THE SUM-of-SQUARES PROBLEM

and

DISSIPATIVE SYSTEMS

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Based in part on joint work with

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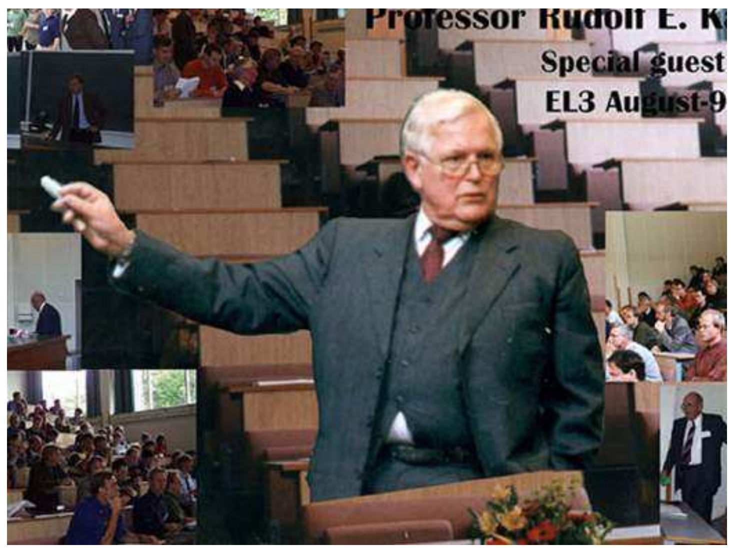
Outline

- Systems
- Dissipative systems
- The storage and the flux
- PDE’s and QDF’s
- The SOS problem

Motto

1. **Get the physics right**
2. **The rest is mathematics**



R.E. Kalman, Opening lecture

IFAC World Congress, Prague, July 4, 2005

Systems

Definition

A system: $\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$, **with** $\mathfrak{B} \subseteq (\mathbb{W})^{\mathbb{T}}$ **the behavior.**

The behavior $\mathfrak{B} \subseteq (\mathbb{W})^{\mathbb{T}}$ **consists of the trajectories $w : \mathbb{T} \rightarrow \mathbb{W}$ that are compatible with the laws of the system, typically the set of sol'ns of an ODE or PDE.**

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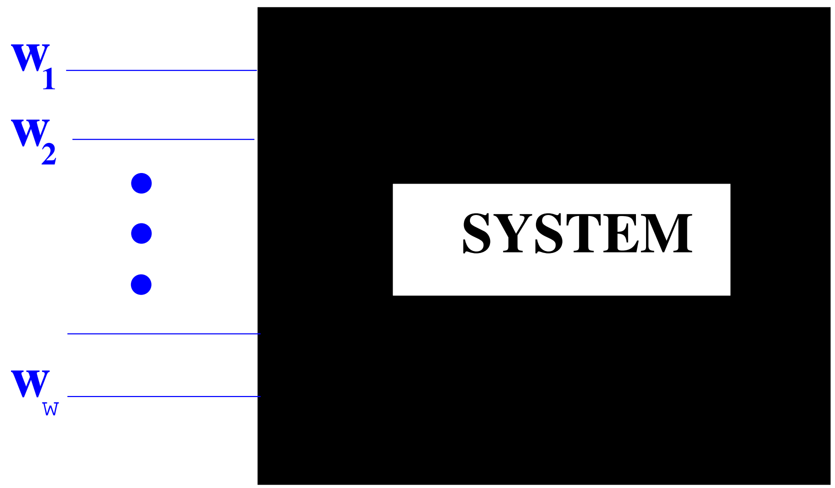
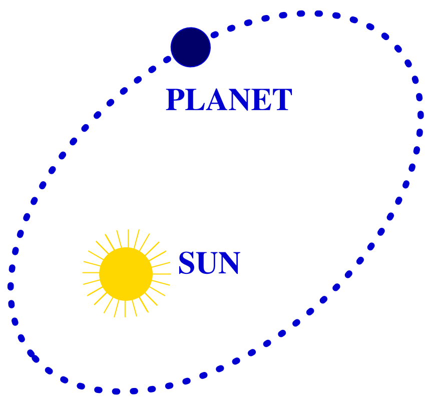


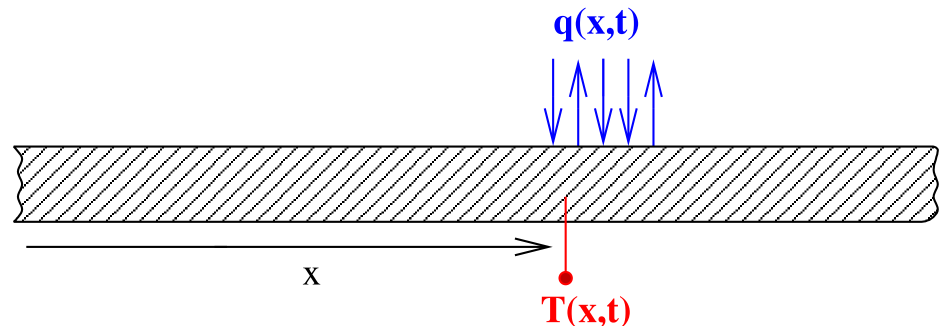
Examples



$\mathbb{T} = \mathbb{R}, \ \mathbb{W} = \mathbb{R}^3, \ \mathfrak{B} =$ all $\mathbb{R} \to \mathbb{R}^3$ satisfying **K.1, K.2, and K.3**

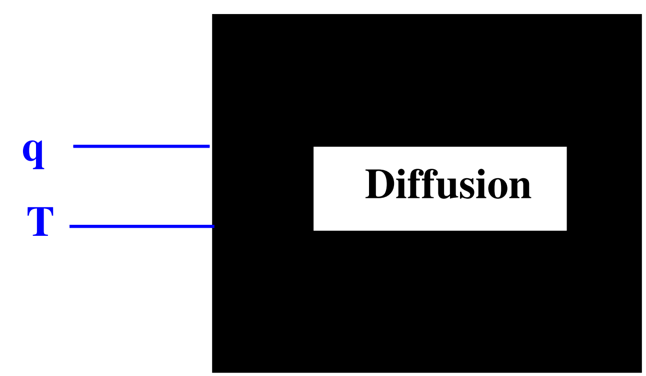
Examples

Heat diffusion



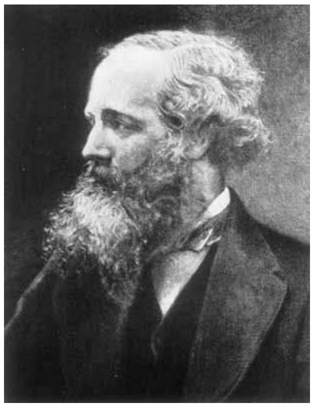
$\mathbb{T} = \mathbb{R}^2$ (x and t), $\mathbb{W} = \mathbb{R}^2$ (q and T), $\mathfrak{B}$ sol'ns of the PDE

$$\frac{\partial}{\partial t} T = \frac{\partial^2}{\partial x^2} T + q$$



Examples

Maxwell's equations

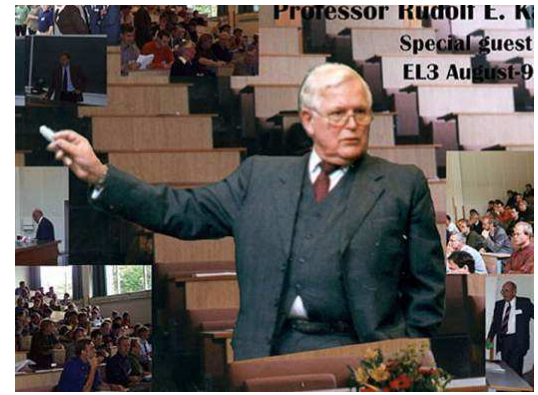


$$
\begin{aligned}
\nabla \cdot \vec{E} &= \frac{1}{\varepsilon_0}\rho\,, \\
\nabla \times \vec{E} &= -\frac{\partial}{\partial t}\vec{B}, \\
\nabla \cdot \vec{B} &= 0\,, \\
c^2 \nabla \times \vec{B} &= \frac{1}{\varepsilon_0}\vec{j} + \frac{\partial}{\partial t}\vec{E}.
\end{aligned}
$$

$\mathbb{T} = \mathbb{R}^4$ **(time and space) ,**
$\mathbb{W} = \mathbb{R}^{10}$ $(\vec{E}, \vec{B}, \vec{j}$ **and** $\rho)$,
$\mathfrak{B}$ = **sol'ns of ME's**

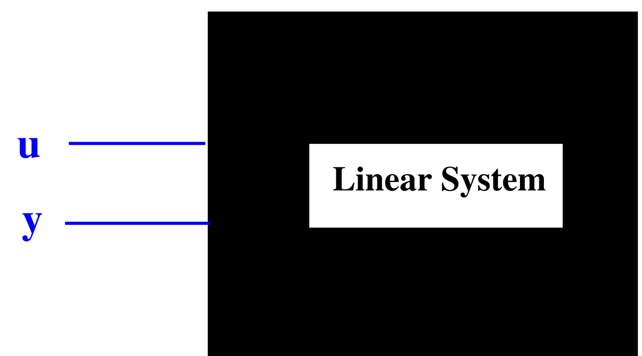
Examples

Linear systems



$$\frac{d}{dt}x = Ax + Bu, y = Cx + Bu, \;\; w = (u, y)$$

$\mathbb{T} = \mathbb{R}($ **time** $), \mathbb{W} = \mathbb{R}^{\mathtt{m}} \times \mathbb{R}^{\mathtt{p}}$ **inputs and outputs** $)$,
$\mathfrak{B} = (u, y) : \mathbb{R} \to \mathbb{R}^{\mathtt{m}} \times \mathbb{R}^{\mathtt{p}} : \exists x : \mathbb{R} \to \mathbb{R}^{\mathtt{n}} \ldots$



Properties

Linearity

Examples: ME, linear systems, diffusion.

Shift-invariance

Examples: Kepler, diffusion, ME, linear systems.

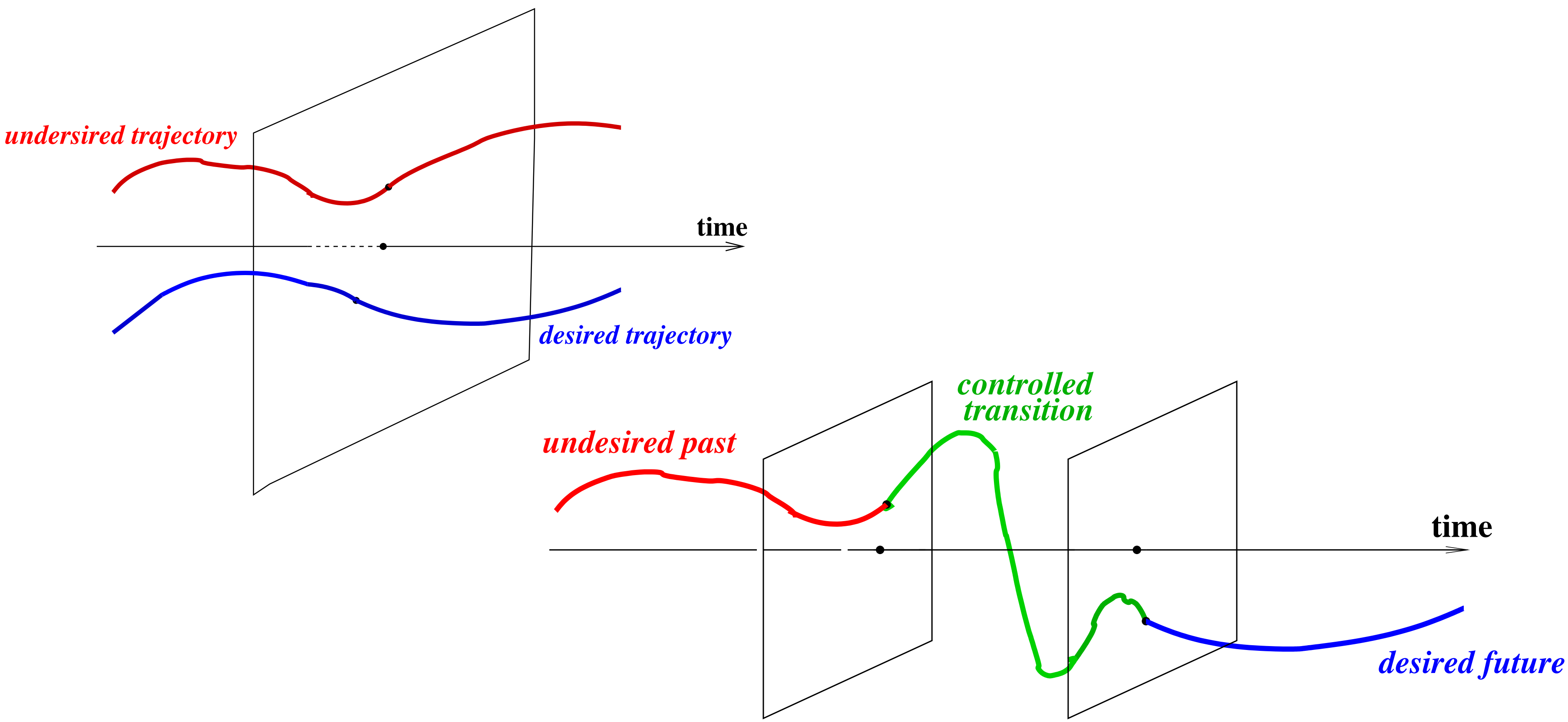
Assumed throughout.

Properties

Controllability

Def'n in pictures:

1-d case: $\mathbb{T} = \mathbb{R}$ **or** $\mathbb{Z}$**.**



$$w_1, w_2 \in \mathfrak{B}.$$

Properties

Controllability

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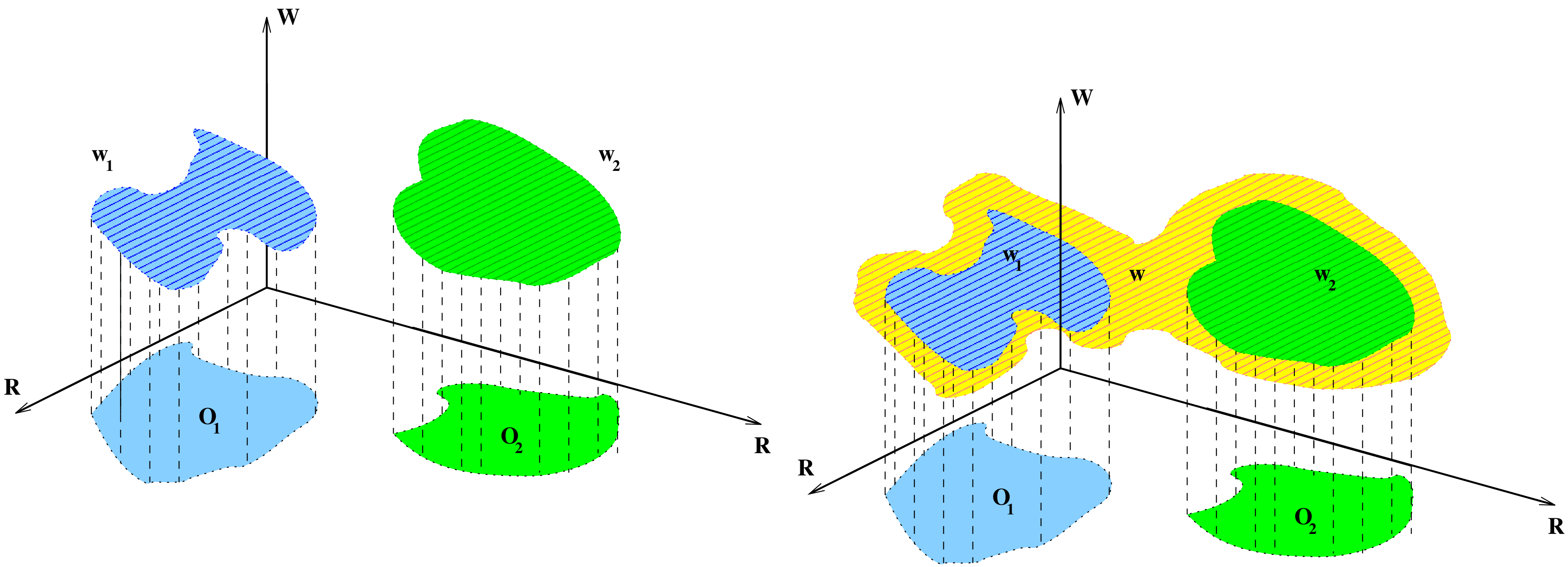


w **'patches'** $w_1, w_2 \in \mathfrak{B}$.

$\forall \; w_1, w_2 \in \mathfrak{B} \; \exists \; w \in \mathfrak{B}$: **Controllability :⇔ 'patchability'.**

Properties

Controllability

Controllability is a typical property of **open** systems.

Open: some variables are left ‘free’.

Open systems interact with their environment.

In contrast with **closed, autonomous** systems.

$:\cong$ ‘Initial conditions’ specify the trajectory uniquely.

Examples:

Kepler: closed, not controllable; QM: idem; flows: idem

diffusion: controllable

ME: controllable

$\frac{d}{dt}x = Ax + Bu$: well-known conditions

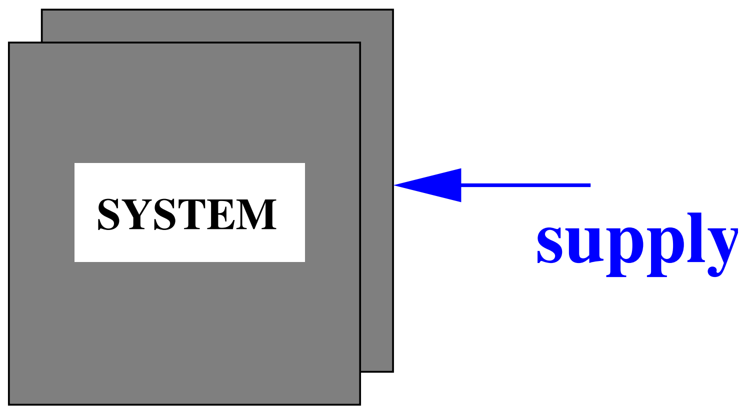
Controllability is assumed where needed. For controllable systems, the compact support or periodic trajectories are ‘representative’ of the whole behavior.

Dissipative Systems

Definition

$\Sigma : (\mathbb{R}^n, \mathbb{R}, \mathfrak{B})$ is dissipative (w = supply rate) $:\Leftrightarrow$

$w \in \mathfrak{B}$ and w periodic (period T) $\Rightarrow$ $$\int_0^T w(t)\,dt \geq 0$$



System absorbs supply, netto

If = holds, the system is called conservative

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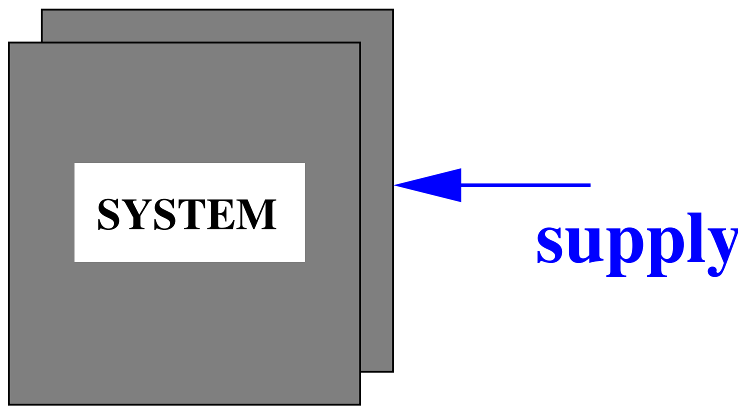

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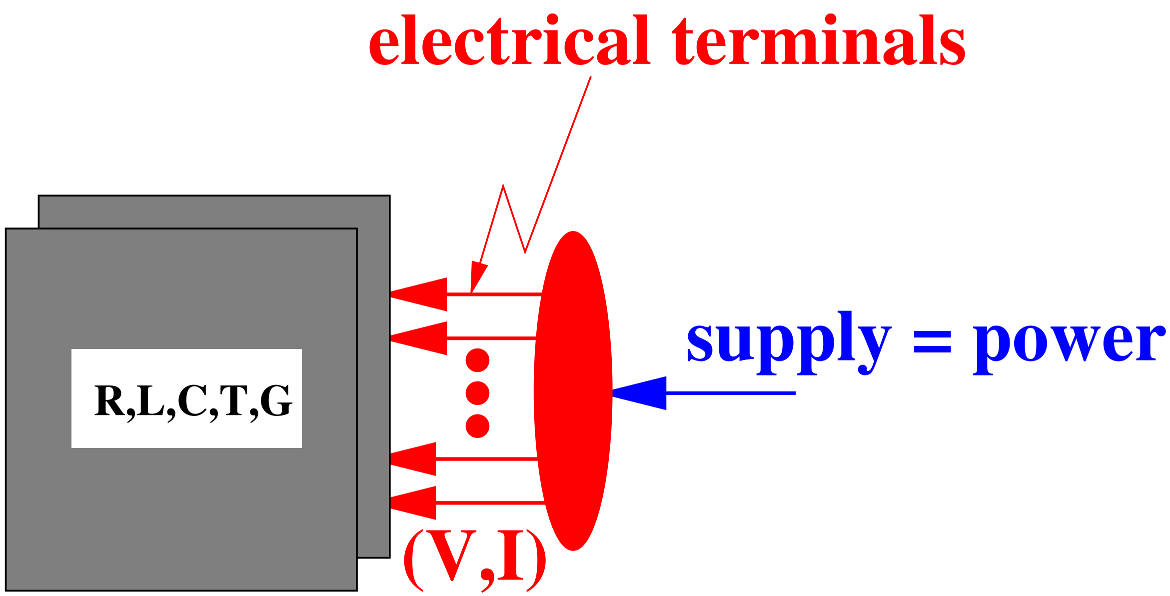
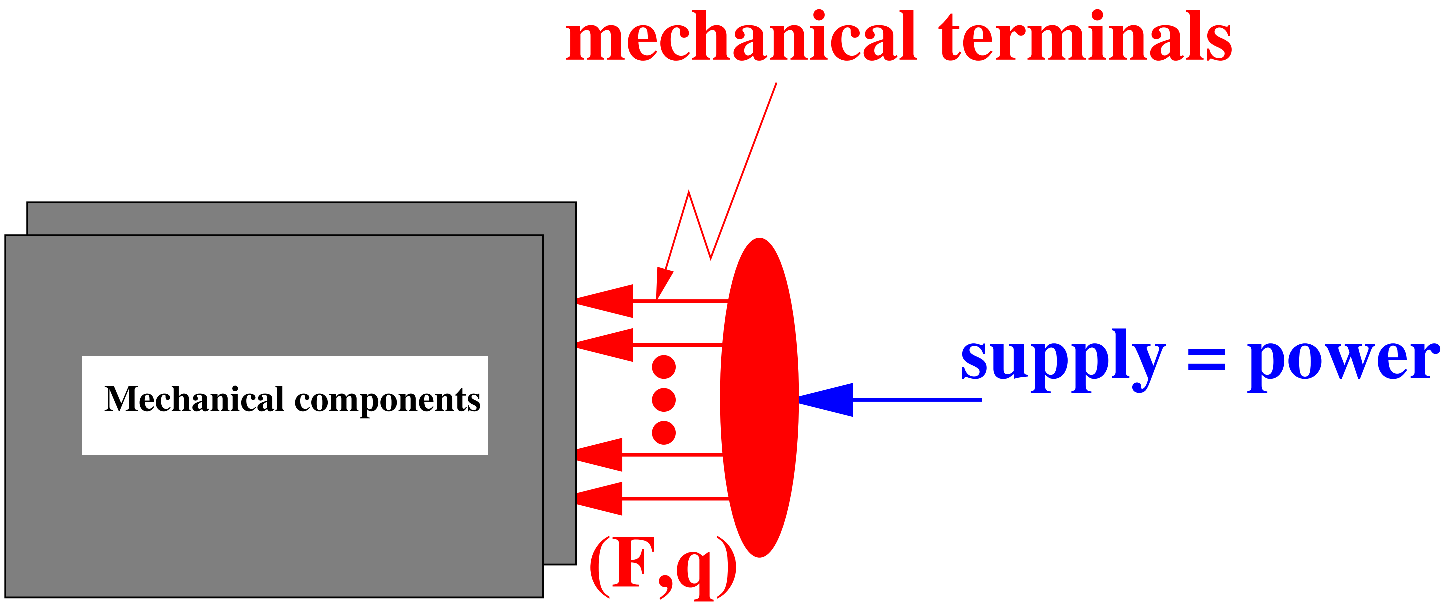
Dissipativity interesting, relevant, for open systems ...

Examples



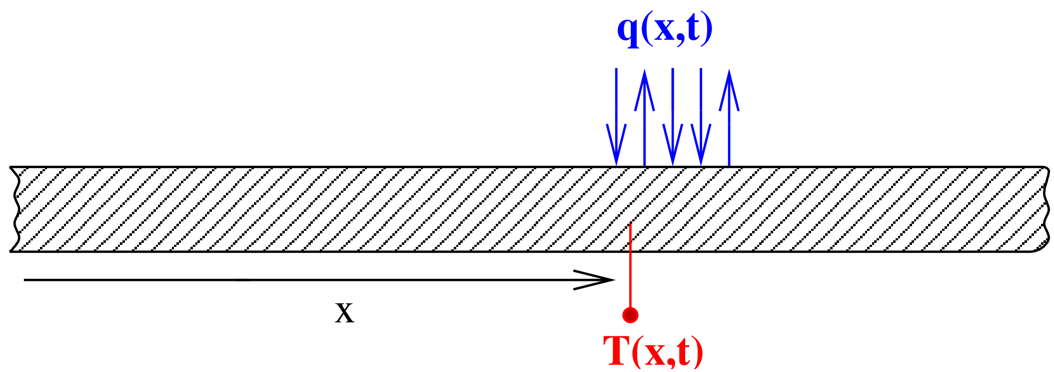
power= $\sum_{\text{terminals}} I_k V_k$

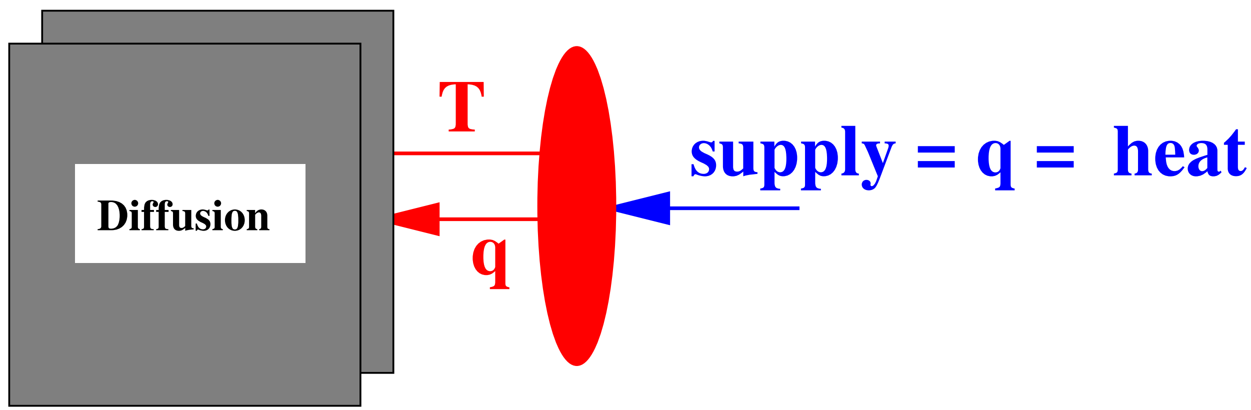
Examples



power= $\sum_{\text{terminals}} F_k \frac{d}{dt} q_k$

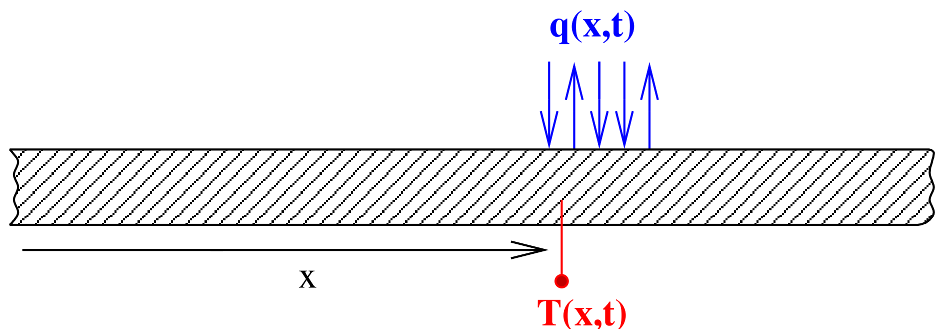
Examples

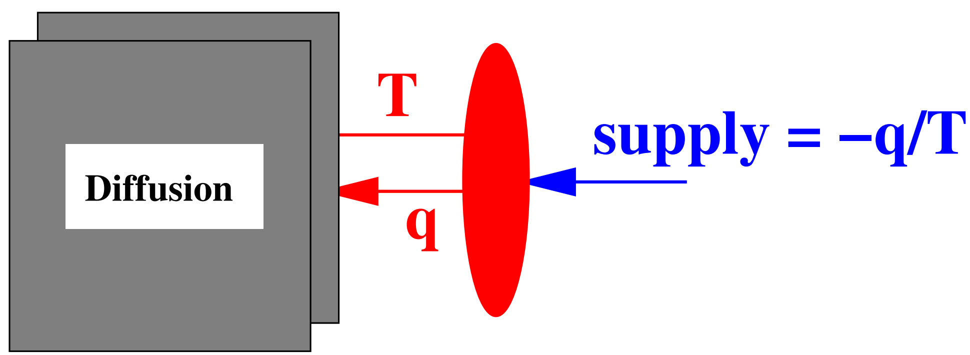




Conservative. $\Leftrightarrow$ **for compact support:** $\int\int_{\mathbb{R}^2} q(x,t)\, dx\, dt = 0$

Examples

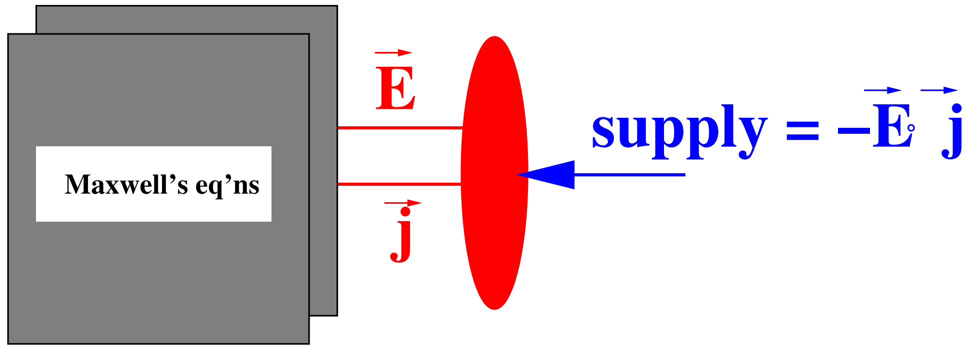




Dissipative. ⇔ for compact support q:

$$\iint_{\mathbb{R}^2} \frac{q(x,t)}{T(x,t)}\, dx\, dt \leq 0$$

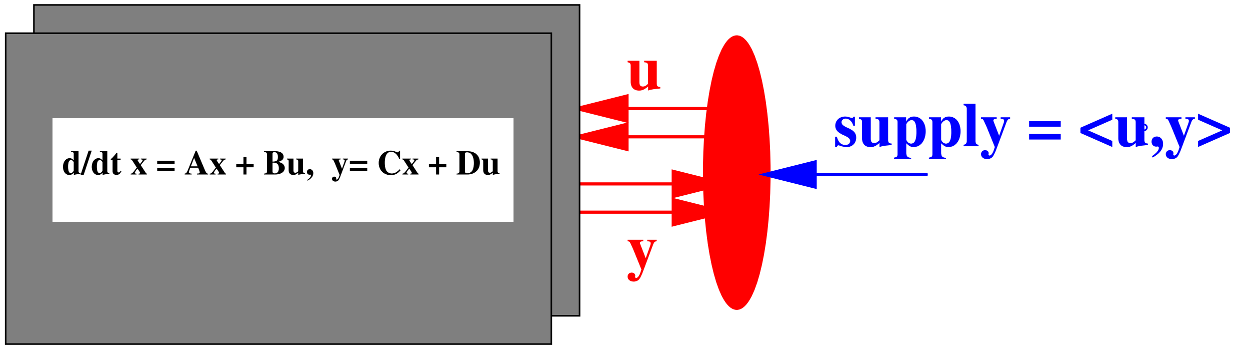
Examples



Conservative. $\Leftrightarrow$ for compact support sol'ns of ME:

$$\iiiint_{\mathbb{R}^4} \vec{E}(x,y,z,t) \cdot \vec{j}(x,y,z,t)\, dx\, dy\, dz\, dt = 0$$

Examples



Dissipative $\Leftrightarrow$

$$G(i\omega) + G^\top(-i\omega) \geq 0 \quad \forall\, \omega \in \mathbb{R}$$

The Storage and the Flux

Definition

Consider the 1-d system $\Sigma' = (\mathbb{R}, \mathbb{R} \times \mathbb{R}, \mathfrak{B}')$.
Each trajectory is a pair (w, V) $\; w : \mathbb{R} \to \mathbb{R}, V : \mathbb{R} \to \mathbb{R}$.
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$$\mathfrak{B} := \{w : \mathbb{R} \to \mathbb{R} \mid (w, V) \in \mathfrak{B}'\}$$

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V is a storage function $:\Leftrightarrow \;\; \forall (w, V) \in \mathfrak{B}'$:

$$\boxed{V(t_1) \leq V(t_0) + \int_{t_0}^{t_1} w(t)\, dt} \quad \forall\, t_0, t_1 \in \mathbb{R}, t_0 \leq t_1$$

$$\frac{d}{dt} V \leq w$$

Implies, reasonable conditions, $\Sigma = (\mathbb{R}, \mathbb{R}, \mathfrak{B})$ dissipative.

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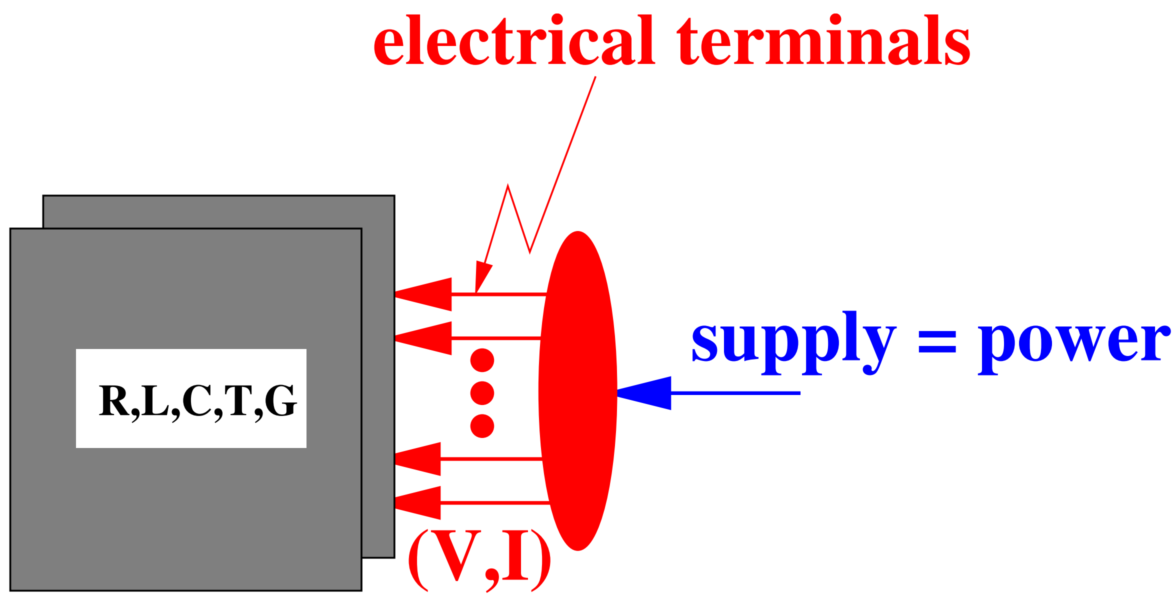
- Given a dissipative system $\Sigma = (\mathbb{R}, \mathbb{R}, \mathfrak{B})$, construct a storage function.
- Is the storage function unique?
- The set of storage functions is obviously convex.
- Does it has an upper/lower bound?

Examples



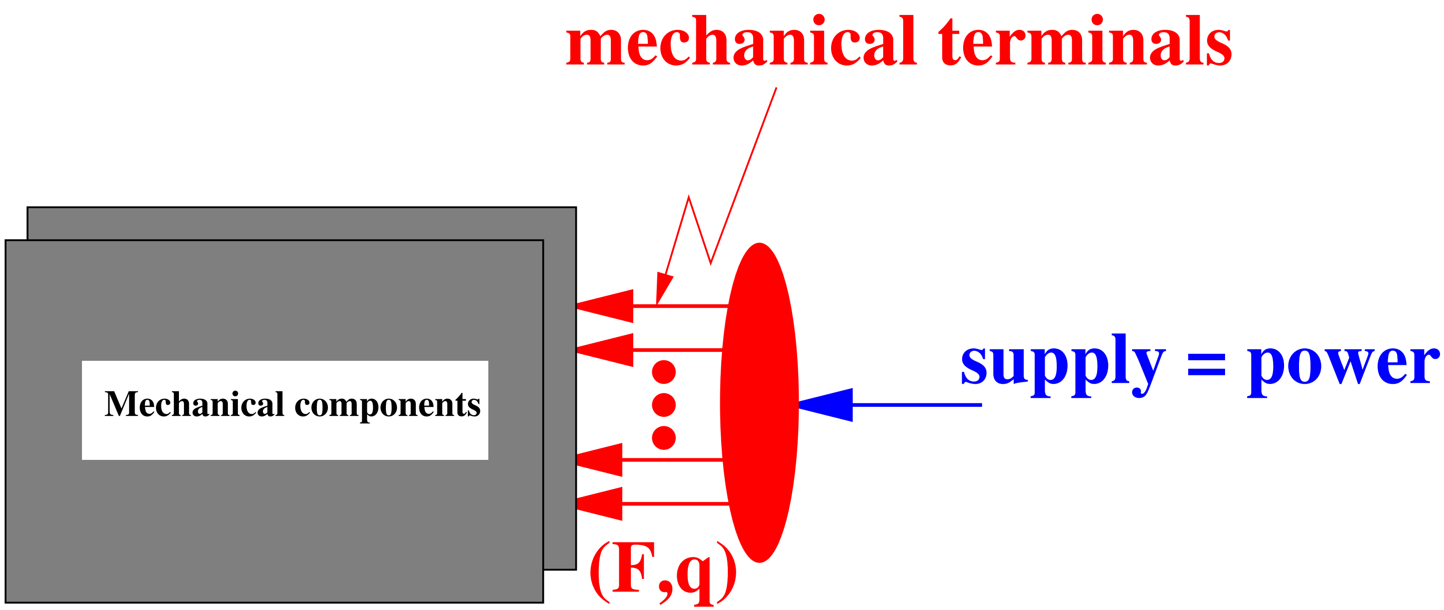
power= $\sum_{\text{terminals}} I_k V_k$

Storage function = **energy stored in L's and C's**

NOT UNIQUE, when viewed from external terminals!

Lower bound: available storage. Upper bound: required supply.

Examples



power= $\sum_{\text{terminals}} F_{\text{k}} \frac{d}{dt} q_{\text{k}}$

Storage function = energy stored in masses and springs

NOT UNIQUE, when viewed from external terminals!

Lower bound: available storage. Upper bound: required supply.

Is energy non-negative?

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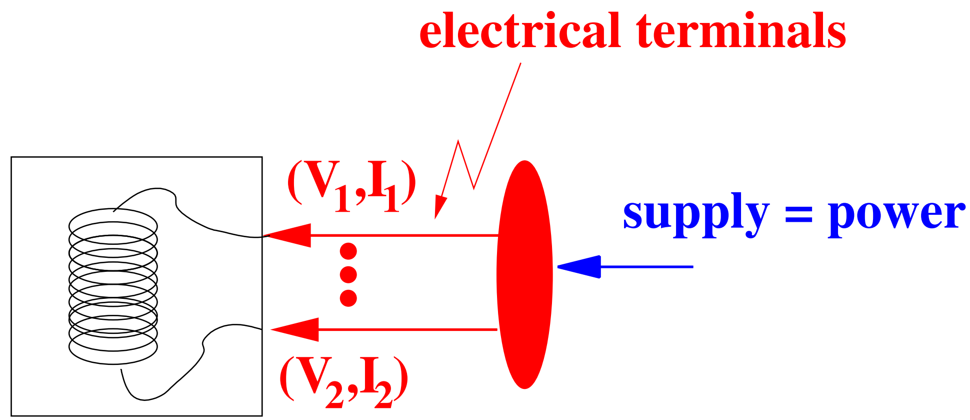
Is the storage function, in the case the supply is the power, bounded from below? Is a negative inductor passive?



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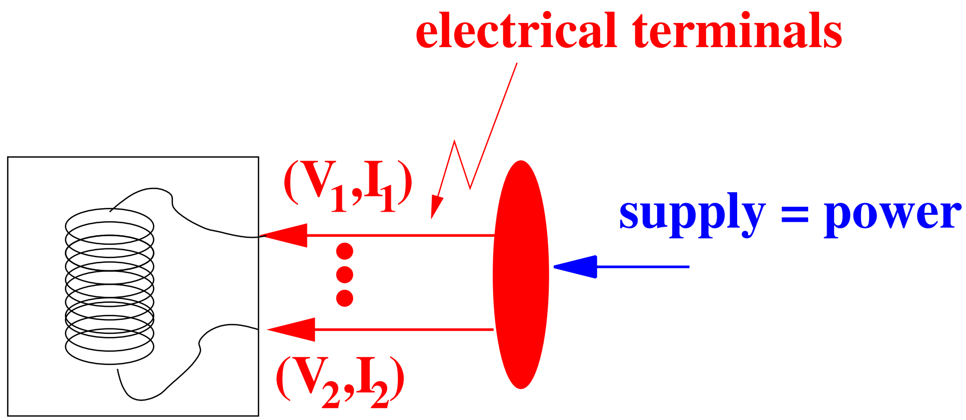

Equations:

$$I_1 + I_2 = 0,\, L\,\frac{d}{dt}I_1 = V_1 - V_2$$

$$\text{power} \;= V_1 I_1 + V_2 I_2 \;\rightsquigarrow\; \frac{d}{dt}\frac{1}{2}LI_1^2 = \text{power}$$

Hence the system is dissipative (in the sense of the periodic sol'ns) regardless of the sign of L).

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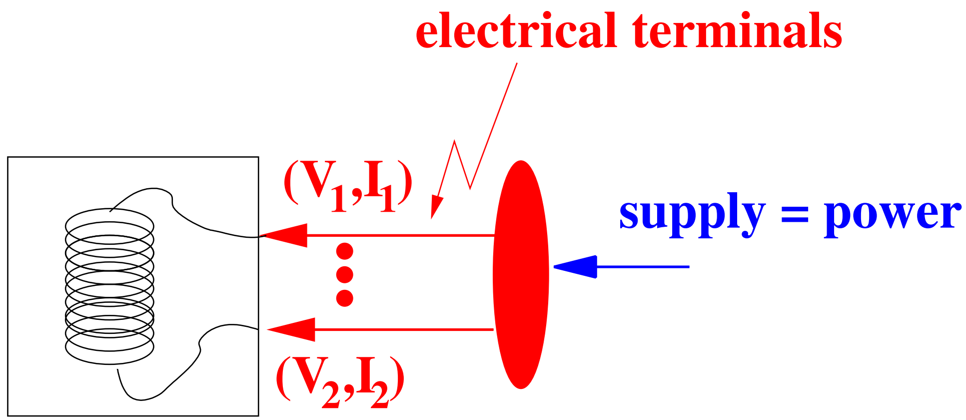

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Is this reasonable? It appears not! But, the answer must lie in electricity, not in physics!

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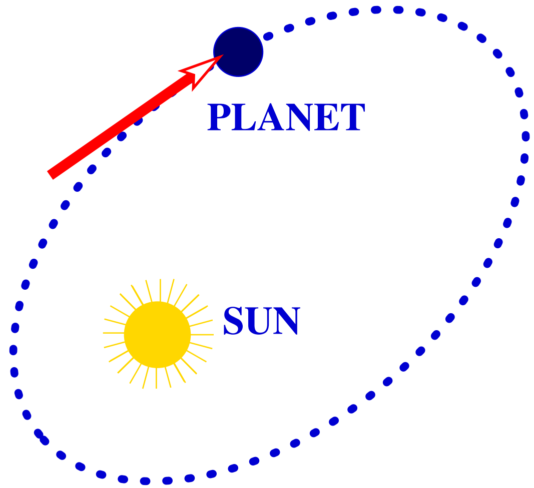
Is the storage function, in the case the supply is the power, bounded from below ? Does the inverse square low define a passive system?



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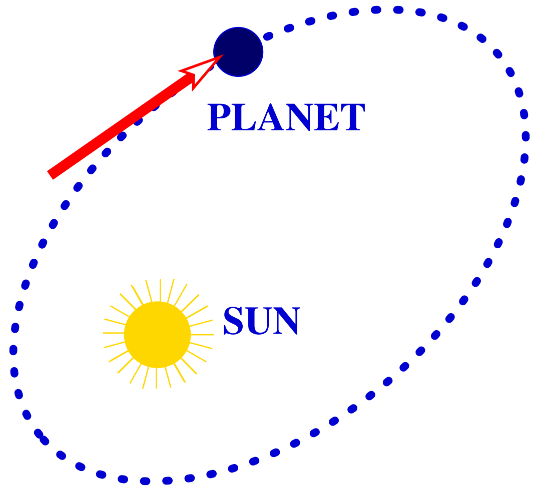

Equations (1 dim., nice numbers):

$$\frac{d^2}{dt^2}q + \frac{1}{q^2} = F$$

$$\frac{1}{2}\left(\frac{d}{dt}q\right)^2 - \frac{1}{q} = F\frac{d}{dt}q$$

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Is the storage function, in the case the supply is the power, bounded from below ? Does the inverse square low define a passive system?   Equations (1 dim., nice numbers):

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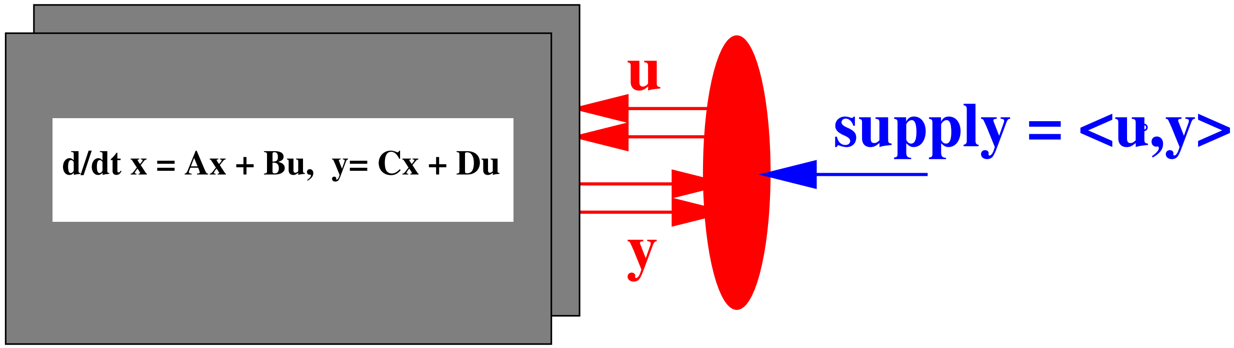
Dissipative (in the sense of the periodic sol'ns),

but the energy $\frac{1}{2}\left(\frac{d}{dt}q\right)^2 - \frac{1}{q}$ **is NOT bounded from below.**

Also physics says this is passive!!

Examples



G := transfer f'n, $G(s) = D + C(Is - A)^{-1}B$. **Equivalent:**

1. **Dissipative**
2. $G(i\omega) + G^\top(-i\omega) \geq 0 \;\forall\, \omega \in \mathbb{R}$
3. $\exists\, Q = Q^\top : \frac{d}{dt}\, x^\top Q x \leq y^\top u$
4. ...

$\leadsto$ **KYP-lemma, AREineq., ARE, LMI's, ...**

Probably the most used circle of ideas in control!

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$$\mathfrak{B} := \{w : \mathbb{R}^{\mathrm{n}} \to \mathbb{R} \mid (w, V) \in \mathfrak{B}'\}$$

V is a **storage/flux function** $:\Leftrightarrow$
$\forall (w, V) \in \mathfrak{B}'$: (case $\mathrm{n} = 4$, variables x, y, z, t)

$$\left(\frac{\partial}{\partial x} F_x + \frac{\partial}{\partial y} F_y + \frac{\partial}{\partial z} F_z + \frac{\partial}{\partial t} S\right)(x, y, z, t) \leq w(x, y, z, t)$$

$\forall\, x, y, z, t \in \mathbb{R}$. **Generally:**

$$\nabla \cdot V \leq w \qquad \nabla \cdot := \frac{\partial}{\partial x_1} + \cdots + \frac{\partial}{\partial x_{\mathrm{n}}}$$

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$\forall \; x, y, z, t \in \mathbb{R}$.

Implies, under reasonable conditions, that Σ is dissipative.

- Given a dissipative system $\Sigma = (\mathbb{R}^{\mathrm{n}}, \mathbb{R}, \mathfrak{B})$, construct V, i.e. a storage S and a flux F.

Local dissipation law

Dissipativity $:\Leftrightarrow$

$$\int_{\mathbb{R}} \int_{\mathbb{R}^3} w(x,y,z,t)\, dxdydz\, dt \geq 0 \qquad \text{for all } w \in \mathfrak{B}.$$

Local dissipation law

Dissipativity :⇔

$$\int_{\mathbb{R}} \int_{\mathbb{R}^3} w(x, y, z, t) \, dxdydz \, dt \geq 0 \qquad \text{for all } w \in \mathfrak{B}.$$

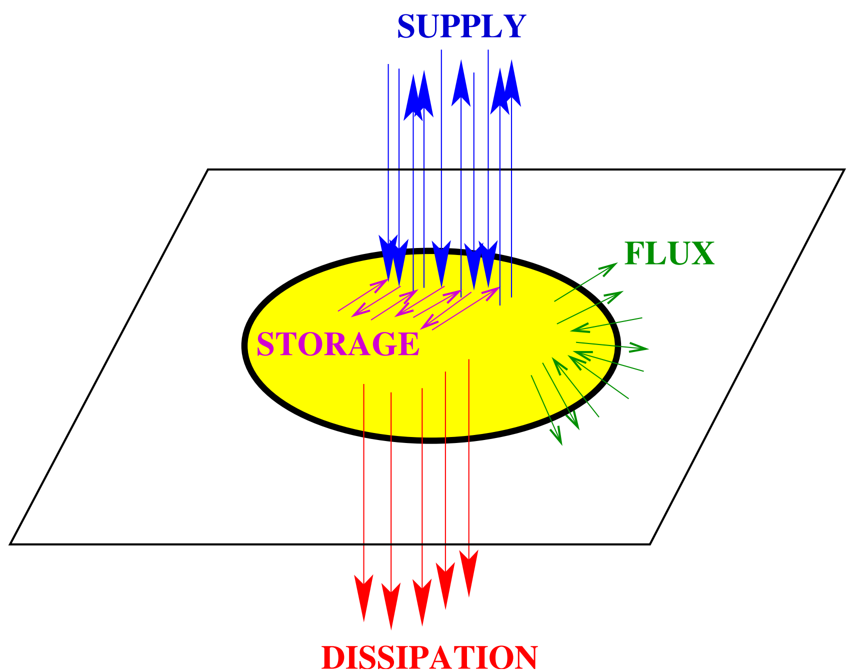
Can this be reinterpreted as:

As the system evolves, some of the supply is locally stored, some locally dissipated, and some redistributed over space?

Local dissipation law

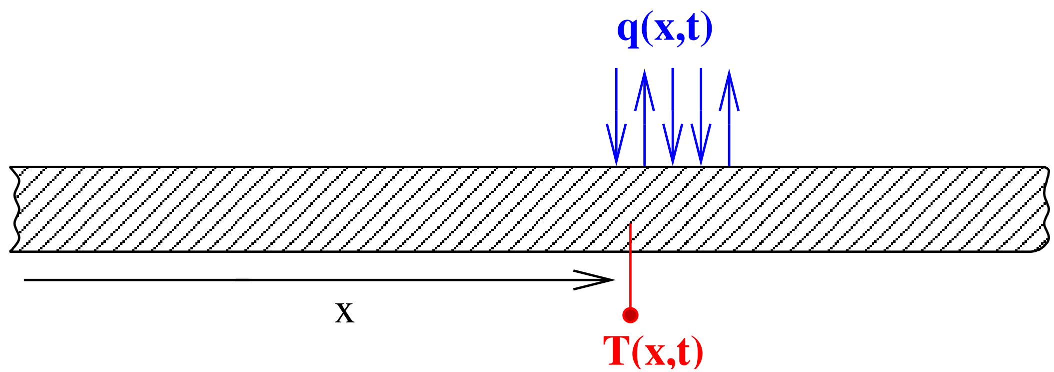
!! Invent storage and flux, locally defined in time and space, such that in every spatial domain there holds:

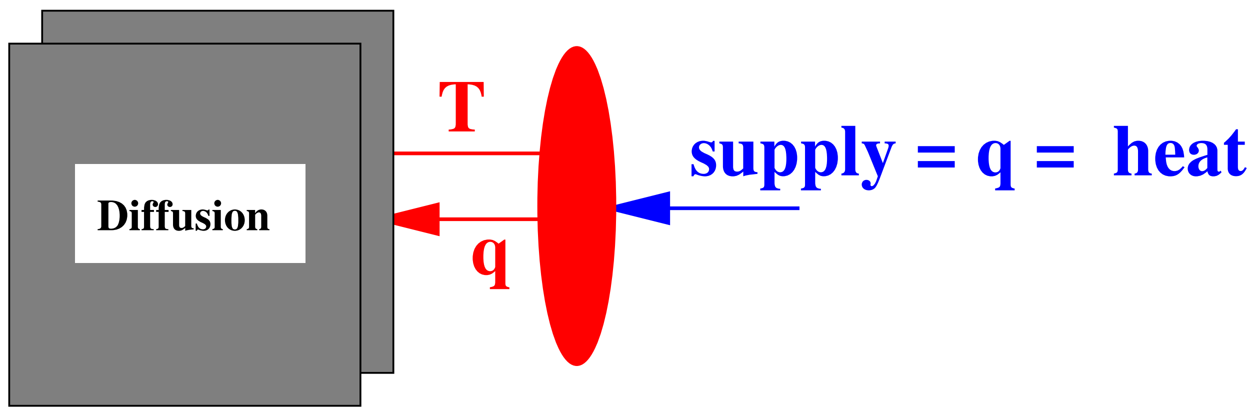
$$\frac{d}{dt}\,\text{Storage} + \text{Spatial flux} \leq \text{Supply.}$$



Supply = partly stored + partly radiated + partly dissipated.

Examples





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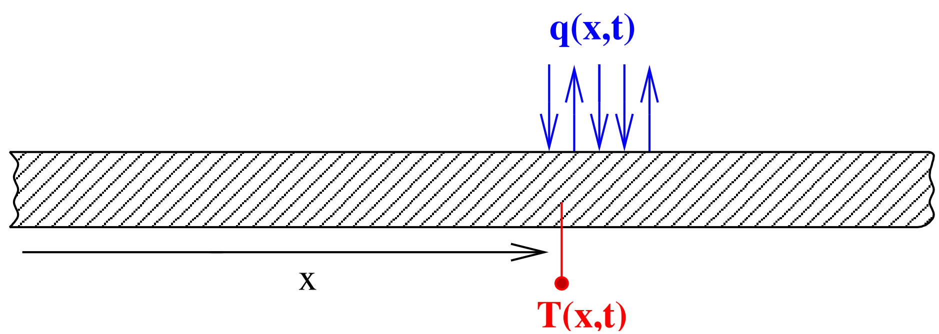
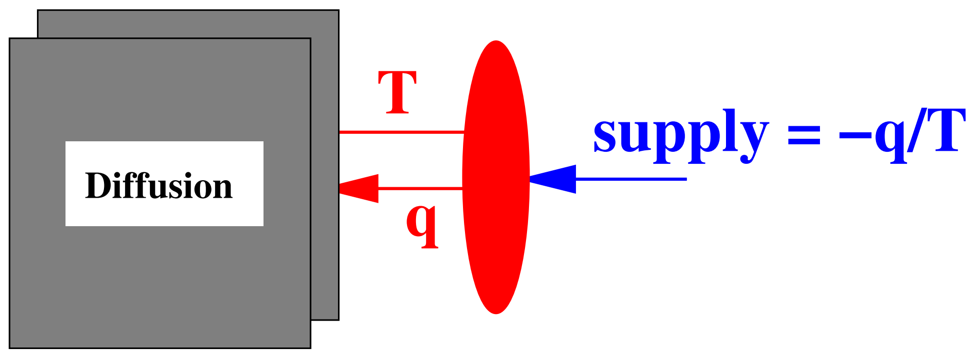
Examples



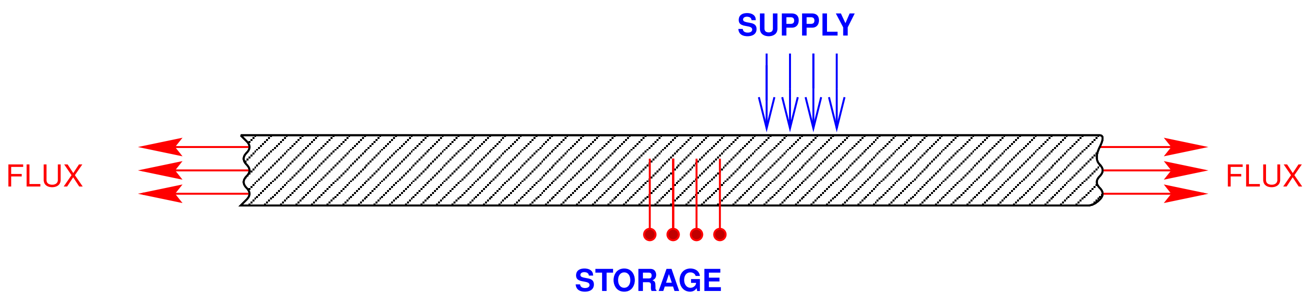


Dissipative. $\Leftrightarrow$ **for compact support** q**:**

$$\iint_{\mathbb{R}^2} \frac{q(x,t)}{T(x,t)}\, dx\, dt \leq 0$$

Examples

Can these 'global' versions be expressed as 'local' laws?



rate of change in storage + spatial flux $\leq$ supply rate

To be invented:

an 'extensive' quantity for the first law: internal energy

an 'extensive' quantity for the second law: entropy

Examples

Can these 'global' versions be expressed as 'local' laws?

Define the following variables:

$$E = T \quad : \text{ the stored energy density,}$$

$$S = \ln(T) \quad : \text{ the entropy density,}$$

$$F_E = -\frac{\partial}{\partial x}T \quad : \text{ the energy flux,}$$

$$F_S = -\frac{1}{T}\frac{\partial}{\partial x}T \quad : \text{ the entropy flux,}$$

$$D_S = (\frac{1}{T}\frac{\partial}{\partial x}T)^2 \quad : \text{ the rate of entropy production.}$$

Examples

Can these 'global' versions be expressed as 'local' laws?

Local versions **of the first and second law:**

rate of change in storage + spatial flux $\leq$ supply rate

Conservation of energy:

$$\frac{\partial}{\partial t}E + \frac{\partial}{\partial x}F_E = q,$$

Entropy production:

$$\frac{\partial}{\partial t}S + \frac{\partial}{\partial x}F_S = \frac{q}{T} + D_S.$$

Since $(D_S \geq 0) \Rightarrow$

$$\frac{\partial}{\partial t}S + \frac{\partial}{\partial x}F_S \geq \frac{q}{T}.$$

Examples

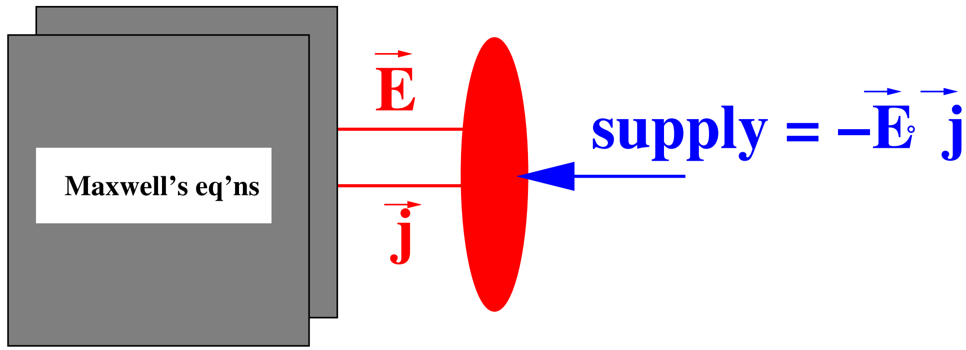
Can these 'global' versions be expressed as 'local' laws?

Problem:

Build a theory behind ad hoc constructions of E, F_E and S, F_S.

Complete as in the 1-d case....

Examples



$$\begin{aligned} \varepsilon_0 \frac{\partial}{\partial t} \nabla \cdot \vec{E} + \nabla \cdot \vec{j} &= 0, \\ \varepsilon_0 \frac{\partial^2}{\partial t^2} \vec{E} + \varepsilon_0 c^2 \nabla \times \nabla \times \vec{E} + \frac{\partial}{\partial t} \vec{j} &= 0. \end{aligned}$$

Conservative. $\Leftrightarrow$ **for compact support sol'ns of ME:**

$$\iiiint_{\mathbb{R}^4} \vec{E}(x,y,z,t) \cdot \vec{j}(x,y,z,t)\, dx\, dy\, dz\, dt = 0$$

There simply isn't a storage function in terms of only $\vec{E}, \vec{j}$!!

PDE’s and QDF’s

Linear differential distributed ($\mathtt{n}$-d) systems

$\mathbb{T} = \mathbb{R}^{\mathtt{n}}$, **the set of independent variables,**

typically $\mathtt{n} = 4$**: time and space,**

$\mathbb{W} = \mathbb{R}^{\mathtt{w}}$, **the set of dependent variables,**

$\mathfrak{B} =$ **the solutions of a linear constant coefficient PDE.**

Linear differential distributed (n-d) systems

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typically $\mathtt{n} = 4$: time and space,

$\mathbb{W} = \mathbb{R}^{\mathtt{w}}$, the set of dependent variables,

$\mathfrak{B}$ = the solutions of a linear constant coefficient PDE.

Let $R \in \mathbb{R}^{\bullet \times \mathtt{w}}[\xi_1, \cdots, \xi_{\mathtt{n}}]$, and consider

$$R\left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathtt{n}}}\right) w = 0. \quad (*)$$

Define the associated behavior

$$\mathfrak{B} = \{ w \in \mathfrak{C}^{\infty}(\mathbb{R}^{\mathtt{n}}, \mathbb{R}^{\mathtt{w}}) \mid (*) \text{ holds} \}.$$

Notation for $\mathtt{n}$-D linear differential systems:

$$(\mathbb{R}^{\mathtt{n}}, \mathbb{R}^{\mathtt{w}}, \mathfrak{B}) \in \mathfrak{L}^{\mathtt{w}}_{\mathtt{n}}, \quad \text{or } \mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\mathtt{n}}.$$

Image representation

$$\boldsymbol{R}\left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathrm{n}}}\right) \boldsymbol{w} = \mathbf{0}$$

is called a kernel representation of the associated $\mathfrak{B} \in \mathfrak{L}_{\mathtt{n}}^{\mathtt{w}}$.

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Another representation: image representation

$$w=M\left(\frac{\partial}{\partial x_1},\cdots,\frac{\partial}{\partial x_{\mathtt{n}}}\right)\ell$$

Elimination thm $\Rightarrow$ $\operatorname{im}\left(M\left(\frac{\partial}{\partial x_1},\cdots,\frac{\partial}{\partial x_{\mathtt{n}}}\right)\right)\in\mathfrak{L}^{\mathtt{w}}_{\mathtt{n}}$!

Do all behaviors of linear constant coefficient PDE's admit an image representation???

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Do all behaviors of linear constant coefficient PDE's admit an image representation???

$\mathfrak{B} \in \mathfrak{L}_{\mathrm{n}}^{\mathrm{w}}$ admits an image representation iff it is 'controllable'.

Are Maxwell's equations controllable ?

Are Maxwell's equations controllable ?

The following equations
in the *scalar potential* $\phi : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$ and
the *vector potential* $\vec{A} : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$
generate exactly the solutions to Maxwell's equations:

$$
\begin{aligned}
\vec{E} &= -\frac{\partial}{\partial t}\vec{A} - \nabla\phi, \\
\vec{B} &= \nabla \times \vec{A}, \\
\vec{j} &= \varepsilon_0 \frac{\partial^2}{\partial t^2}\vec{A} - \varepsilon_0 c^2 \nabla^2 \vec{A} + \varepsilon_0 c^2 \nabla\left(\nabla \cdot \vec{A}\right) + \varepsilon_0 \frac{\partial}{\partial t}\nabla\phi, \\
\rho &= -\varepsilon_0 \frac{\partial}{\partial t}\nabla \cdot \vec{A} - \varepsilon_0 \nabla^2 \phi.
\end{aligned}
$$

Proves controllability. Illustrates the interesting connection

controllability $\Leftrightarrow$ $\exists$ potential!

Observability

Observability **of the image representation**

$$w = M\left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathrm{n}}}\right)\ell$$

is defined as: ℓ **can be deduced from** w**,**
i.e. $M\left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathrm{n}}}\right)$ **should be injective.**

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Not all controllable systems admit an observable im. repr'n.
For $\mathrm{n} = 1$**, they do.** $\Leftrightarrow$ **right co-prime factorization of** G**.**
For $\mathrm{n} > 1$**, exceptionally so.**

The latent variable ℓ **in an im. repr'n may be 'hidden'.**

Example: Maxwell's equations do not allow a potential representation with an observable potential.

Notation

Where convenient, use multi-index notation:

$$x = (x_1, \ldots, x_n)\,,$$
$$\xi = (\xi_1, \cdots, \xi_n)\,, \zeta = (\zeta_1, \ldots, \zeta_n)\,, \eta = (\eta_1, \ldots, \eta_n)\,,$$

$$\frac{d}{dx} = \left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right), \frac{d^k}{dx^k} = \left(\frac{\partial^{k_1}}{\partial x_1^{k_1}}, \ldots, \frac{\partial^{k_n}}{\partial x_n^{k_n}}\right),$$
$$dx = dx_1 dx_2 \ldots dx_n,$$

etc.

QDF's

The quadratic map acting on $w : \mathbb{R}^{\mathtt{n}} \longrightarrow \mathbb{R}^{\mathtt{w}}$ and its derivatives, defined by

$$w \mapsto \sum_{k,\ell} \left(\frac{d^k}{dx^k} w \right)^\top \Phi_{k,\ell} \left(\frac{d^\ell}{dx^\ell} w \right)$$

is called *quadratic differential form* (QDF) on $\mathfrak{C}^\infty\left(\mathbb{R}^{\mathtt{n}}, \mathbb{R}^{\mathtt{w}}\right)$.

$\Phi_{k,\ell} \in \mathbb{R}^{\mathtt{w} \times \mathtt{w}}$; WLOG: $\Phi_{k,\ell} = \Phi_{\ell,k}^\top$.

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$\Phi_{k,\ell} \in \mathbb{R}^{\mathtt{w} \times \mathtt{w}}$; WLOG: $\Phi_{k,\ell} = \Phi_{\ell,k}^\top$.

Introduce the $2\mathtt{n}$-variable polynomial matrix Φ

$$\Phi\left(\zeta, \eta\right) = \sum_{k,\ell} \Phi_{k,\ell} \zeta^k \eta^\ell .$$

Denote the QDF as Q_Φ. QDF's are parameterized by $\mathbb{R}\left[\zeta, \eta\right]$.

Dissipative distributed systems

We henceforth consider only controllable linear differential systems and QDF's for supply rates.

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$\mathfrak{B} \in \mathfrak{L}_{\mathtt{n}}^{\mathtt{w}}$, controllable, is

dissipative with respect to the supply rate Q_{Φ} (a QDF)

$\Leftrightarrow$

$$\int_{\mathbb{R}^{\mathtt{n}}} Q_{\Phi}(w)\ dx \geq 0$$

for all $w \in \mathfrak{B}$ of compact support, i.e., for all $w \in \mathfrak{B} \cap \mathfrak{D}$.

$\mathfrak{D} := \mathfrak{C}^{\infty}$ and 'compact support'.

∃ Storage and Flux

MAIN RESULT (stated for $\mathtt{n} = 4$)

Thm: $\mathtt{n} = 4 : x, y, z; t :$ space/time; $\mathfrak{B} \in \mathfrak{L}_4^{\mathtt{w}}$, controllable.

Then $\int_{\mathbb{R}} \left[\int_{\mathbb{R}^3} Q_{\Phi}(w) \; dxdydz \right] \; dt \geq 0$ for all $w \in \mathfrak{B} \cap \mathfrak{D}$

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and QDF's S, the *storage,* and F_x, F_y, F_z, the *flux,*

such that the *local dissipation law*

$$\frac{\partial}{\partial t}S(\ell) + \frac{\partial}{\partial x}F_x(\ell) + \frac{\partial}{\partial y}F_y(\ell) + \frac{\partial}{\partial z}F_z(\ell) \leq Q_\Phi(w)$$

holds for all (w, ℓ) that satisfy $w = M\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \frac{\partial}{\partial t}\right)\ell$.

Hidden variables

The local law involves

possibly unobservable, - i.e., hidden!

latent variables (the ℓ's).

This gives physical notions as stored energy, entropy, etc., an enigmatic physical flavor.

Energy stored in EM fields

Maxwell's equations are dissipative (in fact, conservative) with respect to $-\vec{E}\cdot\vec{j},$ **the rate of energy supplied.**

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Maxwell's equations are dissipative (in fact, conservative) with respect to $-\vec{E}\cdot\vec{j}$, the rate of energy supplied.

Introduce the *stored energy density*, S, and
the *energy flux density* (the *Poynting vector*), $\vec{F}$,

$$S\left(\vec{E},\vec{B}\right) := \frac{\varepsilon_0}{2}\vec{E}\cdot\vec{E} + \frac{\varepsilon_0 c^2}{2}\vec{B}\cdot\vec{B},$$

$$\vec{F}\left(\vec{E},\vec{B}\right) := \varepsilon_0 c^2 \vec{E}\times\vec{B}.$$

Local conservation law for Maxwell's equations:

$$\boxed{\frac{\partial}{\partial t}S\left(\vec{E},\vec{B}\right) + \nabla\cdot\vec{F}\left(\vec{E},\vec{B}\right) = -\vec{E}\cdot\vec{j}.}$$

Involves $\vec{B}$, unobservable from $\vec{E}$ and $\vec{j}$.

Outline of the proof

Using **controllability** and **image representations**, we may assume, WLOG: $\mathfrak{B} = \mathfrak{C}^{\infty}(\mathbb{R}^{\mathtt{n}}, \mathbb{R}^{\mathtt{w}})$

To be shown

Global dissipation :$\Leftrightarrow$

$$\int_{\mathbb{R}^{\mathtt{n}}} Q_{\Phi}(w) \geq 0 \text{ for all } w \in \mathfrak{D}$$

$$\Updownarrow$$

$$\exists\ \Psi : \quad \nabla \cdot Q_{\Psi}(w) \leq Q_{\Phi}(w) \text{ for all } w \in \mathfrak{C}^{\infty}$$

$\Leftrightarrow$: **Local dissipation**

$$\int_{\mathbb{R}^{\mathrm{n}}} Q_{\Phi}(w) \geq 0 \text{ for all } w \in \mathfrak{D}$$

$\Updownarrow$ **(Parseval)**

$$\Phi(-i\omega, i\omega) \geq 0 \text{ for all } \omega \in \mathbb{R}^{\mathrm{n}}$$

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$$\exists\ \Psi: \quad (\zeta + \eta)^\top \Psi(\zeta, \eta) = \Phi(\zeta, \eta) - D^\top(\zeta) D(\eta)$$

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$\Updownarrow$ **(clearly)**

$$\exists\ \Psi: \quad \nabla \cdot Q_\Psi(w) \leq Q_\Phi(w) \text{ for all } w \in \mathfrak{C}^\infty$$

Outline of the proof

Assuming factorizability, we indeed obtain:

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$\Leftrightarrow$: **Local dissipation**

However, ... this argument is valid only for $\mathrm{n} = 1$**...**

SOS

The factorization equation

Consider

$$X^{\top}(-\xi)\,X(\xi) = Y(\xi) \qquad \textbf{(FE)}$$

with $Y \in \mathbb{R}^{\bullet\times\bullet}[\xi]$ **given, and** X **the unknown.** ***Solvable??***

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Under what conditions on Y does there exist a solution X?

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Under what conditions on Y does there exist a solution X?

Scalar case: write the real polynomial Y as a sum of squares

$$Y = x_1^2 + x_2^2 + \cdots + x_k^2.$$

$$X^\top(\xi)\,X(\xi) = Y(\xi) \quad \textbf{(SOS)}$$

Y given polynomial matrix; X the unknown, $\xi = (\xi_1, \cdots, \xi_{\mathtt{n}})$.

For $\mathtt{n} = 1$ and $Y \in \mathbb{R}[\xi]$, solvable (with $X \in \mathbb{R}^2[\xi]$) iff

$$Y(\alpha) \geq 0 \qquad \text{for all } \alpha \in \mathbb{R}.$$

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For $n = 1$ and $Y \in \mathbb{R}^{\bullet\times\bullet}[\xi]$, it is well-known (but non-trivial) that (SOS) is solvable (with $X \in \mathbb{R}^{\bullet\times\bullet}[\xi]$!) iff

$$Y(\alpha) = Y^\top(\alpha) \geq 0 \qquad \text{for all } \alpha \in \mathbb{R}.$$

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$$Y(\alpha) = Y^\top(\alpha) \geq 0 \qquad \text{for all } \alpha \in \mathbb{R}.$$

For $\mathrm{n} > 1$ and under the symmetry and positivity condition

$$Y(\alpha) = Y^\top(\alpha) \geq 0 \qquad \text{for all } \alpha \in \mathbb{R}^{\mathrm{n}},$$

this equation can nevertheless in general not be solved over the polynomial matrices, for $X \in \mathbb{R}^{\bullet\times\bullet}[\xi]$. The Motzkin polynomial $x^2y^4 + x^4y^2 + 1 - 3x^2y^2$ is non-neg., but not factorizable.

Cases where non-negativity $\Leftrightarrow$ SOS:

or $\mathrm{n} = 1$, or degree = 2, or $\mathrm{n} = 2$ and degree = 4.

$$X^\top(\xi)\,X(\xi) = Y(\xi) \quad \textbf{(SOS)}$$

Y given polynomial matrix; X the unknown, $\xi = (\xi_1, \cdots, \xi_n)$.

For $n = 1$ and $Y \in \mathbb{R}^{\bullet\times\bullet}[\xi]$, it is well-known (but non-trivial) that (SOS) is solvable (with $X \in \mathbb{R}^{\bullet\times\bullet}[\xi]$!) iff

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this equation can nevertheless in general not be solved over the polynomial matrices, for $X \in \mathbb{R}^{\bullet\times\bullet}[\xi]$. But it can be solved over the matrices of rational functions, i.e., for $X \in \mathbb{R}^{\bullet\times\bullet}(\xi)$.

Hilbert's 17-th

This factorizability is a consequence of Hilbert's 17-th pbm!

!! Solve $p = p_1^2 + p_2^2 + \cdots + p_k^2$, p given

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!! Solve $p = p_1^2 + p_2^2 + \cdots + p_k^2$, p given

A polynomial $p \in \mathbb{R}[\xi_1, \cdots, \xi_n]$, with $p(\alpha_1, \ldots, \alpha_n) \geq 0$ for all $(\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$ can in general not be expressed as a SOS of polynomials, with the p_i's $\in \mathbb{R}[\xi_1, \cdots, \xi_n]$.
But a rational function (and hence a polynomial) $p \in \mathbb{R}(\xi_1, \cdots, \xi_n)$, with $p(\alpha_1, \ldots, \alpha_n) \geq 0$, for all $(\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$, can be expressed as a SOS of ($k = 2^n$) rational functions, with the p_i's $\in \mathbb{R}(\xi_1, \cdots, \xi_n)$.

Outline of the proof

$\Rightarrow$ **solvability of the factorization eq'n**

$$\Phi\left(-i\omega, i\omega\right) \geq 0 \text{ for all } \omega \in \mathbb{R}^{\mathrm{n}}$$

$\Updownarrow$ **(Factorization equation)**

$$\exists\ D : \quad \Phi\left(-\xi, \xi\right) = D^{\top}\left(-\xi\right) D\left(\xi\right)$$

over the rational functions, i.e., with D a matrix with elements in $\mathbb{R}\left(\xi_1, \cdots, \xi_{\mathrm{n}}\right)$.

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The need to introduce rational functions in this factorization equation and an image representation of $\mathfrak{B}$ (to reduce the pbm to $\mathfrak{C}^{\infty}$) are the causes of the unavoidable presence of (possibly unobservable, i.e., 'hidden') latent variables in the local dissipation law.

Uniqueness

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2. **of D in the factorization equation**

$$\Phi(-\xi,\xi) = D^\top(-\xi)\,D(\xi)$$

3. **(in the case $\mathtt{n} > 1$) of the solution Ψ of**

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$$\Phi(-\xi,\xi) = D^\top(-\xi)\,D(\xi)$$

3. **(in the case $\mathtt{n} > 1$) of the solution Ψ of**

$$(\zeta+\eta)^\top \Psi(\zeta,\eta) = \Phi(\zeta,\eta) - D^\top(\zeta)\,D(\eta)$$

For conservative systems, $\Phi(-\xi,\xi) = 0$, whence $D = 0$, but, when $\mathtt{n} > 1$, the third source of non-uniqueness remains.

Uniqueness

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The ambiguity of the field energy

... There are, in fact, an infinite number of different possibilities for u **[the internal energy]** ***and*** S **[the flux]** ***... It is sometimes claimed that this problem can be resolved using the theory of gravitation ... as yet nobody has done such a delicate experiment ... So we will follow the rest of the world - besides, we believe that it*** **[our choice]** ***is probably perfectly right.***

The Feynman Lectures on Physics,
Volume II, page 27-6.

Conclusions

What to take home

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- $\exists$ very simple, flexible, general, behavioral def'ns of controllability and observability
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  inputs and outputs OK in signal processing,
  not in physics, not for interconnections
- Physicists and mathematicians should pay (more) attention to open systems

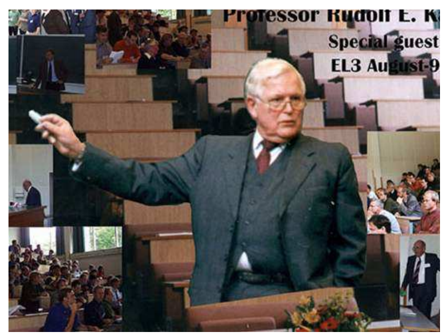
Motto

1. Get the physics right

2. The rest is mathematics

Once you get used to writing $w \in \mathfrak{B}$,

the rest is easy



R.E. Kalman, Opening lecture

IFAC World Congress, Prague, July 4, 2005

Thank you

Thank you

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