RECURSIVE COMPUTATION OF THE MPUM

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Joint paper with Ivan Markovsky & Bart De Moor (K.U. Leuven)

The problem

$\tilde{w} \mapsto$ left kernel

Given the (infinite horizon) vector time-series

$$\tilde{w} = \tilde{w}(1), \tilde{w}(2), \ldots, \tilde{w}(t), \ldots \qquad w(t) \in \mathbb{R}^{\mathtt{w}}$$

Compute the left kernel of the associated Hankel matrix

$$\begin{bmatrix} \tilde{w}(1) & \tilde{w}(2) & \cdots & \tilde{w}(t) & \cdots \\ \tilde{w}(2) & \tilde{w}(3) & \cdots & \tilde{w}(t+1) & \cdots \\ \tilde{w}(3) & \tilde{w}(4) & \cdots & \tilde{w}(t+2) & \cdots \\ \vdots & \vdots & \vdots & & \\ \tilde{w}(\Delta) & \tilde{w}(\Delta+1) & \cdots & \tilde{w}(t+\Delta-1) & \cdots \\ \tilde{w}(\Delta+1) & \tilde{w}(\Delta+2) & \cdots & \tilde{w}(t+\Delta) & \cdots \\ \vdots & \vdots & \vdots & & \end{bmatrix}$$

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Why ? or How ?

The module structure

$\tilde{w} \mapsto$ left kernel

Each left annihilator can be identified with a vector polynomial

$$\begin{bmatrix} a_0 & a_1 & \cdots & a_\Delta & 0 & \cdots \end{bmatrix} \begin{bmatrix} \tilde{w}(1) & \tilde{w}(2) & \cdots & \tilde{w}(t) & \cdots \\ \tilde{w}(2) & \tilde{w}(3) & \cdots & \tilde{w}(t+1) & \cdots \\ \tilde{w}(3) & \tilde{w}(4) & \cdots & \tilde{w}(t+2) & \cdots \\ \vdots & \vdots & \vdots & & \\ \tilde{w}(\Delta) & \tilde{w}(\Delta+1) & \cdots & \tilde{w}(t+\Delta-1) & \cdots \\ \vdots & \vdots & \vdots & & \end{bmatrix} = 0$$

$$a_k \in \mathbb{R}^{1\times \mathtt{w}} \rightsquigarrow a(\xi) = a_0 + a_1\xi + \cdots + a_\Delta \xi^\Delta \in \mathbb{R}[\xi]^{1\times \mathtt{w}}$$

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The set of annihilators forms a $\mathbb{R}[\xi]$-module :

$a'(\xi), a''(\xi)$ **annihilators** $\Rightarrow a'(\xi) + a''(\xi), \xi a'(\xi)$ **annihilators.**

$\rightsquigarrow$ $R(\xi) \in \mathbb{R}[\xi]^{\bullet\times\mathtt{w}}$**, such that the module spanned by the rows ($\leq \mathtt{w}$) generates all these annihilators.**

$\tilde{w} \mapsto$ left kernel

$$\begin{bmatrix} a_0 & a_1 & \cdots & a_\Delta & 0 & \cdots \end{bmatrix} \begin{bmatrix} \tilde{w}(1) & \tilde{w}(2) & \cdots & \tilde{w}(t) & \cdots \\ \tilde{w}(2) & \tilde{w}(3) & \cdots & \tilde{w}(t+1) & \cdots \\ \tilde{w}(3) & \tilde{w}(4) & \cdots & \tilde{w}(t+2) & \cdots \\ \vdots & \vdots & \vdots & & \\ \tilde{w}(\Delta) & \tilde{w}(\Delta+1) & \cdots & \tilde{w}(t+\Delta-1) & \cdots \\ \vdots & \vdots & \vdots & & \end{bmatrix} = 0$$

This left kernel $\cong R(\xi)$ is in a sense always finite dimensional (dimension at most w).

$$R(\sigma)w = 0$$

identifies the most powerful unfalsified model (MPUM) in the class of linear time-invariant complete systems induced by the data $\tilde{w}$.

$\tilde{w} \mapsto$ left kernel

Computing this left kernel has been studied in many aspects

- Recursively in t (Berlekamp-Massey, Kuijper), finite t
- Recursively in Δ (today)

  Compute only a set of generators for the module generated by the left kernel (today)
- Approximately
- $\tilde{w} \mapsto \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ (realization algorithms, subspace ID)

$\tilde{w} \mapsto$ left kernel

Suppose we found the left annihilators of

$$\begin{bmatrix} \tilde{w}(1) & \tilde{w}(2) & \cdots & \tilde{w}(t) & \cdots \\ \tilde{w}(2) & \tilde{w}(3) & \cdots & \tilde{w}(t+1) & \cdots \\ \tilde{w}(3) & \tilde{w}(4) & \cdots & \tilde{w}(t+2) & \cdots \\ \vdots & \vdots & \vdots & & \\ \tilde{w}(\Delta) & \tilde{w}(\Delta+1) & \cdots & \tilde{w}(t+\Delta-1) & \cdots \end{bmatrix}$$

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Can we use this knowledge to simplify finding the left annihilators of

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$$a(\xi) = a_0 + a_1\xi + \cdots + a_{\Delta-1}\xi^{\Delta-1} \in \mathbb{R}[\xi]^{1\times \mathtt{w}}$$

$\rightsquigarrow$ $R_{\Delta-1}(\xi) \in \mathbb{R}[\xi]^{\bullet\times \mathtt{w}}$, **such that the module spanned by the rows generates all these annihilators.**

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Can we use knowledge of $R_{\Delta-1}(\xi)$ to compute $R_{\Delta}(\xi)$?

The completion lemma

Completion lemma

Let $R(\xi) \in \mathbb{R}^{\mathtt{p}\times\mathtt{w}}[\xi]$ be left prime. Then $\exists\ E(\xi) \in \mathbb{R}^{(\mathtt{w}-\mathtt{p})\times\mathtt{w}}[\xi]$ such that

$$\begin{bmatrix} R(\xi) \\ E(\xi) \end{bmatrix} \quad \text{is unimodular}$$

meaning $\det =$ a non-zero constant, invertible as a polynomial matrix.

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Ex.

$\mathtt{p} = 1, \mathtt{w} = 2, R(\xi) = [r_1(\xi)\ \ r_2(\xi)], E(\xi) = [-y(\xi)\ \ x(\xi)]$

Given $r_1(\xi), r_2(\xi) \in \mathbb{R}[\xi]$, find $x(\xi), y(\xi) \in \mathbb{R}[\xi]$ such that

$$x(\xi)r_1(\xi) + y(\xi)r_2(\xi) = 1 \qquad \text{Bézout}$$

Solvable iff r_1, r_2 coprime. $\exists$ algorithms, etc.

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Given $\tilde{w}$, assume $R(\sigma)\tilde{w} = 0$,

R is 'unfalsified': rows of R are in left kernel of Hankel matrix.

Now compute the 'error' $\tilde{e} := E(\sigma)\tilde{w}$

and the MPUM for $\tilde{e}$ $\rightsquigarrow S(\sigma)e = 0.$

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and the **MPUM for $\tilde{e}$** $\rightsquigarrow S(\sigma)e = 0$.

Then

$$\begin{aligned} R(\sigma)w &= 0 \\ S(\sigma)E(\sigma)w &= 0 \end{aligned}$$

defines the MPUM for $\tilde{w}$. Note $\tilde{e}$ has a lower dimension than $\tilde{w}$.

This allows to compute the eq'ns for the MPUM one-by-one.

Each iteration, the time-series is reduced in dimension by one.

Recursive computation

Recursive computation

Assume that we have one of the left annihilators of

$$\begin{bmatrix} \tilde{w}(1) & \tilde{w}(2) & \cdots & \tilde{w}(t) & \cdots \\ \tilde{w}(2) & \tilde{w}(3) & \cdots & \tilde{w}(t+1) & \cdots \\ \vdots & \vdots & \vdots & & \\ \tilde{w}(\Delta+1) & \tilde{w}(\Delta+2) & \cdots & \tilde{w}(t+\Delta) & \cdots \end{bmatrix}$$

$\rightsquigarrow$ $R(\sigma)\tilde{w} = 0$.

Recursive computation

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$\rightsquigarrow$ $R(\sigma)\tilde{w} = 0$. Complete R with E : $\left[\begin{smallmatrix} R \\ E \end{smallmatrix}\right]$ unimodular.

Compute $\tilde{e} = E(\sigma)\tilde{w}$, and one of the left annihilators of

$$\begin{bmatrix} \tilde{e}(1) & \tilde{e}(2) & \cdots & \tilde{e}(t) & \cdots \\ \tilde{e}(2) & \tilde{e}(3) & \cdots & \tilde{e}(t+1) & \cdots \\ \vdots & \vdots & \vdots & & \\ \tilde{e}(\Delta+2) & \tilde{e}(\Delta+3) & \cdots & \tilde{e}(t+\Delta+1) & \cdots \end{bmatrix}$$

$\rightsquigarrow$ $S(\sigma)\tilde{e} = 0$.

Obtain $R \rightarrow \left[\begin{smallmatrix} R \\ SE \end{smallmatrix}\right]$. Proceed with $\tilde{e}$ and S. One relation at the time...

Controllability

At each stage there is the left primeness assumption. How can we guarantee that this will be satisfied?

It is satisfied if the MPUM $R(\sigma)w = 0$ is controllable.

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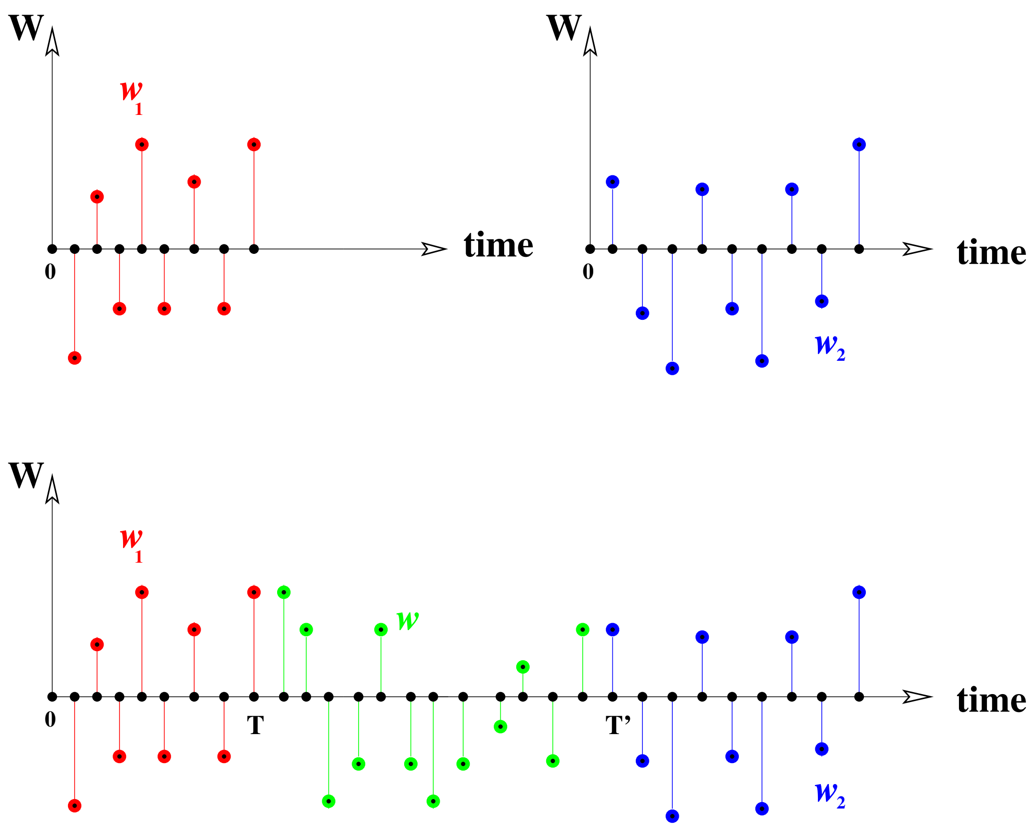

Controllability

At each stage there is the left primeness assumption. How can we guarantee that this will be satisfied?

It is satisfied if the MPUM $R(\sigma)w = 0$ is controllable.

This in turn can be guaranteed if the data $\tilde{w}$ has been generated by a controllable system and an input that is persistently exciting of a suitable order.

Number of iteration steps = number of outputs.

Summary

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Problem: find the left kernel of

$$\begin{bmatrix} \tilde{w}(1) & \tilde{w}(2) & \cdots & \tilde{w}(t) & \cdots \\ \tilde{w}(2) & \tilde{w}(3) & \cdots & \tilde{w}(t+1) & \cdots \\ \tilde{w}(3) & \tilde{w}(4) & \cdots & \tilde{w}(t+2) & \cdots \\ \vdots & \vdots & \vdots & & \\ \tilde{w}(\Delta) & \tilde{w}(\Delta+1) & \cdots & \tilde{w}(t+\Delta-1) & \cdots \\ \vdots & \vdots & \vdots & & \end{bmatrix}$$

This has the structure of a module.

$\Rightarrow$ **suffices to find at most `w` elements.**

Actually = number of outputs of the system that generated the data.

Summary

Check kernel of finite truncations

$$\begin{bmatrix} a_0 & a_1 & \cdots & a_{\Delta-1} \end{bmatrix} \begin{bmatrix} \tilde{w}(1) & \tilde{w}(2) & \cdots & \tilde{w}(t) & \cdots \\ \tilde{w}(2) & \tilde{w}(3) & \cdots & \tilde{w}(t+1) & \cdots \\ \tilde{w}(3) & \tilde{w}(4) & \cdots & \tilde{w}(t+2) & \cdots \\ \vdots & \vdots & \vdots & & \\ \tilde{w}(\Delta) & \tilde{w}(\Delta+1) & \cdots & \tilde{w}(t+\Delta-1) & \cdots \end{bmatrix} = 0$$

by adding one row at the time.

Find an annihilator, complete $\rightsquigarrow E$, compute error $\tilde{e} = E(\sigma)\tilde{w}$, proceed with

$$\begin{bmatrix} \tilde{e}(1) & \tilde{e}(2) & \cdots & \tilde{e}(t) & \cdots \\ \tilde{e}(2) & \tilde{e}(3) & \cdots & \tilde{e}(t+1) & \cdots \\ \vdots & \vdots & \vdots & & \\ \tilde{e}(\Delta) & \tilde{e}(\Delta+1) & \cdots & \tilde{e}(t+\Delta-1) & \cdots \end{bmatrix}$$

Summary

This will get the kernel in exactly $\mathtt{p}$ (= number of outputs) iterations.

Underlying regularity assumptions: controllability, persistency of excitation.

Relevant application: (deterministic) subspace ID.

Details & copies of the lecture frames are available from/at

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