RATIONAL REPRESENTATIONS

of LTID systems

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Preliminaries

Behaviors & all that

A *dynamical system*:⇔ $\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$

$\mathbb{T} \subseteq \mathbb{R}$ **the *time-axis***

$\mathbb{W}$ **the *signal space***

$\mathfrak{B} \subseteq \mathbb{W}^{\mathbb{T}}$ **the *behavior* - a family of trajectories**

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$\mathfrak{B} \subseteq \mathbb{W}^{\mathbb{T}}$	**the *behavior***	**- a family of trajectories**

$\Sigma = (\mathbb{R}, \mathbb{R}^{\mathtt{w}}, \mathfrak{B})$ **is said to be linear** $:\Leftrightarrow$ $\mathfrak{B}$ **is a linear space**

time-invariant $:\Leftrightarrow$ $\mathfrak{B}$ **is shift-invariant**

$w \in \mathfrak{B}$ **and** $t \in \mathbb{R} \quad \Rightarrow \quad \sigma^t w \in \mathfrak{B}$

σ^t **denotes the 'shift':** $\quad (\sigma^t w)(t') = w(t' + t)$

differential $:\Leftrightarrow$ $\mathfrak{B}$ **is the set of sol'ns of an ODE**

Examples

Dynamical system:

$$\Sigma : \quad \dot{x} = f(x, u, t), \quad y = h(x, u, t)$$

$u \in \mathbb{U} = \mathbb{R}^{\mathtt{m}}, y \in \mathbb{Y} = \mathbb{R}^{\mathtt{p}}, x \in \mathbb{X} = \mathbb{R}^{\mathtt{n}}$**: input, output, state.**

Behavior $\mathfrak{B} =$ **all sol'ns** $(u, y, x) : \mathbb{R} \rightarrow \mathbb{U} \times \mathbb{Y} \times \mathbb{X}$.

Time-invariant:

$$\Sigma : \quad \dot{x} = f(x, u), \quad y = h(x, u)$$

Linear time-invariant:

$$\Sigma : \quad \dot{x} = Ax + Bu, \quad y = Cx + Du$$

LTID systems

Linear, time-invariant, differential dynamical system $\Leftrightarrow$

$$R_0 w + R_1 \frac{d}{dt} w + R_2 \frac{d^2}{dt^2} w + \cdots + R_L \frac{d^L}{dt^L} w = 0$$

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Short-hand notation: introduce polynomial matrix

$$R(\xi) = R_0 + R_1 \xi + R_2 \xi^2 + \cdots + R_L \xi^L \in \mathbb{R}[\xi]^{\bullet \times \mathtt{w}}$$

$$R\left(\frac{d}{dt}\right) w = 0$$

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$$\mathfrak{B} = \left\{ w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \mid R\left(\frac{d}{dt}\right) w = 0 \right\}$$

$\mathfrak{B} = \text{kernel}\left(R\left(\frac{d}{dt}\right)\right)$ **'kernel representation', polynomial type.**

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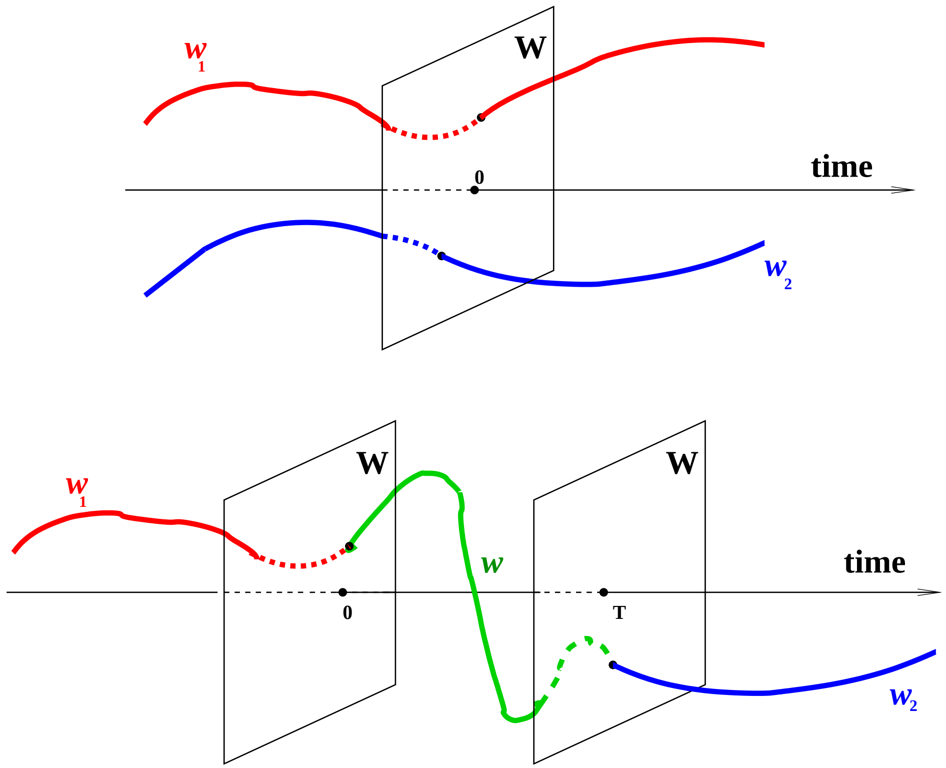
Controllability and stabilizability

Let $\Sigma = (\mathbb{R}, \mathbb{R}^{\mathtt{w}}, \mathfrak{B})$ **be a time-invariant dynamical system**

Σ **is said to be controllable** $:\Leftrightarrow$

$\forall\, w_1, w_2 \in \mathfrak{B}\ \exists\, T \geq 0,$ **and** $w \in \mathfrak{B}$ **such that ...**



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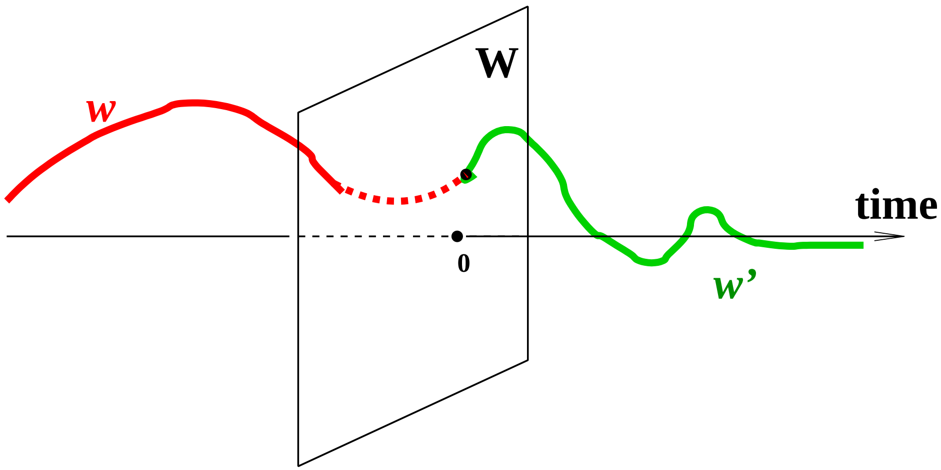

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Theorem: $R\left(\frac{d}{dt}\right)w = 0$ **defines** **a controllable system** $\Leftrightarrow$

$\text{rank}\,(R\,(\lambda))$ **is the same** $\forall\,\lambda \in \mathbb{C}$

a stabilizable system $\Leftrightarrow$

$\text{rank}\,(R\,(\lambda))$ **is the same** $\forall\,\lambda \in \mathbb{C}$ **with real part** ≥ 0

Rational representations

Rational representations

Let $G \in \mathbb{R}(\xi)^{\bullet \times \mathtt{w}}$, and consider the 'differential equation'

$$G\left(\frac{d}{dt}\right) w = 0$$

What do we mean by the solutions, i.e. by the behavior?

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What do we mean by the solutions, i.e. by the behavior?

Let (P,Q) **be a left-coprime polynomial factorization of** G

i.e. $P,Q \in \mathbb{R}[\xi]^{\bullet \times \bullet}, \det(P) \neq 0, G = P^{-1}Q, [P \vdots Q]$ **left-prime.**

$$G(\frac{d}{dt})w = 0 :\Leftrightarrow Q\left(\frac{d}{dt}\right)w = 0$$

E.g., in scalar case, means P **and** Q **have no common roots.**

Rational representations

Let (P, Q) be a left-coprime polynomial factorization of G

$$G(\frac{d}{dt})w = 0 :\Leftrightarrow Q\left(\frac{d}{dt}\right)w = 0$$

Justification:

1. G proper. $G(s) = C(Is - A)^{-1}B + D$ controllable realization. Consider output nulling inputs:

$$\frac{d}{dt}x = Ax + Bw, \quad 0 = Cx + Dw$$

This set of w's are exactly those that satisfy $G\left(\frac{d}{dt}\right)w = 0$.

Same for $\frac{d}{dt}x = Ax + Bw, 0 = Cx + D\left(\frac{d}{dt}\right)w = 0$, $\quad D \in \mathbb{R}[\xi]^{\bullet\times\bullet}$.

Rational representations

Let (P,Q) be a left-coprime polynomial factorization of G

$$G(\frac{d}{dt})w = 0 :\Leftrightarrow Q\left(\frac{d}{dt}\right)w = 0$$

Justification:

2. Consider $y = G(s)u$. View G as a transfer f'n.
Take your usual favorite definition of input/output pairs.

The output nulling inputs are exactly those that satisfy
$G\left(\frac{d}{dt}\right)w = 0$**.**

3. via Laplace transforms...

$G\left(\frac{d}{dt}\right)$ is not a map!

Consider

$$y = G\left(\frac{d}{dt}\right)u$$

We now know what it means that $(u, y) \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^\bullet)$ satisfies this 'ODE'.

Is there a unique y for a given u?

$$P(\frac{d}{dt})y = Q(\frac{d}{dt})u$$

If $P \neq I$ (better, not unimodular), there are many sol'ns y of this ODE for a given RHS.

Representations

Linear time-invariant differential systems $\Sigma = (\mathbb{R}, \mathbb{R}^{\mathtt{w}}, \mathfrak{B})$**.**

$\mathfrak{B} = \ker\left(R\left(\frac{d}{dt}\right)\right)$ **for some** $R \in \mathbb{R}[\xi]^{\bullet\times\mathtt{w}}$ **by definition.**

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But we may as well take the representation $G\left(\frac{d}{dt}\right)w = 0$ **for some** $G \in \mathbb{R}(\xi)^{\bullet \times \mathtt{w}}$ **as the definition.**

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But we may as well take the representation $G\left(\frac{d}{dt}\right) w = 0$ **for some** $G \in \mathbb{R}(\xi)^{\bullet\times\mathtt{w}}$ **as the definition.**
R**: all poles at** ∞**, we can take** G **with no poles at** ∞**, or more generally all poles in some 'fat' set - intersection with** $\mathbb{R}$ **having non-emply interior.**

Theorem: Every linear time-invariant differential systems has a representation

$$G\left(\frac{d}{dt}\right) w = 0$$

with $G \in \mathbb{R}(\xi)^{\bullet\times\mathtt{w}}$ **strictly proper rational stable.**
Proof: Take $G(s) = \frac{R(s)}{(s+\lambda)^{\mathtt{n}}}$**, suitable** $\lambda \in \mathbb{R}, \mathtt{n} \in \mathbb{N}$**.**

Matrices of rational functions

Subrings of $\mathbb{R}(\xi)$

$\mathbb{R}(\xi)$: real rational functions.

Consider 3 subrings:

1. **$\mathbb{R}[\xi]$: polynomials with real coefficients**
2. **$\mathbb{R}(\xi)_{\mathscr{P}}$: proper rational functions**
3. **$\mathbb{R}(\xi)_{\mathscr{S}}$: stable proper rational functions**

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   no poles in RHP or ∞

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Each of these rings has $\mathbb{R}(\xi)$ as its field of fractions.

Unimodular elements (invertible in ring)
1. Non-zero constants.
2. bi-proper.
3. bi-proper and mini-phase.

miniphase: $\Leftrightarrow$ **poles & zeros in LHP**

Matrices over these rings

$\mathbb{R}(\xi)^{\bullet\times\bullet}$: **matrices of real rational functions.**

1. $\mathbb{R}[\xi]^{\bullet\times\bullet}$: **polynomial matrices with real coefficients**

2. $\mathbb{R}(\xi)^{\bullet\times\bullet}_{\mathscr{P}}$: **matrices of proper rational functions**

3. $\mathbb{R}(\xi)^{\bullet\times\bullet}_{\mathscr{S}}$: **of stable proper rational functions**

Matrices over these rings

$\mathbb{R}(\xi)^{\bullet\times\bullet}$: **matrices of real rational functions.**

1. $\mathbb{R}[\xi]^{\bullet\times\bullet}$: **polynomial matrices with real coefficients**
   unimodular: square & determinant = non-zero constant
2. $\mathbb{R}(\xi)^{\bullet\times\bullet}_{\mathscr{P}}$: **matrices of proper rational functions**
   unimodular: square & determinant biproper
3. $\mathbb{R}(\xi)^{\bullet\times\bullet}_{\mathscr{S}}$: **of stable proper rational functions**
   unimodular: square & determinant biproper and miniphase (poles & zeros in LHP)

Prime elements

$M \in \mathbb{R}[\xi]^{\mathtt{n}_1 \times \mathtt{n}_2}$ **is left-prime** $:\Leftrightarrow$
$M = FM', F \in \mathbb{R}[\xi]^{\mathtt{n}_1 \times \mathtt{n}_1}, M' \in \mathbb{R}[\xi]^{\mathtt{n}_1 \times \mathtt{n}_2}$
$\Rightarrow U$ **is uni-modular**

Prime elements

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$M \in \mathbb{R}(\xi)_{\mathscr{P}}^{\mathtt{n}_1 \times \mathtt{n}_2}$ **is left-prime over** $\mathbb{R}(\xi)_{\mathscr{P}}$ $:\Leftrightarrow$
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$M \in \mathbb{R}(\xi)_{\mathscr{S}}^{\mathtt{n}_1 \times \mathtt{n}_2}$ **is left-prime over** $\mathbb{R}(\xi)_{\mathscr{S}}$ $:\Leftrightarrow$
$M = FM', F \in \mathbb{R}(\xi)_{\mathscr{S}}^{\mathtt{n}_1 \times \mathtt{n}_1}, M' \in \mathbb{R}(\xi)_{\mathscr{P}}^{\mathtt{n}_1 \times \mathtt{n}_2}$
$\Rightarrow U$ **is uni-modular over** $\mathbb{R}(\xi)_{\mathscr{S}}$

Prime representations & system properties

Prime representations

Theorem: **a linear time-invariant differential system admits a representation**

$$G\left(\frac{d}{dt}\right)w = 0$$

with

1. $G \in \mathbb{R}(\xi)_{\mathscr{P}}^{\bullet \times \mathtt{w}}$ **left prime over** $\mathbb{R}(\xi)_{\mathscr{P}}$
2. $G \in \mathbb{R}[\xi]^{\bullet \times \mathtt{w}}$ **left prime over** $\mathbb{R}[\xi] \Leftrightarrow$ **it is controllable**
3. $G \in \mathbb{R}(\xi)_{\mathscr{S}}^{\bullet \times \mathtt{w}}$ **left prime over** $\mathbb{R}(\xi)_{\mathscr{S}} \Leftrightarrow$ **it is stabilizable**

The proof of case 3 is not easy!

Controllability and image-like representations

Elimination

Consider

$$G_1\left(\frac{d}{dt}\right)w_1 = G_2\left(\frac{d}{dt}\right)w_2$$

$G_1, G_2 \in \mathbb{R}(\xi)^{\bullet\times\bullet}$. **Behavior** $\mathfrak{B}$. **Eliminate** $w_2 \rightsquigarrow$

$$\mathfrak{B}_1 = \{w_1 \mid \exists\, w_2 \text{ such that } (w_1, w_2) \in \mathfrak{B}\}$$

Then $\mathfrak{B}_1$ **is also a LTID behavior.**

In particular

$$w = H\left(\frac{d}{dt}\right)\ell, \quad H \in \mathbb{R}(\xi)^{\mathtt{w}\times\bullet}.$$

w**-behavior is LTID. Image-like representation.**

Representations of controllable systems

Theorem: **The following are equivalent for LTID systems**

1. $\mathfrak{B}$ **is controllable**
2. $\mathfrak{B}$ **admits an image-like representation**

$$w = M\left(\tfrac{d}{dt}\right)\ell \text{ with } H \in \mathbb{R}[\xi]^{\mathtt{w}\times\bullet}$$

3. $\mathfrak{B}$ **admits an image-like representation**

$$w = H\left(\tfrac{d}{dt}\right)\ell \text{ with } H \in \mathbb{R}(\xi)^{\mathtt{w}\times\bullet}$$

4. **with observability (**ℓ **can be deduced from** w**) added**
5. **with** $M \in \mathbb{R}[\xi]^{\mathtt{w}\times\bullet}$ **right prime over** $\mathbb{R}[\xi]$
6. **with** $H \in \mathbb{R}(\xi)_{\mathscr{S}}^{\mathtt{w}\times\bullet}$ **right prime over** $\mathbb{R}(\xi)_{\mathscr{S}}$

Relations with classical results

Consider system $y = Gu, \;\; G \in \mathbb{R}(\xi)^{\mathtt{p}\times\mathtt{m}}$ **'transfer function'**

Interpret this as

$$y = G\left(\frac{d}{dt}\right)u$$

Automatically controllable!

Only controllable systems covered by tf. f'ns.

Even if G **is i/o unstable or improper,** $\exists$ **stable kernel- and image-like representations!**

Relations with classical results

$$y = G\left(\frac{d}{dt}\right)u$$

Even if G is i/o unstable or improper, $\exists$ stable kernel- and image-like representations!

$$G_1\left(\frac{d}{dt}\right)y = G_2\left(\frac{d}{dt}\right)u,$$

$\begin{bmatrix} G_1 & \vdots & G_2 \end{bmatrix} \in \mathbb{R}(\xi)_{\mathscr{S}}^{\bullet\times\bullet}$ **left prime over** $\mathbb{R}(\xi)_{\mathscr{S}}$.

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$$\begin{bmatrix} u \\ y \end{bmatrix} = \begin{bmatrix} H_1\left(\frac{d}{dt}\right) \\ H_2\left(\frac{d}{dt}\right) \end{bmatrix} \ell,$$

$\begin{bmatrix} H_1 \\ H_2 \end{bmatrix} \in \mathbb{R}(\xi)_{\mathscr{S}}^{\bullet\times\bullet}$ **right prime over** $\mathbb{R}(\xi)_{\mathscr{S}}^{\bullet\times\bullet}$.

Relations with classical results

$$y = G\left(\frac{d}{dt}\right)u$$

$$G = G_1^{-1}G_2 = H_2H_1^{-1}$$

left/right co-prime factorizations over $\mathbb{R}(\xi)_{\mathscr{S}}$. As over $\mathbb{R}[\xi]$.

Classical, but we obtain the representation

$$G_1\left(\frac{d}{dt}\right)y = G_2\left(\frac{d}{dt}\right)u,$$

with $\begin{bmatrix} G_1 & \vdots & G_2 \end{bmatrix} \in \mathbb{R}(\xi)_{\mathscr{S}}^{\bullet\times\bullet}$ left prime over $\mathbb{R}(\xi)_{\mathscr{S}}$
also for stabilzable systems, instead of only controllable ones.

The annihilators

Polynomial and rational annihilators

Let $w \in \mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$**.**

For $R \in \mathbb{R}[\xi]^{\bullet \times \mathtt{w}}$ **it is clear what we mean by**

$$R\left(\frac{d}{dt}\right) w = 0.$$

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For $R \in \mathbb{R}[\xi]^{\bullet \times \mathtt{w}}$ **it is clear what we mean by**

$$R\left(\frac{d}{dt}\right)w = 0.$$

But now we also know what we mean by

$$G\left(\frac{d}{dt}\right)w = 0$$

for $G \in \mathbb{R}(\xi)^{\bullet \times \mathtt{w}}$**.**

This gives us the opportunity to discuss more operators that annihilate a given behavior.

Polynomial and rational annihilators

Let $\mathfrak{B}$ be the behavior of a LTID system.

Call $n \in \mathbb{R}[\xi]^{\mathtt{w}}$ **a polynomial annihilator of** $\mathfrak{B}$ $:\Leftrightarrow$

$$n^\top\left(\frac{d}{dt}\right)\mathfrak{B} = 0.$$

Call $n \in \mathbb{R}(\xi)^{\mathtt{w}}$ **a rational annihilator of** $\mathfrak{B}$ $:\Leftrightarrow$ $n^\top\left(\frac{d}{dt}\right)\mathfrak{B} = 0$.

Polynomial and rational annihilators

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Call $n \in \mathbb{R}(\xi)^{\mathtt{w}}$ **a rational annihilator of** $\mathfrak{B}$ $:\Leftrightarrow$ $n^\top\left(\frac{d}{dt}\right)\mathfrak{B} = 0.$

1. **The polynomial annihilators of $\mathfrak{B}$ form a $\mathbb{R}[\xi]$-module.**
2. **The rational annihilators of $\mathfrak{B}$ form a $\mathbb{R}[\xi]$-module.**
3. **The rational annihilators of a controllable $\mathfrak{B}$ form a $\mathbb{R}(\xi)$-vectorspace.**
4. **There is a one-one relation between the LTID behaviors and the $\mathbb{R}[\xi]$-submodules of $\mathbb{R}[\xi]^{\mathtt{w}}$.**
5. **There is a one-one relation between the controllable LTID behaviors and the $\mathbb{R}(\xi)$-subspaces of $\mathbb{R}(\xi)^{\mathtt{w}}$.**

Why bother with rational rather than just polynomial ‘symbols’?

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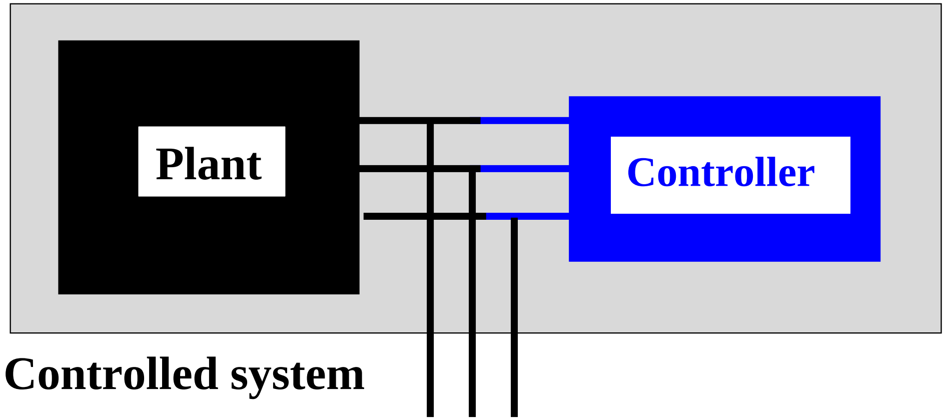
1. Parametrization of all stabilizing controllers
2. Model reduction of behavioral systems

Parametrization of all stabilizing controllers

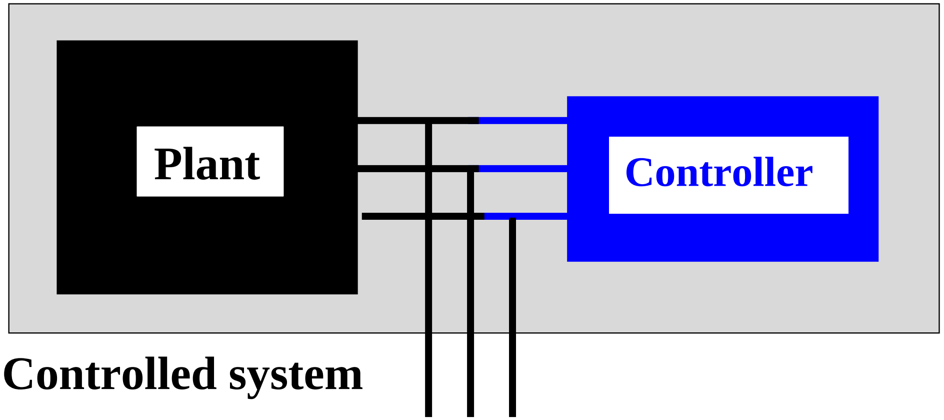
Polynomial characterization



Plant & controller LTID systems, behaviors $\mathfrak{P}$ and $\mathfrak{C}$, resp.

Controlled behavior $\mathfrak{B} = \mathfrak{P} \cap \mathfrak{C}$. Also LTID.

Polynomial characterization



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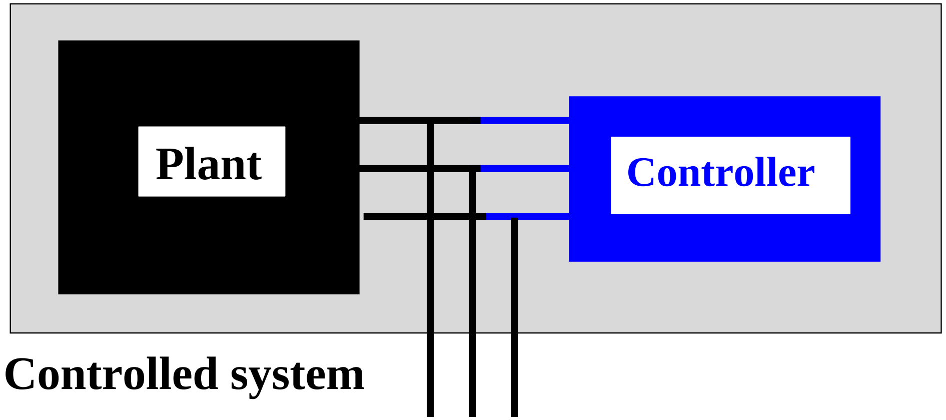
Call $\mathfrak{B}$ stable if $w \in \mathfrak{B} \Rightarrow w(t) \rightarrow 0$ for $t \rightarrow \infty$

Polynomial characterization



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Given plant $\mathfrak{P}$, which controllers $\mathfrak{C}$ are stabilizing?

We will only consider 'regular' controllers.

Polynomial characterization (Kuijper)

Plant $\mathfrak{P}$, assume controllable $\Leftrightarrow\ \exists$ LTID system $\mathfrak{P}'$ such that

$$\mathfrak{P} \oplus \mathfrak{P}' = \mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$$

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Plant $\mathfrak{P}$, assume controllable $\Leftrightarrow$ $\exists$ LTID system $\mathfrak{P}'$ such that

$$\mathfrak{P} \oplus \mathfrak{P}' = \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$$

Now take kernel representations of $\mathfrak{P}$ and $\mathfrak{P}'$

$$P\left(\frac{d}{dt}\right)w = 0, \quad P \in \mathbb{R}[\xi]^{\bullet \times \mathtt{w}}, \text{ **left prime over** } \mathbb{R}[\xi].$$

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Polynomial characterization (Kuijper)

Plant $\mathfrak{P}$, assume controllable $\Leftrightarrow$ $\exists$ LTID system $\mathfrak{P}'$ such that

$$\mathfrak{P} \oplus \mathfrak{P}' = \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$$

$$\mathfrak{P} \cap \mathfrak{P}' = \{0\} \;\; \& \;\; \mathfrak{P} + \mathfrak{P}' = \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \quad \Leftrightarrow \quad \begin{bmatrix} P \\ P' \end{bmatrix} \text{ unimodular}$$

Let $C\left(\frac{d}{dt}\right) w = 0$ be a controller.

Unimodularity $\Rightarrow$ it is of the form $C = FP + F'P'$

since $C = \begin{bmatrix} F & \vdots & F' \end{bmatrix} \begin{bmatrix} P \\ P' \end{bmatrix}$ is solvable for F, F'.

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Stabilizing? Controlled behavior:

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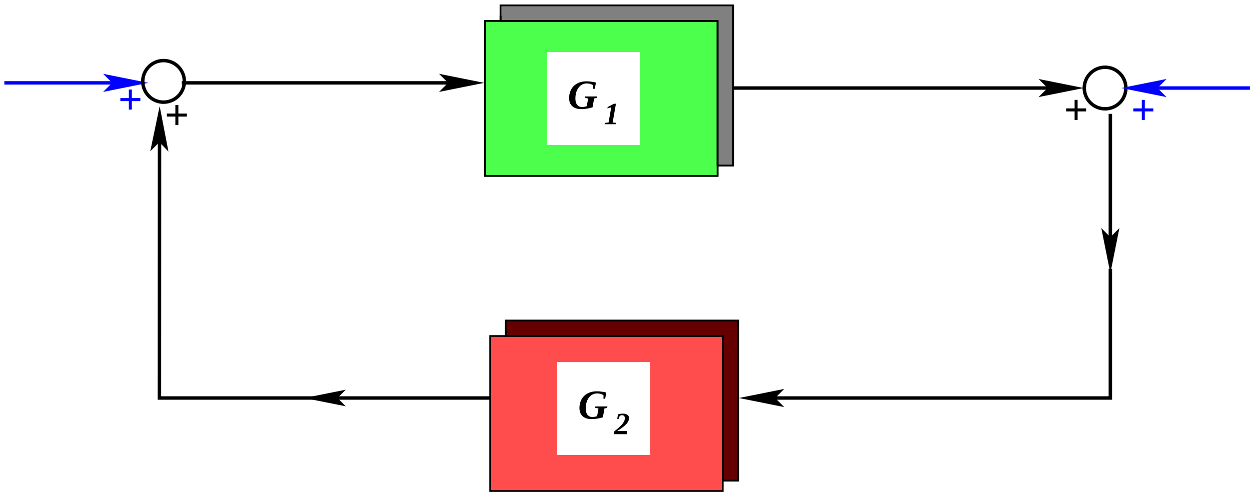
Stabilizing $\Leftrightarrow$ F' is 'Hurwitz' (square, roots det in LHP). All (regular) stab'ing controllers (polynomial parametrization):

$$C = FP + F'P', \quad F \text{ **anything,** } F' \text{ **Hurwitz**}$$

Kucera-Youla type characterization



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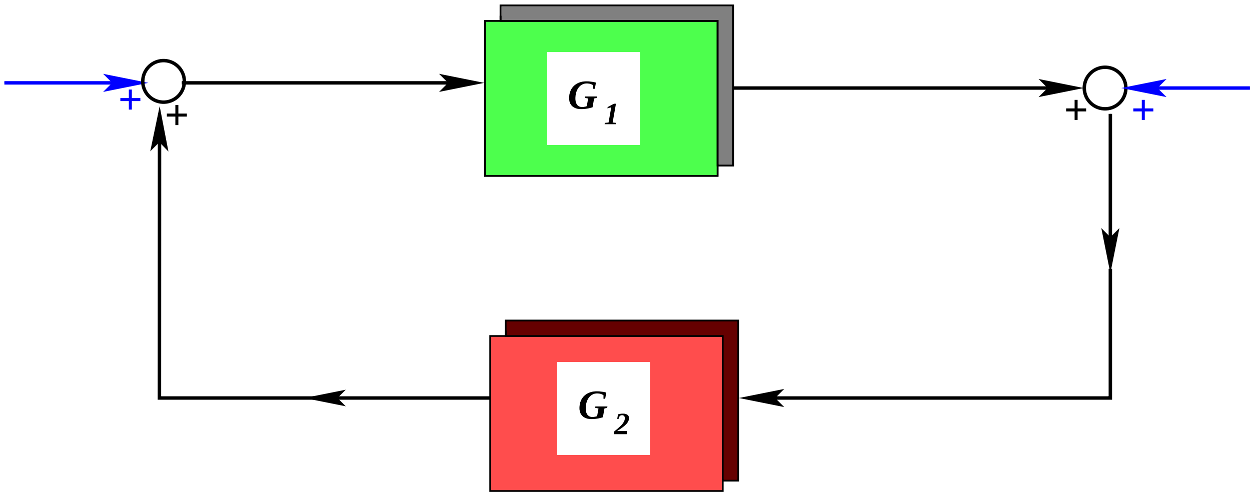

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For simplicity of notation, assume that the signals are scalar.

Represent $G_1 = D_1^{-1}N_1$ **and** $G_2 = D_2^{-1}N_2$ **with** D_1, N_1, D_2, N_2 **proper stable rational.**

Can be shown (Vidyasagar): input/output stability
$\Leftrightarrow N_1N_2 - D_1D_2$ **unimodular over the ring of proper stable rational functions. Bi-proper and miniphase.**

Kucera-Youla type characterization

Given plant, which controllers stabilize?

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Proceed as in the polynomial case: Take kernel representations of $\mathfrak{P}$ and $\mathfrak{P}'$

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All stabilizing controllers (kernel-like representation):

$$C = FP + F'P' \Leftrightarrow C = RP + P' \textbf{ with } R \in \mathbb{R}(\xi)_{\mathscr{S}}$$

Advantages over polynomial case: involves only ring: $\mathbb{R}(\xi)_{\mathscr{S}}$

Model reduction

Unitary representations

It is pedagogically easier to discuss ‘image-like’ representations, hence controllable systems.

Even though it is possible to deal also with ‘kernel-like’ representations. These would only require stabilizability.

Unitary representations

$$w = G\left(\frac{d}{dt}\right)\ell$$

is said to be a unitary representation $:\Leftrightarrow$

$(w,\ell) \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^\bullet)$ **and** $w = G\left(\frac{d}{dt}\right)\ell \quad \Rightarrow$

$$||w||_{\mathscr{L}_2(\mathbb{R},\mathbb{R}^\bullet)} = ||\ell||_{\mathscr{L}_2(\mathbb{R},\mathbb{R}^\bullet)}$$

Easy:

$$\textbf{unitary} \;\Leftrightarrow\; G^\top(-s)G(s) = I \quad \forall s \in \mathbb{C}$$

If in addition G **is stable rational, then norm preserving on** $\mathscr{L}_2(\mathbb{R}_+,\mathbb{R}^\bullet)$.

Unitary representations

A controllable LTID system admits a unitary representation.

Proof: start with any observable representation $w = G\left(\frac{d}{dt}\right)\ell$. Spectral factor

$$G^\top(-s)G(s) = F^\top(-s)F(s).$$

Take $G \to GF^{-1}$. The representation $w = GF^{-1}\left(\frac{d}{dt}\right)\ell$ is unitary. Stability may be added.

This result needs rational symbols - not possible with polynomial models.

Distance between two systems

Usually state space systems

$$\frac{d}{dt}x = Ax + Bu, y = Cx + Du$$

that are moreover stable. Balancing, Hankel norm.

Error bound

$$||G - G_{\text{reduced}}||_{\mathscr{H}_\infty} \leq 2(\textbf{sum of neglected SV's})$$

Is stability needed for model reduction
What can be done with behaviors?

Distance between two systems

In usual input/output approach, the system is (roughly) an input/output map.

Then distance between two systems = induced norm of difference. $\rightsquigarrow \mathscr{H}_\infty$-norms etc.

But this only makes sense if the maps are bounded.
Requires stability!

How do we measure system approximation if a system is given as a behavior?

Distance between two systems

Distance between two LTID behaviors:

Define, for a given $\mathfrak{B}$**, hence** $\subseteq \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$**, the** $\mathscr{L}_2$**-behavior as**

$$\mathfrak{B}_2 = \mathfrak{B} \cap \mathscr{L}_2(\mathbb{R}, \mathbb{R}^{\mathtt{w}}).$$

Easy: $\mathfrak{B}_2$ **is a linear subspace of** $\mathscr{L}_2(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$. **Take closure.**

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Easy: $\mathfrak{B}_2$ is a linear subspace of $\mathscr{L}_2(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$. Take closure.

Define the distance between two controllable LTID behaviors $\mathfrak{B}', \mathfrak{B}''$ as the distance between $\mathfrak{B}_2'$ and $\mathfrak{B}_2''$. $\rightsquigarrow$ distance between 2 closed linear subspaces of $\mathscr{L}_2(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$. Standard notion (Kato): graph metric.

$$d(\mathfrak{B}', \mathfrak{B}'') := ||P_{\mathfrak{B}_2'} - P_{\mathfrak{B}_2''}||$$

where the P's denote the orthogonal projection operators.

Model reduction of behaviors

Consider the LTID $\mathfrak{B}$, controllable (no stability).

Complexity := McMillan degree. Notation: $\mathtt{n}(\mathfrak{B})$.

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Consider the LTID $\mathfrak{B}$, controllable (no stability).

Complexity := McMillan degree. Notation: $\mathtt{n}(\mathfrak{B})$.
This can be defined in many ways. Easiest: dimension of the state space in a minimal state representation of $\mathfrak{B}$

$$\frac{d}{dt}x = Ax + Bw_1, w_2 = Cx + Dw_2, w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}.$$

Model reduction of behaviors

Consider the LTID $\mathfrak{B}$, controllable (no stability).

Complexity := McMillan degree. Notation: $\mathtt{n}(\mathfrak{B})$.

Problem:

Approximate $\mathfrak{B}$ by a LTID $\mathfrak{B}_{\text{reduced}}$ of complexity $\leq \mathtt{k}$ with $\mathtt{k} < \mathtt{n}(\mathfrak{B})$.

Give a bound for $d(\mathfrak{B}, \mathfrak{B}_{\text{reduced}})$ in the graph metric.

Model reduction of behaviors

Algorithm:

1. **Compute a stable unitary representation of $\mathfrak{B}$:**

$$w = G\left(\frac{d}{dt}\right)\ell.$$

   G **is stable!**

2. **Make a balanced reduction of** $G \quad \rightsquigarrow \quad G_{\text{reduced}}$**.**
3. **Define** $\mathfrak{B}_{\text{reduced}}$ **as the system with image-like representation**

$$w = G_{\text{reduced}}\left(\frac{d}{dt}\right)\ell.$$

4. **There holds**

$$d\left(\mathfrak{B}, \mathfrak{B}_{\text{reduced}}\right) \leq 2(\textbf{sum of the neglected SV's})$$

Recapitulation

Conclusion

- **LTID:** $\Sigma = (\mathbb{R}, \mathbb{R}^\bullet, \mathfrak{B})$, $\mathfrak{B} = \ker\left(R\left(\frac{d}{dt}\right)\right)$**,** $R \in \mathbb{R}[\xi]^{\bullet \times \mathtt{w}}$**.**
- **controllability, stabilizability.**
- **Representations: ways to specify** $\mathfrak{B}$**:**
  kernel, image, state space, transfer functions, ...
- **in terms of rational symbols:** $G\left(\frac{d}{dt}\right) w = 0$**, using left co-prime polynomial factorization of** $G \in \mathbb{R}(\xi)^{\bullet \times \mathtt{w}}$**.**

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- **Applications where rational symbols are indispensable:**
  Kucera-Youla parametrization of stabilizing controllers;
  unitary representations and model reduction.

Reference:

JCW and YY

Behaviors defined by rational functions

Linear Algebra and Applications

to appear

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Thank you for your attention