BEHAVIORS

defined by

RATIONAL FUNCTIONS

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Preliminaries

Behaviors & all that

A *dynamical system* $:\Leftrightarrow$ $\Sigma = (\mathbb{T}, \mathbb{W}, \mathscr{B})$

$\mathbb{T} \subseteq \mathbb{R}$ the *time-axis*

$\mathbb{W}$ the *signal space*

$\mathscr{B} \subseteq \mathbb{W}^{\mathbb{T}}$ the *behavior* - a family of trajectories

Behaviors & all that

A *dynamical system* $:\Leftrightarrow$ $\Sigma = (\mathbb{T}, \mathbb{W}, \mathscr{B})$

$\mathbb{T} \subseteq \mathbb{R}$	**the *time-axis***	**today** $\mathbb{T} = \mathbb{R}$
$\mathbb{W}$	**the *signal space***	**today** $\mathbb{W} = \mathbb{R}^{\mathtt{w}}$
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$\mathscr{B} \subseteq \mathbb{W}^{\mathbb{T}}$	**the *behavior***	**- a family of trajectories**

$\Sigma = (\mathbb{R}, \mathbb{R}^{\mathtt{w}}, \mathscr{B})$ **is said to be linear** $:\Leftrightarrow$ $\mathscr{B}$ **is a linear space**

time-invariant $:\Leftrightarrow$ $\mathscr{B}$ **is shift-invariant**

$w \in \mathscr{B}$ **and** $t \in \mathbb{R} \quad \Rightarrow \quad \sigma^t w \in \mathscr{B}$

σ^t **denotes the 'shift':** $\quad (\sigma^t w)(t') = w(t' + t)$

differential $:\Leftrightarrow$ $\mathscr{B}$ **is the set of sol'ns of an ODE**

LTID systems

Linear, time-invariant, differential dynamical system $\Leftrightarrow$

$$R_0 w + R_1 \frac{d}{dt} w + R_2 \frac{d^2}{dt^2} w + \cdots + R_L \frac{d^L}{dt^L} w = 0$$

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Short-hand notation: introduce polynomial matrix

$$R(\xi) = R_0 + R_1 \xi + R_2 \xi^2 + \cdots + R_L \xi^L \in \mathbb{R}[\xi]^{\bullet \times \mathtt{w}}$$

$$R\left(\frac{d}{dt}\right) w = 0$$

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$$\boxed{R\left(\frac{d}{dt}\right) w = 0}$$

Behavior := all solutions, i.e.

$$\boxed{\mathscr{B} = \left\{ w \in \mathscr{C}^{\infty}\left(\mathbb{R}, \mathbb{R}^{\mathtt{w}}\right) \mid R\left(\frac{d}{dt}\right) w = 0 \right\}}$$

$\mathscr{B} = \text{kernel}\left(R\left(\frac{d}{dt}\right)\right)$ **'kernel representation', polynomial type.**

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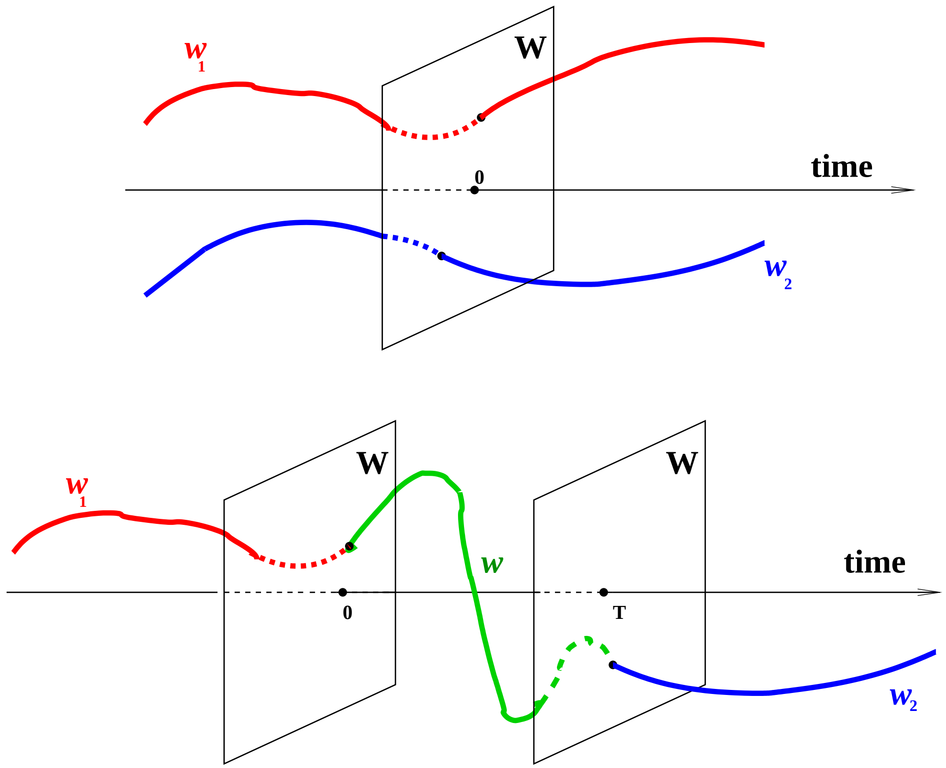
Controllability and stabilizability

Let $\Sigma = (\mathbb{R}, \mathbb{R}^{\mathtt{w}}, \mathscr{B})$ **be a time-invariant dynamical system**

Σ **is said to be controllable** $:\Leftrightarrow$

$\forall\, w_1, w_2 \in \mathscr{B},\ \exists\, T \geq 0,$ **and** $w \in \mathscr{B}$ **such that ...**



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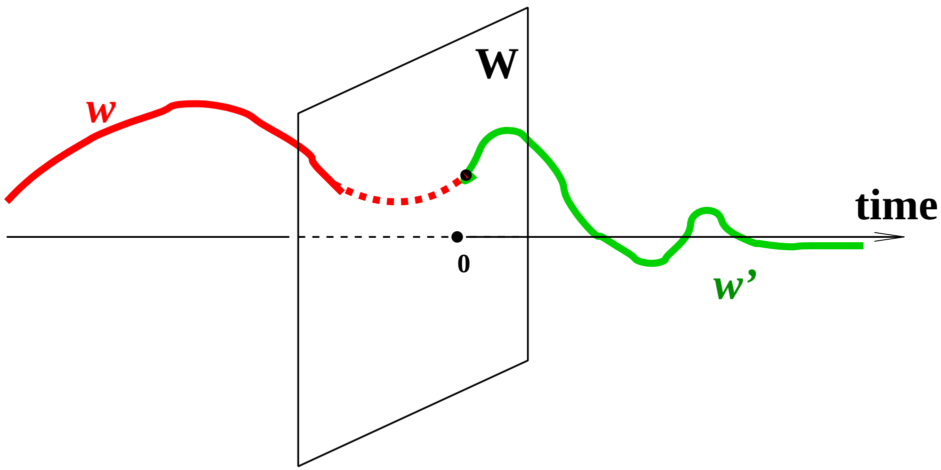

Controllability and stabilizability

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Σ is said to be controllable $:\Leftrightarrow$

Σ is said to be stabilizable $:\Leftrightarrow$

Theorem: $R\left(\frac{d}{dt}\right)w = 0$ **defines** **a controllable system** $\Leftrightarrow$

$\mathrm{rank}\,(R\,(\lambda))$ **is the same** $\forall\,\lambda \in \mathbb{C}$

a stabilizable system $\Leftrightarrow$

$\mathrm{rank}\,(R\,(\lambda))$ **is the same** $\forall\,\lambda \in \mathbb{C}$ **with real part** ≥ 0

Rational representations

Rational representations

Let $G \in \mathbb{R}(\xi)^{\bullet \times \mathtt{w}}$, and consider the 'differential equation'

$$\boxed{G\left(\frac{d}{dt}\right) w = 0}$$

What do we mean by the solutions, i.e. by the behavior?

Rational representations

Let $G \in \mathbb{R}(\xi)^{\bullet \times \mathtt{w}}$, and consider the 'differential equation'

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What do we mean by the solutions, i.e. by the behavior?

Let (P, Q) be a left coprime polynomial factorization of G

i.e. $P, Q \in \mathbb{R}[\xi]^{\bullet \times \bullet}, \det(P) \neq 0, G = P^{-1}Q, [P \vdots Q]$ **left-prime.**

$$G(\frac{d}{dt})w = 0 :\Leftrightarrow \boxed{Q\left(\frac{d}{dt}\right) w = 0}$$

E.g., in scalar case, means P and Q have no common roots.

Rational representations

Let (P,Q) be a left coprime polynomial factorization of G

$$G(\frac{d}{dt})w = 0 :\Leftrightarrow \boxed{Q\left(\frac{d}{dt}\right)w = 0}$$

Justification:

1. G proper. $G(s) = C(Is-A)^{-1}B+D$ controllable realization. Consider output nulling inputs:

$$\frac{d}{dt}x = Ax+Bw, \ \ 0 = Cx+Dw$$

This set of w's are exactly those that satisfy $G\left(\frac{d}{dt}\right)w = 0$.

Same for $\frac{d}{dt}x = Ax+Bw, 0 = Cx+D\left(\frac{d}{dt}\right)w = 0$, $\ \ D \in \mathbb{R}[\xi]^{\bullet\times\bullet}$.

Rational representations

Let (P,Q) be a left coprime polynomial factorization of G

$$G(\frac{d}{dt})w = 0 :\Leftrightarrow \boxed{Q\left(\frac{d}{dt}\right)w = 0}$$

Justification:

2. Consider $y = G(s)u$. View G as a transfer f'n.
Take your usual favorite definition of input/output pairs.

The output nulling inputs are exactly those that satisfy
$G\left(\frac{d}{dt}\right)w = 0$.

3. via Laplace transforms...

$G\left(\frac{d}{dt}\right)$ is not a map!

Consider

$$y = G\left(\frac{d}{dt}\right)u$$

We now know what it means that $(u,y) \in \mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^\bullet)$ **satisfies this 'ODE'.**

Is there a unique y **for a given** u**?**

$$P(\frac{d}{dt})y = Q(\frac{d}{dt})u$$

If $P \neq I$ **(better, not unimodular), there are many sol'ns** y **of this ODE for a given RHS.**

Representations

Linear time-invariant differential systems $\Sigma = (\mathbb{R}, \mathbb{R}^{\mathtt{w}}, \mathscr{B})$**.**
$\mathscr{B} = \text{kernel}\left(R\left(\frac{d}{dt}\right)\right)$ **for some** $R \in \mathbb{R}[\xi]^{\bullet\times\mathtt{w}}$ **by definition.**

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But we may as well take the representation $G\left(\frac{d}{dt}\right)w = 0$ **for some** $G \in \mathbb{R}(\xi)^{\bullet\times\mathtt{w}}$ **as the definition.**
R**: all poles at** ∞**, we can take** G **with no poles at** ∞**, or more generally with all poles in some non-empty set - symmetric w.r.t.** $\mathbb{R}$**. In particular:**

Theorem: Every linear time-invariant differential systems has a representation

$$\boxed{G\left(\frac{d}{dt}\right)w = 0}$$

with $G \in \mathbb{R}(\xi)^{\bullet\times\mathtt{w}}$ **strictly proper stable rational.**
Proof: Take $G(s) = \frac{R(s)}{(s+\lambda)^{\mathtt{n}}}$**, suitable** $\lambda \in \mathbb{R}, \mathtt{n} \in \mathbb{N}$**.**

Matrices of rational functions

Subrings of $\mathbb{R}(\xi)$

$\mathbb{R}(\xi)$: **real rational functions.**

Consider 3 subrings:

1. $\mathbb{R}[\xi]$: **polynomials with real coefficients**
2. $\mathbb{R}(\xi)_{\mathscr{P}}$: **proper rational functions**
3. $\mathbb{R}(\xi)_{\mathscr{S}}$: **stable proper rational functions**

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   no poles in RHP or ∞

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no poles in RHP or ∞

Each of these rings has $\mathbb{R}(\xi)$ as its field of fractions.

Unimodular elements (invertible in ring)

1. **Non-zero constants**
2. **bi-proper**
3. **bi-proper and mini-phase**

miniphase:$\Leftrightarrow$ poles & zeros in LHP

Matrices over these rings

$\mathbb{R}(\xi)^{\bullet\times\bullet}$**: matrices of real rational functions.**

1. $\mathbb{R}[\xi]^{\bullet\times\bullet}$**: polynomial matrices with real coefficients**
2. $\mathbb{R}(\xi)_{\mathscr{P}}^{\bullet\times\bullet}$**: matrices of proper rational functions**
3. $\mathbb{R}(\xi)_{\mathscr{S}}^{\bullet\times\bullet}$**: of stable proper rational functions**

Matrices over these rings

$\mathbb{R}(\xi)^{\bullet\times\bullet}$: **matrices of real rational functions.**

1. $\mathbb{R}[\xi]^{\bullet\times\bullet}$: **polynomial matrices with real coefficients**
   unimodular: square & determinant = non-zero constant
2. $\mathbb{R}(\xi)_{\mathscr{P}}^{\bullet\times\bullet}$: **matrices of proper rational functions**
   unimodular: square & determinant biproper
3. $\mathbb{R}(\xi)_{\mathscr{S}}^{\bullet\times\bullet}$: **of stable proper rational functions**
   unimodular: square & determinant biproper and miniphase (poles & zeros in LHP)

Prime elements

$M \in \mathbb{R}[\xi]^{\mathtt{n}_1 \times \mathtt{n}_2}$ **is left-prime** $:\Leftrightarrow$
$M = FM', F \in \mathbb{R}[\xi]^{\mathtt{n}_1 \times \mathtt{n}_1}, M' \in \mathbb{R}[\xi]^{\mathtt{n}_1 \times \mathtt{n}_2}$
$\Rightarrow U$ **is uni-modular**

Prime elements

$M \in \mathbb{R}[\xi]^{\mathtt{n}_1 \times \mathtt{n}_2}$ **is left-prime over** $\mathbb{R}[\xi]$ $:\Leftrightarrow$

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$M \in \mathbb{R}(\xi)_{\mathscr{P}}^{\mathtt{n}_1 \times \mathtt{n}_2}$ **is left-prime over** $\mathbb{R}(\xi)_{\mathscr{P}}$ $:\Leftrightarrow$
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$M \in \mathbb{R}(\xi)_{\mathscr{S}}^{\mathtt{n}_1 \times \mathtt{n}_2}$ **is left-prime over** $\mathbb{R}(\xi)_{\mathscr{S}}$ $:\Leftrightarrow$

$M = FM', F \in \mathbb{R}(\xi)_{\mathscr{S}}^{\mathtt{n}_1 \times \mathtt{n}_1}, M' \in \mathbb{R}(\xi)_{\mathscr{P}}^{\mathtt{n}_1 \times \mathtt{n}_2}$

$\Rightarrow U$ **is uni-modular over** $\mathbb{R}(\xi)_{\mathscr{S}}$

Prime representations & system properties

Prime representations

Theorem: an LTI differential system admits a representation

$$G\left(\frac{d}{dt}\right)w = 0$$

with

1. $G \in \mathbb{R}(\xi)_{\mathscr{P}}^{\bullet \times \mathtt{w}}$ **left prime over** $\mathbb{R}(\xi)_{\mathscr{P}}$ **always**
2. $G \in \mathbb{R}[\xi]^{\bullet \times \mathtt{w}}$ **left prime over** $\mathbb{R}[\xi] \Leftrightarrow$ **it is controllable**
3. $G \in \mathbb{R}(\xi)_{\mathscr{S}}^{\bullet \times \mathtt{w}}$ **left prime over** $\mathbb{R}(\xi)_{\mathscr{S}} \Leftrightarrow$ **it is stabilizable**

The proof of case 3 is not easy!

Image-like representations

see my website

Elimination

Consider

$$G_1\left(\frac{d}{dt}\right)w_1 = G_2\left(\frac{d}{dt}\right)w_2$$

$G_1, G_2 \in \mathbb{R}(\xi)^{\bullet\times\bullet}$. **Behavior** $\mathscr{B}$. **Eliminate** $w_2 \rightsquigarrow$

$$\mathscr{B}_1 = \{w_1 \mid \exists\, w_2 \textbf{ such that } (w_1, w_2) \in \mathscr{B}\}$$

Then $\mathscr{B}_1$ **is also a LTID behavior.**

In particular

$$w = H\left(\frac{d}{dt}\right)\ell, \quad H \in \mathbb{R}(\xi)^{\mathtt{w}\times\bullet}.$$

w**-behavior is LTID. Image-like representation.**

Representations of controllable systems

Theorem: **The following are equivalent for LTID systems**

1. $\mathscr{B}$ **is controllable**
2. $\mathscr{B}$ **admits an image-like representation**

$$\boxed{w = M\left(\frac{d}{dt}\right)\ell} \text{ with } H \in \mathbb{R}[\xi]^{\mathtt{w}\times\bullet}$$

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4. **with observability (**ℓ **can be deduced from** w**) added**
5. **with** $M \in \mathbb{R}[\xi]^{\mathtt{w}\times\bullet}$ **right prime over** $\mathbb{R}[\xi]$
6. **with** $H \in \mathbb{R}(\xi)^{\mathtt{w}\times\bullet}_{\mathscr{S}}$ **right prime over** $\mathbb{R}(\xi)_{\mathscr{S}}$

Relations with classical results

Consider system $y = Gu, \quad G \in \mathbb{R}(\xi)^{\mathtt{p} \times \mathtt{m}}$ **'transfer function'**

Interpret this as

$$y = G\left(\frac{d}{dt}\right)u$$

Automatically controllable!

Only controllable systems covered by tf. f'ns.

Even if G **is i/o unstable or improper,** $\exists$ **stable kernel- and image-like representations!**

Relations with classical results

$$y = G\left(\frac{d}{dt}\right)u$$

Even if G is i/o unstable or improper, $\exists$ stable kernel- and image-like representations!

$$G_1\left(\frac{d}{dt}\right)y = G_2\left(\frac{d}{dt}\right)u,$$

$\begin{bmatrix} G_1 & \vdots & G_2 \end{bmatrix} \in \mathbb{R}(\xi)_{\mathscr{S}}^{\bullet\times\bullet}$ **left prime over** $\mathbb{R}(\xi)_{\mathscr{S}}$.

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$$\begin{bmatrix} u \\ y \end{bmatrix} = \begin{bmatrix} H_1\left(\frac{d}{dt}\right) \\ H_2\left(\frac{d}{dt}\right) \end{bmatrix} \ell,$$

$\begin{bmatrix} H_1 \\ H_2 \end{bmatrix} \in \mathbb{R}(\xi)_{\mathscr{S}}^{\bullet\times\bullet}$ **right prime over** $\mathbb{R}(\xi)_{\mathscr{S}}^{\bullet\times\bullet}$.

Relations with classical results

$$y = G\left(\frac{d}{dt}\right)u$$

$$\boxed{G = G_1^{-1}G_2 = H_2H_1^{-1}}$$

left/right co-prime factorizations over $\mathbb{R}(\xi)_{\mathscr{S}}$. As over $\mathbb{R}[\xi]$.

Classical, but we obtain the representation

$$G_1\left(\frac{d}{dt}\right)y = G_2\left(\frac{d}{dt}\right)u,$$

with $\begin{bmatrix} G_1 & \vdots & G_2 \end{bmatrix} \in \mathbb{R}(\xi)_{\mathscr{S}}^{\bullet\times\bullet}$ left prime over $\mathbb{R}(\xi)_{\mathscr{S}}$
also for stabilzable systems, instead of only controllable ones.

Why bother with rational rather than just polynomial ‘symbols’?

Why bother with rational rather than just polynomial ‘symbols’?

1. **Parametrization of all stabilizing controllers**
2. **Model reduction of behavioral systems**

Unitary representations

It is pedagogically easier to discuss 'image-like' representations, hence controllable systems.

Even though it is possible to deal also with 'kernel-like' representations. These would only require stabilizability.

Unitary representations

$$w = G\left(\frac{d}{dt}\right)\ell$$

is said to be a unitary representation $:\Leftrightarrow$

$(w,\ell) \in \mathscr{C}^\infty\left(\mathbb{R},\mathbb{R}^\bullet\right)$ **and** $w = G\left(\frac{d}{dt}\right)\ell \quad \Rightarrow$

$$||w||_{\mathscr{L}_2(\mathbb{R},\mathbb{R}^\bullet)} = ||\ell||_{\mathscr{L}_2(\mathbb{R},\mathbb{R}^\bullet)}$$

Easy:

$$\textbf{unitary} \;\Leftrightarrow\; G^\top(-s)G(s) = I \quad \forall s \in \mathbb{C}$$

If in addition G **is stable rational, then norm preserving on** $\mathscr{L}_2\left(\mathbb{R}_+,\mathbb{R}^\bullet\right)$.

Unitary representations

A controllable LTID system admits a unitary representation.

Proof: start with any observable representation $w = G\left(\frac{d}{dt}\right)\ell$**.**
Spectral factor

$$G^\top(-s)G(s) = F^\top(-s)F(s).$$

Take $G \rightarrow GF^{-1}$**. The representation** $w = GF^{-1}\left(\frac{d}{dt}\right)\ell$ **is unitary. Stability may be added.**

This result needs rational symbols - not possible with polynomial models.

Distance between two systems

Usually state space systems

$$\frac{d}{dt}x = Ax + Bu, y = Cx + Du$$

that are moreover stable. Balancing, Hankel norm.

Error bound

$$||G - G_{\text{reduced}}||_{\mathscr{H}_\infty} \leq 2(\textbf{sum of neglected SV's})$$

Is stability needed for model reduction
What can be done with behaviors?

Distance between two systems

In usual input/output approach, the system is (roughly) an input/output map.

Then distance between two systems = induced norm of difference. $\rightsquigarrow \mathscr{H}_\infty$**-norms etc.**

But this only makes sense if the maps are bounded.
Requires stability!

How do we measure system approximation if a system is given as a behavior?

Distance between two systems

Distance between two LTID behaviors:

Define, for a given $\mathscr{B}$, hence $\subseteq \mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$, the $\mathscr{L}_2$-behavior as

$$\mathscr{B}_2 = \mathscr{B} \cap \mathscr{L}_2(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) .$$

Easy: $\mathscr{B}_2$ is a linear subspace of $\mathscr{L}_2(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$. Take closure.

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Define the distance between two controllable LTID behaviors $\mathscr{B}', \mathscr{B}''$ as the distance between $\mathscr{B}'_2$ and $\mathscr{B}''_2$. $\rightsquigarrow$ distance between 2 closed linear subspaces of $\mathscr{L}_2(\mathbb{R},\mathbb{R}^{\mathtt{w}})$. Standard notion (Kato): graph metric.

$$\boxed{d(\mathscr{B}',\mathscr{B}'') := ||P_{\mathscr{B}'_2} - P_{\mathscr{B}''_2}||}$$

where the P's denote the orthogonal projection operators.

Model reduction of behaviors

Consider the LTID $\mathscr{B}$, controllable (no stability).

Complexity := McMillan degree. Notation: $\mathtt{n}(\mathscr{B})$.

Model reduction of behaviors

Consider the LTID $\mathscr{B}$, controllable (no stability).

Complexity := McMillan degree. Notation: $\mathtt{n}(\mathscr{B})$.
This can be defined in many ways. Easiest: dimension of the state space in a minimal state representation of $\mathscr{B}$

$$\frac{d}{dt}x = Ax + Bw_1, w_2 = Cx + Dw_2, w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}.$$

Model reduction of behaviors

Consider the LTID $\mathscr{B}$, controllable (no stability).

Complexity := McMillan degree. Notation: $\mathtt{n}(\mathscr{B})$.

Problem:

Approximate $\mathscr{B}$ by a LTID $\mathscr{B}_{\text{reduced}}$ of complexity $\leq \mathtt{k}$ with $\mathtt{k} < \mathtt{n}(\mathscr{B})$.

Give a bound for $d(\mathscr{B}, \mathscr{B}_{\text{reduced}})$ in the graph metric.

Model reduction of behaviors

Algorithm:

1. **Compute a stable unitary representation of $\mathscr{B}$:**

$$w = G\left(\frac{d}{dt}\right)\ell.$$

   G **is stable!**

2. **Make a balanced reduction of** $G \quad \rightsquigarrow \quad G_{\text{reduced}}$.
3. **Define** $\mathscr{B}_{\text{reduced}}$ **as the system with image-like representation**

$$w = G_{\text{reduced}}\left(\frac{d}{dt}\right)\ell.$$

4. **There holds**

$$d\left(\mathscr{B}, \mathscr{B}_{\text{reduced}}\right) \leq 2(\textbf{sum of the neglected SV's})$$

Recapitulation

Conclusion

- **LTID:** $\Sigma = (\mathbb{R}, \mathbb{R}^{\bullet}, \mathscr{B})\,, \mathscr{B} = \text{kernel}\left(R\left(\frac{d}{dt}\right)\right)$**,** $R \in \mathbb{R}[\xi]^{\bullet \times \mathtt{w}}$**.**
- **controllability, stabilizability.**
- **Representations: ways to specify** $\mathscr{B}$**:**
  kernel, image, state space, transfer functions, ...
- **in terms of rational symbols:** $G\left(\frac{d}{dt}\right) w = 0$**, using left co-prime polynomial factorization of** $G \in \mathbb{R}(\xi)^{\bullet \times \mathtt{w}}$**.**

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- **Left prime representations: over** $\mathbb{R}[\xi] \Leftrightarrow$ **controllable, over proper stable rational** $\Leftrightarrow$ **stabilizable.**

Conclusion

- **LTID:** $\Sigma = (\mathbb{R}, \mathbb{R}^{\bullet}, \mathscr{B})$, $\mathscr{B} = \mathrm{kernel}\left(R\left(\frac{d}{dt}\right)\right)$**,** $R \in \mathbb{R}[\xi]^{\bullet \times \mathtt{w}}$**.**
- **controllability, stabilizability.**
- **Representations: ways to specify** $\mathscr{B}$**:**
  kernel, image, state space, transfer functions, ...
- **in terms of rational symbols:** $G\left(\frac{d}{dt}\right) w = 0$**, using left co-prime polynomial factorization of** $G \in \mathbb{R}(\xi)^{\bullet \times \mathtt{w}}$**.**
- **Left prime representations: over** $\mathbb{R}[\xi] \Leftrightarrow$ **controllable,**
  over proper stable rational $\Leftrightarrow$ **stabilizable.**
- **Applications where rational symbols are indispensable:**
  Kucera-Youla parametrization of stabilizing controllers;
  unitary representations and model reduction.

Reference:

JCW and YY

Behaviors defined by rational functions

Linear Algebra and Applications

to appear

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Thank you for your attention