Open Problem Session

When is a linear system optimal?

Jan C. Willems

K.U. Leuven

Oberwolfach Tagung Regelungstheorie **March 1, 2005**

Re: the title

Plagiarism is the greatest form of compliment

Notation

Consider the QDF

$$w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \mapsto \sum_{\mathtt{k},\ell} \left(\frac{d^{\mathtt{k}}}{dt^{\mathtt{k}}} w\right)^\top \Phi_{\mathtt{k},\ell} \left(\frac{d^{\mathtt{k}}}{dt^{\ell}} w\right)$$

$\Phi_{\mathtt{k},\ell} = \Phi_{\ell,\mathtt{k}}^\top \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$. **Introduce**

$$\Phi(\zeta, \eta) := \sum_{\mathtt{k},\ell} \Phi_{\mathtt{k},\ell} \zeta^{\mathtt{k}} \eta^{\ell},$$

and denote the QDF by $Q_\Phi(w)$.

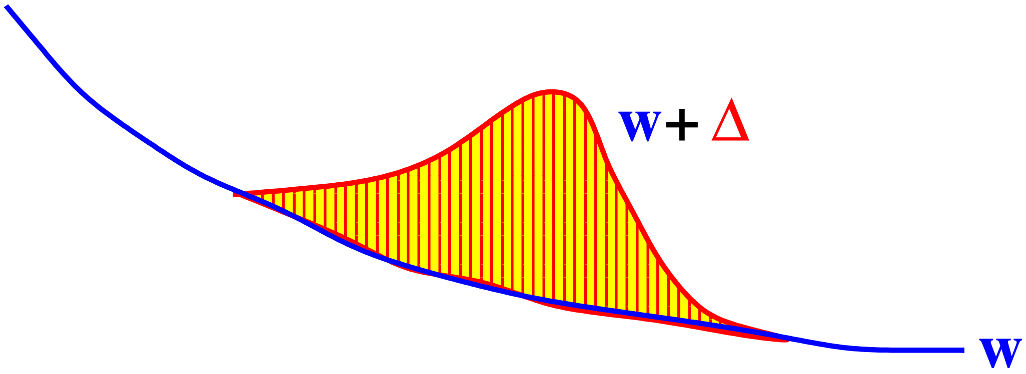
Q_Φ is like a Lagrangian.

Optimality

$w \in \mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$ is an **optimal trajectory** if

$$\int_{-\infty}^{+\infty} \left(\mathbf{Q}_{\Phi}(w + \Delta) - Q_{\Phi}(w) \right) dt \geq 0$$

for all $\Delta \in \mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$ with compact support.



Optimal behavior

Stationarity: $\Leftrightarrow$

$$\Phi(-\tfrac{d}{dt}, \tfrac{d}{dt})w = 0$$

Minimality: $\Leftrightarrow$ in addition

$$\Phi(-i\omega, i\omega) \geq 0 \quad \text{for all } \omega \in \mathbb{R}$$

Note $\Phi(-\frac{d}{dt}, \frac{d}{dt}) = \Phi^{\top}(\frac{d}{dt}, -\frac{d}{dt})$

Importance

Opens up the possibility of describing a behavior very effectively by a single function Q_Φ:

The behavior consists of the trajectories $w : \mathbb{R} \to \mathbb{R}^{\mathtt{w}}$ than minimize, or render stationary, $\int_{-\infty}^{+\infty} Q_\Phi(w)\, dt$.

Open Problem

Consider

$$R(\frac{d}{dt})w = 0.$$

Denote its *behavior* by $\mathfrak{B}$.

Open Problem: ***When is $\mathfrak{B}$ an optimal behavior?***

i.e., Given $\mathfrak{B}$, ¿ $\exists \ \Phi$, with
$\Phi(-i\omega, i\omega) \geq 0$ for all $\omega \in \mathbb{R}$, such that

$$\Phi(-\frac{d}{dt}, \frac{d}{dt})w = 0$$

has also the given behavior $\mathfrak{B}$?

Open Problem

Consider

$$R(\frac{d}{dt})w = 0.$$

Denote its *behavior* by $\mathfrak{B}$.

Open Problem: ***When is $\mathfrak{B}$ an optimal behavior?***

Sufficient:

$[R(\xi) = R^\top(-\xi)] \wedge [R(i\omega) \geq 0$ for all $\omega \in \mathbb{R}]$.

Necessary (autonomous case) and Sufficient:

Given $\exists\ U$ unimodular, such that UR has these properties.

But, we are looking for conditions on $\mathfrak{B}$!

$\mathtt{w} = 1$

The scalar case $\mathtt{w} = 1$ is easy, but not uninteresting.

$$\boldsymbol{R}(\frac{\boldsymbol{d}}{\boldsymbol{dt}})\boldsymbol{w} = 0, \qquad \boldsymbol{R} \in \mathbb{R}[\boldsymbol{\xi}]$$

is stationary iff $R(\xi) = R(-\xi)$, i.e., R is even.

The scalar case $\mathtt{w} = 1$ is easy, but not uninteresting.

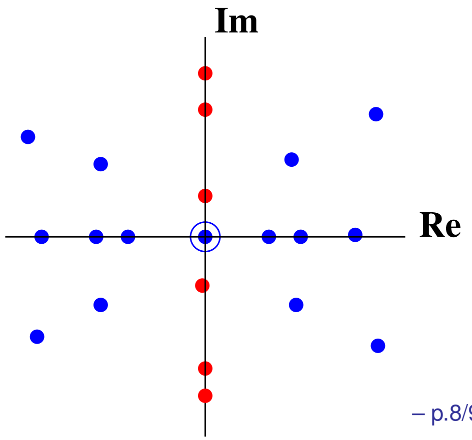
$$R(\frac{d}{dt})w = 0, \qquad R \in \mathbb{R}[\xi]$$

is stationary iff $R(\xi) = R(-\xi)$, i.e., R is even.

Equivalently, iff $\mathfrak{B}$ is

1. time-reversible :=$[w(t) \in \mathfrak{B}] \Leftrightarrow [w(-t) \in \mathfrak{B}]$
2. and of even dimension.

Root pattern:



$\mathtt{w} = 1$

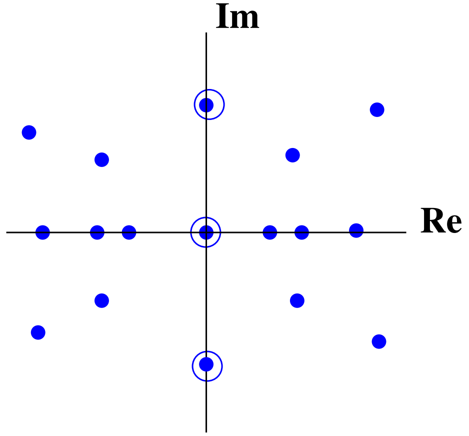
The scalar case $\mathtt{w} = 1$ is easy, but not uninteresting.

$$R(\frac{d}{dt})w = 0, \qquad R \in \mathbb{R}[\xi]$$

is stationary iff $R(\xi) = R(-\xi)$, i.e., R is even.

optimal behavior iff imaginary roots even multiplicity.

Root pattern:



Time-reversible, even dimension, non-oscillatory.

Conclusion

Optimality $\Rightarrow$ non-constant trajectories unbounded.

If, as young Leibniz claimed,

ours is the best of all possible worlds,

there was a Big Bang, and it will end as a Supernova...

Conclusion

Optimality $\Rightarrow$ non-constant trajectories unbounded.

If, as young Leibniz claimed,

ours is the best of all possible worlds,

there was a Big Bang, and it will end as a Supernova...

What $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ are optimal?

Please hand in solutions by noon on Thursday!

Conclusion

Optimality $\Rightarrow$ non-constant trajectories unbounded.

If, as young Leibniz claimed,

ours is the best of all possible worlds,

there was a Big Bang, and it will end as a Supernova...

What $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ are optimal?

Please hand in solutions by noon on Thursday!

Thank you