OPEN DYNAMICAL SYSTEMS

and

THEIR ORIGINS

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Peter Sagirow Lecture **Uni-Stuttgart, November 20, 2006**

Peter Sagirow

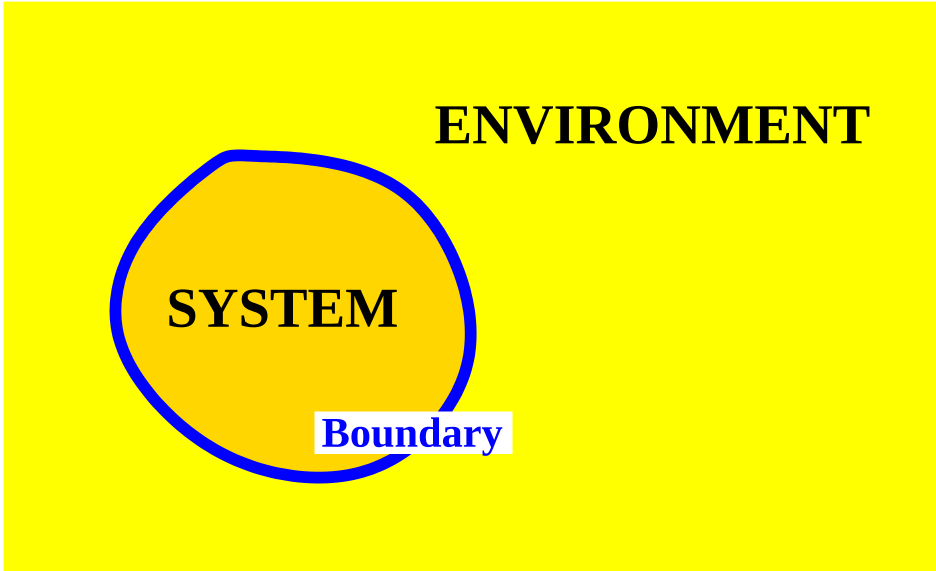
Open and Connected

The central tenets of the field of systems and control:

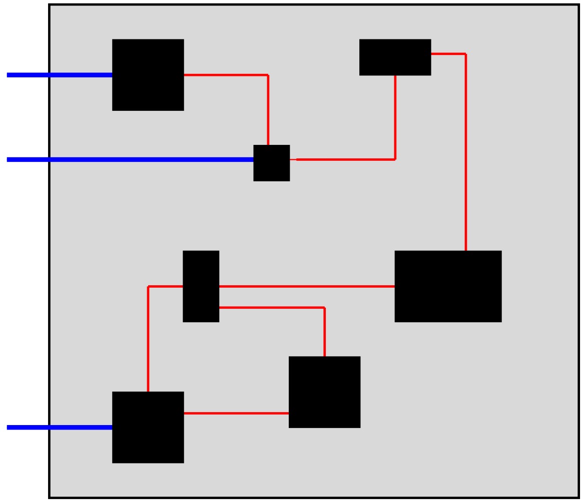
Systems are open and consist of interconnected subsystems.

Synthesis of systems consists of interconnecting subsystems

Open



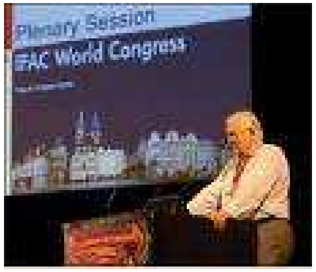
Connected



Architecture with subsystems

Mathematization

1. Get the physics right
2. The rest is mathematics



R.E. Kalman, Opening lecture
IFAC World Congress
Prague, July 4, 2005

Mathematization

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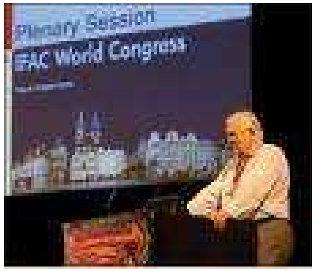

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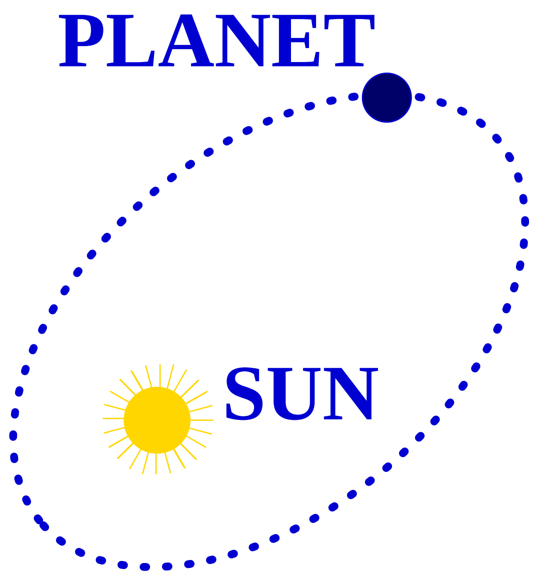
Prima la fisica, poi la matematica

How it all began ...

How, for heaven's sake, does it move?

Kepler’s laws

Johannes Kepler (1571-1630)



Kepler’s laws:

Ellipse, sun in focus; = areas in = times;
(period)^2 $\cong$ (diameter)^3

The equation of the planet

Consequence:

acceleration = function of position and velocity

$$\frac{d^2}{dt^2}w(t) = A(w(t), \frac{d}{dt}w(t))$$

$\rightsquigarrow$ **via calculus and calculation**

$$\frac{d^2}{dt^2}w(t) + \frac{1_{w(t)}}{|w(t)|^2} = 0$$

Isaac Newton (1643-1727)

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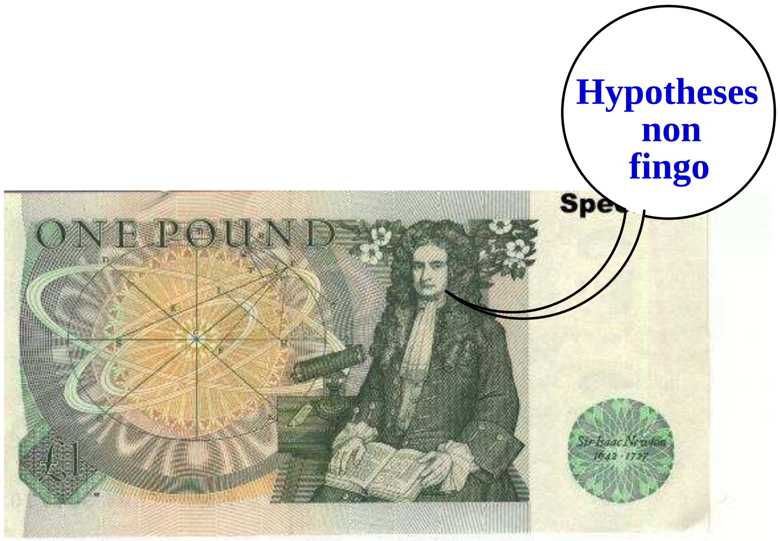


Isaac Newton (1643-1727)

Newton's laws

2-nd law $$F'(t) = m\frac{d^2}{dt^2}w(t)$$

gravity $$F''(t) = m\frac{1_{w(t)}}{|w(t)|^2}$$

3-rd law $$F'(t) + F''(t) = 0$$

$$\Downarrow$$

$$\boxed{\frac{d^2}{dt^2}w(t) + \frac{1_{w(t)}}{|w(t)|^2} = 0}$$

The paradigm of *closed* systems

‘Axiomatization’

K.1, K.2, & K.3

$$\rightsquigarrow \quad \frac{d^2}{dt^2} w(t) + \frac{1_{w(t)}}{|\frac{d}{dt} w(t)|^2} = 0$$

$$\rightsquigarrow \quad \frac{d}{dt} x = f(x)$$

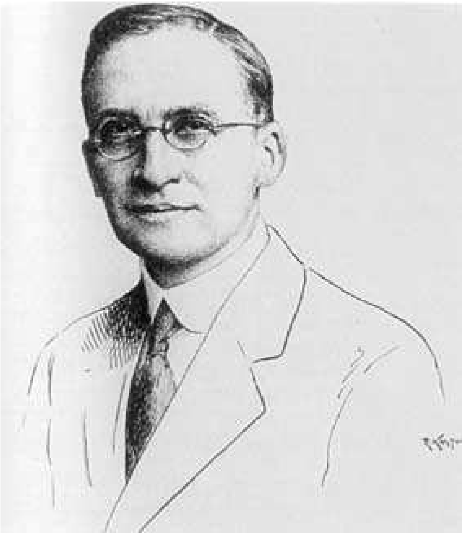
$\rightsquigarrow$ **‘dynamical systems’, flows**

$\rightsquigarrow$ **flows as paradigm of dynamics: closed systems**

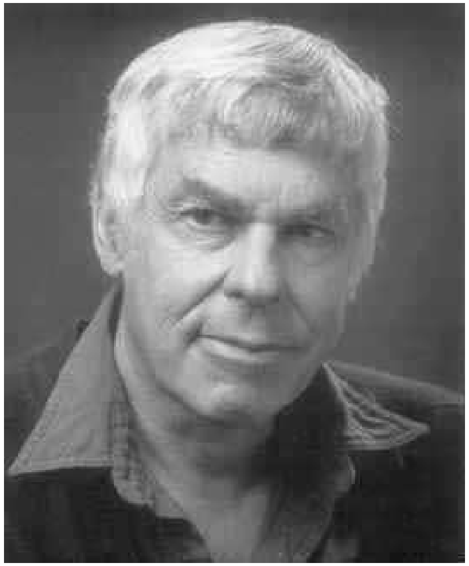
Motion determined by internal initial conditions.

‘Axiomatization’

Henri Poincaré (1854-1912)



George Birkhoff (1884-1944)



Stephen Smale (1930-)

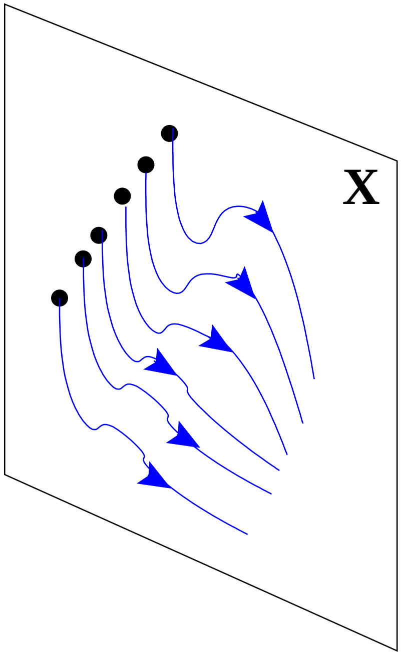
‘Axiomatization’

A *dynamical system* is defined by

a state space X and

a state transition function

$\phi :$ $\cdots$ such that $\cdots$



$\phi(t, \mathrm{x})$ = state at time t starting from state x

This framework of closed systems

is universally used for dynamics

in mathematics and physics

‘Axiomatization’

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$\phi :$ $\cdots$ **such that** $\cdots$

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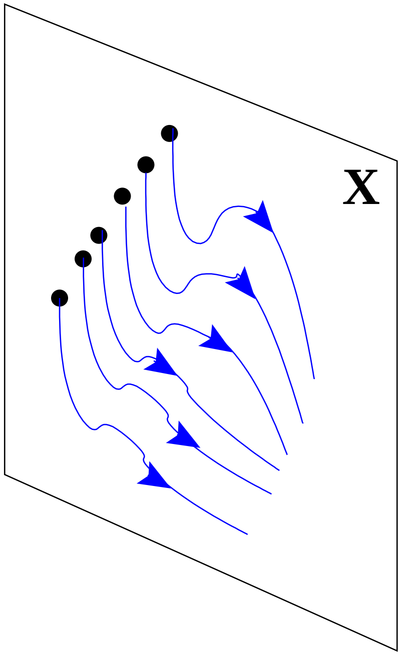


How could they forget Newton’s 2nd law,

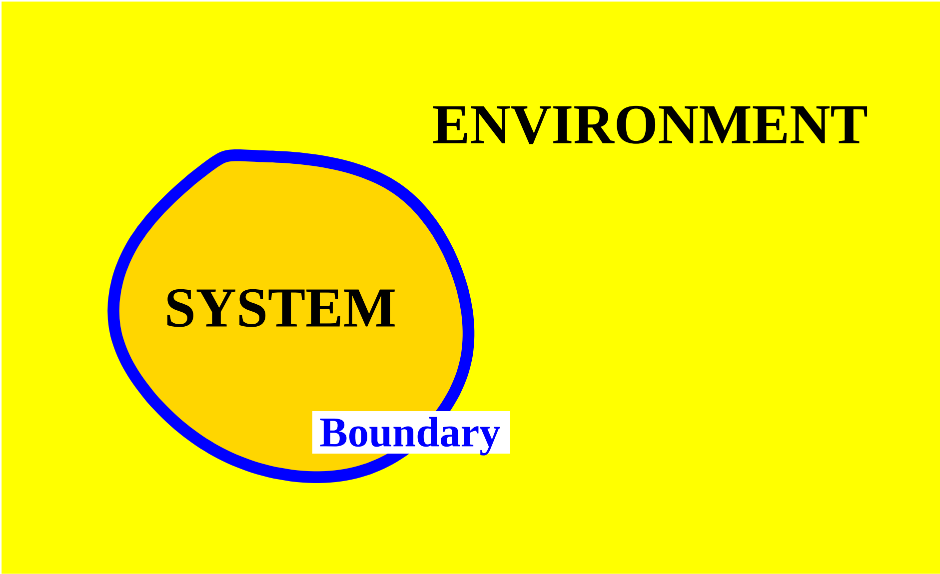
about Maxwell’s eq’ns,

about thermodynamics,

about tearing & zooming & linking,

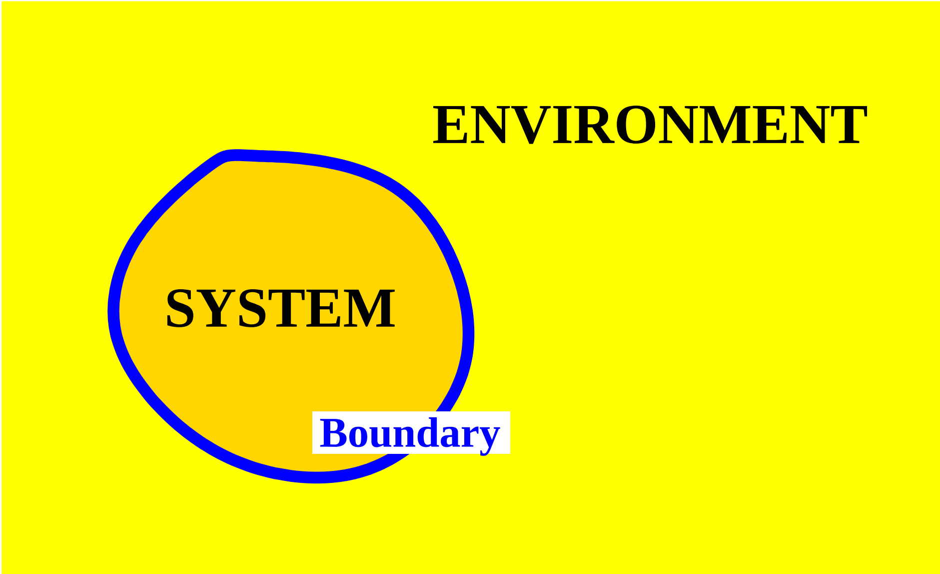
‘Axiomatization’

Reply: assume ‘fixed boundary conditions’



$\leadsto$ to model a system,
we have to model also the environment!

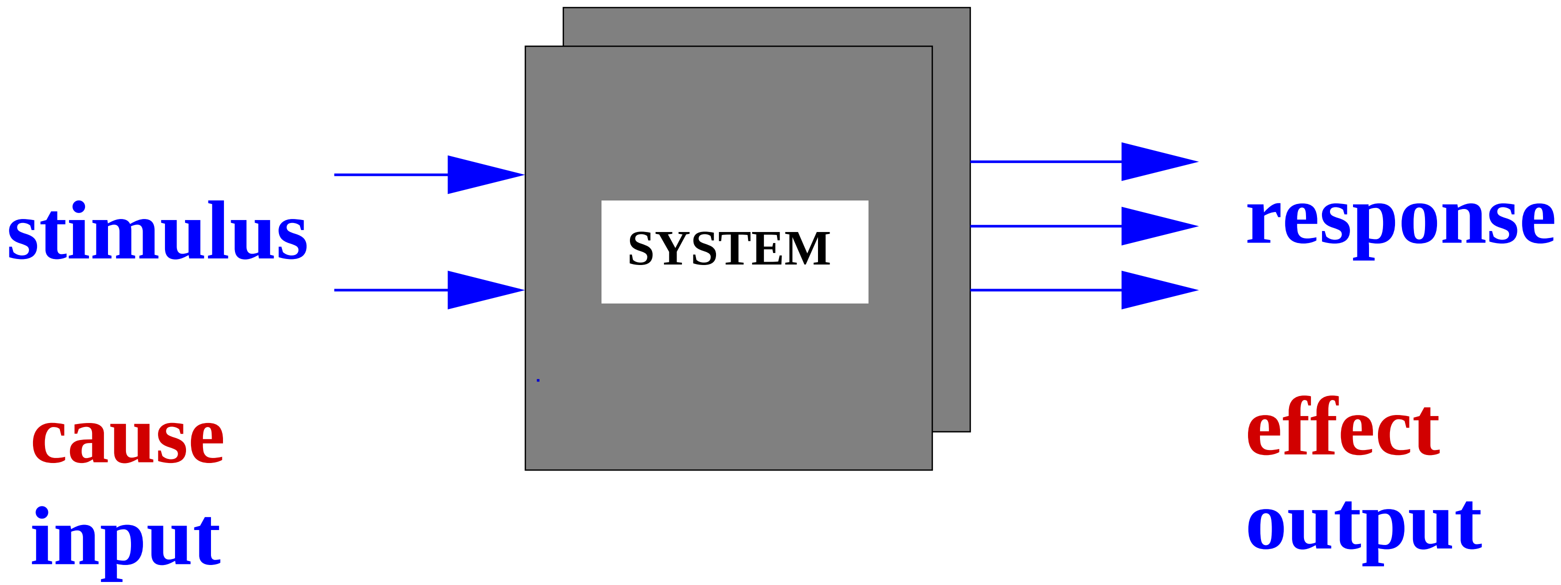
‘Axiomatization’

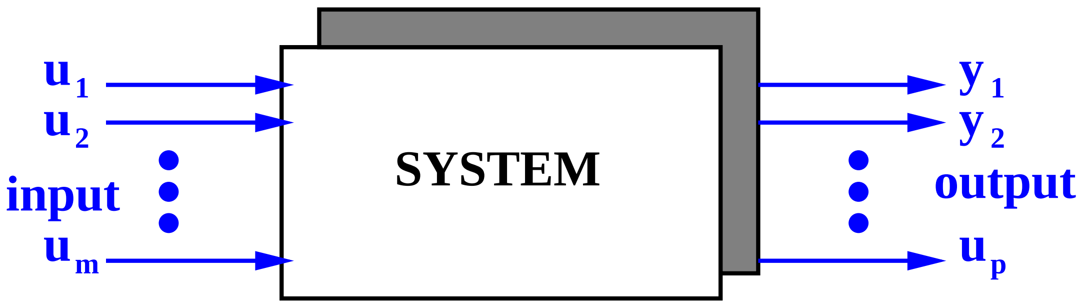


Chaos theory, cellular automata, sync, etc., function in this framework ...

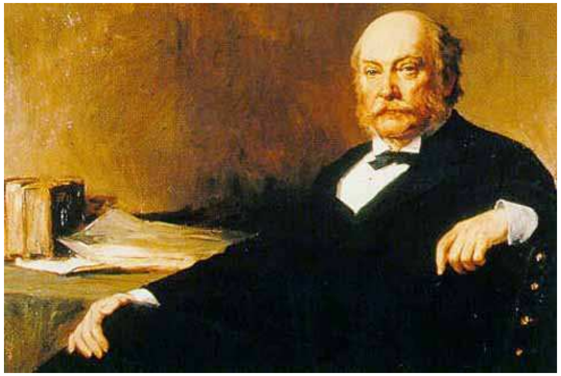
Meanwhile, in engineering, ...

Input/output systems

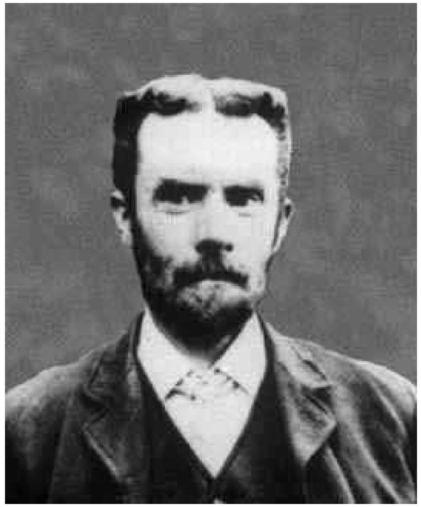




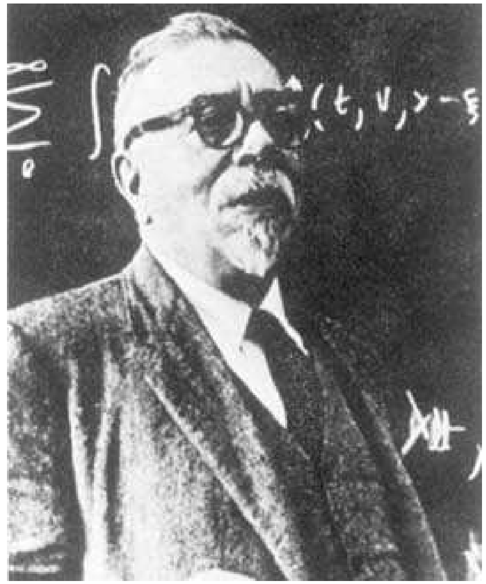
The originators



Lord Rayleigh (1842-1919)



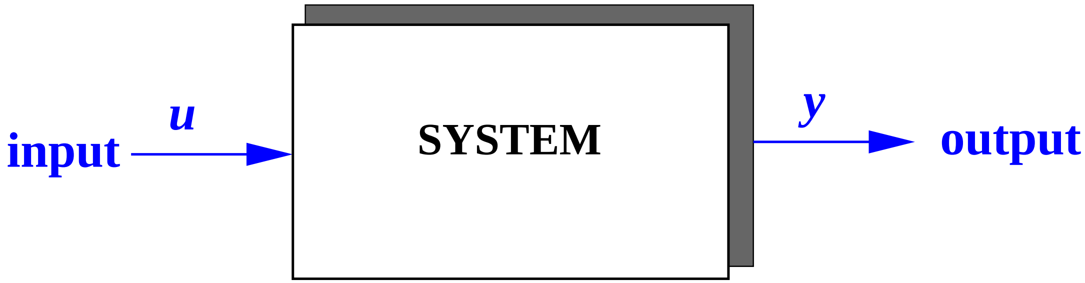
Oliver Heaviside (1850-1925)



Norbert Wiener (1894-1964)

and the many electrical circuit theorists ...

Mathematical description



Classical control $$p\left(\frac{d}{dt}\right) y = q\left(\frac{d}{dt}\right) u$$

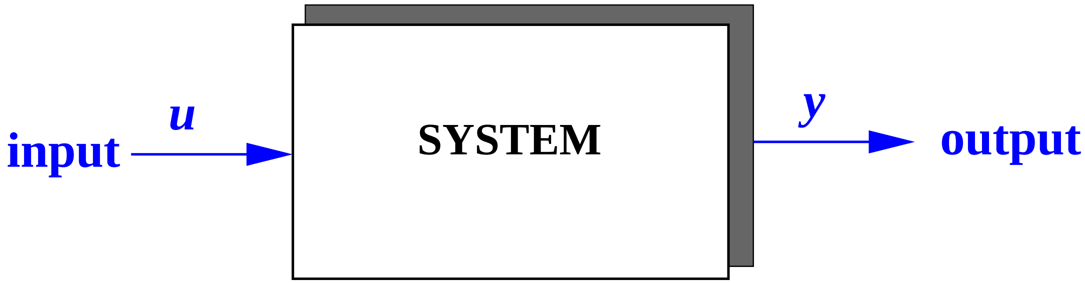
u: input, y: output, p and q polynomials

$G(s) = \frac{q(s)}{p(s)}$ transfer functions, impedances, admittances.

PID rules. Bode, Nyquist, Nichols. Lead-lag. Root-locus.

Also transfer f'n models early on in circuit theory and filtering.

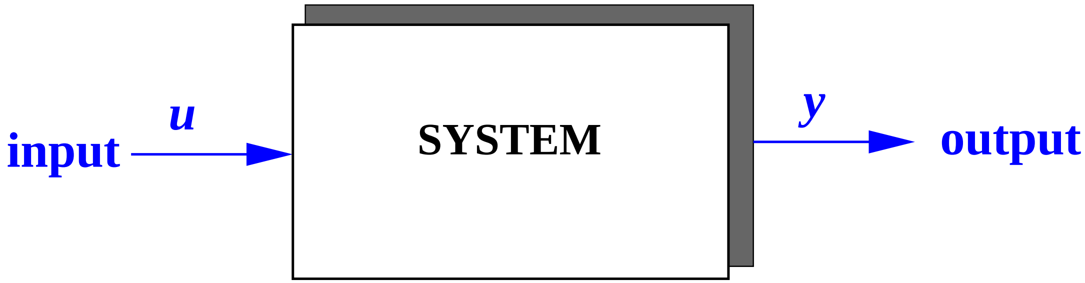
Mathematical description



$$y(t) = \int_{0 \text{ or } -\infty}^{t} H(t-t')u(t')\,dt'$$

$$y(t) = H_0(t) + \int_{-\infty}^{t} H_1(t-t')u(t')\,dt' + \int_{-\infty}^{t}\int_{-\infty}^{t'} H_2(t-t', t'-t'')u(t')u(t'')\,dt'dt'' + \cdots$$

Mathematical description



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These models fail to deal with ‘initial conditions’.

A physical system is SELDOM an i/o map

An input/output map?

A system :⇔ map from inputs u to outputs y. ***Linear*** **:⇔**

$$\alpha u \mapsto \alpha y, \quad (u_1 + u_2) \mapsto (y_1 + y_2)$$

cfr. numerous textbooks and Wikipedia....

Example: $y(t) = \int_{-\infty \text{ or } 0}^{t} H(t - t') u(t') \, dt'.$

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Combine $p\left(\frac{d}{dt}\right) y = q\left(\frac{d}{dt}\right) u$ **with feedback** $u = Ky$.

Both linear...

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Combined $\quad \rightsquigarrow \quad p\left(\frac{d}{dt}\right) y = Kq\left(\frac{d}{dt}\right) y$.

We seem to have left the realm of linear systems.

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Via the (feed)back door??

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Via the (feed)back door??

$$F = \text{ mass } \frac{d^2}{dt^2} q \rightsquigarrow q(t) = \frac{1}{\text{mass}} \int_{-\infty}^{t} (t - t') F(t') \, dt'$$

Combine with inverse square law. ***Eppur NON si muove...***

Around 1960: a paradigm shift

$$\rightsquigarrow \quad \frac{d}{dt}x = f(x,u), \; y = g(x,u)$$

Rudolf Kalman (1930-)

Around 1960: a paradigm shift

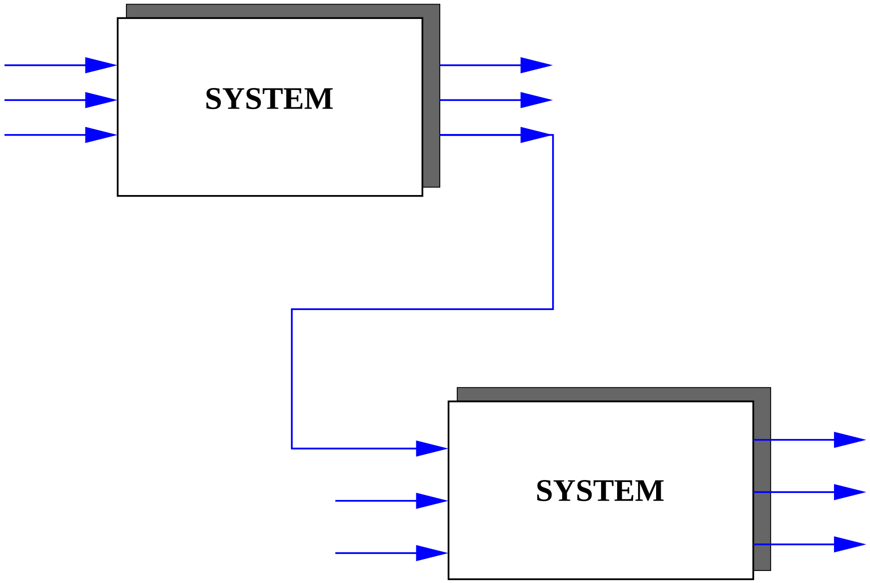
$$\rightsquigarrow \quad \frac{d}{dt}x = f(x,u), \; y = g(x,u)$$

1. open
2. ready to be interconnected
   outputs of one system $\mapsto$ inputs of another
3. deals with initial conditions
4. incorporates nonlinearities, time-variation
5. models many physical phenomena
6. ...

Around 1960: a paradigm shift

$$\rightsquigarrow \quad \frac{d}{dt}x = f(x, u), \; y = g(x, u)$$

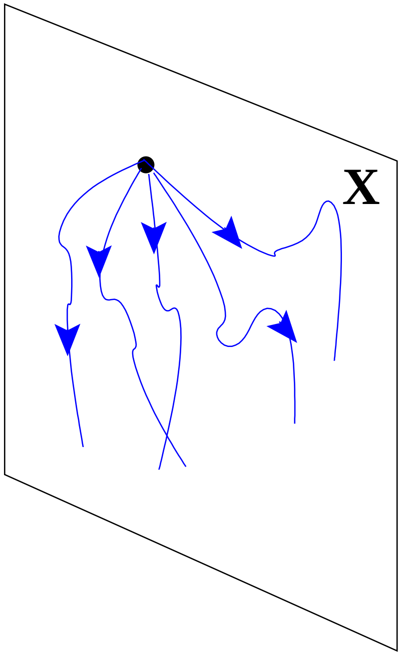
Interconnection:



‘Axiomatization’

State transition function:

$\phi(t, \mathrm{x}, u)$: **state reached at time t from x using input u.**



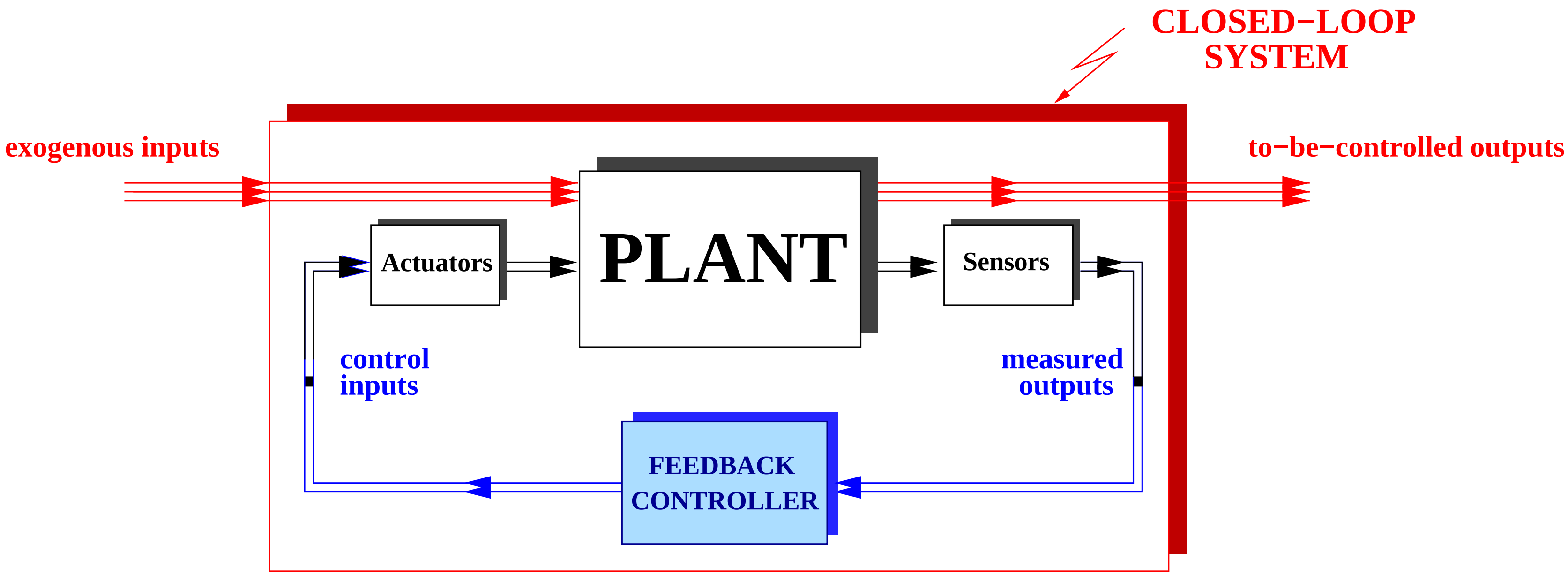
$$\frac{d}{dt}x = f(x, u), \;\; y = g(x, u)$$

Read-out function:

$g(\mathrm{x}, \mathrm{u})$: **output value with state x and input value u.**

The input/state/output view turned out to be very effective and fruitful

- for **modeling**
- for **control** (stabilization, robustness, ...)



The input/state/output view turned out to be very effective and fruitful

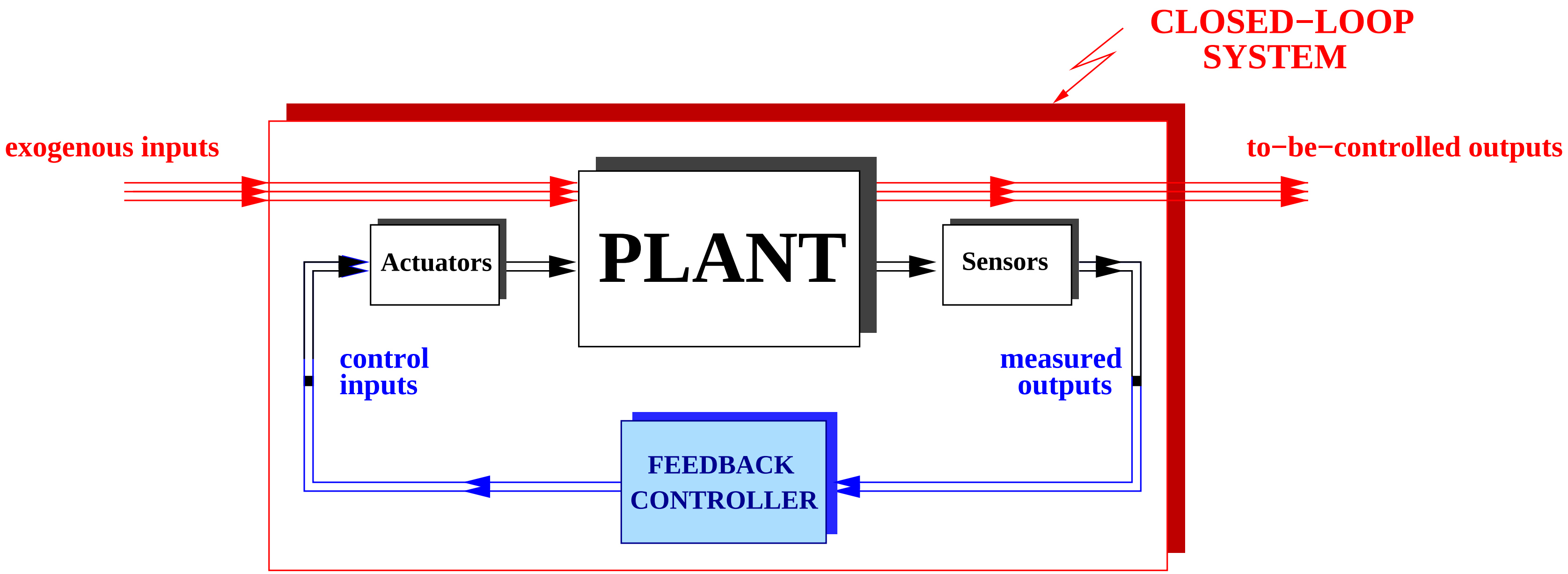
- for modeling
- for control (stabilization, robustness, ...)
- prediction of one signal from another, filtering
- understanding system representations (transfer f'n, input/state/output, etc.)
- model simplification, reduction
- system ID: models from data
- etc., etc., etc.

Let's take a closer look at the i/o framework ...

in control

Difficulties with i/o

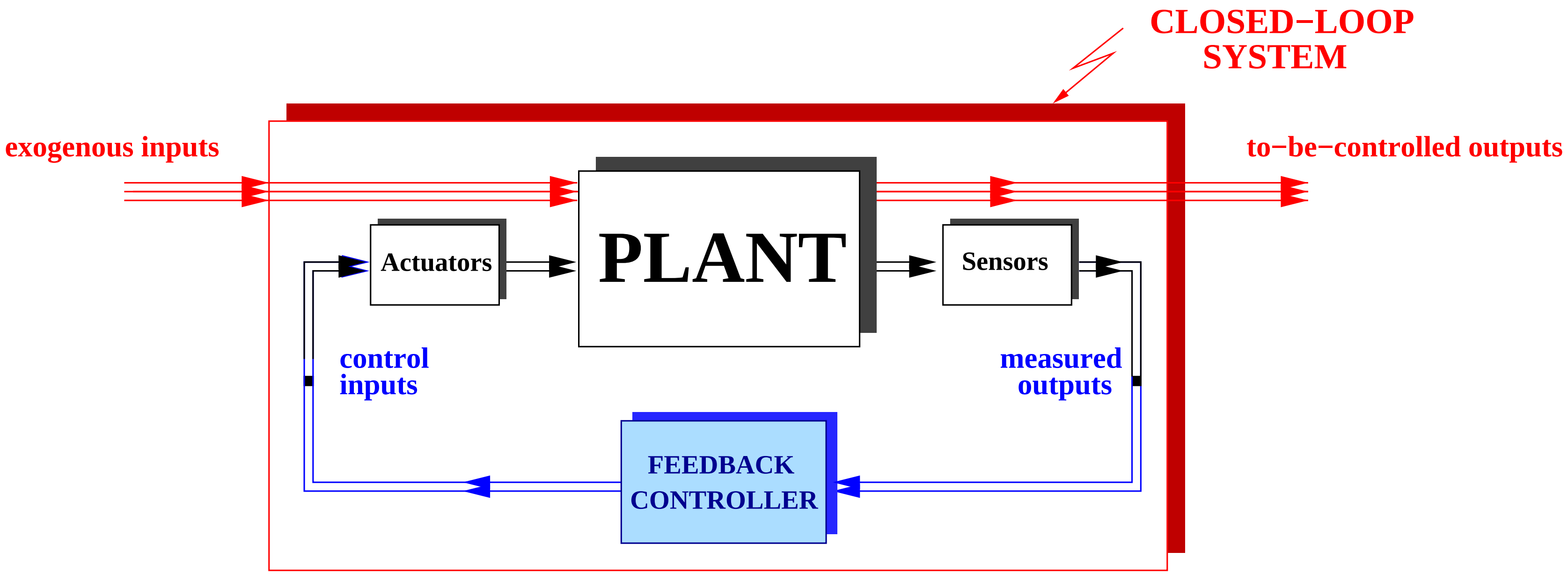
active control



Very intelligent, very useful, ... but general?

Difficulties with i/o

active control



versus passive control

Dampers, heat fins, pressure valves, grooves and strips...

Controllers without sensors and actuators

Difficulties with i/o

active **control versus** **passive** **control**

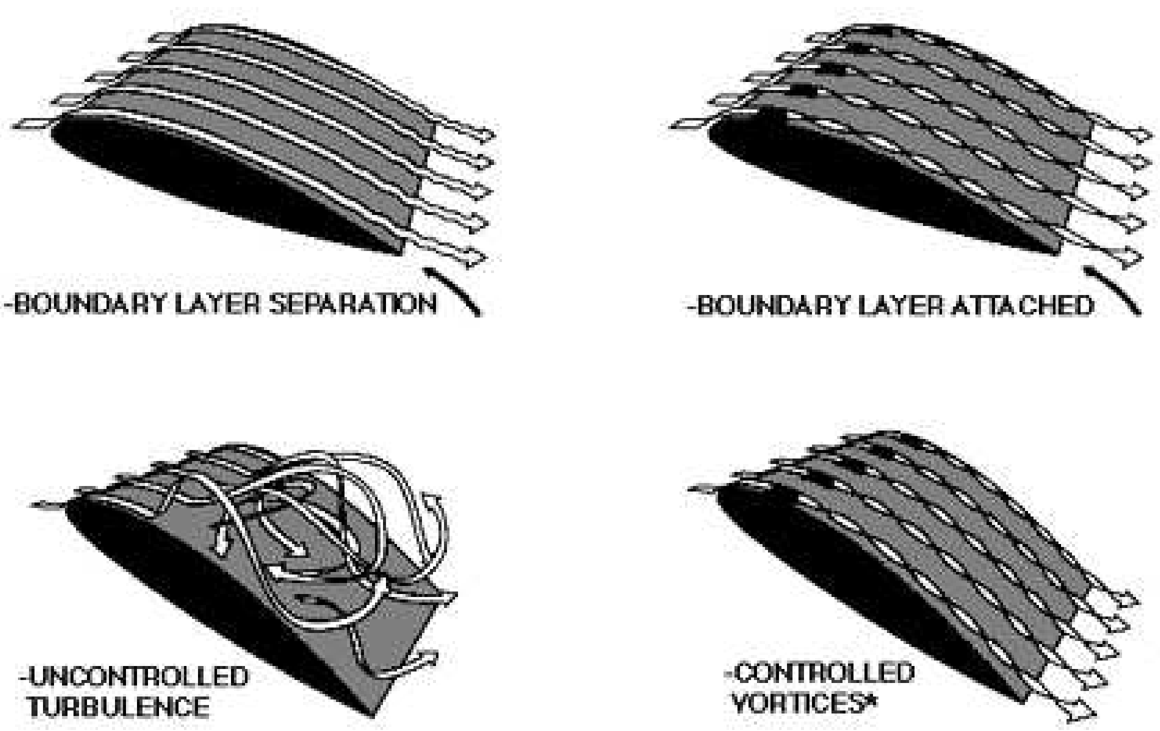
Controlling turbulence

airplanes, sharks, dolphins, golf balls, bicycling helmets, etc.

Difficulties with i/o

active control versus passive control

Controlling turbulence



Difficulties with i/o

active **control versus** **passive** **control**

Controlling turbulence

Nagano 1998

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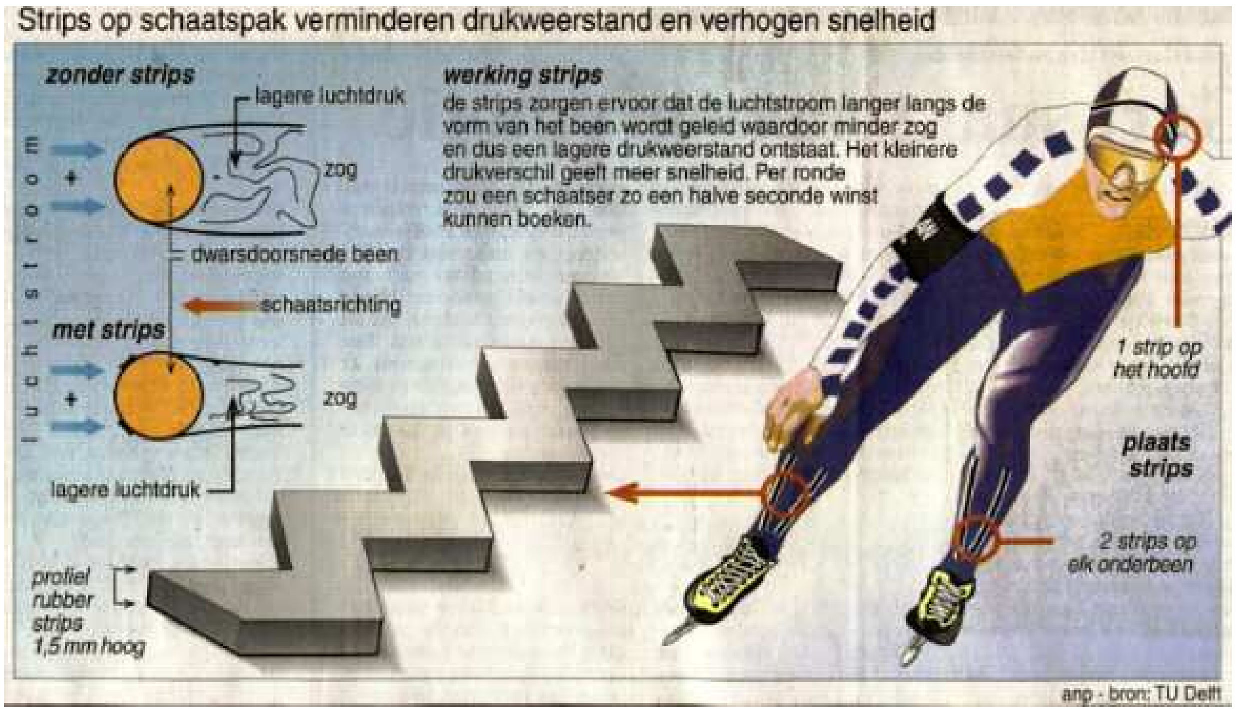


Difficulties with i/o

active control versus passive control

Controlling turbulence

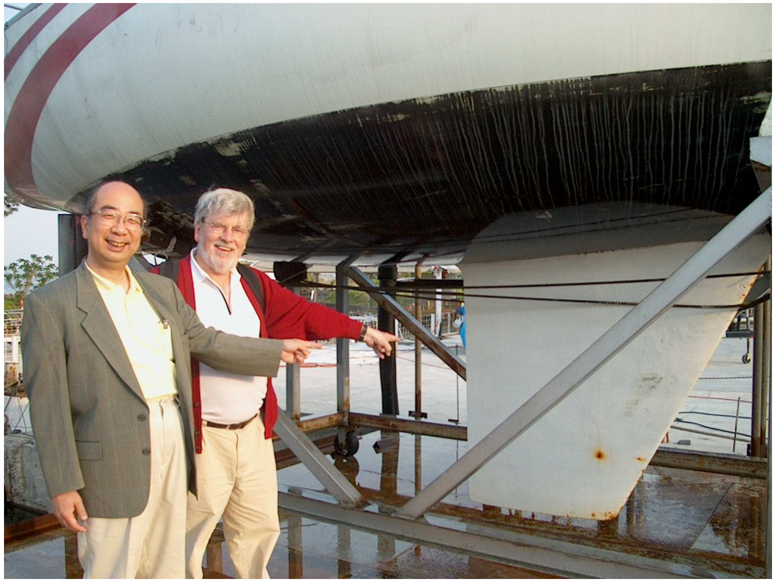
Nagano 1998

These are beautiful controllers! But, the only people not calling this "control", are the control engineers ...

Difficulties with i/o

active control versus passive control

Another example: the stabilizer of a ship



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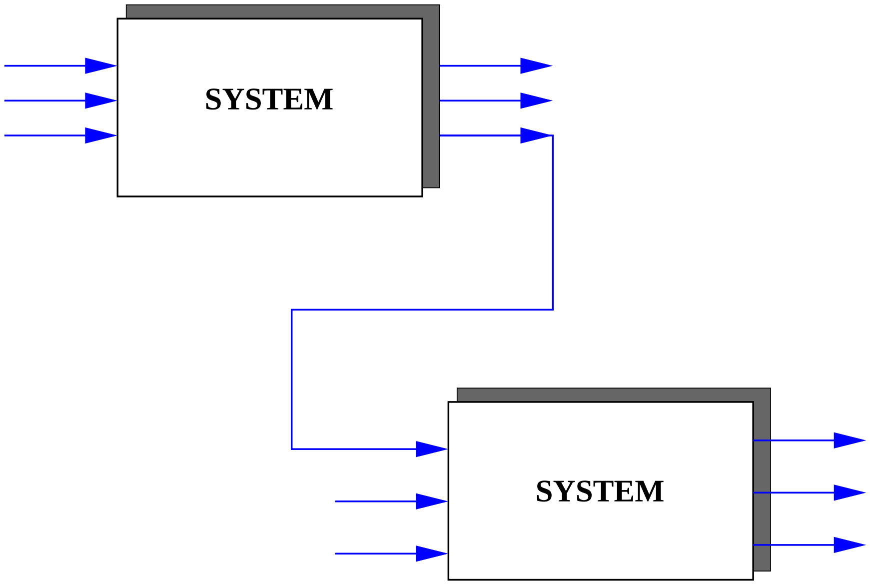
Let's take a closer look at the i/o framework ...

for interconnection

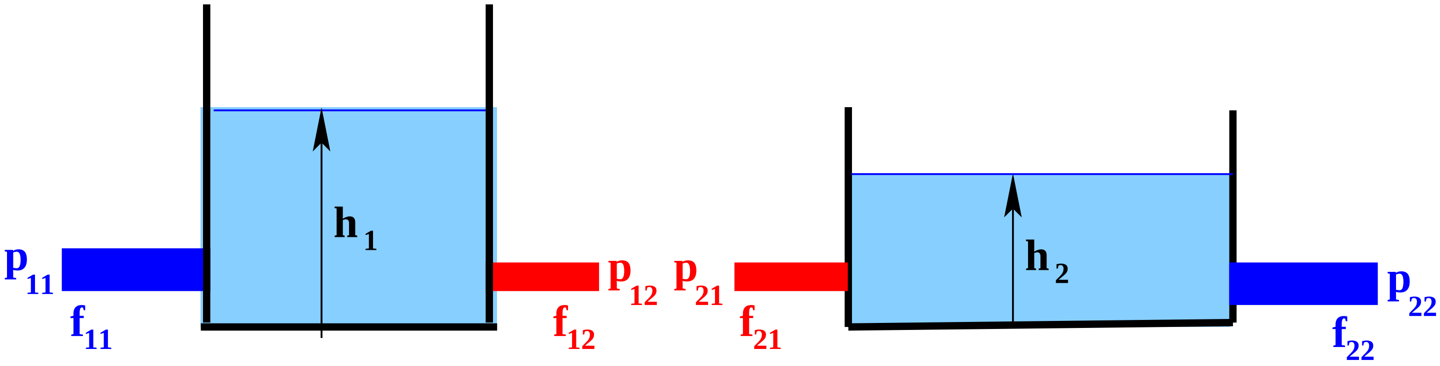
i/o and interconnection

Interconnection:



$\rightsquigarrow$ **SIMULINK©**

i/o and interconnection



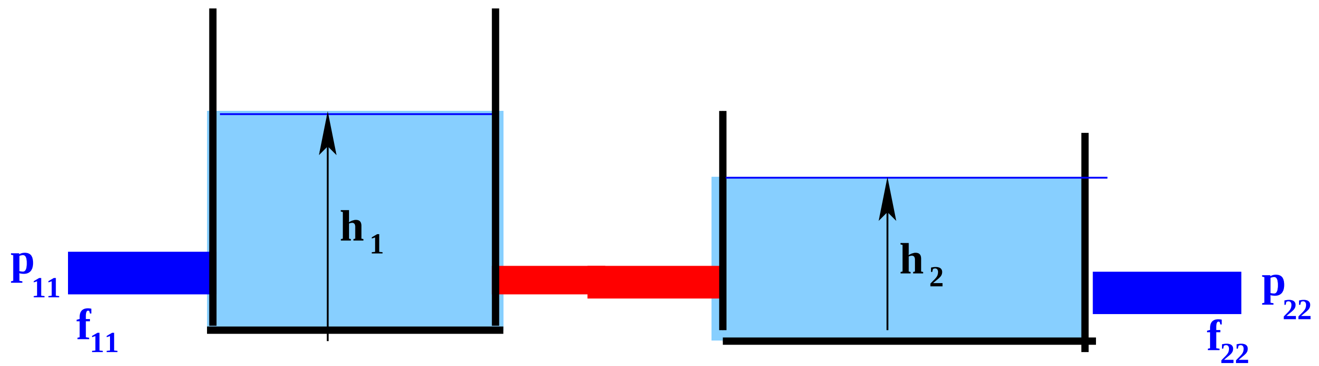
$$\frac{d}{dt}h_1 = F_1(h_1, p_{11}, p_{12}), f_{11} = H_{11}(h_1, p_{11}), f_{12} = H_{12}(h_1, p_{12})$$

$$\frac{d}{dt}h_2 = F_1(h_2, p_{21}, p_{22}), f_{21} = H_{21}(h_2, p_{21}), f_{22} = H_{22}(h_2, p_{22})$$

inputs: the pressures $p_{11}, p_{12}, p_{21}, p_{22}$

outputs: the flows $f_{11}, f_{12}, f_{21}, f_{22}$

i/o and interconnection



$$\frac{d}{dt}h_1 = F_1(h_1, p_{11}, p_{12}), f_{11} = H_{11}(h_1, p_{11}), f_{12} = H_{12}(h_1, p_{12})$$

$$\frac{d}{dt}h_2 = F_1(h_2, p_{21}, p_{22}), f_{21} = H_{21}(h_2, p_{21}), f_{22} = H_{22}(h_2, p_{22})$$

Interconnection: $p_{12} = p_{21}, f_{12} + f_{21} = 0$

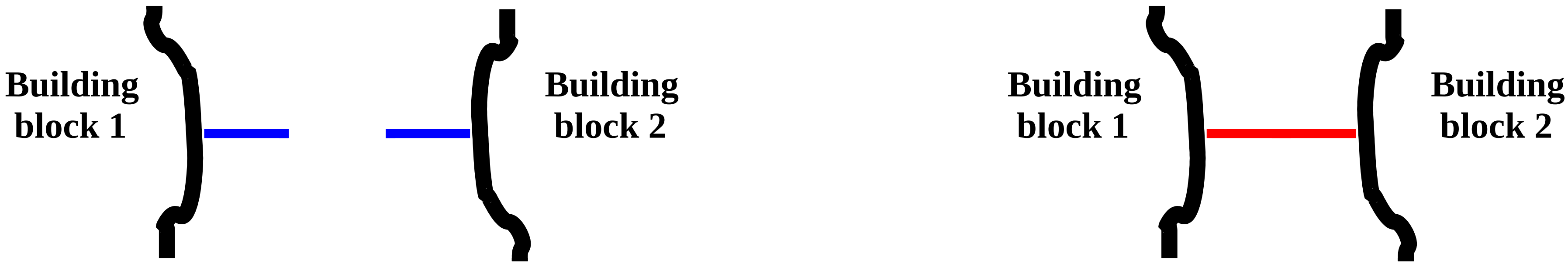
This identifies 2 inputs AND (NOT WITH) 2 outputs,
the sort of thing SIMULINK© forbids.

This is the rule, not the exception (in fluidics, mechanics,...)

Interconnection is not input-to-output assignment!

Sharing variables, not input-to-output assignment,

is the mechanism by which systems interact.



Before interconnection:

variables on interconnected terminals are independent.

After interconnection: they are set equal.

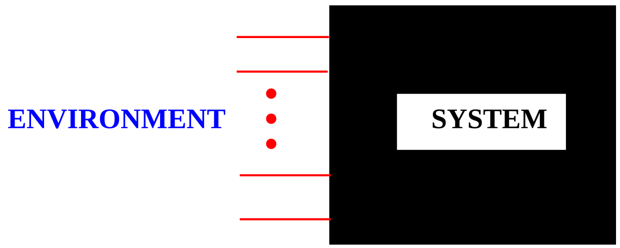
No signal graphs!

Let's take a closer look at the i/o framework ...

for modeling

i/o in modeling

Physical systems often interact with their environment through physical terminals

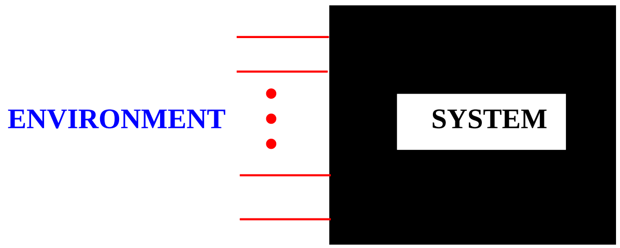


On each of these terminals many variables may 'live':

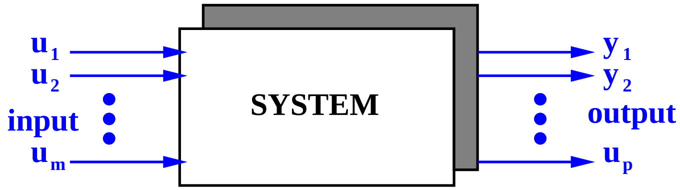
- **voltage & current**
- **position & force**
- **pressure & flow**
- **price & demand**
- **angle & momentum**
- **etc. & etc.**

i/o in modeling

Physical systems often interact with their environment through physical terminals



Usually input and output variables on same terminal:
NOT: on one terminal there is an input, on another there is an output.



This universal picture can be physically very misleading...

Conclusion

The inability of the i/o framework to deal properly with

(i) interconnections

and

(ii) passive control

is lethal.

Just as the state, the input/output partition, if needed, should be constructed from first principles models. Contrary to the state, such a partition may not be useful, or even possible

We need a better, more flexible, universal, simpler framework that properly deals with

open & connected.

General formalism

Generalities

What is a model? As a mathematical concept.

What is a dynamical system?

What is the role of differential equations in thinking about dynamical models?

Generalities

Intuition

We have a ‘phenomenon’ that produces ‘outcomes’ (‘events’).
We wish to model the outcomes that can occur.

Before we model the phenomenon:
the outcomes are in a set, which we call the *universum*.

After we model the phenomenon:
the outcomes are declared (thought, believed)
to belong to the *behavior* of the model,
a subset of this universum.

This subset is what we consider the mathematical model.

Generalities

This way we arrive at the

Definition

A *math. model* is a subset $\mathfrak{B}$ of a universum $\mathfrak{U}$ of outcomes

$$\mathfrak{B} \subseteq \mathfrak{U}.$$

$\mathfrak{B}$ is called the *behavior* of the model.
For example, the ideal gas law states that the temperature T, pressure P, volume V, and quantity (number of moles) N of an ideal gas satisfy

$$\frac{PV}{NT} = R$$

with R a universal constant.

Generalities

So, before Boyle, Charles, and Avogadro got into the act, T, P, V and N may have seemed unrelated, yielding

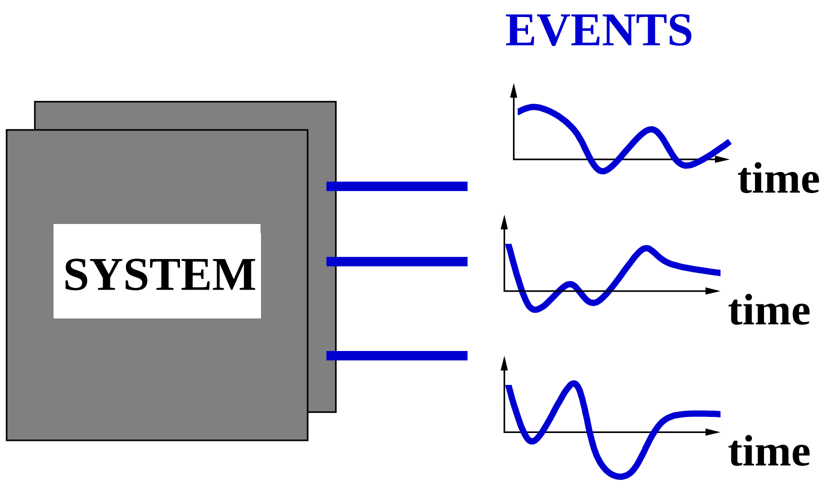
$$\mathfrak{U} = \mathbb{R}_+^4.$$

The ideal gas law restricts the possibilities to

$$\mathfrak{B} = \{(T, P, V, N) \in \mathbb{R}_+^4 \mid PV/NT = R\}$$

Dynamical systems

In dynamics, the outcomes are functions of time $\rightsquigarrow$



Which event trajectories are possible?

Dynamical systems

Definition

A dynamical system = $\boxed{\Sigma := (\mathbb{T}, \mathbb{W}, \mathfrak{B})}$

with $\mathbb{T} \subseteq \mathbb{R}$, the *time-axis* (= the relevant time instances),
$\mathbb{W}$, the *signal space*
(= where the variables take on their values),
$\boxed{\mathfrak{B} \subseteq \mathbb{W}^{\mathbb{T}}}$ ***the behavior*** **(= the admissible trajectories).**

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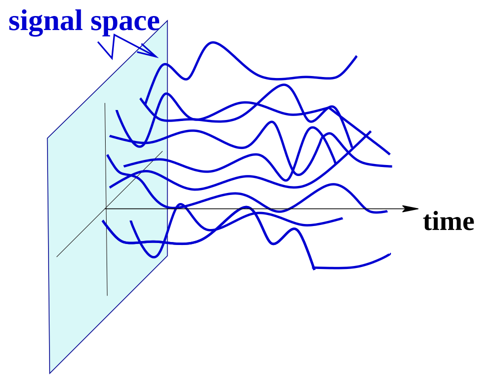

Totality of 'legal' trajectories =: the behavior

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For a trajectory ('an event') $w : \mathbb{T} \to \mathbb{W}$**, we thus have:**

$w \in \mathfrak{B}$ **: the model allows the trajectory** w**,**
$w \notin \mathfrak{B}$ **: the model forbids the trajectory** w**.**

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(= where the variables take on their values),
$\boxed{\mathfrak{B} \subseteq \mathbb{W}^{\mathbb{T}}}$ ***the behavior*** **(= the admissible trajectories).**

Usually,
$\mathbb{T} = \mathbb{R}$, or $[0, \infty)$, etc. (in continuous-time systems),
or $\mathbb{Z}$, or $\mathbb{N}$, etc. (in discrete-time systems).

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$\mathfrak{B} \subseteq \mathbb{W}^{\mathbb{T}}$ *the behavior* (= the admissible trajectories).

Usually,
$\mathbb{W} \subseteq \mathbb{R}^{\mathtt{w}}$ (in lumped systems),
a function space
(in distributed systems, time a distinguished variable),
a finite set (in DES)' etc.

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Emphasis:

$\mathbb{T} = \mathbb{R}$,
$\mathbb{W} = \mathbb{R}^{\mathtt{w}}$,
$\mathfrak{B}$ = **solutions of system of (linear constant coeff.) ODE's, difference eqn's, or PDE's.** $\rightsquigarrow$ **'differential systems'.**

Examples

The behavior is all there is

Equivalence, representations, symmetries, controllability, model simplification, etc. must refer to the behavior.

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Equivalence, representations, symmetries, controllability, model simplification, etc. must refer to the behavior.

Constant coefficient linear ODE's (linear DAE's)

$$R_0 w + R_1 \frac{d}{dt} w + R_2 \frac{d^2}{dt^2} w + \cdots + R_{\mathtt{n}} \frac{d^{\mathtt{n}}}{dt^{\mathtt{n}}} w = 0$$

$\mathbb{T} = \mathbb{R}$, $\mathbb{W} = \mathbb{R}^{\mathtt{w}}$, $\mathfrak{B} =$ **solutions...**

Notation: $R\left(\frac{d}{dt}\right) w = 0, \quad R \in \mathbb{R}[\xi]^{\bullet \times \mathtt{w}}$, **real pol. matrix.**

The solution definition is important. $\mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$ **or** $\mathcal{D}(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$ **different from** $\mathcal{L}_2(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$ **or compact support.**

Not only algebra, also analysis.

Examples

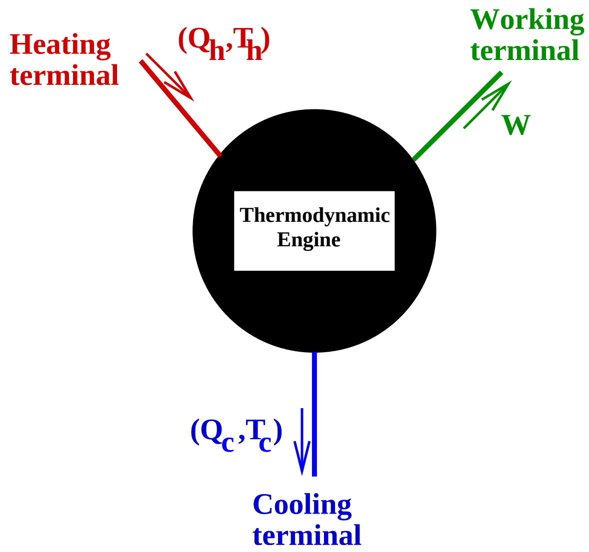
Input / state / output systems

$$\frac{d}{dt}x(t) = f(x(t), u(t), t), \quad y(t) = h(x(t), u(t), t)$$

Behavior:

either the (u, y, x)**'s,** **or the** (u, y)**'s?**

Examples



time-axis: $\mathbb{R}$

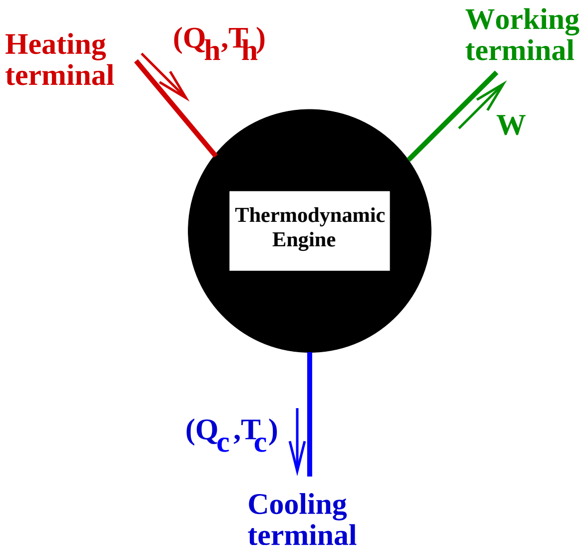
Q: Variables of interest? **A:** Q_h, T_h, Q_c, T_c, W

$\rightsquigarrow$ **signal space:** $\mathbb{W} = \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}$

Behavior $\mathfrak{B}$: a suitable family of trajectories.

But, there are some universal laws that restrict the $\mathfrak{B}$'s that are ‘thermodynamic’.

Examples



First and second law:

$$\oint (Q_h - Q_c - W)\, dt = 0; \quad \oint \left(\frac{Q_h}{T_h} - \frac{Q_c}{T_c}\right) dt \leq 0.$$

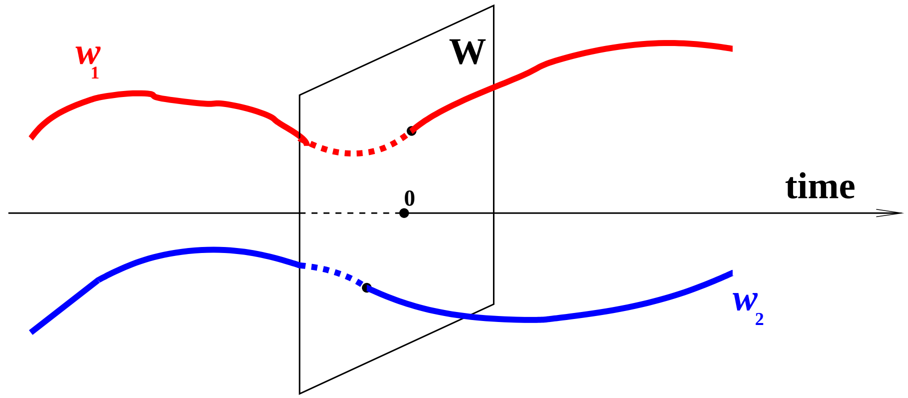
These laws deal with 'open' systems.

But not with input/output systems!

Controllability

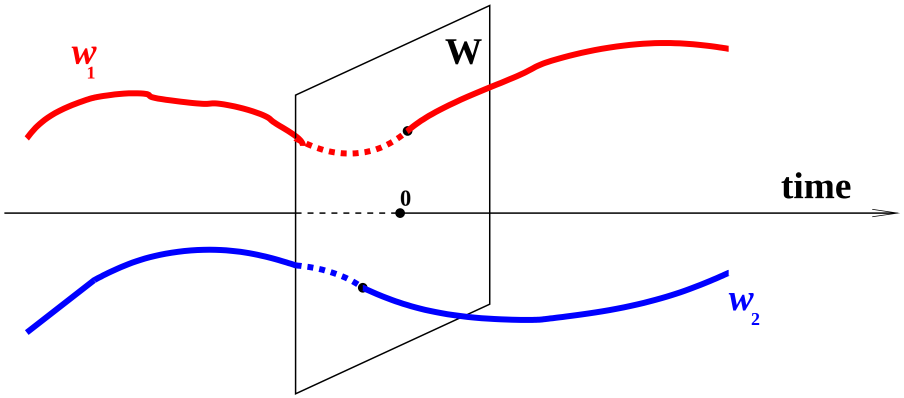
Controllability

Take any two trajectories w_1, $w_2 \in \mathfrak{B}$.

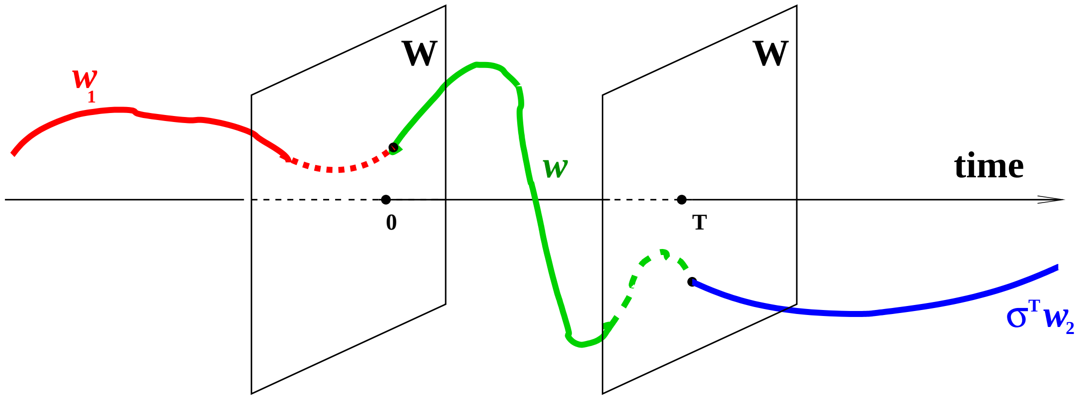


Controllability

Take any two trajectories $w_1, w_2 \in \mathfrak{B}$.



'<u>Controllability</u>':



Controllability

The time-invariant system $\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$ is said to be

controllable

if for all $w_1, w_2 \in \mathfrak{B}$ there exists $w \in \mathfrak{B}$ and $T \geq 0$ such that

$$w(t) = \begin{cases} w_1(t) & t < 0 \\ w_2(t - T) & t \geq T \end{cases}$$

Controllability :$\Leftrightarrow$
legal trajectories must be 'patch-able', 'concatenable'.

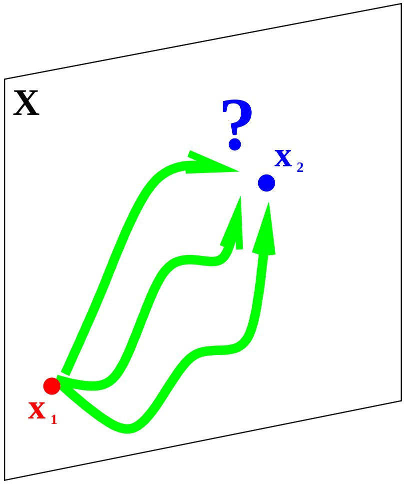
State Controllability

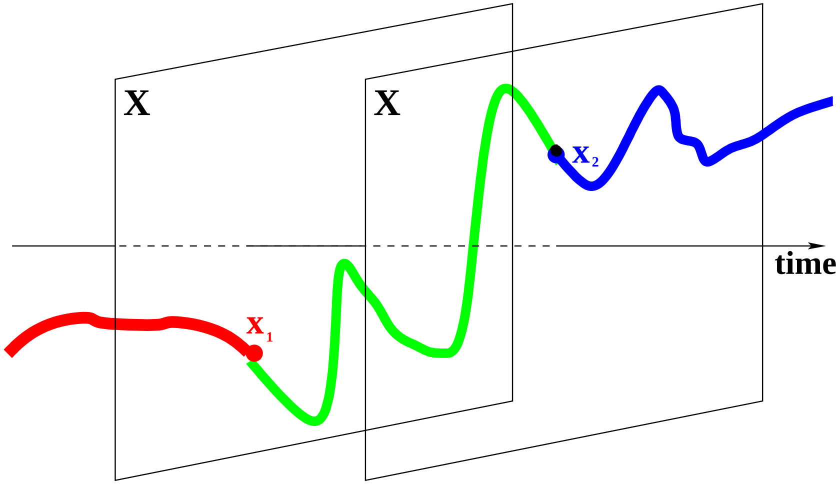
Special case: classical Kalman definitions for

$$\frac{d}{dt}x = f\,(x, u)\,.$$

controllability: variables = state or (input, state)

This is a special case of our controllability:





State Controllability

Special case: classical Kalman definitions for

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controllability: variables = state or (input, state)

Why should we be so concerned with the state?

If a system is not (state) controllable, why is it?

Insufficient influence of the control?

Or bad choice of the state?

Or not properly editing the equations?

Kalman's definition addresses a rather special situation.

Tests

Given a representation, derive algorithms in terms of the parameters for controllability. Consider $\mathfrak{B}$ defined by

$$R\left(\frac{d}{dt}\right)w = 0.$$

R: polynomial matrix. Under what conditions on $R \in \mathbb{R}^{\bullet \times \mathtt{w}}[\xi]$ does it define a controllable system?

Theorem: $R\left(\frac{d}{dt}\right)w = 0$ **defines a controllable system**

$\Leftrightarrow$

$\operatorname{rank}(R(\lambda)) =$ **constant over** $\lambda \in \mathbb{C}$.

Tests

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R**: polynomial matrix. Under what conditions on** $R \in \mathbb{R}^{\bullet \times \mathtt{w}}[\xi]$ **does it define a controllable system?**

Iff it admits an image representation

$$w = M\left(\frac{d}{dt}\right)\ell$$

$$\text{kernel}(R(\tfrac{d}{dt})) = \text{image}(M(\tfrac{d}{dt}))$$

Tests

Note:

- **When is**

$$p\left(\frac{d}{dt}\right)w_1 = q\left(\frac{d}{dt}\right)w_2$$

controllable? $p, q \in \mathbb{R}[\xi]$**, not both zero.**

Controllable $\Leftrightarrow$ **rank**$([p(\lambda) \quad -q(\lambda)] = 1 \ \forall \lambda \in \mathbb{C}$.

Iff p **and** q **are co-prime. No common factors!**

Testable via Sylvester matrix, etc.

Generalizable.

PDE’s

PDE's

Much of the theory also holds for PDE's.

$\mathbb{T} = \mathbb{R}^{\mathtt{n}}$, **the set of independent variables, often** $\mathtt{n} = 4$,
$\mathbb{W} = \mathbb{R}^{\mathtt{w}}$, **the set of dependent variables,**
$\mathfrak{B}$ = **set of maps** $\mathbb{R}^{\mathtt{n}} \longrightarrow \mathbb{R}^{\mathtt{w}}$

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Let $R \in \mathbb{R}^{\bullet \times \mathtt{w}}[\xi_1, \cdots, \xi_{\mathtt{n}}]$, **and consider**

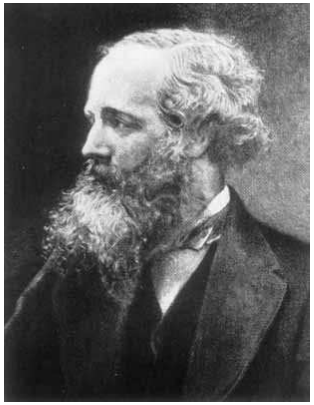
$$R\left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathtt{n}}}\right) w = 0. \quad (*)$$

Define the associated behavior

$$\mathfrak{B} = \{ w \in \mathfrak{C}^{\infty}(\mathbb{R}^{\mathtt{n}}, \mathbb{R}^{\mathtt{w}}) \mid (*) \text{ holds} \}.$$

Example

Maxwell's eq'ns, diffusion eq'n, wave eq'n, . . .



$$
\begin{aligned}
\nabla \cdot \vec{E} &= \frac{1}{\varepsilon_0}\rho\,, \\
\nabla \times \vec{E} &= -\frac{\partial}{\partial t}\vec{B}, \\
\nabla \cdot \vec{B} &= 0\,, \\
c^2 \nabla \times \vec{B} &= \frac{1}{\varepsilon_0}\vec{j} + \frac{\partial}{\partial t}\vec{E}.
\end{aligned}
$$

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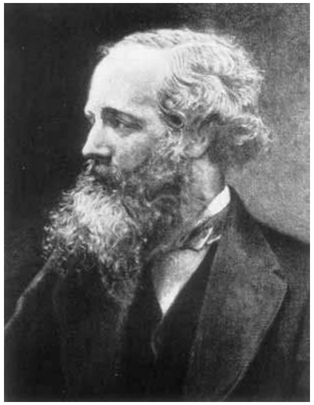


$$\begin{aligned} \nabla \cdot \vec{E} &= \frac{1}{\varepsilon_0}\rho\,, \\ \nabla \times \vec{E} &= -\frac{\partial}{\partial t}\vec{B}, \\ \nabla \cdot \vec{B} &= 0\,, \\ c^2 \nabla \times \vec{B} &= \frac{1}{\varepsilon_0}\vec{j} + \frac{\partial}{\partial t}\vec{E}. \end{aligned}$$

$\mathbb{T} = \mathbb{R} \times \mathbb{R}^3$ **(time and space)** $\mathtt{n} = 4$,

$w = \left(\vec{E}, \vec{B}, \vec{j}, \rho\right)$

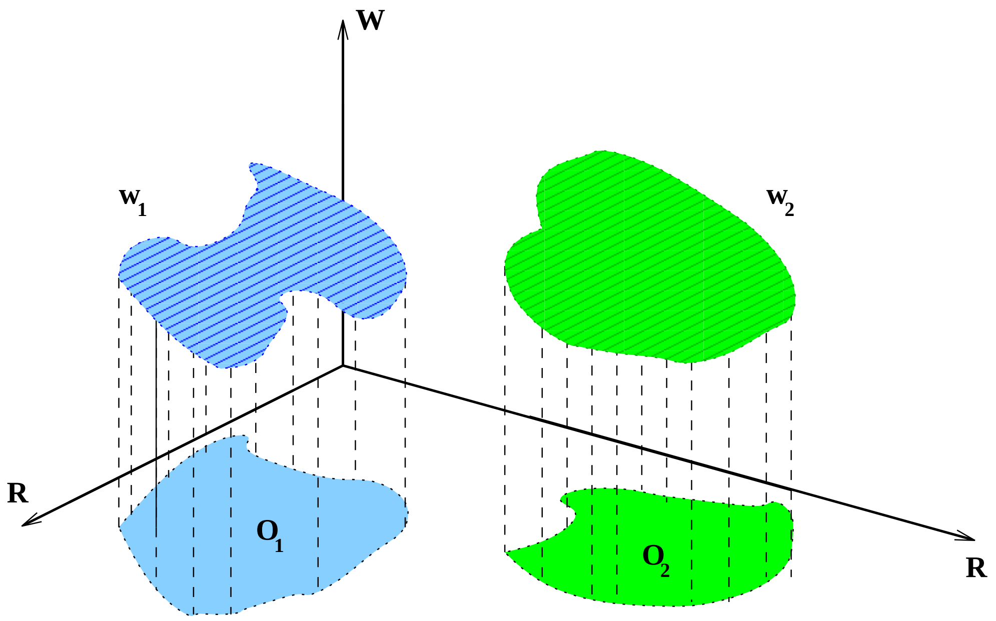
(electric field, magnetic field, current density, charge density),

$\mathbb{W} = \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}, \mathtt{w} = 10,$

$\mathfrak{B}$ = **set of solutions to these PDE's.**

Note: 10 variables, 8 equations! $\Rightarrow$ $\exists$ free variables. 'open'

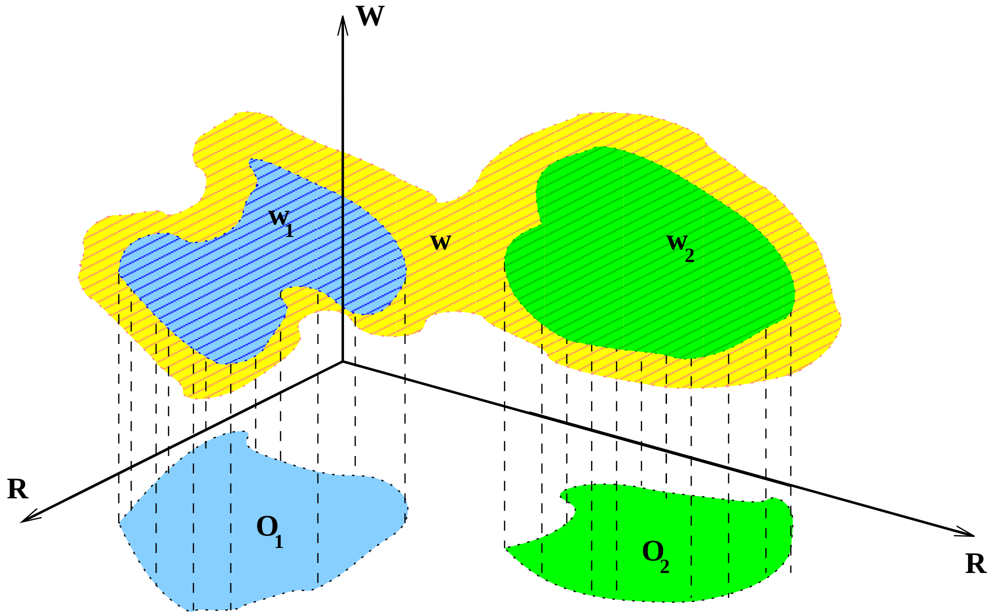
Controllability def'n in pictures:



$$w_1, w_2 \in \mathfrak{B}.$$

Controllability for PDE's

$\exists\, w \in \mathfrak{B}$ **'patches'** $w_1, w_2 \in \mathfrak{B}$.



Controllability :⇔ 'patch-ability'.

Are Maxwell's equations controllable ?

The following equations in the *scalar potential* ϕ : $\mathbb{R} \times \mathbb{R}^3 \longrightarrow \mathbb{R}$ and the *vector potential* $\vec{A} : \mathbb{R} \times \mathbb{R}^3 \longrightarrow \mathbb{R}^3$, generate exactly the solutions to Maxwell's equations:

$$
\begin{aligned}
\vec{E} &= -\frac{\partial}{\partial t}\vec{A} - \nabla \phi, \\
\vec{B} &= \nabla \times \vec{A}, \\
\vec{j} &= \varepsilon_0 \frac{\partial^2}{\partial t^2}\vec{A} - \varepsilon_0 c^2 \nabla^2 \vec{A} + \varepsilon_0 c^2 \nabla \left(\nabla \cdot \vec{A}\right) + \varepsilon_0 \frac{\partial}{\partial t}\nabla \phi, \\
\rho &= -\varepsilon_0 \frac{\partial}{\partial t}\nabla \cdot \vec{A} - \varepsilon_0 \nabla^2 \phi.
\end{aligned}
$$

Proves controllability. Illustrates the interesting connection

controllability $\Leftrightarrow \exists$ potential!

Conclusion

A good theory of dynamics has open systems as the starting point. Allows interconnection and tearing. Closed dynamical systems as used in math and physics very limited.

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Input/state/output models are an excellent paradigm. They model many things!

The flexibility and generality of the behavioral approach in modeling open systems, and their interconnections, is evident. Back-to-basics.
Incorporates wealth of system representations, deals for passive control, generalizes painlessly to PDE's, etc.
Exemplified by the notion of controllability.

Details & copies of the lecture frames are available from/at

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