# The Behavioral Approach to Systems Theory

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# Lecture 3: State and state construction

**Lecturer: Paolo Rapisarda**

# Outline

# Questions

- **Are state representations “natural"?**

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  - **First principles and “tearing and zooming" modelling**
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- **What makes a latent variable a “state"?**

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- **What does that imply for the equations?**
- **How to construct a state from the equations?**
- **How to construct a state representation from the equations?**

# The basic idea

**It's the Mariners' final game in the World Series. You're late...**


**The current score is what matters...**

# The basic idea

- **The state contains all the relevant information about the future behavior of the system**
- **The state is the memory of the system**
- **Independence of past and future given the state**

# The axiom of state

$\boldsymbol{\Sigma} = (\mathbb{T}, \mathbb{W}, \mathbb{X}, \mathfrak{B}_{\text{full}})$ **is a *state system* if**

$$(w_1, x_1), (w_2, x_2) \in \mathfrak{B}_{\text{full}} \text{ and } x_1(T) = x_2(T)$$
$$\Downarrow$$
$$(w_1, x_1) \underset{T}{\wedge} (w_2, x_2) \in \mathfrak{B}_{\text{full}}$$

$\underset{T}{\wedge}$ **is *concatenation at T*:**

$$(f_1 \underset{T}{\wedge} f_2)(t) := \begin{cases} f_1(t) \text{ for } t < T \\ f_2(t) \text{ for } t \geq T \end{cases}$$

# Graphically...

$$(w_1, x_1), (w_2, x_2) \in \mathfrak{B}_{\text{full}} \text{ and } x_1(T) = x_2(T)$$

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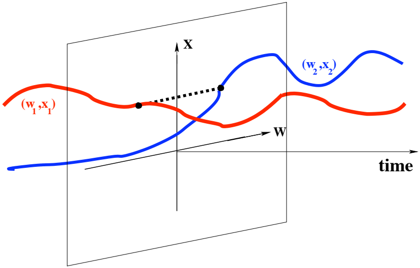

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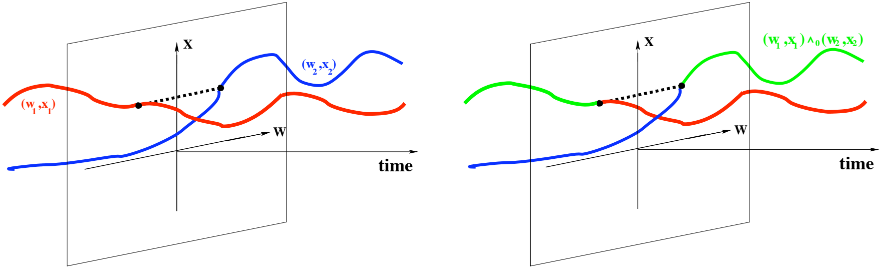

# Example 1: discrete-time system

$\boldsymbol{\Sigma} = (\mathbb{Z}, \mathbb{R}^{\mathtt{w}}, \mathbb{R}^{\mathtt{l}}, \mathfrak{B}_{\text{full}})$, with

$$\mathfrak{B}_{\text{full}} := \{(w, \ell) \mid F \circ (\sigma \ell, \ell, w) = 0\}$$

where

$$\sigma : (\mathbb{R}^{\mathtt{l}})^{\mathbb{Z}} \longrightarrow (\mathbb{R}^{\mathtt{l}})^{\mathbb{Z}}$$
$$(\sigma(\ell))(k) := \ell(k+1)$$

# Example 1: discrete-time system

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**Special case: input-state-output equations**

$$\begin{aligned} \sigma x &= f(x, u) \\ y &= h(x, u) \\ w &= (u, y) \end{aligned}$$

# Example 2: continuous-time system

$\Sigma = (\mathbb{R}, \mathbb{R}^{\mathtt{w}}, \mathbb{R}^{\mathtt{l}}, \mathfrak{B}_{\text{full}})$, with

$$\mathfrak{B}_{\text{full}} := \{(w, \ell) \mid F \circ (\tfrac{d}{dt}\ell, \ell, w) = 0\}$$

**Special case: input-state-output equations**

$$\begin{aligned} \frac{d}{dt}x &= f(x, u) \\ y &= h(x, u) \\ w &= (u, y) \end{aligned}$$

# Outline

The axiom of state

Discrete-time systems

- First-order representations
- State maps
- The shift-and-cut map
- Algebraic characterization

Continuous-time systems

Computation of state-space representations

# First-order discrete-time representations

**Theorem:** **A 'complete' latent variable system**

$$\Sigma = (\mathbb{Z}, \mathbb{R}^{\mathtt{w}}, \mathbb{R}^{\mathtt{x}}, \mathfrak{B}_{\text{full}})$$

**is a state system iff** $\mathfrak{B}_{\text{full}}$ **can be described by**

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**0-th order in *w*, 1st order in *x***

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**1st order in *x* is consequence of state property!**

# Proof (linear case)

$$\mathcal{V} := \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} \mid \exists (x, w) \in \mathfrak{B}_{\text{full}} \text{ s. t. } \begin{bmatrix} x(1) \\ x(0) \\ w(0) \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \right\}$$

$$\mathcal{V} \text{ linear} \Rightarrow \exists\ E, F, G \text{ s.t. } \mathcal{V} = \mathtt{ker}(\begin{bmatrix} E & F & G \end{bmatrix})$$

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$$\Downarrow$$

$$[(x, w) \in \mathfrak{B}_{\text{full}} \Longrightarrow E\sigma x + Fx + Gw = 0]$$

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**Converse by induction, using axiom of state:**

$$E\sigma x + Fx + Gw = 0 \text{ on } [0, k] \Longrightarrow (w, x)_{|[0,k]} \in \mathfrak{B}_{\text{full}|[0,k]}$$

**Then apply completeness of $\mathfrak{B}$**

# State construction: basic idea

**Problem:** **Given kernel or hybrid description, find a state representation**

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**First** **compute polynomial operator in the shift acting on system variables, inducing a state variable:**

$$X(\sigma)w = x \qquad X(\sigma)\begin{bmatrix} w \\ \ell \end{bmatrix} = x$$

# State construction: basic idea

**Problem:** **Given kernel or hybrid description, find a state representation**

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**First** **compute polynomial operator in the shift acting on system variables, inducing a state variable:**

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**Then** **use original eqs. and** $X$ **to obtain 1st order representation.**

# State maps for kernel representations

$X \in \mathbb{R}^{\bullet \times \mathtt{w}}[\xi]$ induces a ***state map*** $X(\sigma)$ for $\mathtt{ker}(R(\sigma))$ if the behavior $\mathfrak{B}_{\text{full}}$ with latent variable $x$, described by

$$\begin{aligned} R(\sigma)w &= 0 \\ X(\sigma)w &= x \end{aligned}$$

satisfies the axiom of state.

# Example

$$\mathfrak{B} = \{w \mid r(\sigma)w = 0\}$$

**where $r \in \mathbb{R}[\xi]$, $\deg(r) = n$.**

**(Minimal) state map induced by**

$$\begin{bmatrix} 1 \\ \xi \\ \vdots \\ \xi^{n-1} \end{bmatrix} \rightsquigarrow \begin{bmatrix} w \\ \sigma w \\ \vdots \\ \sigma^{n-1} w \end{bmatrix}$$

# The axiom of state revisited

**A *linear* system $\Sigma = (\mathbb{T}, \mathbb{W}, \mathbb{X}, \mathfrak{B}_{\text{full}})$ with latent variable $x$ is a state system if**

$$(w, x) \in \mathfrak{B}_{\text{full}} \text{ and } x(T) = 0$$
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$$(0, 0) \underset{T}{\wedge} (w, x) \in \mathfrak{B}_{\text{full}}$$

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- **Time-invariance $\Longrightarrow$ can choose $T = 0$;**
- **Concatenability with zero trajectory is key.**

# When is $w \in \mathfrak{B}$ concatenable with zero?

$$R_0 w + R_1 \sigma w + \ldots + R_L \sigma^L w = 0$$

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| ... | 0 | 0 | $w(0)$ | $w(1)$ | $w(2)$ | $w(3)$ | ... |
| ... | $k = -2$ | $k = -1$ | $k = 0$ | $k = 1$ | $k = 2$ | $k = 3$ | ... |

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| $\ldots$ | $0$ | $R_0$ | $R_1$ | $R_2$ | $R_3$ | $R_4$ | $\ldots$ |
|---|---|---|---|---|---|---|---|
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| $\ldots$ | $R_0$ | $R_1$ | $R_2$ | $R_3$ | $R_4$ | $R_5$ | $\ldots$ |
|---|---|---|---|---|---|---|---|
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$$\begin{aligned} R_0 w(0) + R_1 w(1) + \ldots + R_L w(L) &= 0 \\ R_1 w(0) + R_2 w(1) + \ldots + R_L w(L-1) &= 0 \\ R_2 w(0) + R_3 w(1) + \ldots + R_L w(L-2) &= 0 \\ \vdots &= \vdots \\ R_{L-1} w(0) + R_L w(1) &= 0 \end{aligned}$$

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$$\vdots \quad = \vdots$$

$$R_{L-1} w(0) + R_L w(1) = 0$$

$$R_L w(0) = 0$$

# The shift-and-cut map

$$\sigma_+ : \mathbb{R}[\xi] \to \mathbb{R}[\xi]$$

$$\sigma_+\left(\sum_{i=0}^{n} p_i \xi^i\right) := \sum_{i=0}^{n-1} p_{i+1} \xi^i$$

*"Divide by $\xi$ and take polynomial part"*

**Extended componentwise to vectors and matrices**

# Example

$$R(\xi) = R_0 + R_1\xi + \ldots + R_{L-1}\xi^{L-1} + R_L\xi^L$$

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$$\sigma_+^2(R(\xi)) = R_2 + \ldots + R_{L-1}\xi^{L-3} + R_L\xi^{L-2}$$

$$\vdots = \vdots$$

$$\sigma_+^L(R(\xi)) = R_L$$

# Shift-and-cut and concatenability with zero

**$w$ is concatenable with zero** $\Leftrightarrow$

$$\begin{aligned}
(\sigma_+(R)(\sigma)w)(0) &= 0\\
(\sigma_+^2(R)(\sigma)w)(0) &= 0\\
\vdots &= \vdots\\
(\sigma_+^L(R)(\sigma)w)(0) &= 0
\end{aligned}$$

**$\mathrm{col}((\sigma_+^i(R))_{i=1,\ldots,L}(\sigma)$ is a state map!**

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**Other equations equivalent to shift-and-cut ones $\Longrightarrow$ different state maps are possible!**

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# Example: scalar systems

$$r_0 w + r_1 \sigma w + \ldots + \sigma^n w = 0$$

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**Observe $w$ concatenable with zero iff $w = 0$. Indeed,**

$$\begin{aligned}
\sigma_+^n(r)(\sigma) w &= w \\
\sigma_+^{n-1}(r)(\sigma) w &= r_{n-1} w + \sigma w \\
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\end{aligned}$$

**Zero at $t = 0$ iff $(\sigma^k w)(0) = 0$ for $k = 0, \ldots, n-1$.**

# From kernel representation to state map

**Denote** $\mathrm{col}((\sigma_+^i(R))_{i=1,\ldots,L} =: \Sigma_R$.

**Theorem:** **Let** $\mathfrak{B} = \mathtt{ker}(R(\sigma))$. **Then**

$$\begin{aligned} R(\sigma)w &= 0 \\ \Sigma_R(\sigma)w &= x \end{aligned}$$

**is a state representation of** $\mathfrak{B}$ **with state variable** $x$.

# Algebraic characterization

**Theorem:** Let $\mathfrak{B} = \mathtt{ker}(R(\sigma))$, and define $\Sigma_R$ as above. Then

$$\Xi_R := \{ f \in \mathbb{R}^{1\times \mathtt{w}}[\xi] \mid \exists\, g \in \mathbb{R}^{1\times \bullet}[\xi], \alpha \in \mathbb{R}^{1\times \bullet} \text{ s.t. } f = \alpha \Sigma_R + gR \}$$

is a vector space over $\mathbb{R}$.

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$X \in \mathbb{R}^{\bullet\times \mathtt{w}}[\xi]$ is state map for $\mathfrak{B}$ iff row span$(X) = \Xi_R$

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$X \in \mathbb{R}^{\bullet\times \mathtt{w}}[\xi]$ is state map for $\mathfrak{B}$ iff row span$(X) = \Xi_R$

X is minimal if and only if its rows are a basis for $\Xi_R$.

# Example

$$(\sigma^2 + 2\sigma + 3)y = (\sigma + 3)u$$

$$\begin{bmatrix} \xi^2 + 2\xi + 3 & -\xi - 3 \end{bmatrix}$$

# Example

$$(\sigma^2 + 2\sigma + 3)y = (\sigma + 3)u \qquad \begin{bmatrix} \xi^2 + 2\xi + 3 & -\xi - 3 \end{bmatrix}$$

$$\sigma_+ \rightsquigarrow \begin{bmatrix} \xi + 2 & -1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} \sigma + 2 & -1 \end{bmatrix}$$

# Example

$$(\sigma^2 + 2\sigma + 3)y = (\sigma + 3)u \qquad \begin{bmatrix}\xi^2 + 2\xi + 3 & -\xi - 3\end{bmatrix}$$

$$\sigma_+ \rightsquigarrow \begin{bmatrix}\xi + 2 & -1\end{bmatrix} \rightsquigarrow \begin{bmatrix}\sigma + 2 & -1\end{bmatrix}$$

**If $(y, u) \in \mathfrak{B}$, then for all $g \in \mathbb{R}[\xi]$**

$$\begin{aligned}\begin{bmatrix}\sigma + 2 & -1\end{bmatrix}\begin{bmatrix}y \\ u\end{bmatrix} &= \begin{bmatrix}\sigma + 2 & -1\end{bmatrix}\begin{bmatrix}y \\ u\end{bmatrix} \\ &+ \underbrace{g(\sigma)\begin{bmatrix}\sigma^2 + 2\sigma + 3 & -\sigma - 3\end{bmatrix}}_{=0 \text{ on } \mathfrak{B}}\begin{bmatrix}y \\ u\end{bmatrix}\end{aligned}$$

# Example

$(\sigma^2 + 2\sigma + 3)y = (\sigma + 3)u$ $\quad\quad$ $\begin{bmatrix} \xi^2 + 2\xi + 3 & -\xi - 3 \end{bmatrix}$

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**‘equivalence modulo $R$’**

# Example

$$(\sigma^2 + 2\sigma + 3)y = (\sigma + 3)u \qquad \begin{bmatrix}\xi^2 + 2\xi + 3 & -\xi - 3\end{bmatrix}$$

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$$\begin{aligned}\Xi_R = \{\alpha \begin{bmatrix}\xi + 2 & -1\end{bmatrix} + g(\xi) \begin{bmatrix}\xi^2 + 2\xi + 3 & -\xi - 3\end{bmatrix},\\ \beta \begin{bmatrix}1 & 0\end{bmatrix} + f(\xi) \begin{bmatrix}\xi^2 + 2\xi + 3 & -\xi - 3\end{bmatrix}\\ \alpha, \beta \in \mathbb{R}, f, g \in \mathbb{R}[\xi]\}\end{aligned}$$

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**Any set of generators of $\Xi_R \rightsquigarrow$ a state map**

# Outline

# On the space of solutions

**$\mathfrak{C}^\infty$-solutions to $R(\frac{d}{dt})w = 0$ too small $\rightsquigarrow \mathcal{L}_1^{\text{loc}}$**

**Equality in the sense of distributions:**

$R(\frac{d}{dt})w = 0 \quad \Leftrightarrow \quad \int_{-\infty}^{+\infty} w(t)^\top (R(-\frac{d}{dt})^\top f)(t)dt = 0$

**for all testing functions $f$.**

# The axiom of state revisited

$\Sigma = (\mathbb{T}, \mathbb{W}, \mathbb{X}, \mathfrak{B}_{\text{full}})$ **is a state system if**

$$(w_1, x_1), (w_2, x_2) \in \mathfrak{B}_{\text{full}} \text{ and } x_1(T) = x_2(T)$$

**and $x_1, x_2$ continuous at $T$**

$$\Downarrow$$

$$(w_1, x_1) \underset{T}{\wedge} (w_2, x_2) \in \mathfrak{B}_{\text{full}}$$

**'State map'** $\rightsquigarrow$ $X(\frac{d}{dt})$

# From kernel representation to state map

**Denote** $\mathrm{col}((\sigma_+^i(R)))_{i=1,\ldots,L} =: \Sigma_R$.

**Theorem:** **Let** $\mathfrak{B} = \mathtt{ker}(R(\frac{d}{dt}))$. **Then**

$$\begin{aligned} R(\frac{d}{dt})w &= 0 \\ \Sigma_R(\frac{d}{dt})w &= x \end{aligned}$$

**is a state representation of** $\mathfrak{B}$ **with state variable** $x$.

**¿How to prove it?**

# When is $w \in \mathfrak{B}$ concatenable with zero?

$$0 \underset{0}{\wedge} w \in \mathfrak{B} \iff \int_{-\infty}^{+\infty} (0 \underset{0}{\wedge} w)(t)^\top (R(-\frac{d}{dt})^\top f)(t) dt = 0$$

$$\iff \int_{0}^{+\infty} w(t)^\top (R(-\frac{d}{dt})^\top f)(t) dt = 0$$

for all testing functions $f$

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for all testing functions $f$

Integrating repeatedly by parts on $f$ yields:

$$\sum_{k=1}^{\deg(R)} \sum_{j=k}^{\deg(R)} (-1)^{k-1} (\frac{d^{j-k}}{dt^{j-k}} w)(0)^\top R_j^\top (\frac{d^{k-1}}{dt^{k-1}} f)(0) + \int_0^{+\infty} (R(\frac{d}{dt}) w)(t)^\top f(t) dt = 0$$

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$$+ \int_0^{+\infty} (R(\frac{d}{dt})w)(t)^\top f(t)dt = 0$$

## $w \in \mathfrak{B}$ concatenable with zero if and only if...

$$\sum_{k=1}^{\deg(R)} \sum_{j=k}^{\deg(R)} (-1)^{k-1} \left(\frac{d^{j-k}}{dt^{j-k}} w\right)(0)^\top R_j^\top \left(\frac{d^{k-1}}{dt^{k-1}} f\right)(0) = 0$$

$$\Updownarrow$$

$$\begin{bmatrix} f(0) \\ \left(\frac{d}{dt} f\right)(0) \\ \vdots \\ (-1)^{\deg(R)-1} \left(\frac{d^{\deg(R)-1}}{dt^{\deg(R)-1}} f\right)(0) \end{bmatrix}^\top \left(\Sigma_R\left(\frac{d}{dt}\right) w\right)(0) = 0$$

$$\Updownarrow$$

$$\left(\Sigma_R\left(\frac{d}{dt}\right) w\right)(0) = 0$$

**The shift-and-cut state map!**

# Outline

# From kernel representation to state representation

$$R \in \mathbb{R}^{g \times w}[\xi] \rightsquigarrow \text{state map } X \in \mathbb{R}^{n \times w}[\xi]$$

**Find:**

$E, F \in \mathbb{R}^{(n+g) \times n}$, $G \in \mathbb{R}^{(n+g) \times w}$

$T \in \mathbb{R}^{(n+g) \times g}[\xi]$ with $\text{rank}(T(\lambda)) = g \; \forall \lambda \in \mathbb{C}$

**satisfying**

$$E\xi X(\xi) + FX(\xi) + G = T(\xi)R(\xi)$$

**Linear equations, Gröbner bases computations!**

# From I/O representation to I/O/S representation

**I/O representation** $R = \begin{bmatrix} P & -Q \end{bmatrix}$ $\rightsquigarrow$ **state map** $\begin{bmatrix} X_y & X_u \end{bmatrix}$

**Find:**

$A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times p}$, $D \in \mathbb{R}^{p\times m}$

$T \in \mathbb{R}^{(n+p)\times p}[\xi]$ with $\text{rank}(T(\lambda)) = g \;\forall \lambda \in \mathbb{C}$

**satisfying**

$$\begin{bmatrix} \xi X_y(\xi) & \xi X_u(\xi) \\ I_p & 0 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} X_y(\xi) & X_u(\xi) \\ 0 & I_m \end{bmatrix} + T(\xi)R(\xi)$$

# On the choice of state map

**State map**
**+**
**system equations** $\rightsquigarrow$ **state-space equations**

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$$\left(\frac{d^2}{dt^2} + 2\frac{d}{dt} + 3\right)y = \left(\frac{d}{dt} + 3\right)u$$

$$\begin{bmatrix} \xi^2 + 2\xi + 3 & -\xi - 3 \end{bmatrix}$$

# On the choice of state map

**State map**
**+**
**system equations** $\rightsquigarrow$ **state-space equations**

$$\left(\frac{d^2}{dt^2} + 2\frac{d}{dt} + 3\right)y = \left(\frac{d}{dt} + 3\right)u \qquad \begin{bmatrix}\xi^2 + 2\xi + 3 & -\xi - 3\end{bmatrix}$$

**Take** $X(\xi) = \begin{bmatrix} 1 & 0 \\ \xi + 2 & -1 \end{bmatrix}$ **('reverse shift-and-cut').**

**Then**

$$A = \begin{bmatrix} -2 & 1 \\ -3 & 0 \end{bmatrix} \quad B = \begin{bmatrix} -1 \\ -3 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 \end{bmatrix} \qquad D = \begin{bmatrix} 0 \end{bmatrix}$$

**'observer canonical form'**

# On the choice of state map

**State map**
**+**
**system equations** $\rightsquigarrow$ **state-space equations**

$$\left(\frac{d^2}{dt^2} + 2\frac{d}{dt} + 3\right)y = \left(\frac{d}{dt} + 3\right)u \qquad \begin{bmatrix}\xi^2 + 2\xi + 3 & -\xi - 3\end{bmatrix}$$

**Take** $X(\xi) = \begin{bmatrix}1 & 0\\ \xi & -1\end{bmatrix}$**. Then**

$$A = \begin{bmatrix}0 & 1\\ -3 & -2\end{bmatrix} \quad B = \begin{bmatrix}1\\ 1\end{bmatrix}$$

$$C = \begin{bmatrix}1 & 0\end{bmatrix} \quad D = \begin{bmatrix}0\end{bmatrix}$$

**‘observable canonical form’**

# Summary

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- **Algorithms!**