DISSIPATIVE DISTRIBUTED SYSTEMS

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Lyapunov functions

Lyapunov functions

Consider the classical dynamical system, the ***'flow'***

$$\Sigma : \frac{d}{dt}x = f\,(x)$$

with $x \in \mathbb{X} = \mathbb{R}^{\mathrm{n}}$ **the** ***state*** **and** $f : \mathbb{X} \to \mathbb{X}$ **the** ***vectorfield*****.**

Denote the set of solutions $x : \mathbb{R} \to \mathbb{X}$ **by** $\mathfrak{B}$**, the** ***'behavior'.***

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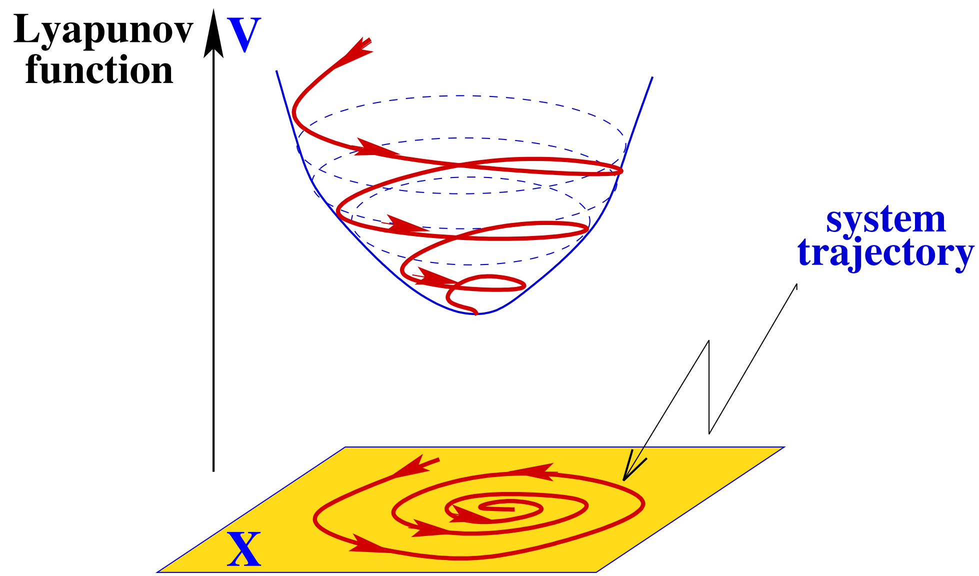
$$V : \mathbb{X} \to \mathbb{R}$$

is said to be a *Lyapunov function* for Σ if along $x \in \mathfrak{B}$

$$\frac{d}{dt} V(x(\cdot)) \leq 0$$

Equivalently, if $\overset{\bullet}{V}^{\Sigma} := \nabla V \cdot f \leq 0.$

Typical Lyapunov theorem



$$V(x) > 0 \text{ and } \overset{\bullet}{V}^{\Sigma}(x) < 0 \text{ for } 0 \neq x \in \mathbb{X}$$

$$\Rightarrow$$

$$\forall\ x \in \mathfrak{B},\ \text{there holds } x(t) \to 0 \text{ for } t \to \infty$$ **'global stability'**

Lyapunov

Lyapunov f'ns play a remarkably central role in the field.

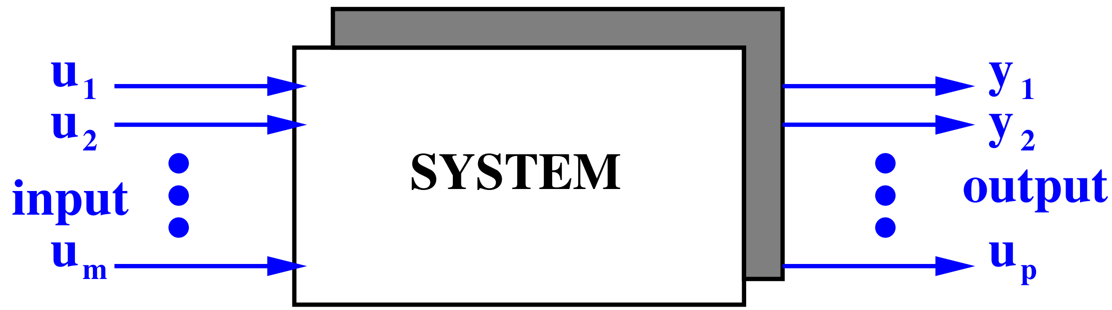
Aleksandr Mikhailovich Lyapunov (1857-1918)

Introduced Lyapunov's 'second method' in his thesis (1899).

Dissipative systems

Open systems

'Open' systems are a much more appropriate starting point for the study of dynamics. For example,



$\rightsquigarrow$ **the dynamical system**

$$\Sigma: \quad \frac{d}{dt}x = f(x,u), \quad y = h(x,u).$$

$u \in \mathbb{U} = \mathbb{R}^{\mathtt{m}}, y \in \mathbb{Y} = \mathbb{R}^{\mathtt{p}}, x \in \mathbb{X} = \mathbb{R}^{\mathtt{n}}$: **input, output, state.**

Behavior $\mathfrak{B} =$ **all sol'ns** $(u,y,x): \mathbb{R} \to \mathbb{U} \times \mathbb{Y} \times \mathbb{X}$.

Dissipative dynamical systems

Let $s : \mathbb{U} \times \mathbb{Y} \to \mathbb{R}$ **be a function, called the *supply rate*.**

Σ is said to be *dissipative* w.r.t. the supply rate s if $\exists$

$$V : \mathbb{X} \to \mathbb{R},$$

called the *storage function*, such that

$$\frac{d}{dt} V(x(\cdot)) \leq s(u(\cdot), y(\cdot))$$

$\forall \ (u(\cdot), y(\cdot), x(\cdot)) \in \mathfrak{B}$.

Dissipation inequality

$$\frac{d}{dt} V(x(\cdot)) \leq s(u(\cdot), y(\cdot))$$

$\forall \quad (u(\cdot), y(\cdot), x(\cdot)) \in \mathfrak{B}$.

This inequality is called the *dissipation inequality.*

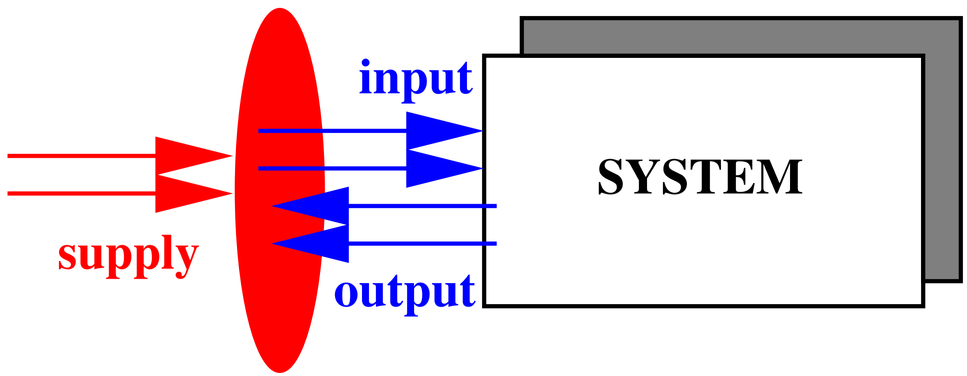
Equivalent to

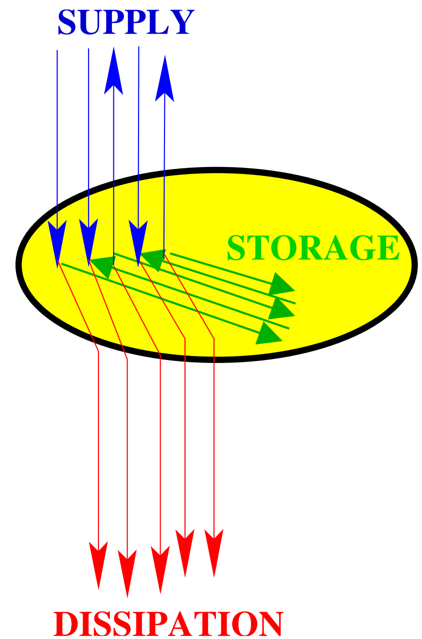
$$\overset{\bullet}{V}^{\Sigma}(x,u) := \nabla V(x) \cdot f(x,u) \leq s(u, h(x,u))$$

for all $(u,x) \in \mathbb{U} \times \mathbb{X}$.

If equality holds: 'conservative' system.

Dissipation inequality





$s(u, y)$ models something like the power delivered to the system when the input value is u and output value is y.

$V(x)$ then models the internally stored energy.

Dissipativity $:\Leftrightarrow$

rate of increase of internal energy $\leq$ power delivered.

Dissipation inequality

<u>Special case</u>: 'closed' system: $s = 0$ then

dissipativeness $\leftrightarrow$ V is a Lyapunov function.

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Stability for closed systems $\simeq$ Dissipativity for open systems.

The construction of storage functions

Basic question:

Given (a representation of) Σ, the dynamics,
and given s, the supply rate,
is the system dissipative w.r.t. s, i.e.
does there exist a storage function V such that
the dissipation inequality holds?

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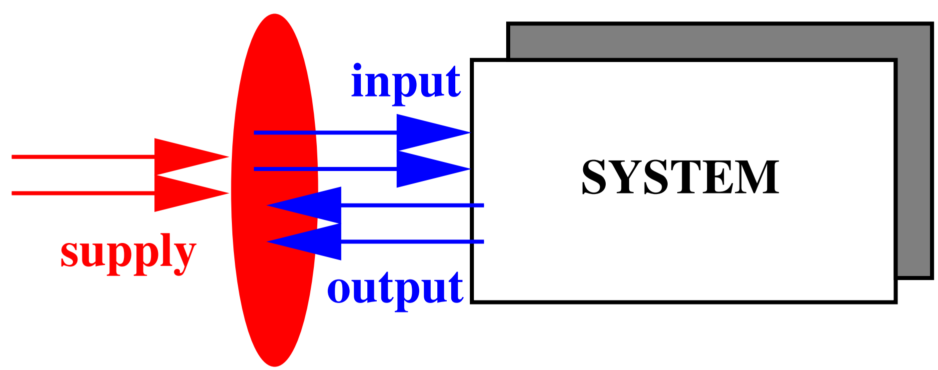

Monitor power in, known dynamics, what is the stored energy?

The construction of storage functions

The construction of storage f'ns is very well understood, particularly for finite dimensional linear systems and quadratic supply rates.

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Leads to the KYP-lemma, LMI's, ARIneq, ARE, semi-definite programming, spectral factorization, Lyapunov functions, $\mathcal{H}_\infty$ and robust control, positive and bounded real functions, electrical circuit synthesis, stochastic realization theory.

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The storage function V is in general far from unique. There are two 'canonical' storage functions:

the available storage and the required supply.

For conservative systems, V is unique.

Dissipative systems

Dissipative systems and storage functions play a remarkably central role in the field.

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The construction of storage functions is the question which we shall discuss today for systems described by PDE's.

PDE's

PDE's: polynomial notation

Consider, for example, the PDE:

$$\begin{aligned} w_1(x_1,x_2) + \frac{\partial^2}{\partial x_2^2} w_1(x_1,x_2) + \frac{\partial}{\partial x_1} w_2(x_1,x_2) &= 0 \\ w_2(x_1,x_2) + \frac{\partial^3}{\partial x_2^3} w_1(x_1,x_2) + \frac{\partial^4}{\partial x_1^4} w_2(x_1,x_2) &= 0 \end{aligned}$$

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$$\updownarrow$$

Notation:

$$\xi_1 \leftrightarrow \frac{\partial}{\partial x_1},\ \xi_2 \leftrightarrow \frac{\partial}{\partial x_2}, w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix},\ R(\xi_1,\xi_2) = \begin{bmatrix} 1+\xi_2^2 & \xi_1 \\ \xi_2^3 & 1+\xi_1^4 \end{bmatrix}$$

$$\boxed{R\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right) w = 0.}$$

Linear differential distributed ($\mathtt{n}$-D) systems

$\mathbb{T} = \mathbb{R}^{\mathtt{n}}$, **the set of independent variables,**

typically $\mathtt{n} = 4$**: time and space,**

$\mathbb{W} = \mathbb{R}^{\mathtt{w}}$, **the set of dependent variables,**

$\mathfrak{B} =$ **the solutions of a linear constant coefficient PDE.**

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$\mathfrak{B}$ = the solutions of a linear constant coefficient PDE.

Let $R \in \mathbb{R}^{\bullet \times \mathtt{w}}[\xi_1, \cdots, \xi_{\mathtt{n}}]$, and consider

$$R\left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathtt{n}}}\right) w = 0. \quad (*)$$

Define the associated behavior

$$\mathfrak{B} = \{ w \in \mathfrak{C}^{\infty}(\mathbb{R}^{\mathtt{n}}, \mathbb{R}^{\mathtt{w}}) \mid (*) \text{ holds } \}.$$

Notation for n-D linear differential systems:

$$(\mathbb{R}^{\mathtt{n}}, \mathbb{R}^{\mathtt{w}}, \mathfrak{B}) \in \mathfrak{L}^{\mathtt{w}}_{\mathtt{n}}, \quad \text{or } \mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\mathtt{n}}.$$

1-D systems

Case $\mathtt{n} = 1$:

$$R(\tfrac{d}{dt})w = 0$$

$R \in \mathbb{R}^{\bullet \times \mathtt{w}}[\xi]$: **real polynomial matrix. For example,**

$$P(\frac{d}{dt})y = Q(\frac{d}{dt})u, \quad w = \begin{bmatrix} u \\ y \end{bmatrix}.$$

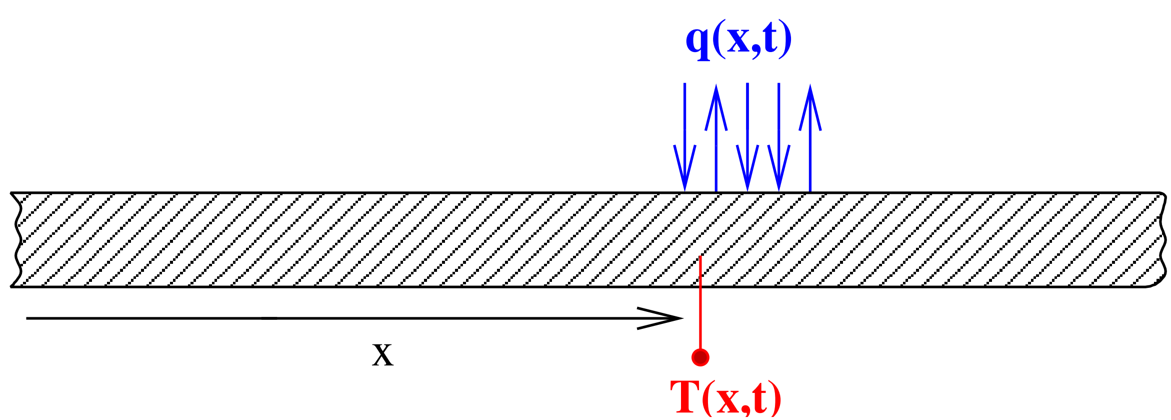
$$\rightsquigarrow \quad \text{transfer f'n } G = P^{-1}Q$$

For example,

$$\frac{d}{dt}x = Ax + Bu,\ y = Cx + Du\ , w = \begin{bmatrix} u \\ x \\ y \end{bmatrix},\ \text{or } w = \begin{bmatrix} u \\ y \end{bmatrix}.$$

Examples

Heat diffusion in a bar



$\rightsquigarrow$ **the PDE**

$$\boxed{\frac{\partial}{\partial t} T = \frac{\partial^2}{\partial x^2} T + q}$$

($x \in \mathbb{R}$, position, $t \in \mathbb{R}$, time), (2-D system)
describes the evolution of the temperature $T(x, t)$
and the heat $q(x, T)$ supplied to / radiated away.

Examples

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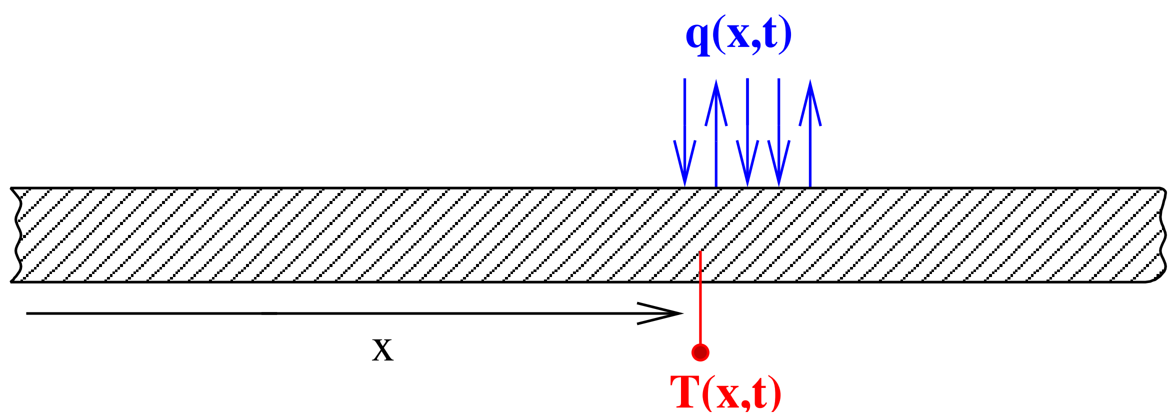


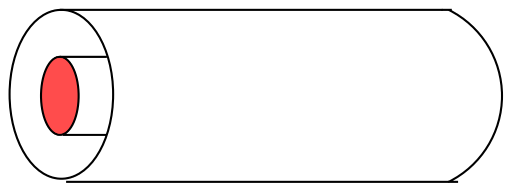
$\rightsquigarrow$ **the PDE**

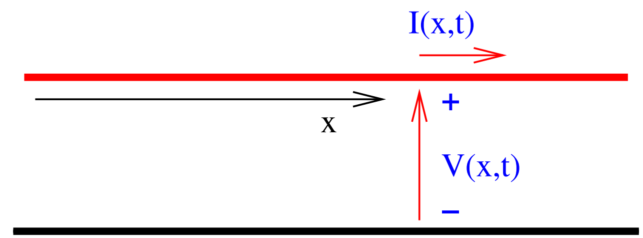
$$\boxed{\frac{\partial}{\partial t} T = \frac{\partial^2}{\partial x^2} T + q}$$

$$\rightsquigarrow w = \begin{bmatrix} T \\ q \end{bmatrix}, R = \begin{bmatrix} \xi_1 - \xi_2^2 & -1 \end{bmatrix}.$$

Examples

The voltage $V(x,t)$ and current $I(x,t)$ in a *coaxial cable*





$$\frac{\partial}{\partial x}V = \rho I - L\frac{\partial}{\partial t}I,$$
$$\frac{\partial}{\partial x}I = \gamma V - C\frac{\partial}{\partial t}V.$$

ρ the resistance, L the inductance, C the capacitance of the cable, γ the conductance of the dielectric medium, all per unit length.

(2-D system)

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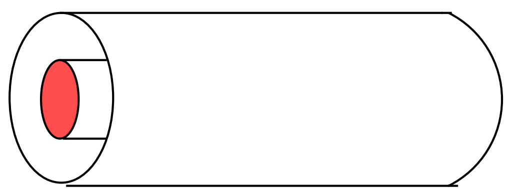
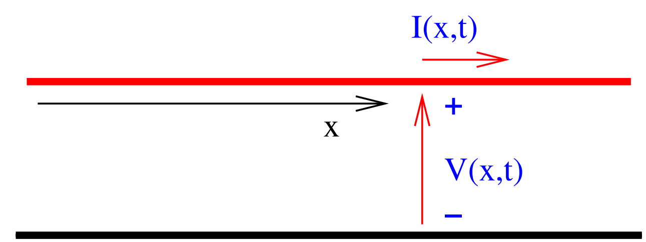




$$\begin{aligned} \frac{\partial}{\partial x} V &= \rho I - L \frac{\partial}{\partial t} I, \\ \frac{\partial}{\partial x} I &= \gamma V - C \frac{\partial}{\partial t} V. \end{aligned}$$

$$\rightsquigarrow w = \begin{bmatrix} V \\ I \end{bmatrix}, R = \begin{bmatrix} \xi_x & -\rho + L\xi_t \\ -\gamma + C\xi_t & \xi_x \end{bmatrix}.$$

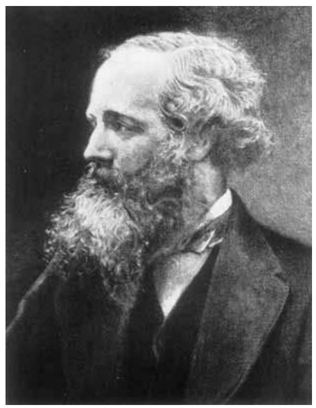
Examples

Maxwell's equations

$$
\begin{aligned}
\nabla \cdot \vec{E} &= \frac{1}{\varepsilon_0}\rho \,, \\
\nabla \times \vec{E} &= -\frac{\partial}{\partial t}\vec{B}, \\
\nabla \cdot \vec{B} &= 0 \,, \\
c^2 \nabla \times \vec{B} &= \frac{1}{\varepsilon_0}\vec{j} + \frac{\partial}{\partial t}\vec{E}.
\end{aligned}
$$

Examples

Maxwell's equations



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$\mathbb{T} = \mathbb{R} \times \mathbb{R}^3$ **(time and space)** $\rightsquigarrow \mathtt{n} = 4$ **(4-D system)** ,
$w = (\vec{E}, \vec{B}, \vec{j}, \rho)$
(electric field, magnetic field, current density, charge density),
$\mathbb{W} = \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}, \rightsquigarrow \mathtt{w} = 10,$
$\mathfrak{B}$ = **set of solutions to these PDE's.**

Note: 10 variables, 8 equations! $\Rightarrow \exists$ free variables.

Elimination theorem

Theorem:

If the behavior of $(w_1, \ldots, w_{\mathtt{k}}, w_{\mathtt{k}+1}, \ldots, w_{\mathtt{w}})$ obeys a constant coefficient linear PDE, then so does the behavior of $(w_1, \ldots, w_{\mathtt{k}})$!

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Which PDE's describe $(\rho, \vec{E}, \vec{j})$ in Maxwell's equations ?

Eliminate $\vec{B}$ from Maxwell's equations $\rightsquigarrow$

$$\begin{aligned} \nabla \cdot \vec{E} &= \frac{1}{\varepsilon_0}\rho\,, \\ \varepsilon_0 \frac{\partial}{\partial t}\nabla \cdot \vec{E} + \nabla \cdot \vec{j} &= 0, \\ \varepsilon_0 \frac{\partial^2}{\partial t^2}\vec{E} + \varepsilon_0 c^2 \nabla \times \nabla \times \vec{E} + \frac{\partial}{\partial t}\vec{j} &= 0. \end{aligned}$$

Image representation

$$\boldsymbol{R}\left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathrm{n}}}\right) w = 0$$

is called a kernel representation of the associated $\mathfrak{B} \in \mathfrak{L}_{\mathtt{n}}^{\mathtt{w}}$.

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Another representation: image representation

$$w = M\left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathrm{n}}}\right) \ell$$

Elimination thm $\Rightarrow$ $\mathrm{im}\left(M\left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathrm{n}}}\right)\right) \in \mathfrak{L}_{\mathrm{n}}^{\mathrm{w}}$ **!**
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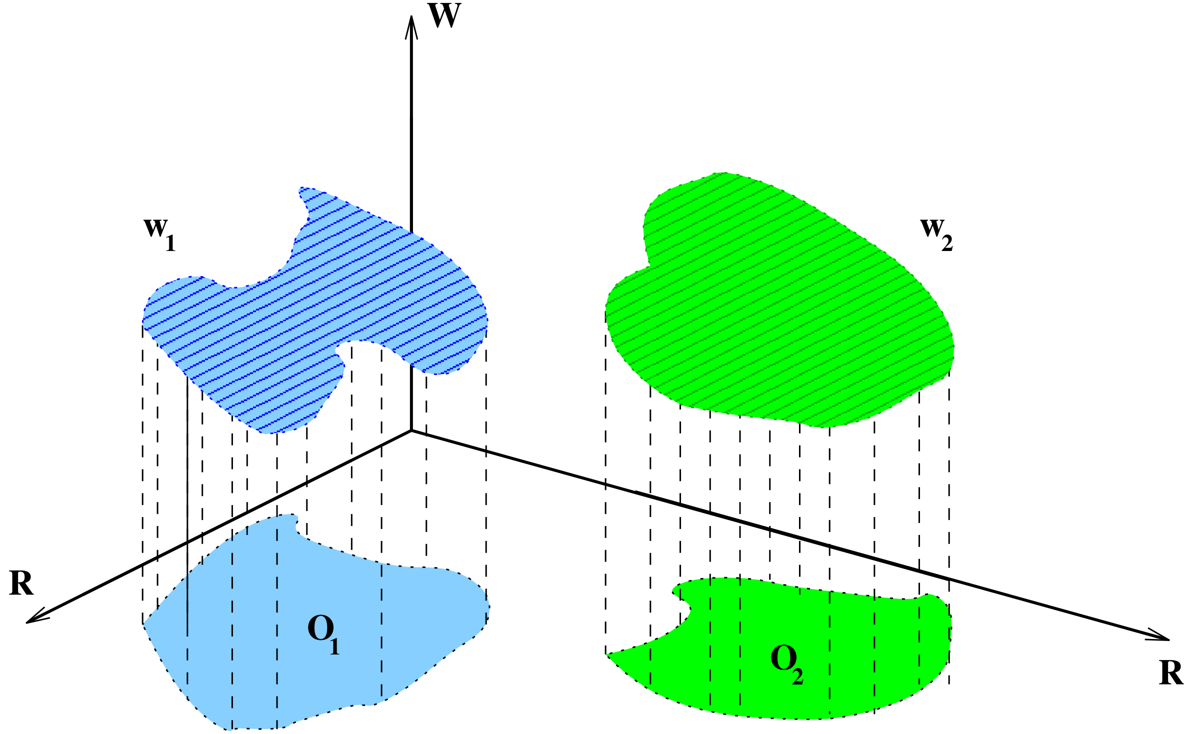
Do all behaviors of linear constant coefficient PDE's admit an image representation???

$\mathfrak{B} \in \mathfrak{L}_{\mathtt{n}}^{\mathtt{w}}$ admits an image representation iff it is 'controllable'.

Controllability

Def'n in pictures:



$$w_1, w_2 \in \mathfrak{B}.$$

Controllability

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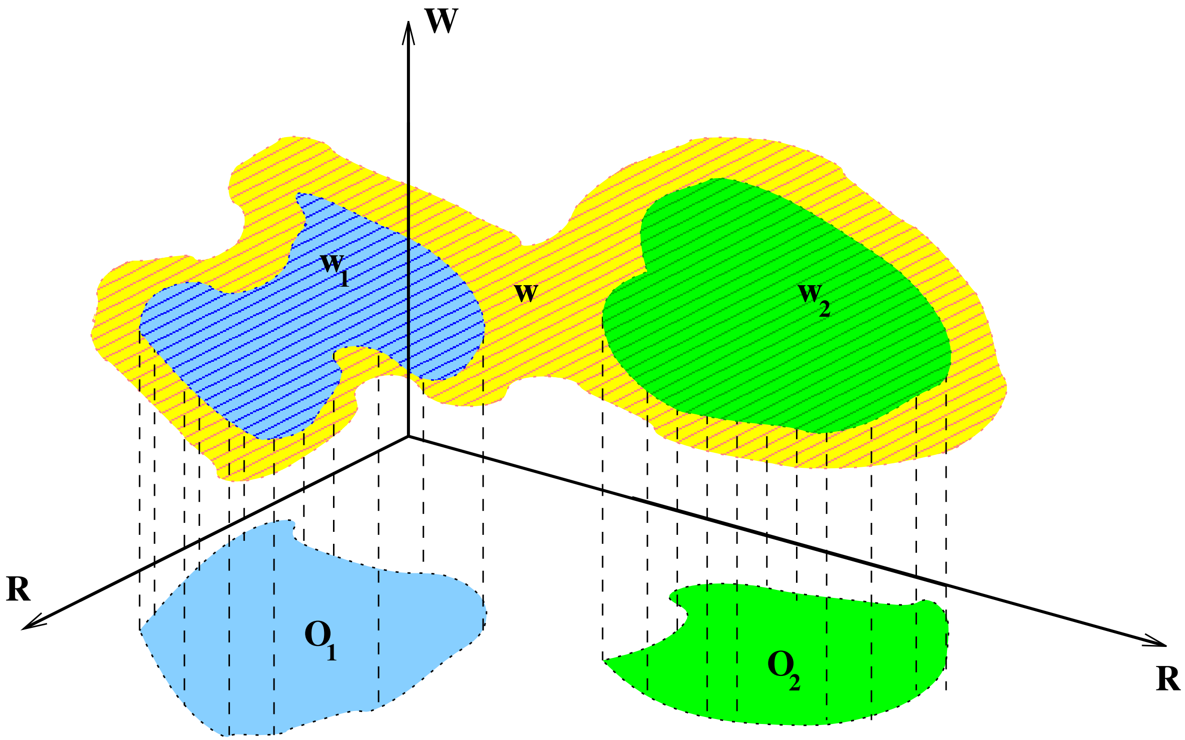
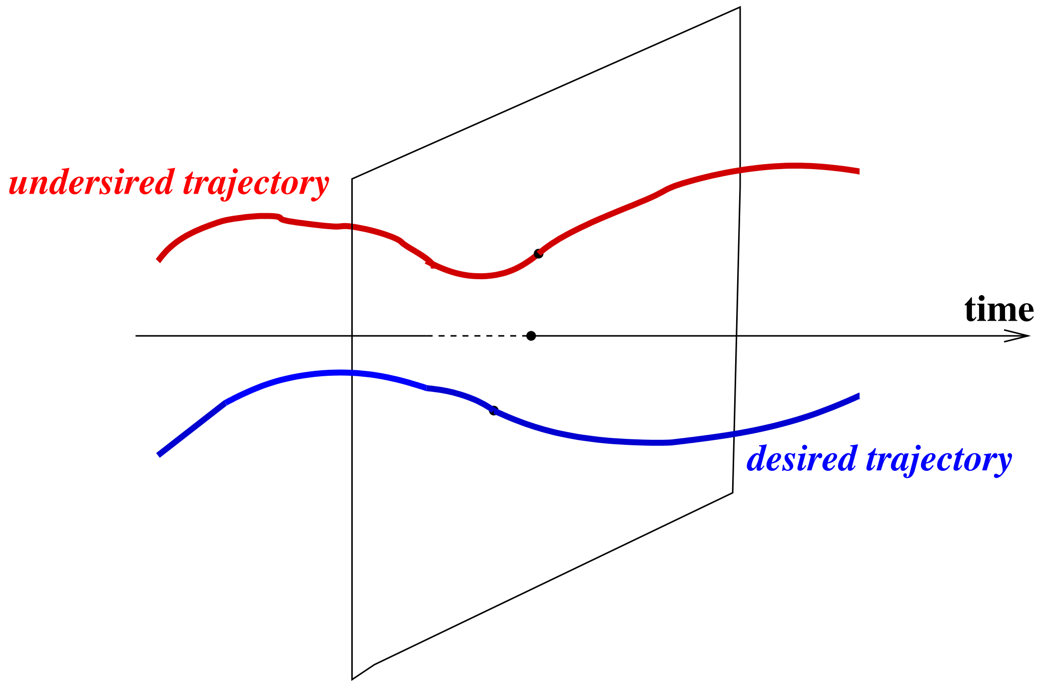


w **'patches'** $w_1, w_2 \in \mathfrak{B}$.

$\exists \; w \in \mathfrak{B} \; \forall \; w_1, w_2 \in \mathfrak{B}$: **Controllability :⇔ 'patchability'.**

case $\mathtt{n} = 1$



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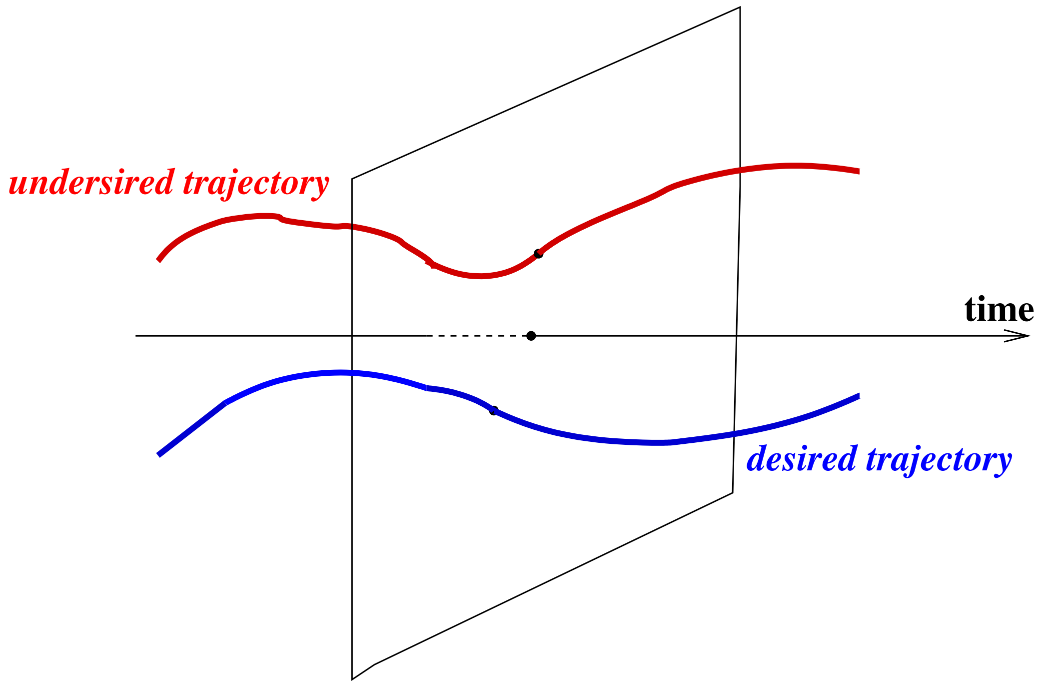

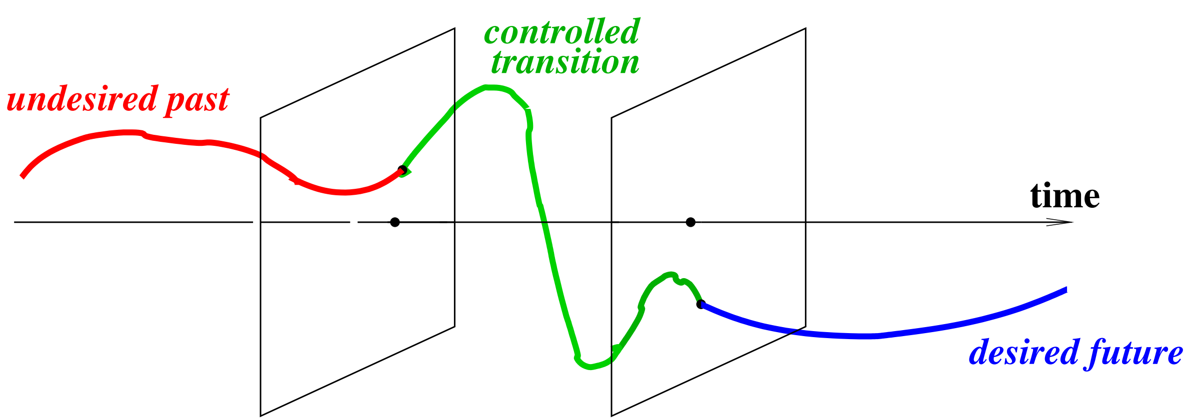

Controllability

<u>Theorem</u>: The following are equivalent:

1. $\mathfrak{B} \in \mathfrak{L}_{\mathtt{n}}^{\mathtt{w}}$ is **controllable**
2. $\mathfrak{B}$ admits an **image representation**
3. $\cdots$

Controllability

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Case $\mathtt{n} = 1$:

$$P(\frac{d}{dt})y = Q(\frac{d}{dt})u, \quad w = \begin{bmatrix} u \\ y \end{bmatrix}.$$

Controllable iff P and Q are left co-prime. Representation

$$w = \begin{bmatrix} u \\ y \end{bmatrix} = \begin{bmatrix} D(\frac{d}{dt}) \\ N(\frac{d}{dt}) \end{bmatrix} \ell$$

$G = P^{-1}Q = ND^{-1}$.

Are Maxwell's equations controllable ?

Are Maxwell's equations controllable ?

The following equations
in the *scalar potential* $\phi : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$ and
the *vector potential* $\vec{A} : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$
generate exactly the solutions to Maxwell's equations:

$$\begin{aligned}
\vec{E} &= -\frac{\partial}{\partial t}\vec{A} - \nabla\phi, \\
\vec{B} &= \nabla \times \vec{A}, \\
\vec{j} &= \varepsilon_0 \frac{\partial^2}{\partial t^2}\vec{A} - \varepsilon_0 c^2 \nabla^2 \vec{A} + \varepsilon_0 c^2 \nabla \left(\nabla \cdot \vec{A} \right) + \varepsilon_0 \frac{\partial}{\partial t} \nabla \phi, \\
\rho &= -\varepsilon_0 \frac{\partial}{\partial t} \nabla \cdot \vec{A} - \varepsilon_0 \nabla^2 \phi.
\end{aligned}$$

Proves controllability. Illustrates the interesting connection

controllability $\Leftrightarrow \exists$ potential!

Observability

Observability **of the image representation**

$$w = M\left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathrm{n}}}\right)\ell$$

is defined as: ℓ **can be deduced from** w**,**
i.e. $M\left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathrm{n}}}\right)$ **should be injective.**

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Not all controllable systems admit an observable im. repr'n.
For $\mathtt{n} = 1$, they do. $\Leftrightarrow$ right co-prime factorization of G.
For $\mathtt{n} > 1$, exceptionally so.

The latent variable ℓ in an im. repr'n may be 'hidden'.

Example: Maxwell's equations do not allow a potential representation with an observable potential.

Dissipative distributed systems

Notation

Multi-index notation:

$$x = (x_1, \ldots, x_{\mathrm{n}}), k = (k_1, \ldots, k_{\mathrm{n}}), \ell = (\ell_1, \ldots, \ell_{\mathrm{n}}),$$
$$\xi = (\xi_1, \cdots, \xi_{\mathrm{n}}), \zeta = (\zeta_1, \ldots, \zeta_{\mathrm{n}}), \eta = (\eta_1, \ldots, \eta_{\mathrm{n}}),$$

$$\frac{d}{dx} = \left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_{\mathrm{n}}}\right), \frac{d^k}{dx^k} = \left(\frac{\partial^{k_1}}{\partial x_1^{k_1}}, \ldots, \frac{\partial^{k_{\mathrm{n}}}}{\partial x_{\mathrm{n}}^{k_{\mathrm{n}}}}\right),$$
$$dx = dx_1 dx_2 \ldots dx_{\mathrm{n}},$$

$$R\left(\frac{d}{dx}\right) w = 0 \quad \text{for} \quad R\left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathrm{n}}}\right) w = 0,$$
$$w = M\left(\frac{d}{dx}\right) \ell \quad \text{for} \quad w = M\left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathrm{n}}}\right) \ell,$$

etc.

Notation

$$\nabla \cdot := \frac{\partial}{\partial x_1} + \cdots + \frac{\partial}{\partial x_{\mathtt{n}}}.$$

For simplicity of notation, and for concreteness, we often take $\mathtt{n} = 4$, independent variables, t, time, and x, y, z, space.

$$\nabla \cdot := \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z},$$

'spatial flux'

QDF's

The quadratic map acting on $w : \mathbb{R}^{\mathtt{n}} \to \mathbb{R}^{\mathtt{w}}$ and its derivatives, defined by

$$w \mapsto \sum_{k,\ell} \left(\frac{d^k}{dx^k} w \right)^\top \Phi_{k,\ell} \left(\frac{d^\ell}{dx^\ell} w \right)$$

is called *quadratic differential form* (QDF) on $\mathfrak{C}^\infty(\mathbb{R}^{\mathtt{n}}, \mathbb{R}^{\mathtt{w}})$.
$\Phi_{k,\ell} \in \mathbb{R}^{\mathtt{w} \times \mathtt{w}}$; WLOG: $\Phi_{k,\ell} = \Phi_{\ell,k}^\top$.

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$$w \mapsto \sum_{k,\ell} \left(\frac{d^k}{dx^k} w \right)^\top \Phi_{k,\ell} \left(\frac{d^\ell}{dx^\ell} w \right)$$

is called *quadratic differential form* (QDF) on $\mathfrak{C}^\infty\left(\mathbb{R}^{\mathtt{n}}, \mathbb{R}^{\mathtt{w}}\right)$.
$\Phi_{k,\ell} \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$; WLOG: $\Phi_{k,\ell} = \Phi_{\ell,k}^\top$.

Introduce the $2\mathtt{n}$-variable polynomial matrix Φ

$$\Phi\left(\zeta, \eta\right) = \sum_{k,\ell} \Phi_{k,\ell} \zeta^k \eta^\ell.$$

Denote the QDF as Q_Φ. QDF's are parametrized by $\mathbb{R}[\zeta, \eta]$.

Dissipative distributed systems

We henceforth consider only controllable linear differential systems and QDF's for supply rates.

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Definition: $\mathfrak{B} \in \mathfrak{L}_{\mathtt{n}}^{\mathtt{w}}$, **controllable, is said to be**

***dissipative* with respect to the supply rate Q_Φ**

(a QDF) if

$$\int_{\mathbb{R}^{\mathtt{n}}} Q_\Phi(w)\ dx \geq 0$$

for all $w \in \mathfrak{B}$ of compact support, i.e., for all $w \in \mathfrak{B} \cap \mathfrak{D}$.

$\mathfrak{D} := \mathfrak{C}^\infty$ and 'compact support'.

Dissipative distributed systems

Assume $\mathtt{n} = 4$**:**
independent variables $x, y, z; t$ **:** **space and time.**

Idea: $Q_\Phi(w)(x, y, z; t) \;\; dxdydz \;\; dt$ **:**

'energy' supplied to the system
in the space-cube $[x, x + dx] \times [y, y + dy] \times [z, z + dz]$
during the time-interval $[t, t + dt]$**.**

Dissipativity $:\Leftrightarrow$

$$\boxed{\int_{\mathbb{R}} \left[\int_{\mathbb{R}^3} Q_\Phi(w)(x, y, z, \; t) \;\; dxdydz\right] \;\; dt \geq 0} \quad \forall \, w \in \mathfrak{B} \cap \mathfrak{D}.$$

A dissipative system absorbs net energy.

Example: EM fields

Maxwell's eq'ns define a dissipative (in fact, a conservative) system w.r.t. the QDF $-\vec{E}\cdot\vec{j}$

Indeed, if $\vec{E},\vec{j}$ **are of compact support and satisfy**

$$\begin{aligned} \varepsilon_0\frac{\partial}{\partial t}\nabla\cdot\vec{E} + \nabla\cdot\vec{j} &= 0, \\ \varepsilon_0\frac{\partial^2}{\partial t^2}\vec{E} + \varepsilon_0 c^2\nabla\times\nabla\times\vec{E} + \frac{\partial}{\partial t}\vec{j} &= 0, \end{aligned}$$

then

$$\int_{\mathbb{R}}\left[\int_{\mathbb{R}^3}\left(-\vec{E}\cdot\vec{j}\right)\,dxdydz\right]dt = 0.$$

The storage and the flux

Local dissipation law

Dissipativity $:\Leftrightarrow$

$$\int_{\mathbb{R}} \left[\int_{\mathbb{R}^3} Q_{\Phi}(w) \; dxdydz\right] \; dt \geq 0 \qquad \text{for all } w \in \mathfrak{B} \cap \mathfrak{D}.$$

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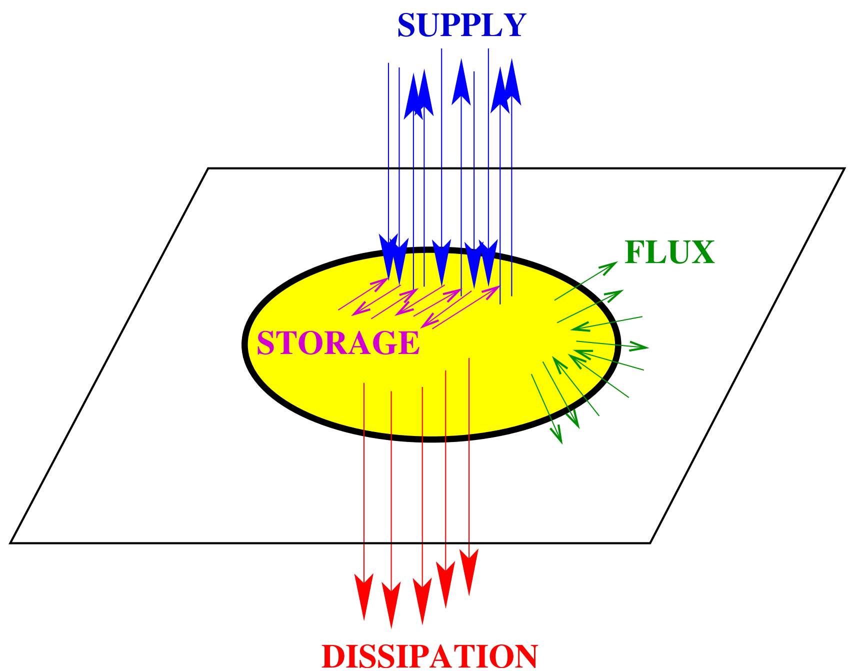
Can this be reinterpreted as:

As the system evolves, some of the energy supplied is locally stored, some locally dissipated, and some redistributed over space?

Local dissipation law

!! Invent storage and flux, locally defined in time and space, such that in every spatial domain there holds:

$$\frac{d}{dt}\ \text{Storage} + \text{Spatial flux} \leq \ \text{Supply.}$$



Supply = partly stored + partly radiated + partly dissipated.

MAIN RESULT (stated for $\mathtt{n} = 4$)

<u>Thm</u>: $\mathtt{n} = 4 : x, y, z; t :$ **space/time;** $\mathfrak{B} \in \mathfrak{L}_4^{\mathtt{w}}$, **controllable.**

Then $\int_{\mathbb{R}} \left[\int_{\mathbb{R}^3} Q_{\Phi}(w) \ dxdydz \right] \ dt \geq 0$ **for all** $w \in \mathfrak{B} \cap \mathfrak{D}$

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$\exists$ an im. repr. $w = M\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \frac{\partial}{\partial t}\right) \ell$ of $\mathfrak{B}$, and QDF's S, the *storage*, and F_x, F_y, F_z, the *flux*, such that the *local dissipation law*

$$\boxed{\frac{\partial}{\partial t} S(\ell) + \frac{\partial}{\partial x} F_x(\ell) + \frac{\partial}{\partial y} F_y(\ell) + \frac{\partial}{\partial z} F_z(\ell) \leq Q_\Phi(w)}$$

holds for all (w, ℓ) that satisfy $w = M\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \frac{\partial}{\partial t}\right) \ell$.

Hidden variables

The local law involves
possibly unobservable, - i.e., hidden!
latent variables (the ℓ's).

This gives physical notions as stored energy, entropy, etc., an enigmatic physical flavor.

Energy stored in EM fields

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Maxwell's equations are dissipative (in fact, conservative) with respect to $-\vec{E}\cdot\vec{j},$ the rate of energy supplied.

Introduce the *stored energy density,* S, and the *energy flux density* (the *Poynting vector*), $\vec{F}$,

$$S\left(\vec{E},\vec{B}\right) := \frac{\varepsilon_0}{2}\vec{E}\cdot\vec{E} + \frac{\varepsilon_0 c^2}{2}\vec{B}\cdot\vec{B},$$

$$\vec{F}\left(\vec{E},\vec{B}\right) := \varepsilon_0 c^2 \vec{E}\times\vec{B}.$$

Local conservation law for Maxwell's equations:

$$\boxed{\frac{\partial}{\partial t}S\left(\vec{E},\vec{B}\right) + \nabla\cdot\vec{F}\left(\vec{E},\vec{B}\right) = -\vec{E}\cdot\vec{j}.}$$

Involves $\vec{B}$, unobservable from $\vec{E}$ and $\vec{j}$.

The proof

Outline of the proof

Using controllability and image representations, we may assume, WLOG: $\mathfrak{B} = \mathfrak{C}^{\infty}(\mathbb{R}^{\mathtt{n}}, \mathbb{R}^{\mathtt{w}})$

To be shown

Global dissipation :$\Leftrightarrow$

$$\int_{\mathbb{R}^{\mathtt{n}}} Q_{\Phi}(w) \geq 0 \text{ for all } w \in \mathfrak{D}$$

$$\Updownarrow$$

$$\exists \ \Psi : \quad \nabla \cdot Q_{\Psi}(w) \leq Q_{\Phi}(w) \ \text{ for all } w \in \mathfrak{C}^{\infty}$$

$\Leftrightarrow$: Local dissipation

$$\int_{\mathbb{R}^{\mathtt{n}}} Q_{\Phi}(w) \geq 0 \text{ for all } w \in \mathfrak{D}$$

$$\Updownarrow \quad \text{(Parseval)}$$

$$\Phi(-i\omega, i\omega) \geq 0 \text{ for all } \omega \in \mathbb{R}^{\mathtt{n}}$$

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$\Updownarrow$ **(Parseval)**

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$$\exists\ \Psi : \quad \nabla \cdot Q_\Psi(w) \leq Q_\Phi(w) \ \text{ for all } w \in \mathfrak{C}^\infty$$

Outline of the proof

Assuming factorizability, we indeed obtain:

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$\Leftrightarrow$: **Local dissipation**

However, ... this argument is valid only for $\mathtt{n} = 1$**...**

The factorization equation (FE)

The factorization equation

Consider

$$X^{\top}(-\xi)\,X(\xi) = Y(\xi) \quad \textbf{(FE)}$$

with $Y \in \mathbb{R}^{\bullet\times\bullet}[\xi]$ given, and X the unknown. ***Solvable??***

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Under what conditions on Y does there exist a solution X?

Scalar case: write the real polynomial Y as a sum of squares

$$Y = x_1^2 + x_2^2 + \cdots + x_k^2.$$

$$X^\top(\xi)\,X(\xi) = Y(\xi) \quad \textbf{(FE)}$$

Y is a given polynomial matrix; X is the unknown.

For $\mathtt{n} = 1$ and $Y \in \mathbb{R}[\xi]$, solvable (with $X \in \mathbb{R}^2[\xi]$) iff

$$Y(\alpha) \geq 0 \qquad \textbf{for all } \alpha \in \mathbb{R}.$$

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For $\mathtt{n} > 1$ and under the symmetry and positivity condition

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this equation can nevertheless in general not be solved over the polynomial matrices, for $X \in \mathbb{R}^{\bullet\times\bullet}[\xi]$,

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this equation can nevertheless in general not be solved over the polynomial matrices, for $X \in \mathbb{R}^{\bullet\times\bullet}[\xi]$,
but it can be solved over the matrices of rational functions, i.e., for $X \in \mathbb{R}^{\bullet\times\bullet}(\xi)$.

Hilbert's 17-th

This factorizability is a consequence of Hilbert's 17-th pbm!

!! Solve $p = p_1^2 + p_2^2 + \cdots + p_k^2$, **$p$ given**

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!! Solve $p = p_1^2 + p_2^2 + \cdots + p_k^2$, **$p$ given**

A polynomial $p \in \mathbb{R}[\xi_1, \cdots, \xi_n]$, **with** $p(\alpha_1, \ldots, \alpha_n) \geq 0$ **for all** $(\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$ **can in general not be expressed as a SOS of polynomials, with the** p_i**'s** $\in \mathbb{R}[\xi_1, \cdots, \xi_n]$.

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!! Solve $p = p_1^2 + p_2^2 + \cdots + p_k^2$, p **given**

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But a rational function (and hence a polynomial) $p \in \mathbb{R}(\xi_1, \cdots, \xi_n)$, **with** $p(\alpha_1, \ldots, \alpha_n) \geq 0$, **for all** $(\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$, **can be expressed as a SOS of** $(k = 2^n)$ **rational functions, with the** p_i**'s** $\in \mathbb{R}(\xi_1, \cdots, \xi_n)$.

Outline of the proof

$\Rightarrow$ **solvability of the factorization eq'n**

$$\Phi\left(-i\omega, i\omega\right) \geq 0 \text{ for all } \omega \in \mathbb{R}^{\mathrm{n}}$$

$$\Updownarrow$$

(Factorization equation)

$$\exists \ D : \quad \Phi\left(-\xi, \xi\right) = D^{\top}\left(-\xi\right) D\left(\xi\right)$$

over the rational functions, i.e., with D a matrix with elements in $\mathbb{R}\left(\xi_1, \cdots, \xi_{\mathrm{n}}\right)$.

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The need to introduce rational functions in this factorization equation and an image representation of $\mathfrak{B}$ (to reduce the pbm to $\mathfrak{C}^{\infty}$) are the causes of the unavoidable presence of (possibly unobservable, i.e., 'hidden') latent variables in the local dissipation law.

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1. **The non-uniqueness of the latent variable ℓ in various (non-observable) image representations of $\mathfrak{B}$.**
2. **of D in the factorization equation**

$$\Phi(-\xi,\xi) = D^\top(-\xi)\,D(\xi)$$

3. **(in the case $\mathtt{n} > 1$) of the solution Ψ of**

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$$\left(\zeta + \eta\right)^{\top} \Psi\left(\zeta, \eta\right) = \Phi\left(\zeta, \eta\right) - D^{\top}\left(\zeta\right) D\left(\eta\right)$$

For conservative systems, $\Phi\left(-\xi, \xi\right) = 0$, whence $D = 0$, but, when $\mathtt{n} > 1$, the third source of non-uniqueness remains.

Uniqueness

The non-uniqueness is very real, even for EM fields.

Uniqueness

The non-uniqueness is very real, even for EM fields. Cfr.

The ambiguity of the field energy

... There are, in fact, an infinite number of different possibilities for u* [the internal energy] *and S* [the flux] ... *It is sometimes claimed that this problem can be resolved using the theory of gravitation ... as yet nobody has done such a delicate experiment ... So we will follow the rest of the world - besides, we believe that it* [our choice] *is probably perfectly right.

**The Feynman Lectures on Physics,
Volume II, page 27-6.**

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SUMMARY

- **The theory of dissipative systems centers around the construction of the storage function**
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- **The proof $\cong$ Hilbert's 17-th problem**

SUMMARY

- **The theory of dissipative systems centers around the construction of the storage function**
- **global dissipation $\Leftrightarrow \exists$ local dissipation law**
- **Involves possibly hidden latent variables**
  (e.g. $\vec{B}$ in Maxwell's eq'ns)
- **The proof $\cong$ Hilbert's 17-th problem**
- **Neither controllability nor observability are good generic system theoretic assumptions for physical models**

Thank you

Thank you

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