

The Behavioral Approach to Systems Theory

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Lecture 1: General Introduction

Lecturer: Jan C. Willems

Questions

- What is a **mathematical model**, really?

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 - From basic laws: **'first principles' modeling**
 - Combined with interconnection:
tearing, zooming, & linking
 - From measured data: **SYSID** (system identification)

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 - From basic laws: **'first principles' modeling**
 - Combined with interconnection:
tearing, zooming, & linking
 - From measured data: **SYSID** (system identification)
- What is the role of (differential) **equations**?
- Importance of **latent** variables

Static models

The seminal idea

Consider a 'phenomenon'; produces 'outcomes', 'events'.

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$\Rightarrow \Rightarrow$ **a mathematical model, with *behavior* \mathfrak{B}** $\Leftarrow \Leftarrow$

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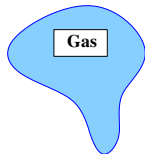
Before modeling: events in \mathcal{U} are possible

After modeling: **only** events in \mathfrak{B} are possible

Sharper model \rightsquigarrow smaller \mathfrak{B} .

Examples

Gas law

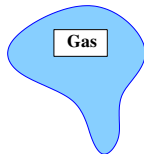


Phenomenon: A balloon filled with a gas

ii Model the relation between
volume, quantity, pressure, & temperature !!

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Gas law



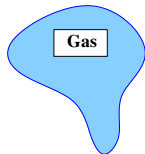
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Charles  Boyle  and Avogadro 

\rightsquigarrow model $\frac{PV}{NT} = \text{a universal constant} =: R$

$\Rightarrow \Rightarrow \mathfrak{B} = \left\{ (T, P, V, N) \in \mathbb{R}_+^4 \mid \frac{PV}{NT} = R \right\} \Leftarrow \Leftarrow$

Examples

An economy

Phenomenon: trading of a product

ii Model the relation between

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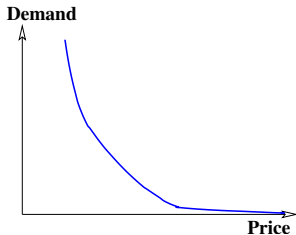
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Typical model: $\mathfrak{B} = \text{graph of a curve}$



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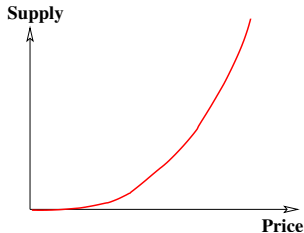
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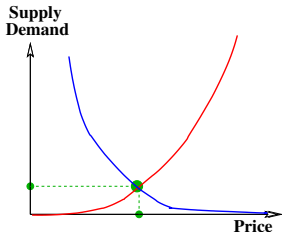
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Examples

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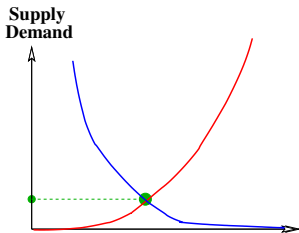
Phenomenon: trading of a product

ii Model the relation between
sales & production !! Price only to explain mechanism

Event: (demand D , supply S) $\rightsquigarrow \mathcal{U} = \mathbb{R}_+^2$

\mathfrak{B} = intersection of two graphs : \rightsquigarrow usually point(s)

The price P becomes a 'hidden' variable. Modeling using 'hidden', 'auxiliary', 'latent' intermediate variables is very common.

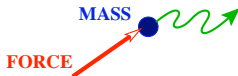


How shall we deal with such variables?

Examples

Newton's 2-nd law

Phenomenon: A moving mass

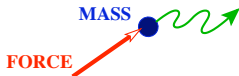


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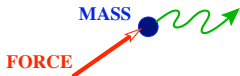
Event: (force F , mass m , acceleration a)

$$\leadsto \mathcal{U} = \mathbb{R}^3 \times \mathbb{R}_+ \times \mathbb{R}^3$$

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Model due to Newton:



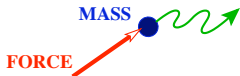
$$F = ma$$

$$\Rightarrow \Rightarrow \mathcal{B} = \{ (F, m, a) \in \mathbb{R}^3 \times \mathbb{R}_+ \times \mathbb{R}^3 \mid F = ma \} \Leftarrow \Leftarrow$$

Examples

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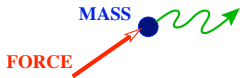
But, the aim of Newton's law is really:

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Newton's 2-nd law

Phenomenon: A moving mass



But, the aim of Newton's law is really:

ii Model the relation between
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Event: (force F , mass m , position q)

$$F = ma, \quad a = \frac{d^2}{dt^2} q$$

not 'instantaneous' relation between $F, m, q \rightsquigarrow$ dynamics

How shall we deal with this?

Dynamic models

Dynamical systems

Phenomenon produces 'events' that are **functions of time**

Mathematization: It is convenient to distinguish

domain ('independent' variables) $\mathbb{T} \subseteq \mathbb{R}$ 'time-axis'
co-domain ('dependent' variables) \mathbb{W} 'signal space'

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$$\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B}) \quad \mathfrak{B} \subseteq (\mathbb{W})^{\mathbb{T}} \quad \text{the behavior}$$

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$\mathbb{T} = \mathbb{R}, \mathbb{R}_+$, or interval in \mathbb{R} : **continuous-time** systems

$\mathbb{T} = \mathbb{Z}, \mathbb{N}$, etc.: **discrete-time** systems

Later: set of independent variables = \mathbb{R}^n , $n > 1$, PDE's.

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$\mathbb{W} = \mathbb{R}^w$, etc. **lumped** systems

\mathbb{W} = finite: **finitary** systems

$\mathbb{T} = \mathbb{Z}$ or \mathbb{N} , \mathbb{W} finite: **DES** (discrete event systems)

\mathbb{W} = function space: **DPS** (distributed parameter systems)

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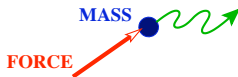
$$\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B}) \quad \mathfrak{B} \subseteq (\mathbb{W})^{\mathbb{T}} \quad \text{the behavior}$$

\mathbb{W} vector space, $\mathfrak{B} \subset (\mathbb{W})^{\mathbb{T}}$ linear subspace: **linear** systems
controllability, observability, stabilizability, dissipativity,
stability, symmetry, reversibility, (equivalent) representa-
tions, etc.: to be defined in terms of the behavior \mathfrak{B}

THE BEHAVIOR IS ALL THERE IS!

Examples

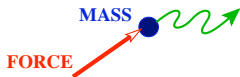
Newton's 2-nd law



ii Model the relation between
force & position of a pointmass !!

Examples

Newton's 2-nd law

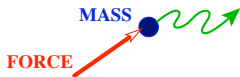


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Model:

$$F = ma, \quad a = \frac{d^2}{dt^2} q$$

$$\leadsto \Sigma = (\mathbb{R}, \mathbb{R}^3 \times \mathbb{R}^3, \mathfrak{B})$$

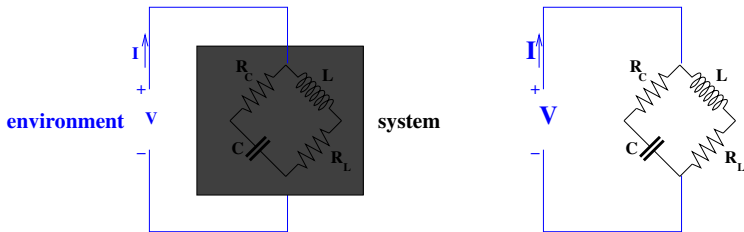
with

$$\Rightarrow \Rightarrow \mathfrak{B} = \left\{ (F, q) : \mathbb{R} \rightarrow \mathbb{R}^3 \times \mathbb{R}^3 \mid F = m \frac{d^2}{dt^2} q \right\} \Leftarrow \Leftarrow$$

Examples

RLC circuit

Phenomenon: the port voltage and current, f'n's of time



Model voltage/current histories as a f'n of time !

Examples

RLC circuit

$$\rightsquigarrow \Sigma = (\mathbb{R}, \mathbb{R}^2, \mathfrak{B})$$

behavior \mathfrak{B} specified by:

Case 1: $CR_C \neq \frac{L}{R_L}$

$$\left(\frac{R_C}{R_L} + \left(1 + \frac{R_C}{R_L} \right) CR_C \frac{d}{dt} + CR_C \frac{L}{R_L} \frac{d^2}{dt^2} \right) \mathbf{v} = \left(1 + CR_C \frac{d}{dt} \right) \left(1 + \frac{L}{R_L} \frac{d}{dt} \right) R_C I$$

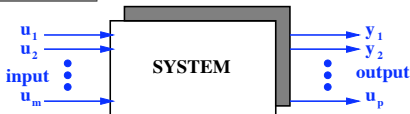
Case 2: $CR_C = \frac{L}{R_L}$

$$\left(\frac{R_C}{R_L} + CR_C \frac{d}{dt} \right) \mathbf{v} = (1 + CR_C) \frac{d}{dt} R_C I$$

\rightsquigarrow behavior **all solutions $(V, I) : \mathbb{R} \rightarrow \mathbb{R}^2$ of this ODE**

Examples

input/output models



$$y(t) = f(y(t-1), \dots, y(t-n), u(t), u(t-1), u(t-n)), \quad w = \begin{bmatrix} u \\ y \end{bmatrix}$$

Differential equation analogue

$$P\left(\frac{d}{dt}\right)y = P\left(\frac{d}{dt}\right)u, \quad w = \begin{bmatrix} u \\ y \end{bmatrix}, \quad P, Q : \text{polynomial matrices}$$

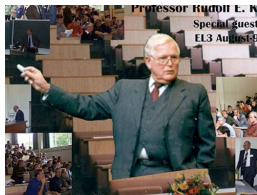
or matrices of rational functions as in $y = G(s)u$

How shall we define the behavior with the rational f'ns ?

Examples

input/output models

State models



R.E. Kalman

$$\frac{d}{dt}x = Ax + Bu, y = Cx + Du; \quad \frac{d}{dt}x = f \circ (x, u), y = h \circ (x, u)$$

¿¿ What is the behavior of this system ??

Examples

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State models

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?? What is the behavior of this system ??

In applications, we care foremost about i/o pairs u, y

$$\leadsto \Sigma = (\mathbb{R}, \mathbb{U} \times \mathbb{Y}, \mathfrak{B})$$

$$\mathfrak{B} = \{(u, y) : \mathbb{R} \rightarrow \mathbb{U} \times \mathbb{Y} \mid$$

$$\exists x : \mathbb{R} \rightarrow \mathbb{X} \text{ such that } x = f \circ (x, u), y = h \circ (x, u)\}$$

So, here again, we meet auxiliary variables, the state x .

Latent variables

Latent variables

Auxiliary variables. We call them **'latent'**. They are ubiquitous:

- states in dynamical systems
- prices in economics
- the wave function in QM
- the basic probability space Ω
- potentials in mechanics, in EM
- interconnection variables
- driving variables in linear system theory
- etc., etc.

Their importance in applications merits formalization.

Latent variables

Latent variable model := $(\mathcal{U}, \mathcal{L}, \mathcal{B}_{\text{full}})$ with $\mathcal{B}_{\text{full}} \subseteq (\mathcal{U} \times \mathcal{L})$

\mathcal{U} : space of **manifest** variables

\mathcal{L} : space of **latent** variables

$\mathcal{B}_{\text{full}}$: 'full behavior'

$\mathcal{B} = \{u \in \mathcal{U} \mid \exists \ell \in \mathcal{L} : (u, \ell) \in \mathcal{B}_{\text{full}}\}$: 'manifest behavior'.

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This is readily generalized to dynamical systems.

A latent variable dynamical system :=

$(\mathbb{T}, \mathbb{W}, \mathbb{L}, \mathcal{B}_{\text{full}})$ with $\mathcal{B}_{\text{full}} \subseteq (\mathbb{W} \times \mathbb{L})^{\mathbb{T}}$

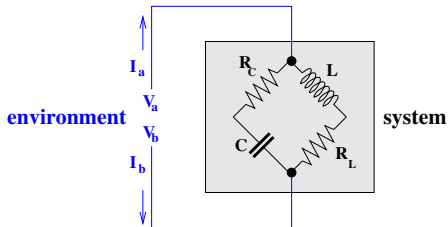
etc.

Example

The price in our economic example

Example

RLC circuit



Model voltage/current histories as a f'n of time !

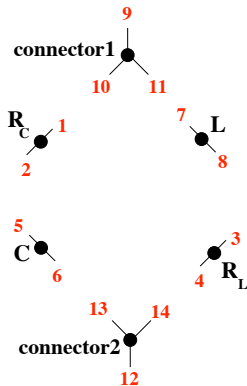
How do we actually go about this modeling ?

Emergence of latent variables.

Example

RLC circuit

TEARING



Example

RLC circuit

ZOOMING

The list of the modules & the associated terminals:

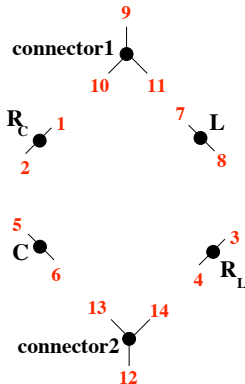
Module	Type	Terminals	Parameter
R_C	resistor	(1, 2)	in ohms
R_L	resistor	(3, 4)	in ohms
C	capacitor	(5, 6)	in farad
L	inductor	(7, 8)	in henry
connector1	3-terminal connector	(9, 10, 11)	
connector2	3-terminal connector	(12, 13, 14)	

Example

RLC circuit

TEARING

The interconnection architecture:



Pairing

{10, 1}

{11, 7}

{2, 5}

{8, 3}

{6, 13}

{4, 14}

Example

RLC circuit

Manifest variable assignment:

the variables

$$V_9, I_9, V_{12}, I_{12}$$

on the **external** terminals **{9, 12}**, i.e.,

$$V_a = V_9, I_a = I_9, V_b = V_{12}, I_b = I_{12}.$$

The **internal** terminals are

{1, 2, 3, 4, 5, 6, 7, 8, 10, 11, 13, 14}

The variables (currents and voltages) on these terminals are our **latent variables**.

Example

RLC circuit

Equations for the full behavior:



Faraday



Ohm



Henry



Coulomb



Kirchhoff

Modules	Constitutive equations	
R_C	$I_1 + I_2 = 0$	$V_1 - V_2 = R_C I_1$
R_L	$I_7 + I_8 = 0$	$V_7 - V_8 = R_L I_7$
C	$I_5 + I_6 = 0$	$C \frac{d}{dt} (V_5 - V_6) = I_5$
L	$I_7 + I_8 = 0$	$V_7 - V_8 = L \frac{d}{dt} I_7$
connector1	$I_9 + I_{10} + I_{11} = 0$	$V_9 = V_{10} = V_{11}$
connector2	$I_{12} + I_{13} + I_{14} = 0$	$V_{12} = V_{13} = V_{14}$

Interconnection pair	Interconnection equations	
{10, 1}	$V_{10} = V_1$	$I_{10} + I_1 = 0$
{11, 7}	$V_{11} = V_7$	$I_{11} + I_7 = 0$
{2, 5}	$V_2 = V_5$	$I_2 + I_5 = 0$
{8, 3}	$V_8 = V_3$	$I_8 + I_3 = 0$
{6, 13}	$V_6 = V_{13}$	$I_6 + I_{13} = 0$
{4, 14}	$V_4 = V_{14}$	$I_4 + I_{14} = 0$

Example

RLC circuit

All these eq'ns combined define a latent variable system in the **manifest 'external' variables**

$$w = (V_a, I_a, V_b, I_b)$$

with **'internal' latent variables**

$$\ell = (V_1, I_1, V_2, I_2, V_3, I_3, V_4, I_4, V_5, I_5, V_6, I_6, V_7, I_7, \\ V_8, I_8, V_{10}, I_{10}, V_{11}, I_{11}, V_{13}, I_{13}, V_{14}, I_{14}).$$

The manifest behavior \mathfrak{B} is given by

$$\mathfrak{B} = \{(V_a, I_a, V_b, I_b) : \mathbb{R} \rightarrow \mathbb{R}^4 \mid \exists \ell : \mathbb{R} \rightarrow \mathbb{R}^{24} \dots\}$$

Example

RLC circuit

Elimination:

Case 1: $CR_C \neq \frac{L}{R_L}$.

$$\left(\frac{R_C}{R_L} + \left(1 + \frac{R_C}{R_L}\right)CR_C \frac{d}{dt} + CR_C \frac{L}{R_L} \frac{d^2}{dt^2}\right)(V_a - V_b) = \left(1 + CR_C \frac{d}{dt}\right)\left(1 + \frac{L}{R_L} \frac{d}{dt}\right)R_C I_a.$$

$$I_a + I_b = 0$$

Case 2: $CR_C = \frac{L}{R_L}$.

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$$I_a + I_b = 0$$

Perhaps 'port' variables: $V = V_a - V_b, I = I_a = -I_b$

Example

RLC circuit

Note: the eliminated equations are differential equations!
Does this follow from some general principle?

Algorithms for elimination?

The modeling of this RLC circuit is an example of **tearing, zooming & linking**. It is the most prevalent way of modeling. See my website for formalization. Crucial role of latent variables.

Note: no input/output thinking;
systems in nodes, connections in edges.

Controllability & Observability

System properties

In this framework, system theoretic notions like

Controllability, observability, stabilizability,...

become simpler, more general, more convincing.

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For simplicity, we consider only

time-invariant, continuous-time systems with $\mathbb{T} = \mathbb{R}$

time-invariant := $\llbracket w \in \mathfrak{B} \rrbracket \Rightarrow \llbracket w(t' + \cdot) \in \mathfrak{B} \forall t' \in \mathbb{R} \rrbracket$.

Controllability

The time-invariant system $\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$ is said to be

controllable

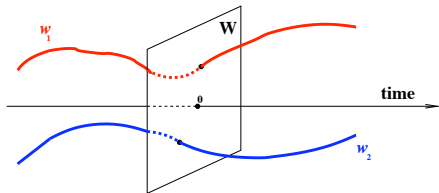
if for all $w_1, w_2 \in \mathfrak{B} \exists w \in \mathfrak{B}$ and $T \geq 0$ such that

$$w(t) = \begin{cases} w_1(t) & t < 0 \\ w_2(t - T) & t \geq T \end{cases}$$

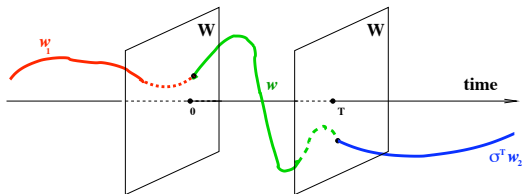
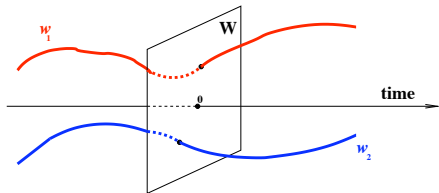
Controllability $:\Leftrightarrow$

legal trajectories must be **'patch-able', 'concatenable'**.

Controllability



Controllability



Examples

$$\frac{d}{dt}x = Ax + Bu; \quad \frac{d}{dt}x = f \circ (x, u)$$

with $w = (x, u)$, controllable \Leftrightarrow 'state point' controllable.

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likewise \Leftrightarrow with $w = x$

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RLC circuit

Case 2: $CR_C = \frac{L}{R_L}$

$$\left(\frac{R_C}{R_L} + CR_C \frac{d}{dt}\right)(V_a - V_b) = \left(1 + CR_C \frac{d}{dt}\right)R_C I_a$$

$$I_a + I_b = 0$$

Assume also $R_C = R_L$. Controllable?

$V_a - V_b = R_C I_a + \text{constant} \cdot e^{-\frac{t}{CR_C}}$. Not controllable.

Examples

$$\frac{d}{dt}x = Ax + Bu; \quad \frac{d}{dt}x = f \circ (x, u)$$

with $w = (x, u)$, controllable \Leftrightarrow 'state point' controllable.

$$p\left(\frac{d}{dt}\right)y = q\left(\frac{d}{dt}\right)u$$

controllable $\Leftrightarrow p, q$ co-prime

Examples

$$\frac{d}{dt}x = Ax + Bu; \quad \frac{d}{dt}x = f \circ (x, u)$$

with $w = (x, u)$, controllable \Leftrightarrow 'state point' controllable.

$$w = M\left(\frac{d}{dt}\right)\ell$$

M a polynomial matrix, always has a controllable manifest behavior.

In fact, this characterizes the controllable linear time-invariant differentiable systems ('image representation').

Note emergence of latent variables, ℓ .

Examples

$$w = M\left(\frac{d}{dt}\right)\ell$$

M a polynomial matrix, always has a controllable manifest behavior. Likewise,

$$w = F\left(\frac{d}{dt}\right)\ell$$

F matrix of **rat. f'ns** has controllable manifest behavior. But we need to give this 'differential equation' a meaning.

Whence

$$y = G\left(\frac{d}{dt}\right)u, \quad w = \begin{bmatrix} u \\ y \end{bmatrix}$$

is always controllable.

Observability



¿ Is it possible to deduce w_2 from w_1 and the model \mathfrak{B} ?

Observability

Consider the system $\Sigma = (\mathbb{T}, \mathbb{W}_1 \times \mathbb{W}_2, \mathfrak{B})$. Each element of \mathfrak{B} hence consists of a pair of trajectories (w_1, w_2) :

w_1 : observed;

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Definition: w_2 is said to be

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if $\llbracket (w_1, w_2') \in \mathfrak{B}, \text{ and } (w_1, w_2'') \in \mathfrak{B} \rrbracket \Rightarrow \llbracket (w_2' = w_2'') \rrbracket$,
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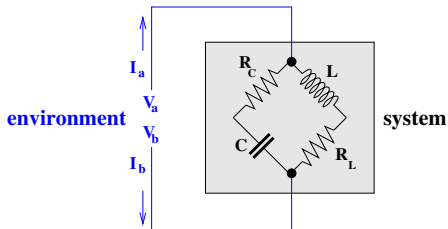
Very often **manifest** = observed, **latent** = to-be-deduced.
We then speak of an **observable (latent variable) system**.

Examples

$$\frac{d}{dt}x = Ax + Bu, y = Cx + Du; \quad \frac{d}{dt}x = f_{\circ}(x, u), y = h_{\circ}(x, u)$$

with $w_1 = (u, y)$, $w_2 = x$, observable \Leftrightarrow 'state' observable.

Examples



Controllability of this system (referring to external terminal variables) is a well-defined question.

Observability is not! **No duality on the system's level.**
Of course, there is a notion of \mathfrak{B}^\perp , and results connecting controllability of \mathfrak{B} to state observability of \mathfrak{B}^\perp .

Examples

Equations for the full behavior:



Faraday



Ohm



Henry



Coulomb



Kirchhoff

Modules	Constitutive equations	
R_C	$I_1 + I_2 = 0$	$V_1 - V_2 = R_C I_1$
R_L	$I_7 + I_8 = 0$	$V_7 - V_8 = R_L I_7$
C	$I_5 + I_6 = 0$	$C \frac{d}{dt} (V_5 - V_6) = I_5$
L	$I_7 + I_8 = 0$	$V_7 - V_8 = L \frac{d}{dt} I_7$
connector1	$I_9 + I_{10} + I_{11} = 0$	$V_9 = V_{10} = V_{11}$
connector2	$I_{12} + I_{13} + I_{14} = 0$	$V_{12} = V_{13} = V_{14}$

Interconnection pair	Interconnection equations	
{10, 1}	$V_{10} = V_1$	$I_{10} + I_1 = 0$
{11, 7}	$V_{11} = V_7$	$I_{11} + I_7 = 0$
{2, 5}	$V_2 = V_5$	$I_2 + I_5 = 0$
{8, 3}	$V_8 = V_3$	$I_8 + I_3 = 0$
{6, 13}	$V_6 = V_{13}$	$I_6 + I_{13} = 0$
{4, 14}	$V_4 = V_{14}$	$I_4 + I_{14} = 0$

Examples

All these eq'ns combined define a latent variable system in the **manifest variables**

$$w = (V_a, I_a, V_b, I_b)$$

with **latent variables**

$$\ell = (V_1, I_1, V_2, I_2, V_3, I_3, V_4, I_4, V_5, I_5, V_6, I_6, V_7, I_7, \\ V_8, I_8, V_{10}, I_{10}, V_{11}, I_{11}, V_{13}, I_{13}, V_{14}, I_{14}).$$

The manifest behavior \mathfrak{B} is given by

$$\mathfrak{B} = \{(V_a, I_a, V_b, I_b) : \mathbb{R} \rightarrow \mathbb{R}^4 \mid \exists \ell : \mathbb{R} \rightarrow \mathbb{R}^{24} \dots\}$$

Are the latent variables observable from the manifest ones?

$$\Leftrightarrow CR_C \neq L/R_L$$

Examples

$$p\left(\frac{d}{dt}\right)y = q\left(\frac{d}{dt}\right)u$$

u is observable from $y \Leftrightarrow q = \text{non-zero constant}$
(**no zeros**).

A controllable linear time-invariant differential system always has an **observable** 'image' representation

$$w = M\left(\frac{d}{dt}\right)\ell.$$

In fact, this again characterizes the controllable linear time-invariant differentiable systems.

Kalman definitions

Special case: classical Kalman definitions for

$$\frac{d}{dt}\mathbf{x} = \mathbf{f} \circ (\mathbf{x}, \mathbf{u}), \quad \mathbf{y} = \mathbf{h} \circ (\mathbf{x}, \mathbf{u}).$$



R.E. Kalman

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R.E. Kalman

controllability: variables = (input, state)

If a system is not (state) controllable, why is it?

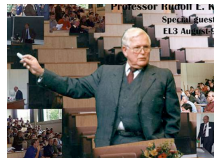
Insufficient influence of the control?

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Or bad choice of the state?

observability: \rightsquigarrow observed = (input, output),
to-be-deduced = state.

Why is it so interesting to try to deduce the state, of all things? The state is a derived notion, not a 'physical' one.

Stabilizability

The system $\Sigma = (\mathbb{T}, \mathbb{R}^w, \mathfrak{B})$ is said to be **stabilizable** if, for all $w \in \mathfrak{B}$, there exists $w' \in \mathfrak{B}$ such that

$$w(t) = w'(t) \text{ for } t < 0 \quad \text{and} \quad w'(t) \xrightarrow[t \rightarrow \infty]{} 0.$$

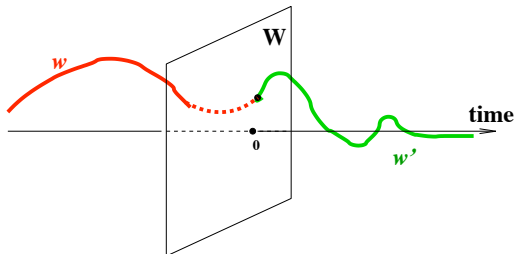
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Stabilizability $:\Leftrightarrow$

legal trajectories can be steered to a desired point.



Detectability



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Detectability



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Definition: w_2 is said to be

detectable from w_1 if

$$\begin{aligned} & \llbracket (w_1, w_2') \in \mathfrak{B}, \text{ and } (w_1, w_2'') \in \mathfrak{B} \rrbracket \\ & \Rightarrow \llbracket (w_2' - w_2'') \rightarrow 0 \text{ for } t \rightarrow \infty \rrbracket \end{aligned}$$

Summary

Btw

- **A model is not a map, but a relation.**

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- A flow

$$\frac{d}{dt}x = f(x) \text{ with or without } y = h(x)$$

is a very limited model class.

~> closed dynamical systems.

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- A model is not a map, but a relation.
- A flow is a very limited model class.
 \rightsquigarrow closed dynamical systems.
- An **open** dynamical system is not an input/output **map**.



Heaviside



Wiener



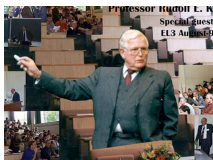
Nyquist



Bode

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 \rightsquigarrow closed dynamical systems.
- An **open** dynamical system is not an input/output **map**.
- input/state/output systems, although still limited, are the first class of suitably general models



R.E. Kalman

Btw

- A model is not a map, but a relation.
- A flow is a very limited model class.
 \rightsquigarrow closed dynamical systems.
- An **open** dynamical system is not an input/output **map**.
- input/state/output systems, although still limited, are the first class of suitably general models
- Behaviors, including latent variables, are the first suitable general model class for physical applications and modeling by tearing, zooming, and linking

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Summary

- A mathematical model = a subset
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- Latent variables are ubiquitous in models
- Important properties of dynamical systems
 - **Controllability** : concatenability of trajectories
 - **Observability** : deducing one trajectory from another
 - **Stabilizability** : driving a trajectory to zero
- The behavior is all there is. All properties in terms of the behavior. Equivalence, representations also.

Stochastic models

We only consider **deterministic** models. **Stochastic models:**



Laplace



Kolmogorov

there is a map P (the '**probability**')

$$P : \mathfrak{A} \rightarrow [0, 1]$$

with \mathfrak{A} a ' σ -algebra' of subsets of \mathfrak{X} .

$P(\mathfrak{B}) =$ '**degree of certainty**' (relative frequency, propensity, plausibility, belief) that outcomes are in \mathfrak{B} ;
 \cong the degree of validity of \mathfrak{B} as a model.

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 \cong the degree of validity of \mathfrak{B} as a model.

Determinism: P is a '**{0, 1}-law**'

$$\mathfrak{A} = \{\emptyset, \mathfrak{B}, \mathfrak{B}^{\text{complement}}, \mathfrak{U}\}, P(\mathfrak{B}) = 1.$$

Fuzzy models



L. Zadeh

Fuzzy models: there is a map μ ('membership f'n')

$$\mu : \mathcal{X} \rightarrow [0, 1]$$

$\mu(x) =$ 'the extent to which x belongs to
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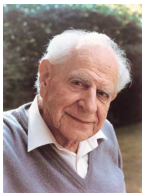
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Determinism: μ is 'crisp':

$$\text{image}(\mu) = \{0, 1\},$$

$$\mathfrak{B} = \mu^{-1}(\{1\}) := \{x \in \mathfrak{X} \mid \mu(x) = 1\}$$

***Every 'good' scientific theory is prohibition: it forbids certain things to happen...
The more a theory forbids, the better it is.***



Karl Popper (1902-1994)

**Replace 'scientific theory'
by 'mathematical model' !**

