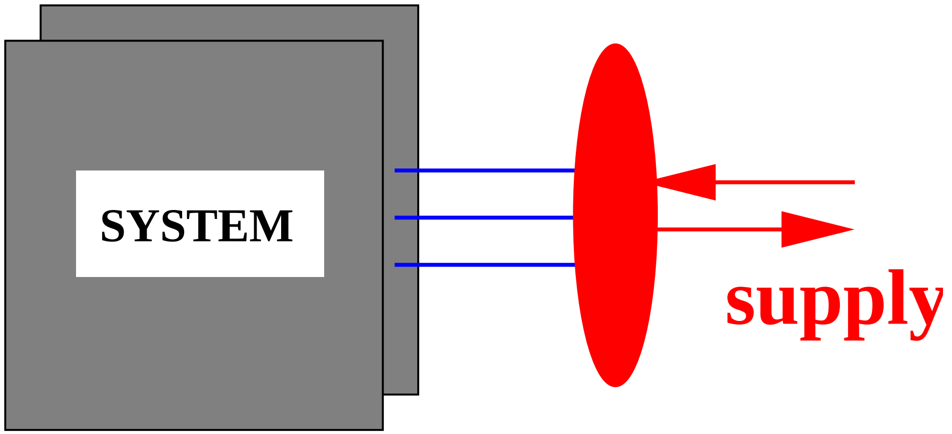

Monitor power in, known dynamics, what is the stored energy?

The construction of storage functions

The construction of storage f'ns is very well understood, particularly for finite dimensional linear systems and quadratic supply rates.

The construction of storage functions

Leads to the KYP-lemma, LMI's, ARIneq, ARE, semi-definite programming, spectral factorization, Lyapunov functions, $\mathscr{H}_\infty$ and robust control, positive and bounded real functions, electrical circuit synthesis, stochastic realization theory.

Example:

$$\frac{d}{dt}x = Ax + Bu, y = Cx, \ \ s \rightsquigarrow ||u||^2 - ||y||^2, \ \ V \rightsquigarrow x^\top Q x, \ \ Q = Q^\top.$$

$$[\![\ \frac{d}{dt}x^\top Qx \leq ||u||^2 - ||y||^2 \ ]\!] \ \Leftrightarrow \ [\![\begin{bmatrix} A^\top Q + QA - C^\top C & QB \\ B^\top Q & -I \end{bmatrix} \preccurlyeq 0]\!]$$

form $\alpha_1 A_1 + \alpha_2 A_2 + \cdots + \alpha_n A_n \succcurlyeq 0 \quad \rightsquigarrow$ **! acronym LMI**

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The storage function V is in general far from unique. There are two 'canonical' storage functions:

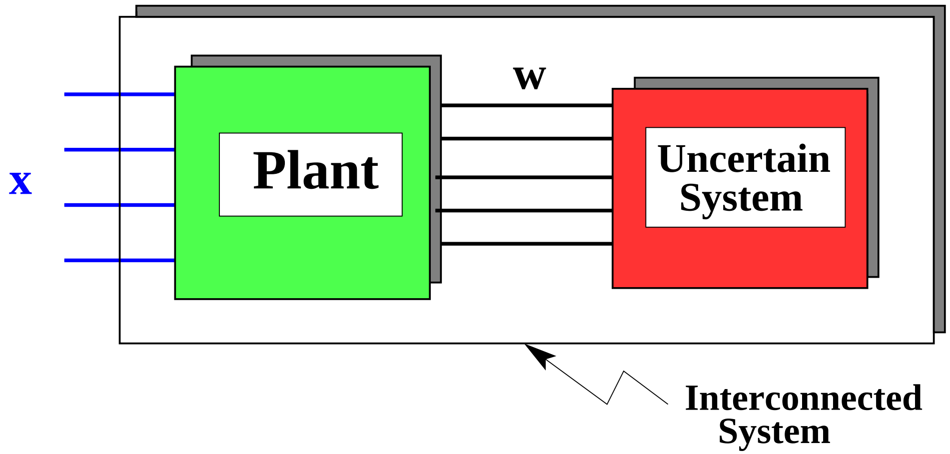
the available storage and the required supply.

For conservative systems, V is unique.
Dissipative systems play an important role in the field.

How good is this notion?

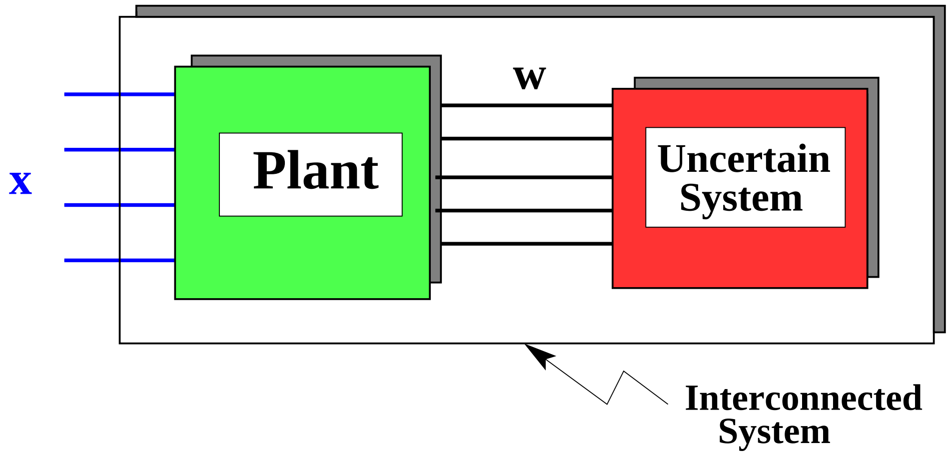
Stability of dissipative interconnections

Construction of Lyapunov functions



Is this uncertain system stable?

Construction of Lyapunov functions



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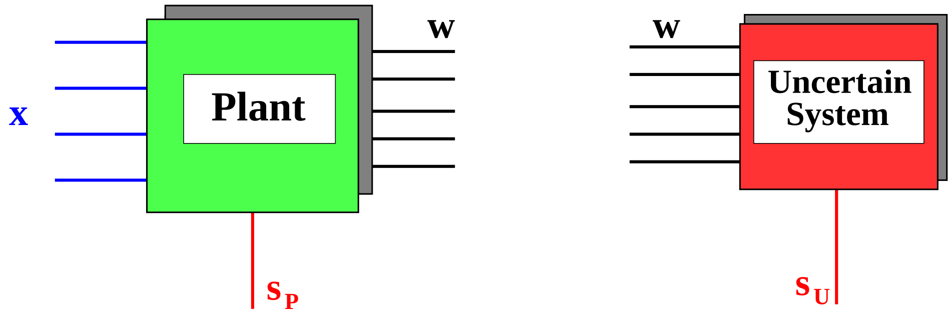
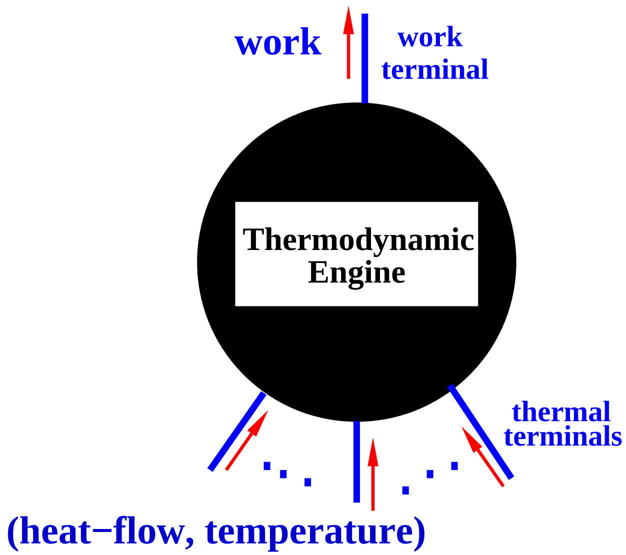


Yes, if both systems are dissipative and $s_P + s_U = 0$

$\rightsquigarrow$ **Lyapunov f'n = sum of storage f'ns.** $\Rightarrow$ **stability.**
This requires the state, also for the uncertain system.

Thermodynamics

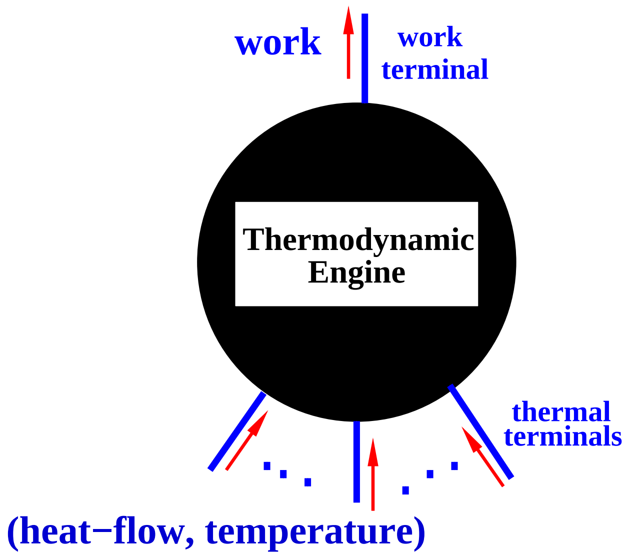
Thermodynamics



Conservative w.r.t. $- \text{work} + \Sigma_{\text{heat terminals}} \ \text{heat flow}$

Dissipative w.r.t. $-\Sigma_{\text{heat terminals}} \frac{\text{heat flow}}{\text{temperature}}$

Thermodynamics



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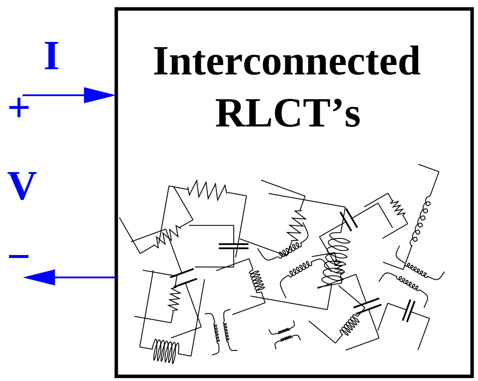
Dissipative w.r.t. $-\Sigma_{\text{heat terminals}} \dfrac{\text{heat flow}}{\text{temperature}}$

Input/output setting is hopeless!

Electrical circuit synthesis

Circuit synthesis

Consider the relation between the voltage across and the current into a one-port electrical circuit containing (positive) resistors, capacitors, inductors, and transformers.



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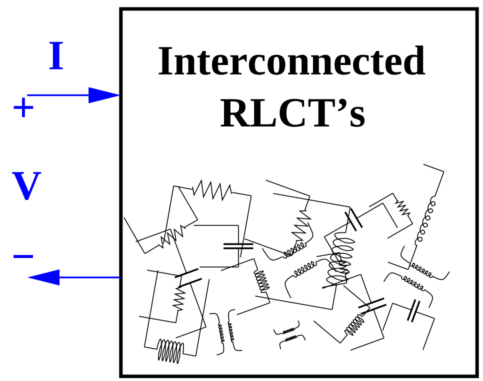
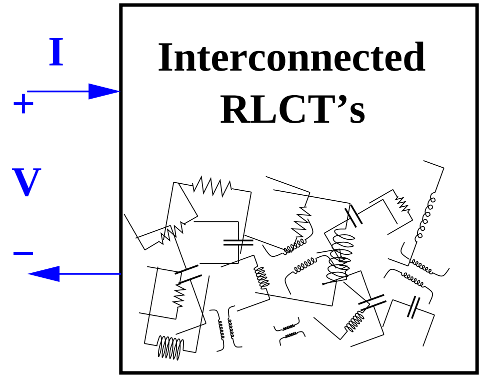


This relation is an FDLS (assume properness, etc.)

$$\frac{d}{dt}x = Ax + BI, \quad V = Cx + DI.$$

The transfer function $G(s) = C(Is - A)^{-1}B + D$ **is called the driving point impedance.**

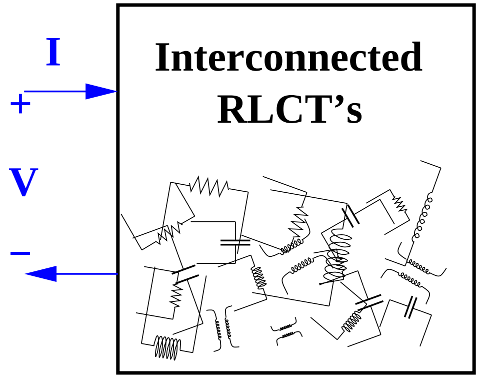
Circuit synthesis



Synthesis problem:

When is a rational f'n $G \in \mathbb{R}(\xi)$ realizable as the driving point impedance of an electrical circuit containing (positive) resistors, capacitors, inductors, and transformers?

Circuit synthesis



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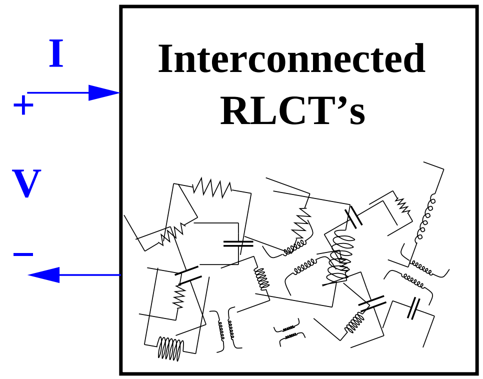
Iff G is 'positive real'

[p.r.] $:\Leftrightarrow \quad [\mathrm{Re}(s) > 0 \Rightarrow \mathrm{Re}\,(G(s)) > 0]$ **Otto Brune, 1932**

Circuit synthesis



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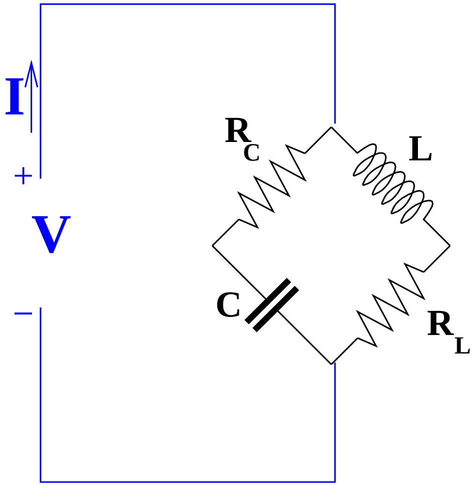
Trafos are not needed **Raoul Bott & Richard Duffin, 1949**

Synthesis of behaviors

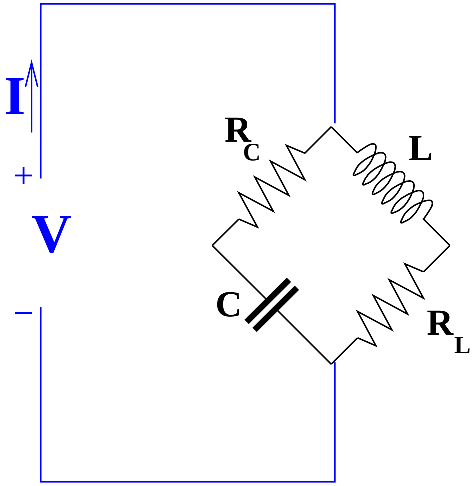


$$\frac{d}{dt}I_L = -\frac{R_L}{L}I_L + \frac{1}{L}V$$

$$\frac{d}{dt}V_C = -\frac{1}{R_C C}V_C + \frac{1}{R_C C}V$$

$$R_C I - V = R_C I_L - V_C$$

Synthesis of behaviors



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$$R_C I - V = \ R_C I_L - V_C$$

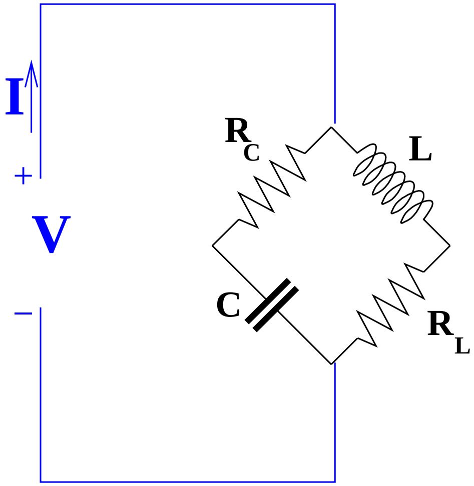
Take $R_L = R_C = 1, C = 1, L = 1$**, and eliminate** $V_C, I_L \ \rightsquigarrow$

$$\frac{d}{dt}V + V = \frac{d}{dt}I + I$$

Uncontrollable system with unobservable storage function

$$S = \frac{1}{2}\left(I_L^2 + V_C^2\right) \qquad \frac{d}{dt}S = VI - \frac{1}{2}\left(R_L I_L^2 + R_C I_C^2\right) \leq VI$$

Synthesis of behaviors



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Synthesis of behaviors

What can we conclude from this example?

1. **The behavior is uncontrollable**

$$V(t) = I(t) + Ae^{-t}, \;\; A \in \mathbb{R}$$

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   The physical storage function is not a f'n of just any state representation.

Synthesis of behaviors

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2. **The storage f'n is unobservable**

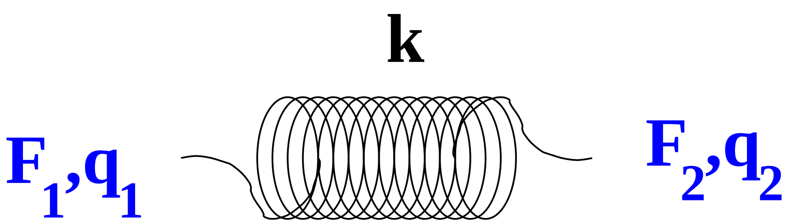
   The physical storage function is not a f'n of just any state representation.

3. **RLC(T) synthesis of behaviors (rather than impedances) is an open problem.**

Dynamics in the supply rate

In some examples (part of) the dynamics comes from the supply rate.

Consider a spring



Dynamical variables: positions q_1, q_2**, forces** F_1, F_2**. Eq'ns**

$$F_1 = -F_2, \quad F_1 = k(q_2 - q_1).$$

Memoryless system. But stores energy.

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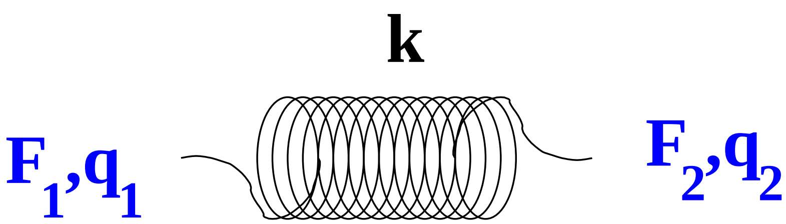

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Memoryless system. But stores energy.

Supply rate: $s \rightsquigarrow F_1 \frac{d}{dt} q_1 + F_2 \frac{d}{dt} q_2$**.**

Stored energy $V \rightsquigarrow \frac{1}{2}(q_1 - q_2)^2$.

i/s/o systems

As is often observed, the input/state/output framework models many things, is better than anything that came before it, but it has some shortcomings...

For the analysis of physical systems,

it does not really fit well

A new definition of dissipativity

Behaviors

Dynamical system: $\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$**, with** $\mathbb{T} \subseteq \mathbb{R}$ **the time-set,** $\mathbb{W}$ **the signal space, and** $\mathfrak{B} \subseteq \mathbb{W}^{\mathbb{T}}$ **the *behavior*.**

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Latent variable dynamical system **is a refinement, with behavior represented with the aid of** *latent variables***.**

$\Sigma_L = (\mathbb{T}, \mathbb{W}, \mathbb{L}, \mathfrak{B}_{\text{full}})$ **with** $\mathbb{L}$ **the space of latent variables, and** $\mathfrak{B}_{\text{full}} \subseteq (\mathbb{W} \times \mathbb{L})^{\mathbb{T}}$ **the** ***full behavior*****.**

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Σ_L **induces** $\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$ **with** *manifest behavior*

$$\mathfrak{B} = \left\{ w : \mathbb{T} \to \mathbb{W} \;\middle|\; \exists\, \ell : \mathbb{T} \to \mathbb{L} \text{ such that } (w, \ell) \in \mathfrak{B}_{\text{full}} \right\}.$$

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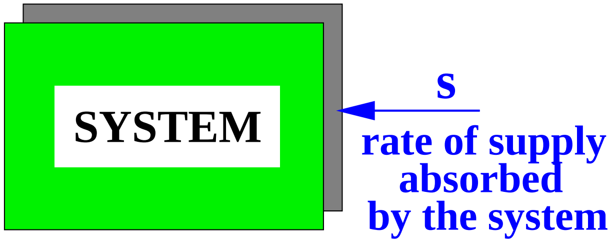
The behavior is all there is. Linearity, time-invariance, ...

Dissipativity & Behaviors



Dissipativeness restricts the way supply goes in and out.

Start with $\Sigma = (\mathbb{R}, \mathbb{R}, \mathfrak{B})$ **dynamical system,**
where $s : \mathbb{R} \to \mathbb{R}$, $s \in \mathfrak{B}$, **models rate of supply *absorbed*.**

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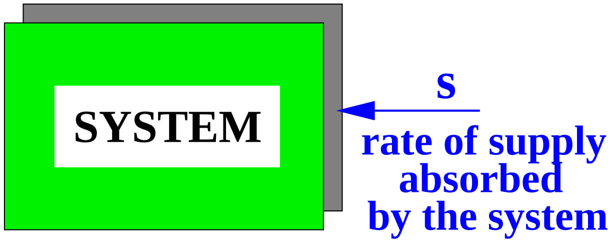

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Add $\Sigma_L = (\mathbb{R}, \mathbb{R}, \mathbb{R}, \mathfrak{B}_{\text{full}})$ **a latent variable representation.**
$(s, V) \in \mathfrak{B}_{\text{full}}$, $V : \mathbb{R} \to \mathbb{R}$ **models the supply** ***stored*****; assume time-invariant.**

Dissipativity & Behaviors

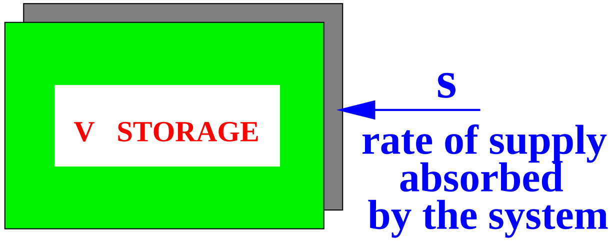
Start with $\Sigma = (\mathbb{R}, \mathbb{R}, \mathfrak{B})$ dynamical system, where $s : \mathbb{R} \to \mathbb{R}$, $s \in \mathfrak{B}$, models rate of supply *absorbed*.

Add $\Sigma_L = (\mathbb{R}, \mathbb{R}, \mathbb{R}, \mathfrak{B}_{\text{full}})$ a latent variable representation. $(s, V) \in \mathfrak{B}_{\text{full}}$, $V : \mathbb{R} \to \mathbb{R}$ models the supply *stored*; assume time-invariant.

V is said to be a *storage* w.r.t. the supply rate s if the *dissipation inequality*

$$V(t_1) - V(t_0) \leq \int_{t_0}^{t_1} s(t)\, dt$$

holds $\forall (s, V) \in \mathfrak{B}_{\text{full}}$ and $\forall t_0 \leq t_1$,



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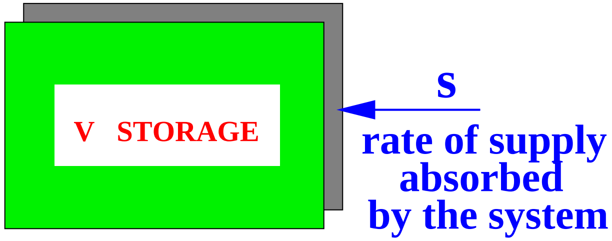

Given $\Sigma = (\mathbb{R}, \mathbb{R}, \mathfrak{B})$, time-invariant, does there exists a representation $\Sigma_L = (\mathbb{R}, \mathbb{R}, \mathbb{R}, \mathfrak{B}_{\text{full}})$, time-invariant, such that the dissipation inequality holds?

Nonnegative storage

Simple existence result for non-negative storage functions.

THEOREM

$\Sigma = (\mathbb{R}, \mathbb{R}, \mathfrak{B})$ **is** ***dissipative with non-negative storage*** $\Leftrightarrow$

$\forall s \in \mathfrak{B}$ **and** $\forall t_0 \in \mathbb{R}, \ \exists K \in \mathbb{R},$

such that $-\int_{t_0}^{T} s(t)\, dt \leq K$ **for** $T \geq t_0$

'Available storage' is finite. N.a.s.c.!

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A n.a.s.c. for the existence of $\mathfrak{B}_{\text{full}}$ and V (in terms of $\mathfrak{B}$) is ?
$\exists$ sufficient conditions in terms of periodic trajectories assuming *observability* of V from s.

Quadratic supply rates

QDF's

A ***quadratic differential form (QDF)*** **is a quadratic expression in the components of** $w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$ **and its derivatives:**

$$\Sigma_{\mathtt{k},\ell} \left(\frac{d^{\mathtt{k}}}{dt^{\mathtt{k}}} w\right)^\top \Phi_{\mathtt{k},\ell} \left(\frac{d^\ell}{dt^\ell} w\right)$$

with the $\Phi_{\mathtt{k},\ell} \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$**. Map from** $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$ **to** $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R})$. **Compact notation and a convenient calculus.**

$$\Phi(\zeta, \eta) = \Sigma_{\mathtt{k},\ell}\, \Phi_{\mathtt{k},\ell} \zeta^{\mathtt{k}} \eta^\ell$$

Notation QDF $Q_\Phi(w)$**.**

Q_Φ **is said to be** ***non-negative*** **(denoted** $Q_\Phi \geq 0$**)** $:\Leftrightarrow$
$Q_\Phi(w) \geq 0$ **for all** $w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$**.**

Dissipativity of QDF's

Consider $\Sigma_\Phi = \left(\mathbb{R}, \mathbb{R}, \mathrm{im}\left(Q_\Phi\right)\right)$**: supply rate is QDF.**

$s : \mathbb{R} \to \mathbb{R}$ **is in** $\mathfrak{B} \Leftrightarrow \exists\, w$ **such that**

$$s = Q_\Phi(w) = \Sigma_{\mathtt{k},\ell} \left(\frac{d^{\mathtt{k}}}{dt^{\mathtt{k}}} w\right)^\top \Phi_{\mathtt{k},\ell} \left(\frac{d^\ell}{dt^\ell} w\right)$$

Very general, 'Linear systems, quadratic functionals', controllability. Examples: linear circuits, t'f f'n with supply rate quadratic form in input and output, linear mechanical systems, ... Interesting special cases:

$$\frac{d}{dt}x = Ax + Bu, y = Cx + Du, s = ||u||^2 - ||y|^2, \text{ or } s = u^\top y$$

(u,y)**-behavior:** $\begin{bmatrix} u \\ y \end{bmatrix} = M(\frac{d}{dt})w, \quad \Phi \to M(\zeta)^\top \Sigma M(\eta)$

Dissipativity of QDF's

Σ_Φ is dissipative ($\exists$ storage) if

$$\int_{-\infty}^{+\infty} Q_\Phi(w)\, dt \geq 0$$

$\forall\, w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$ **compact support.**

Equivalently, if

$$\Phi(i\omega, -i\omega) + \Phi^\top(-i\omega, i\omega) \geq 0 \quad \forall\, \omega \in \mathbb{R}$$

Equivalently, if

$$\exists\, \Psi : \; \frac{d}{dt} Q_\Psi \leq Q_\Phi \qquad \textbf{(LMI)}$$

Dissipativity of QDF's

For a non-negative storage function, we obtain instead

$$\int_{-\infty}^{0} Q_{\Phi}(w)\, dt \geq 0$$

$\forall\, w \in \mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$ **of compact support.**

In 1-D case storage f'n of w **'observability'.**
Not so in n-D case, as Maxwell's eq'ns.

Some open problems

cfr. my website

Intrinsic characterization of dissipativity

Let $\Sigma = (\mathbb{R}, \mathbb{R}, \mathfrak{B})$ be time-invariant. When is it dissipative?

I.e., when does there exists a time-invariant latent variable representation $\Sigma_L = (\mathbb{R}, \mathbb{R}, \mathbb{R}, \mathfrak{B}_{\text{full}})$, time-invariant, such that the dissipation inequality holds?

$\exists$ sufficient conditions in terms of periodic behavior, controllability, observability, equilibrium points, ...

Characterization of QDF's

Given $\mathfrak{B} \subseteq \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R})$, **shift-invariant.**

When does $\exists\ \Phi \in \mathbb{R}^{\mathtt{w} \times \mathtt{w}}[\zeta, \eta]$ **such that** $\mathfrak{B} = \text{image}(Q_\Phi)$ **?**

Characterization of positive storage f'ns for QDF's

Conjecture:

The following are equivalent for $\Phi \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$**:**

1. $\int_{-\infty}^{0} Q_\Phi(w)\,dt \geq 0 \quad \forall w \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}})$ **of compact support,**
2. $\forall w \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}}),\ \exists K \in \mathbb{R}$**,**
   such that $-\int_0^T Q_\Phi(w)\,dt \leq K \quad \forall T \geq 0$**.**

1. $\Rightarrow$ **2. is easy.**

Characterization of quadratic storage functions

Conjecture:

A QDF has a storage iff it has a QDF as a storage

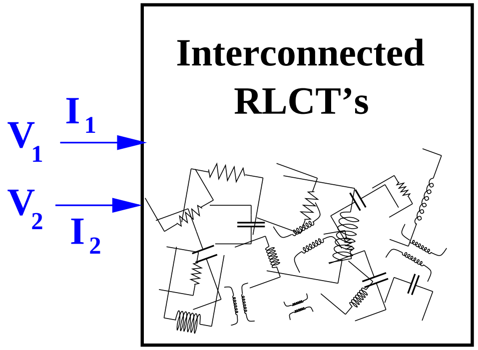
Without signature conditions (as small gain, positive operator, conicity).

Passive behavior synthesis

Stated for single input/single output systems. Consider

$$p(\frac{d}{dt})V = q(\frac{d}{dt})I.$$

When realizable as behavior of the port var. of a circuit with (positive) resistors, capacitors, inductors, and transformers?



Necessary: $\frac{p}{q}$ **p.r.** **p.r. n.a.s.c. when** p **and** q **co-prime.**

What conditions does dissipativity impose on common factors?

Transformerless synthesis

Bott-Duffin synthesis realizes the impedance, not the behavior. They do not use minimal realization, common factors are introduced. Uncontrollable parts are added in the behavior.

Is a synthesizable SISO behavior ... without transformers?

Suspect: NOT.
Transformerless synthesis of behaviors more open than ever.

Summary

Conclusion

The notion of dissipativeness, while subtle, allows an adequate formulation in the setting of behaviors (*s*, the supply) and latent variables (*V*, the storage).

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The notion of dissipativeness, while subtle, allows an adequate formulation in the setting of behaviors (s, the supply) and latent variables (V, the storage).

Also for dissipative systems, this means backing off from input/output thinking!

Reference:

JCW and K. Takaba

Dissipativity and Stability of Interconnections

International Journal of Nonlinear and Robust Control

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Thank you for your attention