DISSIPATIVE SYSTEMS:

Where do we stand?

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GianniFest **October 27, 2006**

On the occasion of the 70-th birthday of

Gianni Marchesini

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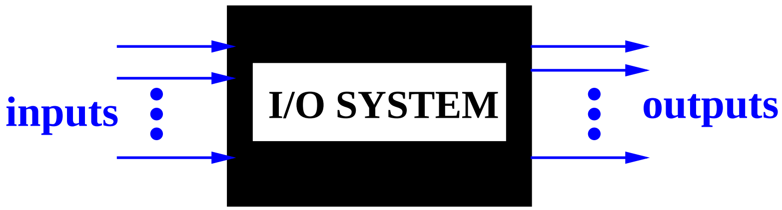
Dissipative systems

Open systems

'Open' systems are an appropriate starting point for the study of dynamics. For example,



$\rightsquigarrow$ **the dynamical system**

$$\Sigma : \quad \dot{x} = f(x,u), \quad y = h(x,u).$$

$u \in \mathbb{U} = \mathbb{R}^{\mathtt{m}}, y \in \mathbb{Y} = \mathbb{R}^{\mathtt{p}}, x \in \mathbb{X} = \mathbb{R}^{\mathtt{n}}$**: input, output, state.**

Behavior $\mathfrak{B} =$ **all sol'ns** $(u,y,x) : \mathbb{R} \to \mathbb{U} \times \mathbb{Y} \times \mathbb{X}$.

Dissipative dynamical systems

$s : \mathbb{U} \times \mathbb{Y} \rightarrow \mathbb{R}$ called the ***supply rate*,**

$V : \mathbb{X} \rightarrow \mathbb{R}$ called the ***storage functon*.**

Σ is said to be

***dissipative* w.r.t. the supply rate s and with storage V**

if

$$\frac{d}{dt} V(x(\cdot)) \leq s(u(\cdot), y(\cdot))$$

for all $(u, y, x) \in \mathfrak{B}$**.**

Dissipation inequality

$$\frac{d}{dt} V(x(\cdot)) \leq s(u(\cdot), y(\cdot))$$

for all $(u, y, x) \in \mathfrak{B}$**.**

This inequality is called the ***dissipation inequality.***

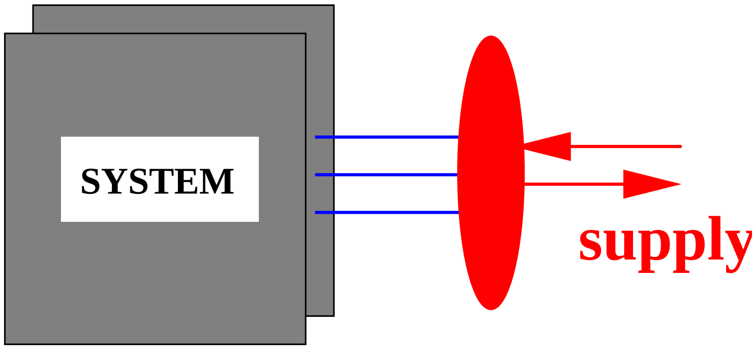
Equivalent to

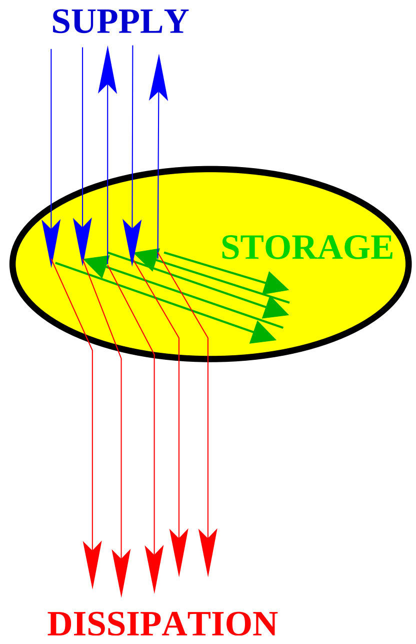
$$\overset{\bullet}{V}^{\Sigma}(\mathrm{x}, \mathrm{u}) := \nabla V(\mathrm{x}) \cdot f(\mathrm{x}, \mathrm{u}) \leq s(\mathrm{x}, h(\mathrm{x}, \mathrm{u}))$$

for all $(\mathrm{u}, \mathrm{x}) \in \mathbb{U} \times \mathbb{X}$**.**

If equality holds: 'conservative' system.

Dissipation inequality





$s(\mathrm{u}, \mathrm{y})$ **models something like the power delivered to the system when the input value is** u **and output value is** y**.**

$V(\mathrm{x})$ **then models the internally stored energy.**

Dissipativity $:\Leftrightarrow$

rate of increase of internal energy $\leq$ **power delivered**

Lyapunov function

Special case: 'closed' system:

$$\Sigma : \dot{\mathrm{x}} = f(\mathrm{x}) \quad \textbf{and} \quad s = 0$$

then dissipativity with $V : \mathbb{X} \to \mathbb{R}$

$\rightsquigarrow$ ***Lyapunov function*** $\frac{d}{dt} V(x(\cdot)) \leq 0$

Lyapunov function

Special case: 'closed' system:

$$\Sigma : \dot{x} = f(x) \quad \textbf{and} \quad s = 0$$

then dissipativity with $V : \mathbb{X} \to \mathbb{R}$

$\leadsto$ ***Lyapunov function*** $\quad \frac{d}{dt} V(x(\cdot)) \leq 0$

dissipativity $\leftrightarrow$ V is a Lyapunov function.

Dissipativity is the natural generalization to open systems of Lyapunov theory.

Stability for closed systems $\simeq$ Dissipativity for open systems.

The construction of storage functions

Basic question:

Given (a representation of) Σ, the dynamics,
and given s, the supply rate,
does there exist a storage function V such that
the dissipation inequality holds?

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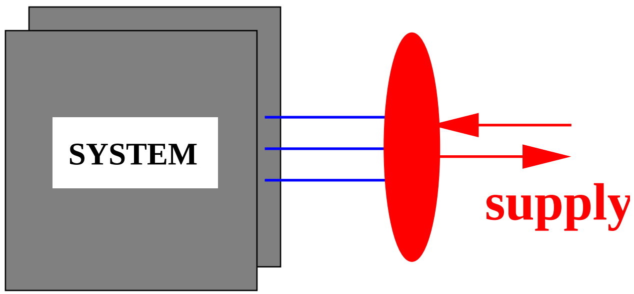

Monitor power in, known dynamics, what is the stored energy?

The construction of storage functions

The construction of storage f'ns is very well understood, particularly for finite dimensional linear systems and quadratic supply rates.

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The storage function V is in general far from unique. There are two 'canonical' storage functions:

the available storage and the required supply.

For conservative systems, V is unique.

From storage to LMI's

$$\Sigma: \quad \frac{d}{dt}x = Ax + Bu, y = Cx + Du, \quad s(u,y) = \textbf{quadratic}$$

From storage to LMI's

$$\Sigma : \quad \frac{d}{dt}x = Ax + Bu, y = Cx, \qquad s(u,y) = ||u||^2 - ||y||^2$$

If storage f'n exists, quadratic one exists $V(x) = \frac{1}{2}x^\top Kx$

WLOG $K = K^\top$**, possibly** $K \geq 0$**.**

$$\frac{d}{dt}V(x) \leq s(u,y) \quad \rightsquigarrow \quad \begin{bmatrix} A^\top K + AK + C^\top C & KB \\ B^\top K & -I \end{bmatrix} \leq 0$$

solvable?

$$\Leftrightarrow \quad A^\top K + AK + KBB^\top K + C^\top C \leq 0 \quad \textbf{solvable?}$$

$$\Leftrightarrow \quad A^\top K + AK + KBB^\top K + C^\top C = 0 \quad \textbf{solvable?}$$

ARIneq, ARE, ...

From storage to LMI's

$$\begin{bmatrix} A^\top K + AK + C^\top C & KB \\ B^\top K & -I \end{bmatrix} \leq 0$$ **and, possibly,** $K \geq 0 \rightsquigarrow$

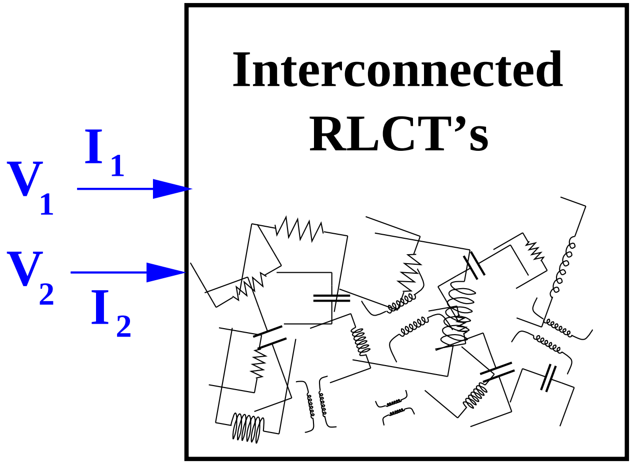
$A_0 + x_1 A_1 + x_2 A_2 + \cdots x_{\mathtt{n}} A_{\mathtt{n}} \geq 0$ **feasible?, etc.**

LMI's, SDP, ...

How good is this notion?

Circuit synthesis

Is $G \in \mathbb{R}(\xi)$ realizable as the driving point impedance of an electrical circuit containing (positive) resistors, capacitors, inductors, and transformers?



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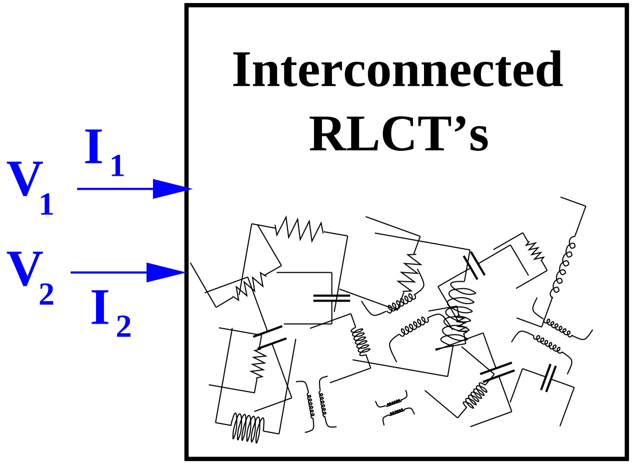

Iff G **is 'positive real'**

[$\mathrm{Re}(s) > 0 \Rightarrow \mathrm{Re}(G(s)) > 0$] **Otto Brune, 1932**

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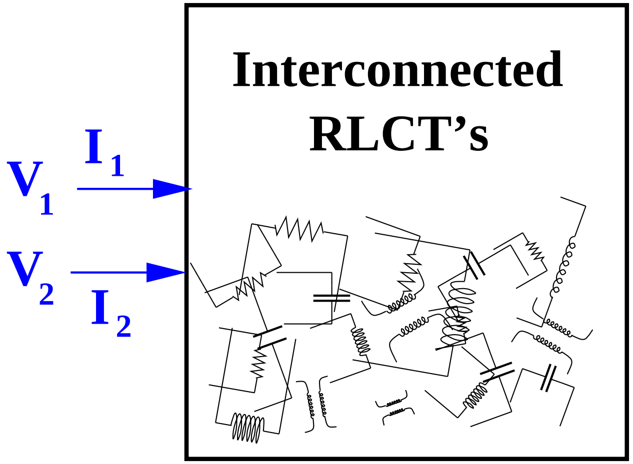

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[$\mathrm{Re}(s) > 0 \Rightarrow \mathrm{Re}\,(G(s)) > 0$**]** **Otto Brune, 1932**

Trafos not needed **Raoul Bott & Richard Duffin, 1949**

Circuit synthesis

Central idea of proof, using storage functions:

Let $\frac{d}{dt}x = Ax + Bu, \;\; y = Cx + Du$ **be a minimal realization of** G**.**
G **p.r.** $\Leftrightarrow$ **dissipative w.r.t.** $u^\top y$**, storage f'n** $\frac{1}{2}x^\top Kx$**,** $K = K^\top > 0$**.**
Choice of basis $\Rightarrow K = I$**.** $\rightsquigarrow \; \frac{d}{dt}\frac{1}{2}x^\top x \leq u^\top y \Leftrightarrow$

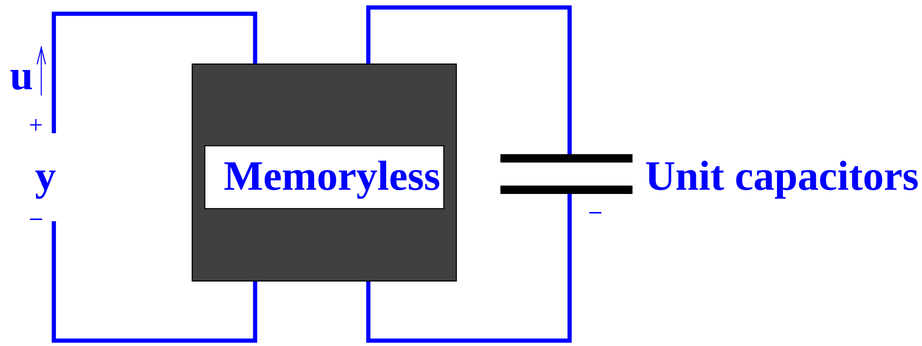
$$\begin{bmatrix} -A & -B \\ C & D \end{bmatrix} + \begin{bmatrix} -A & -B \\ C & D \end{bmatrix}^\top \geq 0$$

Circuit synthesis

Central idea of proof, using storage functions:

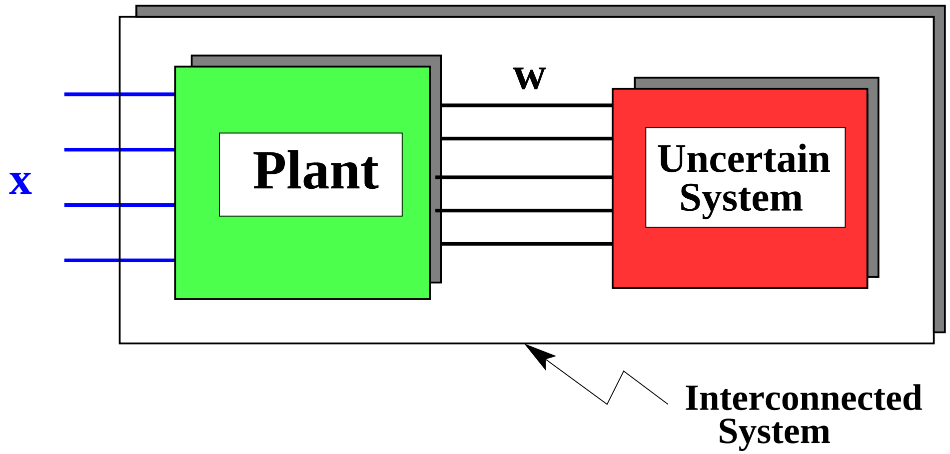
Now, interconnect $\begin{bmatrix} V \\ y \end{bmatrix} = \begin{bmatrix} -A & -B \\ C & D \end{bmatrix} \begin{bmatrix} I \\ u \end{bmatrix}$ **with** $\frac{d}{dt}I = -V$

The storage f'n and the LMI takes the dynamics out.
Terminate a memoryless system with unit capacitors.



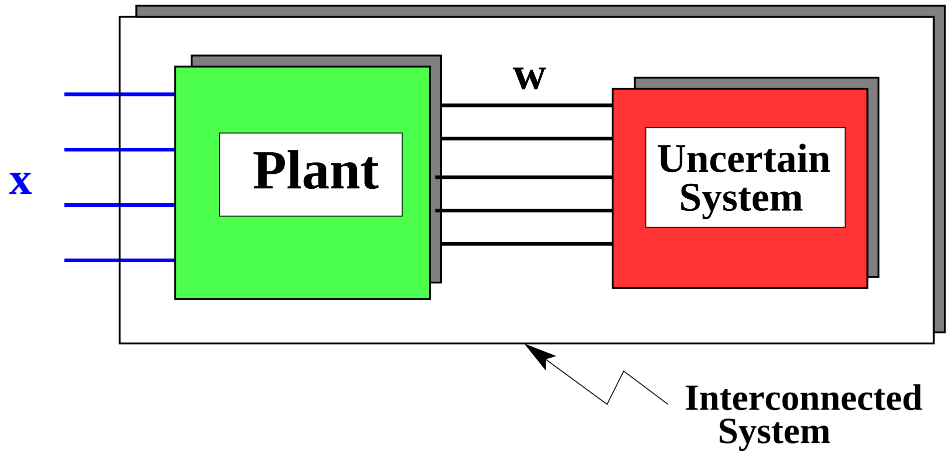
Enforce reciprocity, etc.

Stability of dissipative interconnections



Is this uncertain system stable?

Stability of dissipative interconnections



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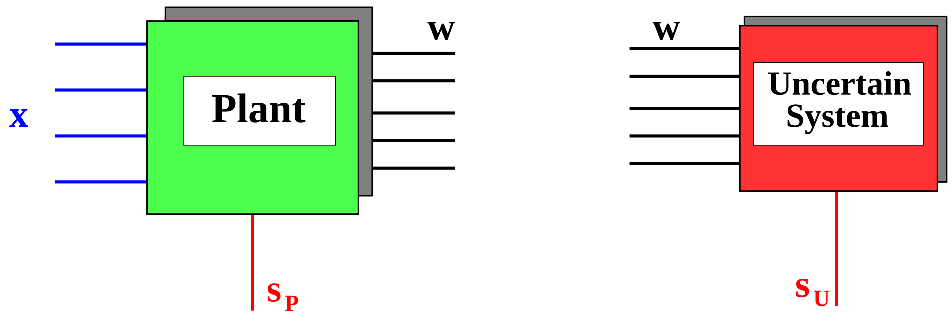
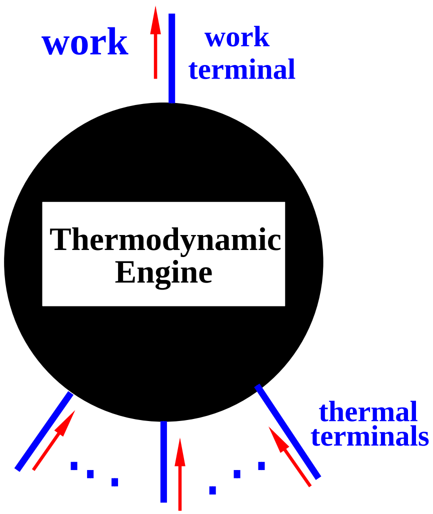


Yes, if both systems are dissipative and $s_P + s_U = 0$

$\rightsquigarrow$ **Lyapunov f'n = sum of storage f'ns.** $\Rightarrow$ **stability.**
This requires the state, also for the uncertain system.

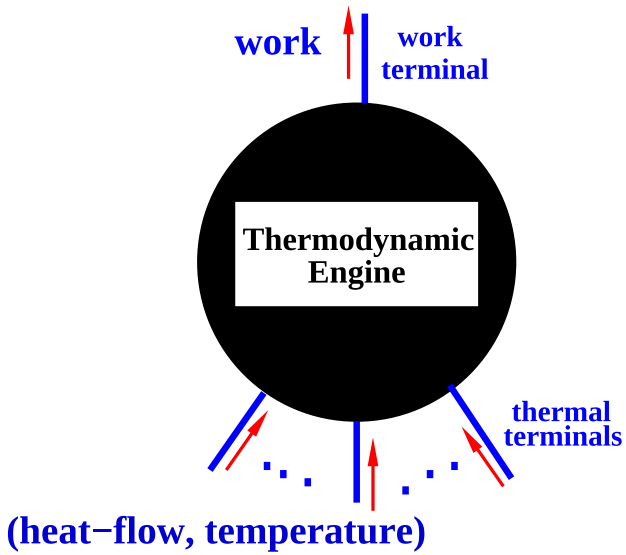
Thermodynamics



Conservative w.r.t. $- \textit{work} + \Sigma_{\textit{heat terminals}} \; \textit{heat flow}$

Dissipative w.r.t. $-\Sigma_{\textit{heat terminals}} \; \frac{\textit{heat flow}}{\textit{temperature}}$

Thermodynamics



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Dissipative w.r.t. $-\Sigma_{\mathit{heat\ terminals}} \dfrac{\mathit{heat\ flow}}{\mathit{temperature}}$

Input/output setting is hopeless!

Back to basics

Behaviors

Dynamical system: $\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$, **with** $\mathbb{T} \subseteq \mathbb{R}$ **the time-set,** $\mathbb{W}$ **the signal space, and** $\mathfrak{B} \subseteq \mathbb{W}^{\mathbb{T}}$ **the** ***behavior*****.**

Latent variable dynamical system **is a refinement, with behavior represented with the aid of** *latent variables***.**

$\Sigma_L = (\mathbb{T}, \mathbb{W}, \mathbb{L}, \mathfrak{B}_{\text{full}})$ **with** $\mathbb{L}$ **the space of latent variables, and** $\mathfrak{B}_{\text{full}} \subseteq (\mathbb{W} \times \mathbb{L})^{\mathbb{T}}$ **the** ***full behavior*****.**

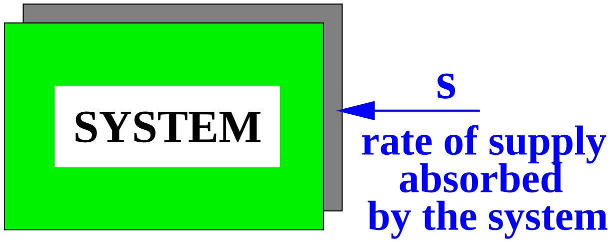
Σ_L **induces** $\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$ **with** *manifest behavior*

$$\mathfrak{B} = \left\{ w : \mathbb{T} \to \mathbb{W} \;\middle|\; \exists \ell : \mathbb{T} \to \mathbb{L} \textbf{ such that } (w, \ell) \in \mathfrak{B}_{\text{full}} \right\}.$$

Example: $\frac{d}{dt}\ell = A\ell + Bu, \;\; y = C\ell + Du$**.**

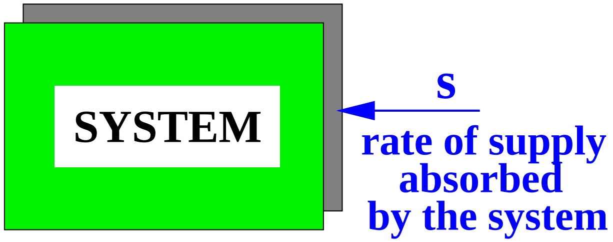
The behavior is all there is. Linearity, time-invariance, ...

Dissipativity & Behaviors



Dissipativeness restricts the way supply goes in and out.

Dissipativity & Behaviors



Dissipativeness restricts the way supply goes in and out.

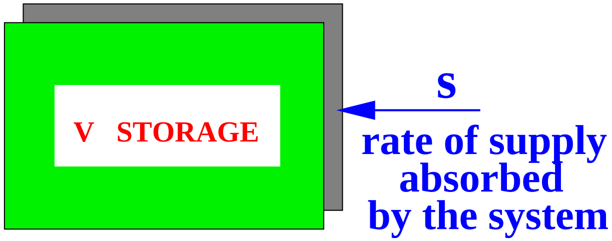
$\Sigma = (\mathbb{R}, \mathbb{R}, \mathfrak{B})$ **dynamical system.**
$s : \mathbb{R} \to \mathbb{R}$, $s \in \mathfrak{B}$**, models rate of supply *absorbed*.**

$\Sigma_L = (\mathbb{R}, \mathbb{R}, \mathbb{R}, \mathfrak{B}_{\text{full}})$ **a latent variable representation.**
$(s, V) \in \mathfrak{B}_{\text{full}}$**,** $V : \mathbb{R} \to \mathbb{R}$ **models the supply *stored*.**

Dissipativity & Behaviors

V **is said to be a** ***storage*** **if** $\forall\,(s,V) \in \mathfrak{B}_{\text{full}}$ **and** $\forall\, t_0 \leq t_1$**,**
the ***dissipation inequality*** **holds**

$$V(t_1) - V(t_0) \leq \int_{t_0}^{t_1} s(t)\, dt$$



$\Sigma = (\mathbb{R}, \mathbb{R}, \mathfrak{B})$**, time-invariant, is said to be** **dissipative** **if**
there exists $\Sigma_L = (\mathbb{R}, \mathbb{R}, \mathbb{R}, \mathfrak{B}_{\text{full}})$**, time-invariant,**
such that the dissipation inequality holds.

Nonnegative storage

Simple existence result for non-negative storage functions.

<u>THEOREM</u>

Σ is *dissipative with non-negative storage* $\Leftrightarrow$

$\forall s \in \mathfrak{B}$ **and** $\forall t_0 \in \mathbb{R}, \ \exists K \in \mathbb{R},$

such that $-\int_{t_0}^{T} s(t)\, dt \leq K$ **for** $T \geq t_0$

'Available storage' is finite. N.a.s.c.!

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'Available storage' is finite. N.a.s.c.!

A n.a.s.c. for the existence of $\mathfrak{B}_{\text{full}}$ **and** V **(in terms of** $\mathfrak{B}$**) is ?**
$\exists$ **sufficient conditions in terms of periodic trajectories assuming** ***observability*** **of** V **from** s**.**

Quadratic supply rates

QDF's

A *quadratic differential form (QDF)* is a quadratic expression in the components of $w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$ and its derivatives:

$$\Sigma_{\mathtt{k},\ell} \left(\frac{d^{\mathtt{k}}}{dt^{\mathtt{k}}} w\right)^\top \Phi_{\mathtt{k},\ell} \left(\frac{d^\ell}{dt^\ell} w\right)$$

with the $\Phi_{\mathtt{k},\ell} \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$. Map from $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$ to $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R})$.
Compact notation and a convenient calculus.

$$\Phi(\zeta, \eta) = \Sigma_{\mathtt{k},\ell}\, \Phi_{\mathtt{k},\ell} \zeta^{\mathtt{k}} \eta^\ell$$

Notation QDF $Q_\Phi(w)$.

Q_Φ is said to be *non-negative* (denoted $Q_\Phi \geq 0$) $:\Leftrightarrow$
$Q_\Phi(w) \geq 0$ for all $w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$.

Dissipativity of QDF's

The system $\Sigma_\Phi = \left(\mathbb{R}, \mathbb{R}, \mathrm{im}\left(Q_\Phi\right)\right)$: supply rate is QDF.

Quite general, 'LQ':

1. **Linear time-invariant differential system**

$$R\left(\frac{d}{dt}\right) w = 0$$

   perhaps including latent variables.

2. **Controllable (in the behavioral sense: patchability).**
3. **QDF for the supply rate.**

Extendable to rat. f'ns, both in system eq'ns and supply rate.

Examples: linear circuits, t'f f'n with supply rate quadratic form in input and output, linear mechanical systems, ...

Dissipativity of $\Sigma_\Phi = \left(\mathbb{R}, \mathbb{R}, \mathrm{im}\,(\mathrm{Q}_\Phi)\right)$

6 statements concerning a supply rate defined by a QDF.

(i) Σ_Φ **is dissipative (**$\exists$ **storage)**

(ii) Σ_Φ **admits a ... with a QDF as storage**

(iii) Σ_Φ **admits a ... with a memoryless state f'n as storage**

(iv) Σ_Φ **admits a ... with a m'ess quadr. state f'n as storage**

(v) $\int_{-\infty}^{+\infty} Q_\Phi(w)\, dt \geq 0 \quad \forall w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$ **compact support**

(vi) $\Phi(i\omega, -i\omega) + \Phi^\top(-i\omega, i\omega) \geq 0 \quad \forall\, \omega \in \mathbb{R}$

(i)$\Leftarrow$**(ii)**$\Leftrightarrow$**(iii)**$\Leftrightarrow$**(iv)**$\Leftrightarrow$**(v)**$\Leftrightarrow$**(vi)**

Under certain 'signature conditions' (i)$\Leftrightarrow$**(ii).**

Dissipativity of $\Sigma_\Phi = \left(\mathbb{R}\,,\mathbb{R}\,,\mathrm{im}\,(\mathrm{Q}_\Phi)\,\right)$

With a non-negative storage function, we obtain instead

(i) **Available storage for Σ_Φ is finite**

(ii) **Σ_Φ admits a latent var. with non-negative storage**

(iii) **Σ_Φ ... with a non-negative QDF as storage**

(iv) **Σ_Φ ... with a ≥ 0 memoryless state f'n as storage**

(v) **Σ_Φ ... with a ≥ 0 ... quadr. state f'n as storage**

(vi) $\int_{-\infty}^{0} Q_\Phi(w)\,dt \geq 0$ $\quad \forall w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$ **of compact support**

(vii) **RHP frequency-domain & Pick matrix condition on Φ**

(i)$\Leftrightarrow$(ii)$\Leftarrow$(iii)$\Leftrightarrow$(iv)$\Leftrightarrow$(v)$\Leftrightarrow$(vi)$\Leftrightarrow$(vii)

Under certain 'signature conditions' (ii)$\Leftrightarrow$(iii).

Dissipativity of $\Sigma_\Phi = \big(\mathbb{R}, \mathbb{R}, \mathrm{im}(Q_\Phi)\big)$

The existence of a QDF as storage is an LMI.

$\Phi \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ **is given,** $\Psi \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ **is unknown.**

$$\frac{d}{dt} Q_\Psi(w) \leq Q_\Phi(w) \quad \forall w \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}})$$

$$\Updownarrow$$

$$(\zeta+\eta)\Psi(\zeta,\eta) \leq \Phi(\zeta,\eta)$$

Remains LMI if $\Psi \geq 0$ **is added.**

In 1-D case storage f'n of w **'observability'.**
Not so in n-D case, as Maxwell's eq'ns.

Some open problems

Intrinsic characterization of dissipativity

Let $\Sigma = (\mathbb{R}, \mathbb{R}, \mathfrak{B})$ be time-invariant. When is it dissipative?

I.e., when does there exists a time-invariant latent variable representation $\Sigma_L = (\mathbb{R}, \mathbb{R}, \mathbb{R}, \mathfrak{B}_{\text{full}})$, time-invariant, such that the dissipation inequality holds?

$\exists$ sufficient conditions in terms of periodic behavior, controllability, observability, equilibrium points, ...

Characterization of QDF's

Given $\mathfrak{B} \subseteq \mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R})$**, shift-invariant.**

When does $\exists\ \Phi \in \mathbb{R}^{\mathtt{w} \times \mathtt{w}}[\zeta, \eta]$ **such that** $\mathfrak{B} = \mathrm{image}(Q_{\Phi})$ **?**

Characterization of positive storage f'ns for QDF's

Conjecture:

The following are equivalent for $\Phi \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$**:**

1. $\int_{-\infty}^{0} Q_\Phi(w)\,dt \geq 0 \quad \forall w \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}})$ **of compact support,**
2. $\forall w \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}}),\ \exists K \in \mathbb{R}$**,**
   such that $-\int_0^T Q_\Phi(w)\,dt \leq K \quad \forall T \geq 0$**.**

1. $\Rightarrow$ **2. is easy.**

Characterization of quadratic storage functions

Conjecture:

A QDF has a storage iff it has a QDF as a storage

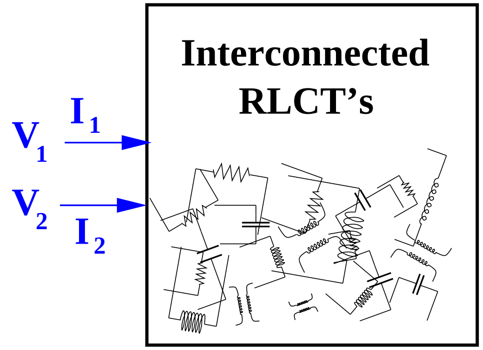
Without signature conditions (as small gain, positive operator, conicity).

Passive behavior synthesis

Stated for single input/single output systems. Consider

$$p(\frac{d}{dt})V = q(\frac{d}{dt})I.$$

When realizable as behavior of the port var. of a circuit with (positive) resistors, capacitors, inductors, and transformers?



Necessary: $\frac{p}{q}$ **p.r.** **p.r. n.a.s.c. when** p **and** q **co-prime.**

What conditions does dissipativity impose on common factors?

Transformerless synthesis

Bott-Duffin synthesis realizes the impedance, not the behavior. They do not use minimal realization, common factors are introduced. Uncontrollable parts are added in the behavior.

Is a synthesizable SISO behavior ... without transformers?

Suspect: NOT.
Transformerless synthesis of behaviors more open than ever.

Conclusion

Let us get the physics right!

The rest is mathematics

Also for dissipative systems, this means backing off from input/output thinking!

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Prima la fisica, poi la matematica

Thank you for your attention

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&

Happy Birthday, Gianni