The Behavioral Approach to Systems Theory

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Lecture 4: Bilinear and quadratic differential forms

Lecturer: Paolo Rapisarda

Part I: Basics

Outline

Dynamics and functionals in systems and control

Instances: **Lyapunov theory, performance criteria, etc.**

Linear case $\Longrightarrow$ *quadratic* and *bilinear* functionals.

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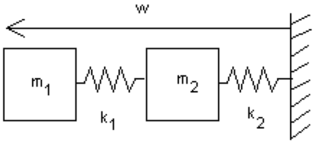
¡High-order differential equations!

...involving also *latent variables*...

Example : a mechanical system



$$\begin{aligned} m_1 \frac{d^2 w_1}{dt^2} + k_1 w_1 - k_1 w_2 &= 0 \\ -k_1 w_1 + m_2 \frac{d^2 w_2}{dt^2} + (k_1 + k_2) w_2 &= 0 \end{aligned}$$

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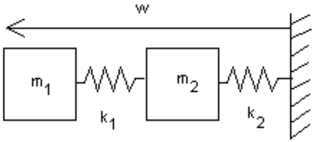

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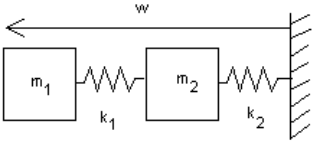

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¿Stability, stored energy, conservation laws?

Aim

An effective algebraic representation
of bilinear and quadratic functionals
of the system variables and their derivatives:

Operations/properties of functionals
$\Updownarrow$
algebraic operations/properties of representation

...a calculus of these functionals!

Outline

Bilinear differential forms (BDFs)

$$\Phi := \left\{ \Phi_{k,\ell} \in \mathbb{R}^{w_1 \times w_2} \right\}_{k,\ell=0,\dots,L}$$

$$L_\Phi : \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{w_1}) \times \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{w_2}) \rightarrow \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R})$$

$$L_\Phi(w_1, w_2) := \begin{bmatrix} w_1^\top & \frac{dw_1}{dt}^\top & \dots \end{bmatrix} \begin{bmatrix} \Phi_{0,0} & \Phi_{0,1} & \cdots \\ \Phi_{1,0} & \Phi_{1,1} & \cdots \\ \vdots & \vdots & \cdots \\ \Phi_{k,0} & \Phi_{k,1} & \cdots \\ \vdots & \vdots & \cdots \end{bmatrix} \begin{bmatrix} w_2 \\ \frac{dw_2}{dt} \\ \vdots \end{bmatrix}$$

$$= \sum_{k,\ell} \left(\frac{d^k}{dt^k} w_1 \right)^\top \Phi_{k,\ell} \left(\frac{d^\ell}{dt^\ell} w_2 \right)$$

Quadratic differential forms (QDFs)

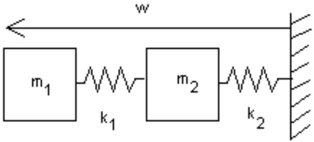
$$\Phi := \left\{\Phi_{k,\ell} \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}\right\}_{k,\ell=0,\ldots,L} \text{ symmetric, i.e. } \Phi_{k,\ell} = \Phi_{\ell,k}^{\top}$$

$$Q_\Phi : \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}}) \to \mathfrak{C}^\infty(\mathbb{R},\mathbb{R})$$

$$Q_\Phi(w) := \begin{bmatrix} w^\top & \frac{dw}{dt}^\top & \ldots \end{bmatrix} \begin{bmatrix} \Phi_{0,0} & \Phi_{0,1} & \cdots \\ \Phi_{1,0} & \Phi_{1,1} & \cdots \\ \vdots & \vdots & \cdots \\ \Phi_{k,0} & \Phi_{k,1} & \cdots \\ \vdots & \vdots & \cdots \end{bmatrix} \begin{bmatrix} w \\ \frac{dw}{dt} \\ \vdots \end{bmatrix}$$

$$= \sum_{k,\ell=0}^{L} \left(\frac{d^k}{dt^k} w\right)^\top \Phi_{k,\ell} \left(\frac{d^\ell}{dt^\ell} w\right)$$

Example: total energy in mechanical system



$$\frac{1}{2}\left[\left(\frac{d}{dt}w_1\right)^2 + \left(\frac{d}{dt}w_2\right)^2\right] + \frac{1}{2}\left[k_1 w_1^2 + k_2 w_2^2\right]$$

$$\begin{bmatrix} w_1 & w_2 & \frac{d}{dt}w_1 & \frac{d}{dt}w_2 \end{bmatrix} \begin{bmatrix} \frac{1}{2}k_1 & 0 & 0 & 0 \\ 0 & \frac{1}{2}k_2 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \frac{d}{dt}w_1 \\ \frac{d}{dt}w_2 \end{bmatrix}$$

Outline

Two-variable polynomial matrices for BDFs

$$\left\{\Phi_{k,\ell} \in \mathbb{R}^{\mathtt{w}_1 \times \mathtt{w}_2}\right\}_{k,\ell=0,\ldots,L}$$

$$L_\Phi(w_1, w_2) = \sum_{k,\ell=0}^{L} \left(\frac{d^k}{dt^k} w_1\right)^\top \Phi_{k,\ell} \frac{d^\ell}{dt^\ell} w_2$$

$$\Phi(\zeta, \eta) = \sum_{k,\ell=0}^{L} \Phi_{k,\ell}\, \zeta^k\, \eta^\ell$$

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2-variable polynomial matrix associated with L_Φ

Two-variable polynomial matrices for QDFs

$$\{\Phi_{k,\ell} \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}\}_{k,\ell=0,\dots,L} \text{ symmetric } (\Phi_{k,\ell} = \Phi_{\ell,k}^\top)$$

$$Q_\Phi(w) = \sum_{k,\ell=0}^{L} (\frac{d^k}{dt^k} w)^\top \Phi_{k,\ell} \frac{d^\ell}{dt^\ell} w$$

$$\Phi(\zeta,\eta) = \sum_{k,\ell=0}^{L} \Phi_{k,\ell} \zeta^k \eta^\ell$$

symmetric: $\Phi(\zeta,\eta) = \Phi(\eta,\zeta)^\top$

Example: total energy in mechanical system

$$Q_E(w_1, w_2) = \begin{bmatrix} w_1 & w_2 & \frac{d}{dt}w_1 & \frac{d}{dt}w_2 \end{bmatrix} \begin{bmatrix} \frac{1}{2}k_1 & 0 & 0 & 0 \\ 0 & \frac{1}{2}k_2 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \frac{d}{dt}w_1 \\ \frac{d}{dt}w_2 \end{bmatrix}$$

$$E(\zeta, \eta) = \begin{bmatrix} \frac{1}{2}k_1 & 0 \\ 0 & \frac{1}{2}k_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{2}\zeta\eta & 0 \\ 0 & \frac{1}{2}\zeta\eta \end{bmatrix}$$

Historical intermezzo



Historical intermezzo

stability tests ('60s)

Historical intermezzo

path integrals ('60s)

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Historical intermezzo

Lyapunov functionals ('80s)

path integrals ('60s)

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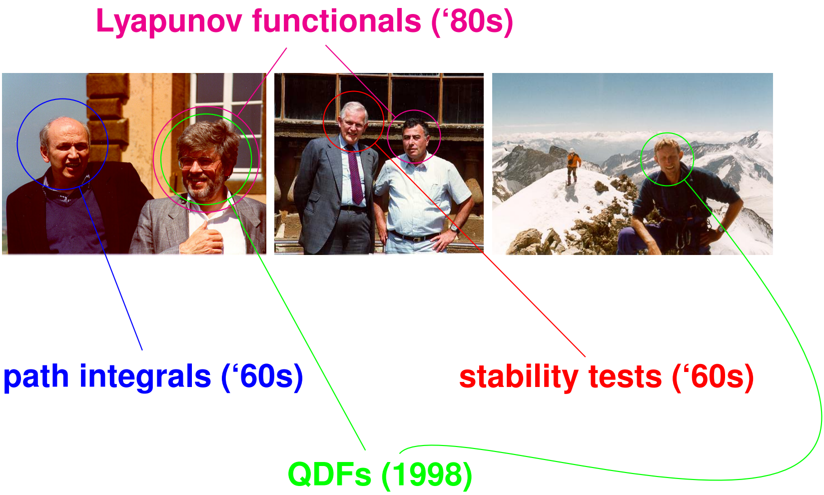
Historical intermezzo



Outline

The calculus of B/QDFs

Using powers of ζ and η as placeholders,

B/QDF $\leftrightsquigarrow$ two-variable polynomial matrix

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Using powers of ζ and η as placeholders,

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Operations and properties of B/QDF	$\leftrightsquigarrow$	**algebraic operations/properties on two-variable matrix**

Differentiation

$\Phi \in \mathbb{R}_s^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$. $\overset{\bullet}{\Phi}$ **derivative** of Q_Φ:

$$Q_{\overset{\bullet}{\Phi}} : \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}}) \to \mathfrak{C}^\infty(\mathbb{R},\mathbb{R})$$

$$Q_{\overset{\bullet}{\Phi}}(w) := \frac{d}{dt}(Q_\Phi(w))$$

$$\overset{\bullet}{\Phi}(\zeta,\eta) = (\zeta+\eta)\Phi(\zeta,\eta)$$

Two-variable version of Leibniz's rule

Integration

$\mathfrak{D}(\mathbb{R}, \mathbb{R}^{\bullet})$ $\mathfrak{C}^{\infty}$**-compact-support trajectories**

$$L_{\Phi} : \mathfrak{D}(\mathbb{R}, \mathbb{R}^{w_1}) \times \mathfrak{D}(\mathbb{R}, \mathbb{R}^{w_2}) \longrightarrow \mathfrak{D}(\mathbb{R}, \mathbb{R})$$

$$\int L_{\Phi} : \mathfrak{D}(\mathbb{R}, \mathbb{R}^{w_1}) \times \mathfrak{D}(\mathbb{R}, \mathbb{R}^{w_2}) \longrightarrow \mathbb{R}$$

$$\int L_{\Phi}(w_1, w_2) := \int_{-\infty}^{+\infty} L_{\Phi}(w_1, w_2)dt$$

Analogous for QDFs

Part II: Applications

Outline

Lyapunov theory

Dissipativity theory

Balancing and model reduction

Nonnegativity and positivity along a behavior

$$Q_\Phi \overset{\mathfrak{B}}{\geq} 0 \text{ if } Q_\Phi(w) \geq 0 \;\forall\, w \in \mathfrak{B}$$

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Prop.: Let $\mathfrak{B} = \mathtt{ker} R(\frac{d}{dt})$. Then $Q_\Phi \overset{\mathfrak{B}}{\geq} 0$ iff there exist $D \in \mathbb{R}^{\bullet \times \mathtt{w}}[\xi]$, $X \in \mathbb{R}^{\bullet \times \mathtt{w}}[\zeta, \eta]$ such that

$$\Phi(\zeta, \eta) = \underbrace{D(\zeta)^\top D(\eta)}_{\geq 0 \text{ for all } w} + \underbrace{R(\zeta)^\top X(\zeta, \eta) + X(\eta, \zeta)^\top R(\eta)}_{=0 \text{ if evaluated on } \mathfrak{B}}$$

Lyapunov theory

$\mathfrak{B}$ **autonomous is *asymptotically stable***
if $\lim_{t\to\infty} w(t) = 0 \;\forall\; w \in \mathfrak{B}$

$$\mathfrak{B} = \mathtt{ker}\, R(\tfrac{d}{dt}) \text{ stable} \iff \det(R) \text{ Hurwitz}$$

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Theorem: $\mathfrak{B}$ asymptotically stable iff exists Q_Φ such that $Q_\Phi \overset{\mathfrak{B}}{\geq} 0$ and $Q_{\overset{\bullet}{\Phi}} \overset{\mathfrak{B}}{<} 0$

Example

$$\mathfrak{B} = \texttt{ker}\left(\frac{d^2}{dt^2} + 3\frac{d}{dt} + 2\right)$$

$$r(\xi) = \xi^2 + 3\xi + 2$$

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Find $\Phi(\zeta, \eta)$ s.t. $\frac{d}{dt} Q_\Phi(w) = Q_\Psi(w)$ for all $w \in \mathfrak{B}$:

$$(\zeta + \eta)\Phi(\zeta, \eta) = \Psi(\zeta, \eta) + \underbrace{r(\zeta)x(\eta) + x(\zeta)R(\eta)}_{=0 \text{ on } \mathfrak{B}}$$

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Equivalent to solving polynomial Lyapunov equation

$$0 = \underbrace{\Psi(-\xi,\xi)}_{\xi^2} + \underbrace{r(-\xi)}_{\xi^2-3\xi+2}x(\xi) + x(-\xi)\underbrace{r(\xi)}_{\xi^2+3\xi+2}$$

$\rightsquigarrow x(\xi) = \frac{1}{6}\xi$

Example

$\mathfrak{B} = \texttt{ker}\left(\frac{d^2}{dt^2} + 3\frac{d}{dt} + 2\right)$ $\quad$ $r(\xi) = \xi^2 + 3\xi + 2$

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$$\begin{aligned}\Phi(\zeta,\eta) &= \frac{-\zeta\eta + (\zeta^2+3\zeta+2)\frac{1}{6}\eta + \frac{1}{6}\zeta(\eta^2+3\eta+2)}{\zeta+\eta} \\ &= \frac{1}{6}\zeta\eta + \frac{1}{3} > 0\end{aligned}$$

State-space case

$$\left(\frac{d}{dt}I_{\mathtt{x}} - A\right)x = 0 \quad \rightsquigarrow \quad R(\xi) = \xi I_{\mathtt{x}} - A$$

- Choose $Q < 0$;
- Solve polynomial Lyapunov equation

$$(\xi I_{\mathtt{x}} - A)^\top P + P(\xi I_{\mathtt{x}} - A) = -A^\top P - PA = Q$$

  equivalent with *matrix* Lyapunov equation!
- Lyapunov functional is

$$x^\top(-P)x$$

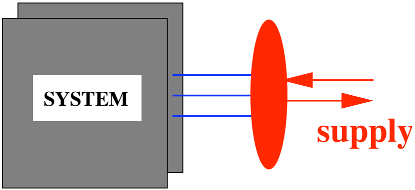
Outline

Lyapunov theory

Dissipativity theory

Balancing and model reduction

Dissipativity theory



Power is supplied

$\rightsquigarrow$ **energy is stored**

RLC circuits **Power** $V^\top I$

Storage in capacitors and inductors

Mechanical system **Power** $F^\top v + (\frac{d}{dt}\vartheta)^\top T$

Potential+kinetic

Setting the stage

Controllable system

$$w = M(\tfrac{d}{dt})\ell \rightsquigarrow M(\xi)$$

Power ('supply rate')

$$Q_\Phi(w) \rightsquigarrow \Phi(\zeta, \eta)$$

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$$Q_\Phi(w) = Q_\Phi(M(\tfrac{d}{dt})\ell)$$

$$\Phi'(\zeta,\eta) := M(\zeta)^\top \Phi(\zeta,\eta) M(\eta)$$

$Q_{\Phi'}$ acts on free variable ℓ, i.e. $\mathfrak{C}^\infty$

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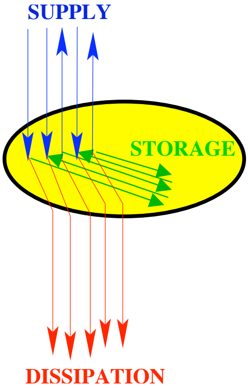
Dissipation inequality

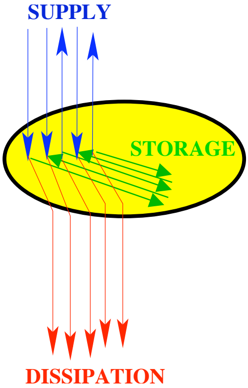


Dissipation inequality

Q_Ψ is **storage function** for the supply Q_Φ if

$$\frac{d}{dt} Q_\Psi \leq Q_\Phi$$

Rate of storage increase $\leq$ supply



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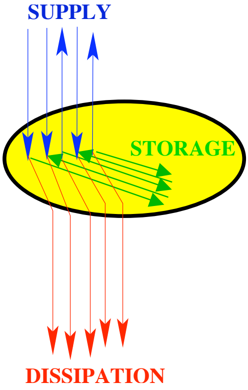
Q_Δ is **dissipation function** for Q_Φ if

$$Q_\Delta \geq 0 \text{ and } \int Q_\Delta dt = \int Q_\Phi dt$$



Characterizations of dissipativity

Theorem: **The following conditions are equivalent:**

- $\int_{-\infty}^{+\infty} Q_\Phi(\ell)dt \geq 0$ **for all** $\mathfrak{C}^\infty$ **compact-support** ℓ;
- Q_Φ **admits a storage function;**
- Q_Φ **admits a dissipation function**

Also, storage and dissipation functions are one-one:

$$\begin{aligned} \frac{d}{dt} Q_\Psi &= Q_\Phi - Q_\Delta \\ (\zeta + \eta)\Psi(\zeta, \eta) &= \Phi(\zeta, \eta) - \Delta(\zeta, \eta) \end{aligned}$$

Example: mechanical systems

$$M\frac{d^2}{dt^2}q + D\frac{d}{dt}q + Kq = F \qquad \begin{bmatrix} F \\ q \end{bmatrix} = \begin{bmatrix} M\frac{d^2}{dt^2} + D\frac{d}{dt} + K \\ I_1 \end{bmatrix} \ell$$

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Supply rate: power

$$F^\top \left(\frac{d}{dt}q\right) = \left(M\frac{d^2}{dt^2}\ell + D\frac{d}{dt}\ell + K\ell\right)^\top \left(\frac{d}{dt}\ell\right)$$

corresponding to

$$\Phi(\zeta,\eta) = \frac{1}{2}(M\zeta^2 + D\zeta + K)^\top \eta + \frac{1}{2}\zeta(M\eta^2 + D\eta + K)$$

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If dissipation inequality

$$\Phi(\zeta,\eta) = (\zeta+\eta)\Psi(\zeta,\eta) + \Delta(\zeta,\eta)$$

holds, then

$$\Phi(-\xi,\xi) = -\frac{1}{2}\xi^2(D^\top + D) = \Delta(-\xi,\xi)$$

$$\Longrightarrow \Delta(\zeta,\eta) = \frac{1}{2}(D^\top + D)\zeta\eta$$

Spectral factorization of $\Phi(-\xi,\xi)$ is key

Example: mechanical systems

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$$\Delta(\zeta,\eta) = \tfrac{1}{2}(D^\top + D)\zeta\eta$$

Storage function

$$\Psi(\zeta,\eta) = \frac{\Phi(\zeta,\eta) - \Delta(\zeta,\eta)}{\zeta+\eta} = \frac{1}{2}M\zeta\eta + \frac{1}{2}K$$

Total energy

Outline

Lyapunov theory

Dissipativity theory

Balancing and model reduction

Balancing

A minimal and stable realization (A, B, C, D) is balanced if exist $\sigma_i \in \mathbb{R}$ such that $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n > 0$ and moreover

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\begin{aligned}
A\Sigma + \Sigma A^\top + BB^\top &= 0 \\
A^\top\Sigma + \Sigma A + C^\top C &= 0
\end{aligned}
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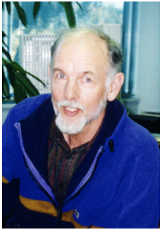
where $\Sigma := \mathrm{diag}(\sigma_1, \sigma_2, \ldots, \sigma_n)$



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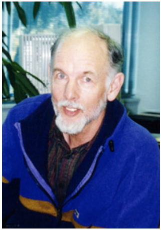

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Balancing $\equiv$ choice of basis of state space
diagonalizing the Gramians

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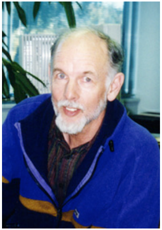

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Balancing $\equiv$ choice of basis of state space diagonalizing the Gramians

$\equiv$ choice of state map!

The controllability Gramian K

$$p(\tfrac{d}{dt})y = q(\tfrac{d}{dt})u \qquad\qquad \begin{bmatrix} y \\ u \end{bmatrix} = \begin{bmatrix} q(\frac{d}{dt}) \\ p(\frac{d}{dt}) \end{bmatrix} \ell$$

where $GCD(p,q) = 1$, p stable, $\deg(q) \leq \deg(p) =: n$

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In state-space framework, K is defined as

$$\inf_u \int_{-\infty}^{0} u(t)^2 dt =: x_0^\top K x_0$$

where u is such that $x(-\infty) \rightsquigarrow x(0) = x_0$

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In our framework: let $\ell \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R})$. Then Q_K is QDF such that

$$\inf_{\ell'} \int_{-\infty}^{0} \left(p(\frac{d}{dt})\ell' \right) dt =: Q_K(\ell)(0)$$

where $\ell' \in \mathfrak{C}^\infty(\mathbb{R}_+, \mathbb{R})$ is such that $\ell'_{|[0,+\infty)} = \ell_{|[0,+\infty)}$

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Computation of $K(\zeta, \eta)$

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Since $p(-\xi)p(\xi) = p(\xi)p(-\xi)$, exists $K' \in \mathbb{R}[\zeta, \eta]$ such that

$$p(\zeta)p(\eta) - p(-\zeta)p(-\eta) = (\zeta + \eta)K(\zeta, \eta)$$

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Consequently,

$$\int_{-\infty}^{0} \left(p(\frac{d}{dt})\ell' \right) dt = \int_{-\infty}^{0} \left(p(-\frac{d}{dt})\ell' \right) dt + Q_K(\ell')(0)$$

minimized for the ℓ' in `ker` $p(-\frac{d}{dt})$ with the given initial conditions.

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$\Longrightarrow Q_K$ is quadratic function of $\frac{d^j \ell}{dt^j}$, $j = 0, \ldots, n-1$

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where y is free response emanating from $x(0) = x_0$

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then

$$Q_W(\ell)(0) = \int_0^{+\infty} \left(q(\frac{d}{dt})\ell \right)^2 dt$$

for all $\ell \in \mathtt{ker}\; p(\frac{d}{dt})$

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- $Q_W(\ell_1)(0) \geq Q_W(\ell_2)(0) \geq \ldots \geq Q_W(\ell_{\mathtt{n}})(0) > 0$

  or equivalently

  $$0 < Q_K(\ell_1)(0) \leq Q_K(\ell_2)(0) \leq \ldots \leq Q_K(\ell_{\mathtt{n}})(0)$$

  'first who contributes most'

Balancing with QDFs

Linear algebra $\Longrightarrow$ **there is basis** $\{x_i^b \in \mathbb{R}_{n-1}[\xi]\}_{i=1,\ldots,n}$ **and** $\sigma_i \in \mathbb{R}$ **such that** $\sigma_1 \geq \sigma_2 \geq \ldots \sigma_n$ **such that**

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$\sigma_i \rightsquigarrow$ (classical) Hankel singular values

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Model reduction by balancing follows

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- **Differentiation, integration, positivity;**
- **Lyapunov theory, dissipativity, model reduction by balancing.**