THE BEHAVIORAL APPROACH

to

SYSTEMS THEORY

Jan C. Willems

K.U. Leuven, Belgium

SCTW 2005 **Gebze Yüksek Teknoloji Enstitüsü, September 7, 2005**

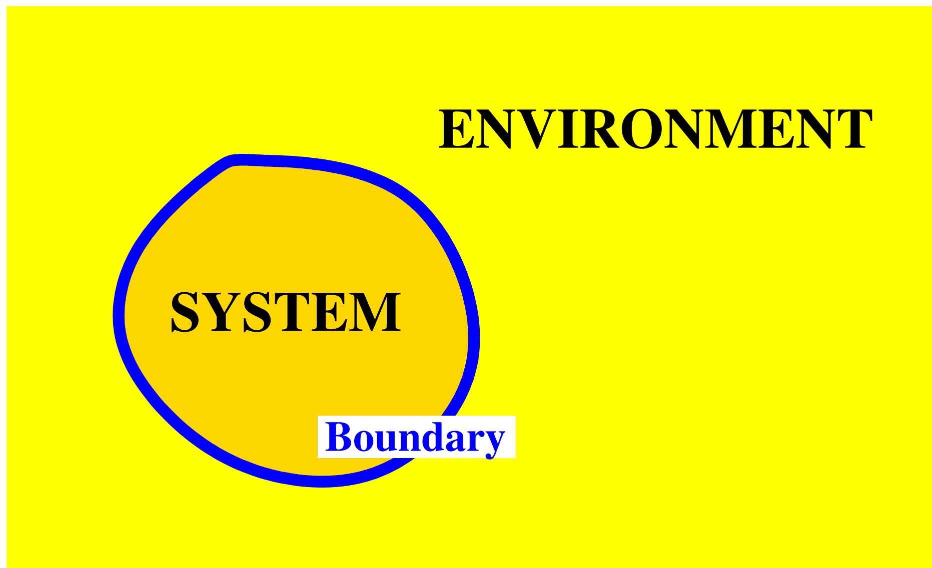
Open and Connected

The central tenets of our field:

Systems are open and consist of interconnected subsystems.

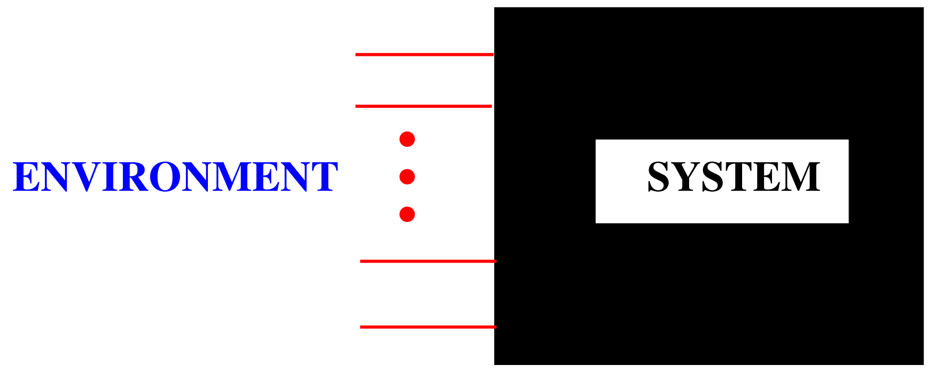
Synthesis of systems consists of interconnecting subsystems

Open



Open

In this lecture, we think of this interaction boundary as ‘terminals’



electrical components with ‘wires’

mechanical components with ‘pins’

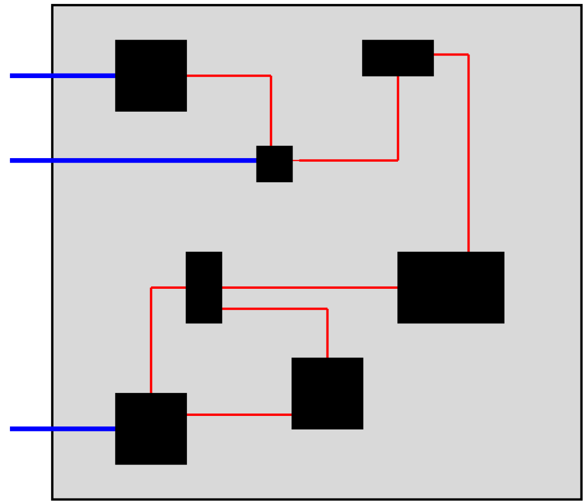
fluidic components with ‘ducts’

signal processors with inputs and outputs

motors with terminals & pins

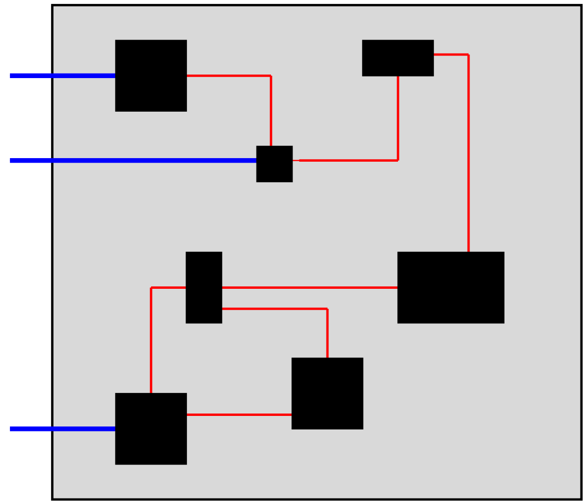
computer terminal, etc., etc., etc.

Connected



An interconnection architecture with subsystems

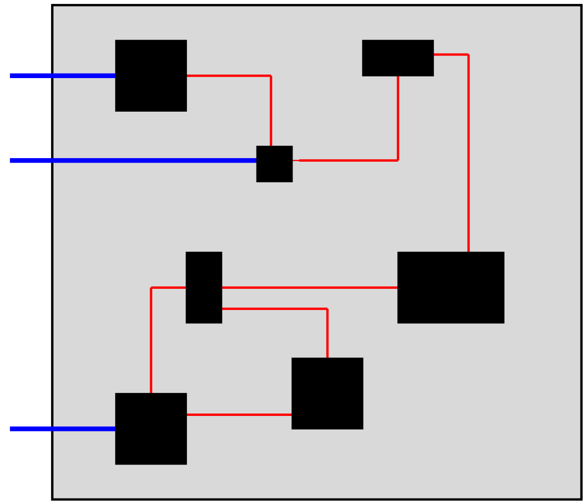
Connected



Think of:

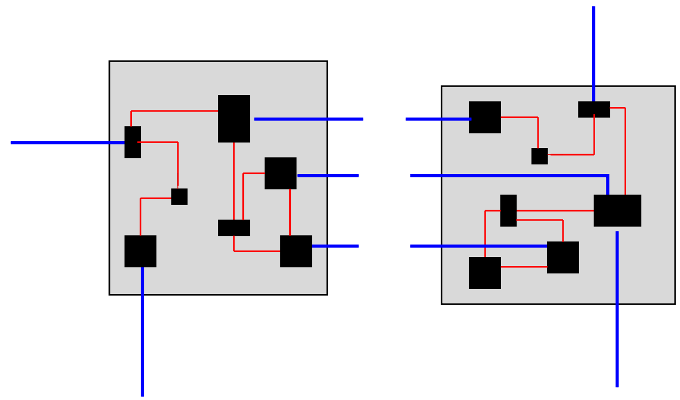
- **electrical circuits**
- **mechanical constructions**
- **fluidic systems**
- **networks of signal processors**
- **computers**
- **essentially all engineering systems**

Connected

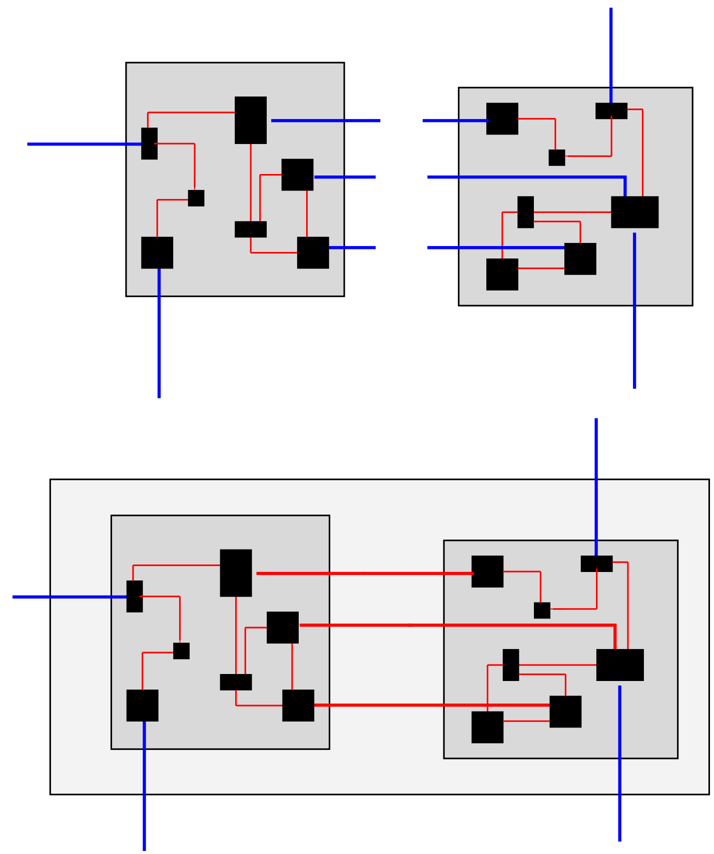


Observe the hierarchical nature

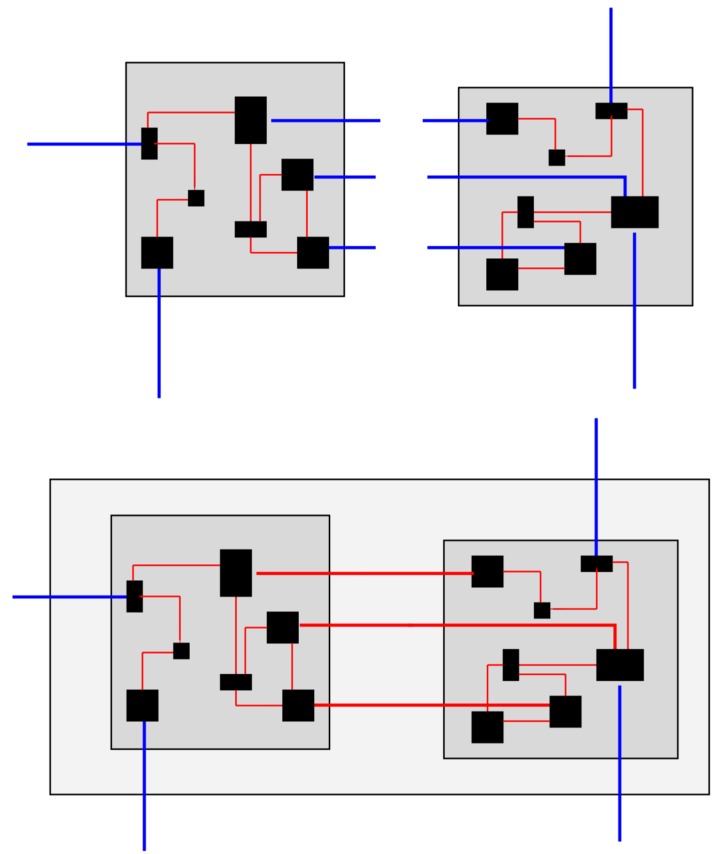
Interconnect



Interconnect



Interconnect



Reverse process: ‘tearing’ & ’zooming’ & ’linking’:

very useful in modeling.

Mathematization

What are the appropriate concepts / mathematization?

What is an open dynamical system?

How do we deal with interconnections?

How does control fit in?

Mathematization

1. **Get the physics right**
2. **The rest is mathematics**

R.E. Kalman, Opening lecture
IFAC World Congress, Prague, July 4, 2005

THEMES

First part:

1. Open and connected
2. A brief history of systems theory
3. Why are better framework is needed
4. Models and behaviors

Second part:

5 Linear time-invariant differential systems

6 Controllability and stabilizability

7 Representations of linear differential systems

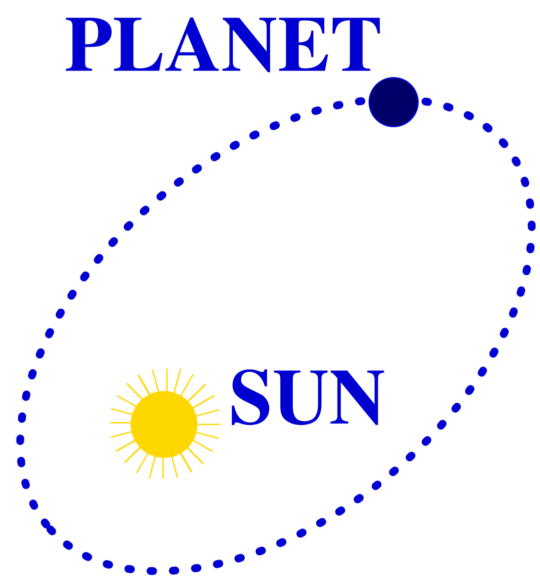
8 PDE's

How it all began ...

How does it move?

Kepler's laws

Johannes Kepler (1571-1630)



Kepler's laws:

Ellipse, sun in focus; = areas in = times; (period)^2 $\cong$ (diameter)^3

The equation of the planet

Consequence:

acceleration = function of position and velocity

$$\frac{d^2}{dt^2}w(t) = A\left(w(t), \frac{d}{dt}w(t)\right)$$

$\rightsquigarrow$ **via calculus and calculation**

$$\frac{d^2}{dt^2}w(t) + \frac{1_{w(t)}}{|w(t)|^2} = 0$$

The equation of the planet

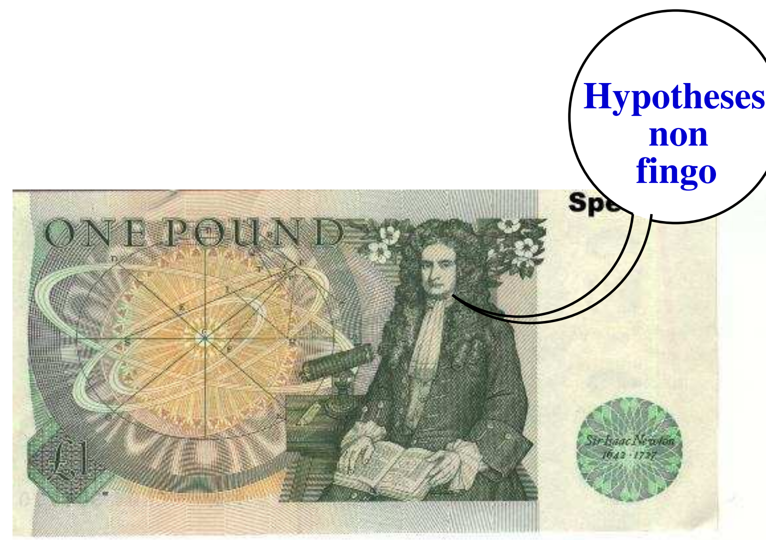
Consequence:

acceleration = function of position and velocity

$$\frac{d^2}{dt^2}w(t) = A\left(w(t), \frac{d}{dt}w(t)\right)$$

$\leadsto$ via calculus and calculation

$$\frac{d^2}{dt^2}w(t) + \frac{1_{w(t)}}{|w(t)|^2} = 0$$

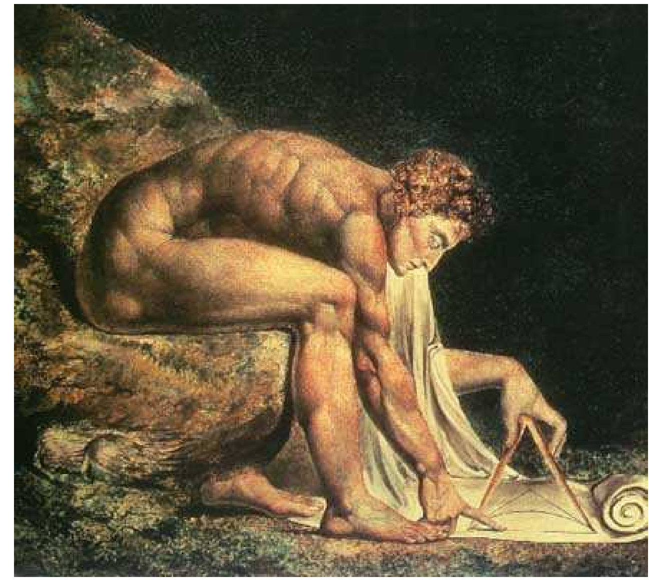


Newton's laws

2-nd law $F'(t) = m\frac{d^2}{dt^2}w(t)$

gravity $F''(t) = m\frac{1_{w(t)}}{|w(t)|^2}$

3-rd law $F'(t) + F''(t) = 0$



$\Downarrow$

$$\frac{d^2}{dt^2}w(t) + \frac{1_{w(t)}}{|w(t)|^2} = 0$$

The paradigm of closed systems

‘Axiomatization’

K.1, K.2, & K.3

$\rightsquigarrow \quad \frac{d^2}{dt^2}w(t) + \frac{1_{w(t)}}{|\frac{d}{dt}w(t)|^2} = 0$

$\rightsquigarrow \quad \frac{d}{dt}x = f(x)$

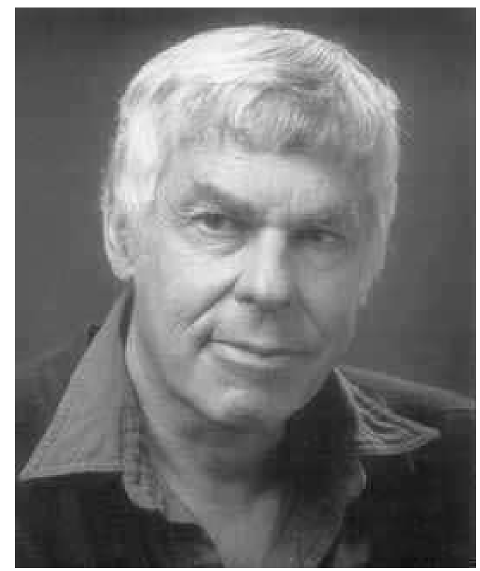
$\rightsquigarrow$ **‘dynamical systems’, flows**

$\rightsquigarrow$ **closed systems as paradigm of dynamics**

‘Axiomatization’

Henri Poincaré (1854-1912)

George Birkhoff (1884-1944)



Stephen Smale (1930-)

‘Axiomatization’

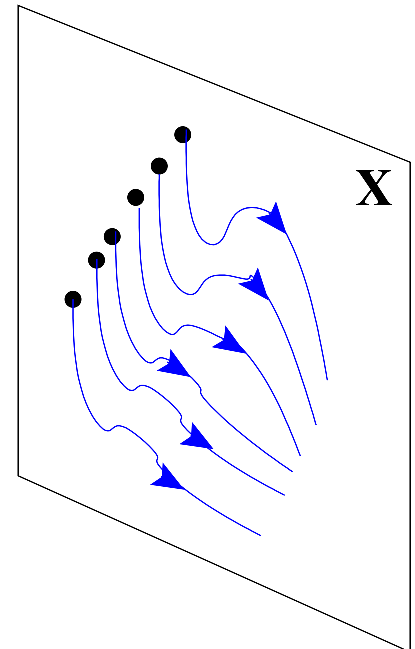
A *dynamical system* is defined by

a state space X and

a state transition function

ϕ : $\cdots$ such that $\cdots$

$\phi(t, \mathrm{x})$ = state at time t starting from state x



How could they forget about Newton’s second law,

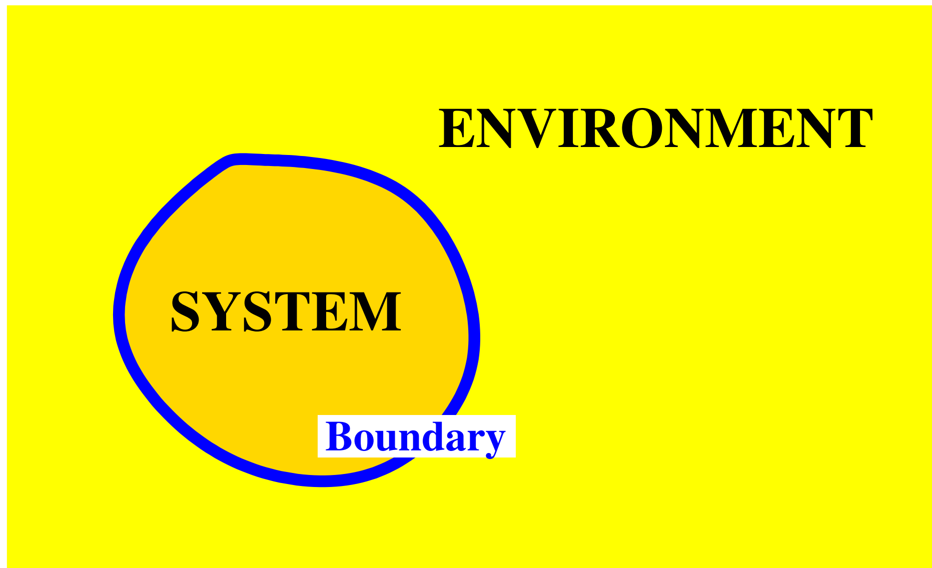
about Maxwell’s eq’ns,

about thermodynamics,

about tearing & zooming & linking, ...?

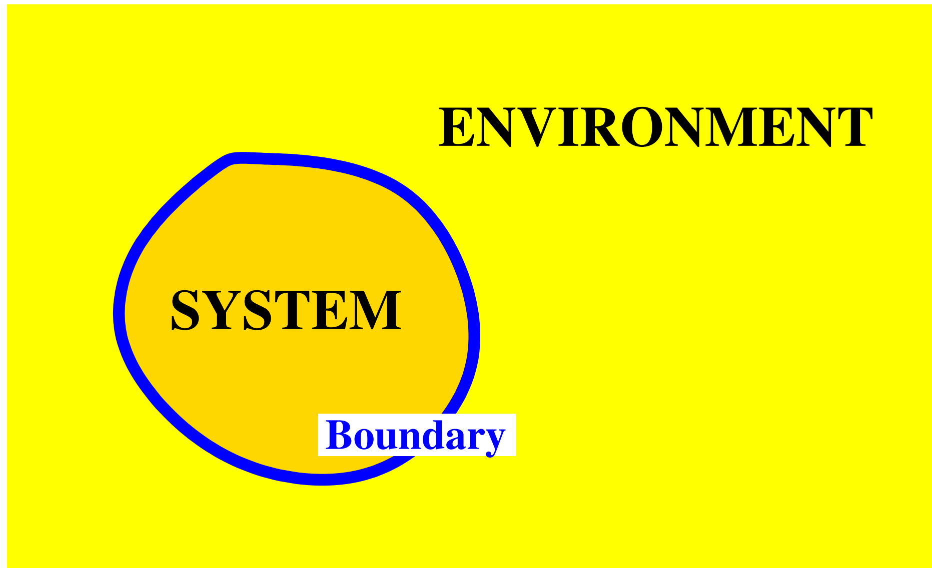
‘Axiomatization’

Reply: assume ‘fixed boundary conditions’



⇝ an absurd situation: to model a system,
we have to model also the environment!

'Axiomatization'

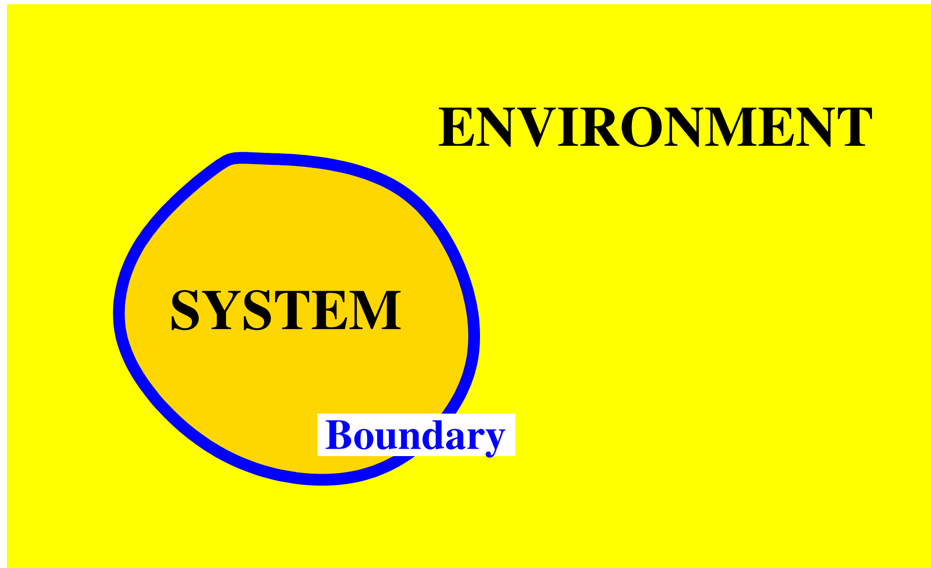


Chaos theory, cellular automata, sync, etc.,

'function' in this framework ...

Chaos: not a property of the physical laws,
but just as much of what the system is
interconnected to.

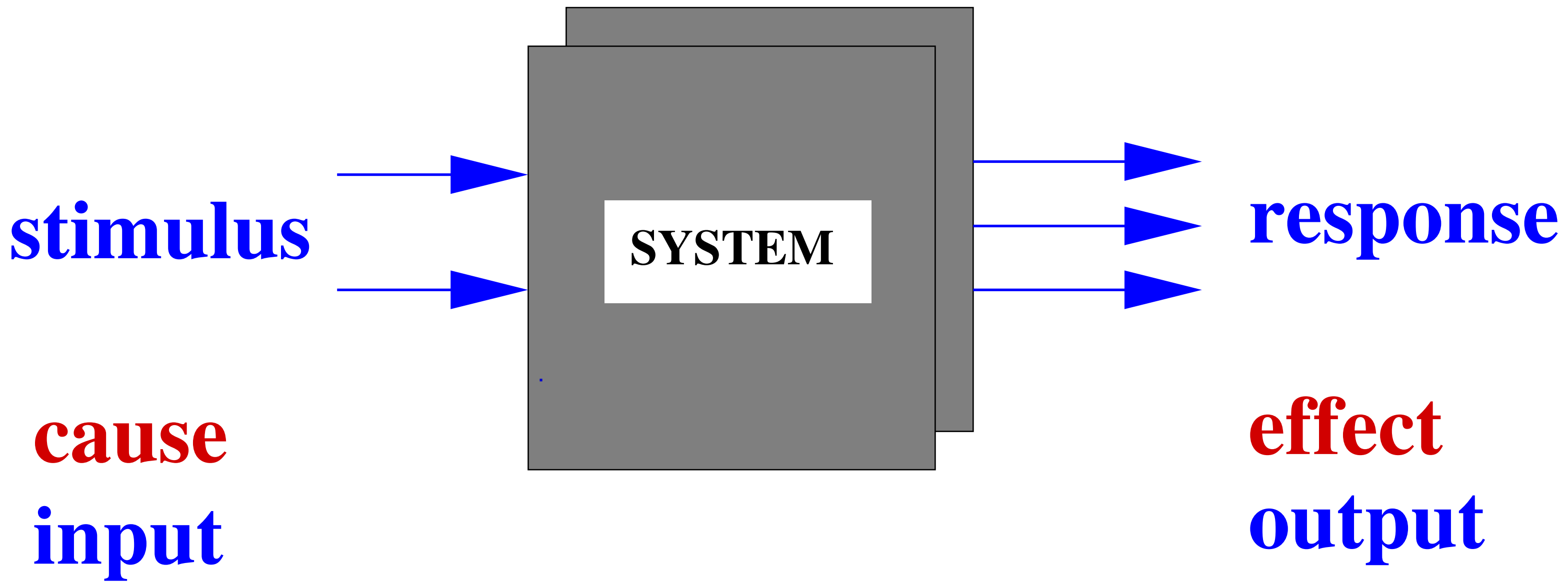
‘Axiomatization’

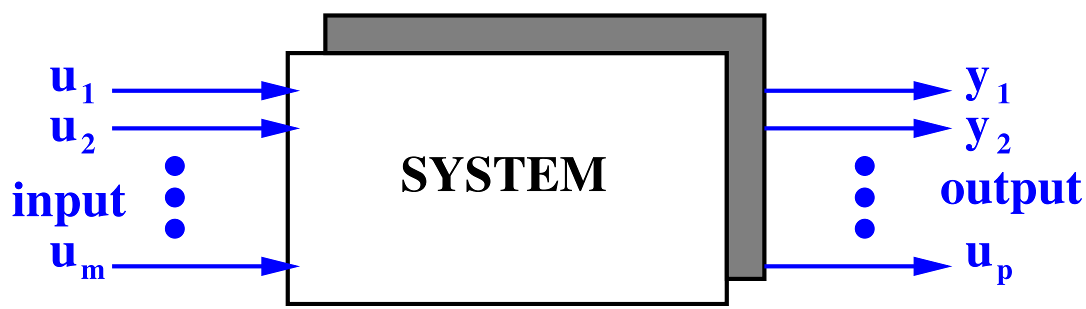


Turbulence may not be a property of Navier-Stokes, but just as much of the boundary conditions.

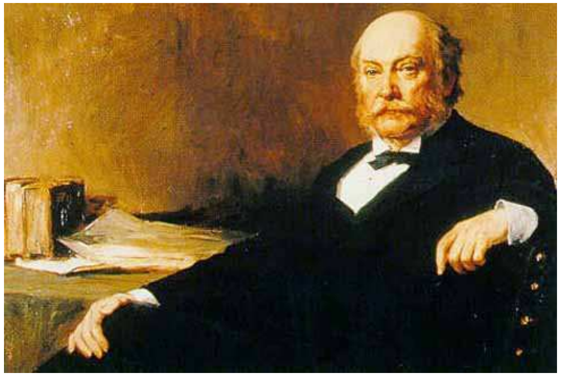
Meanwhile, in engineering, ...

Input/output systems

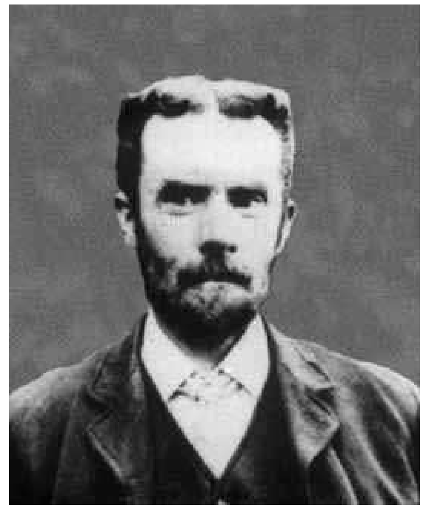




The originators



Lord Rayleigh (1842-1919)

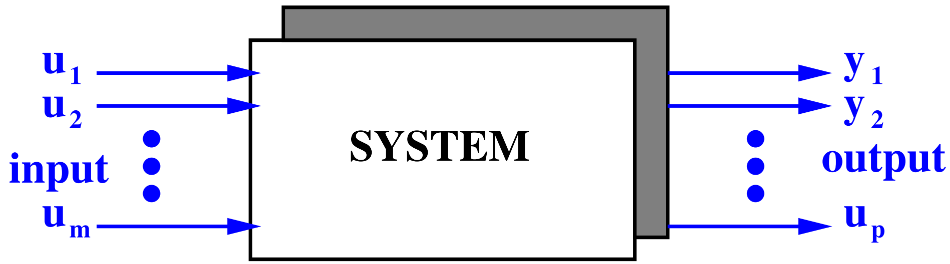


Oliver Heaviside (1850-1925)

Norbert Wiener (1894-1964)

and the many electrical circuit theorists ...

Mathematical description



$$y(t) = \int_{0 \text{ or } -\infty}^{t} H(t-t')u(t')\,dt'$$

$$y(t) = H_0(t) + \int_{-\infty}^{t} H_1(t-t')u(t')\,dt' + \int_{-\infty}^{t}\int_{-\infty}^{t'} H_2(t-t', t'-t'')u(t')u(t'')\,dt'dt'' + \cdots$$

These models fail to deal with ‘initial conditions’.

A physical system is SELDOM an i/o map

Input/state/output systems

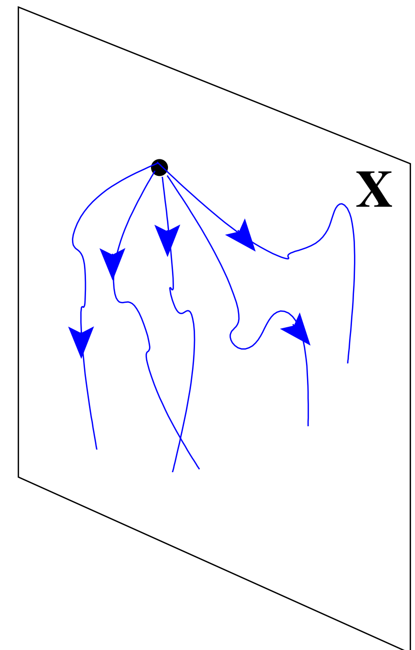
$$\rightsquigarrow \quad \frac{d}{dt}x = f(x,u), \; y = g(x,u)$$

Rudolf Kalman (1930-)

‘Axiomatization’

State transition function:

$\phi(t, \mathrm{x}, u)$: state reached at time t from x using input u.

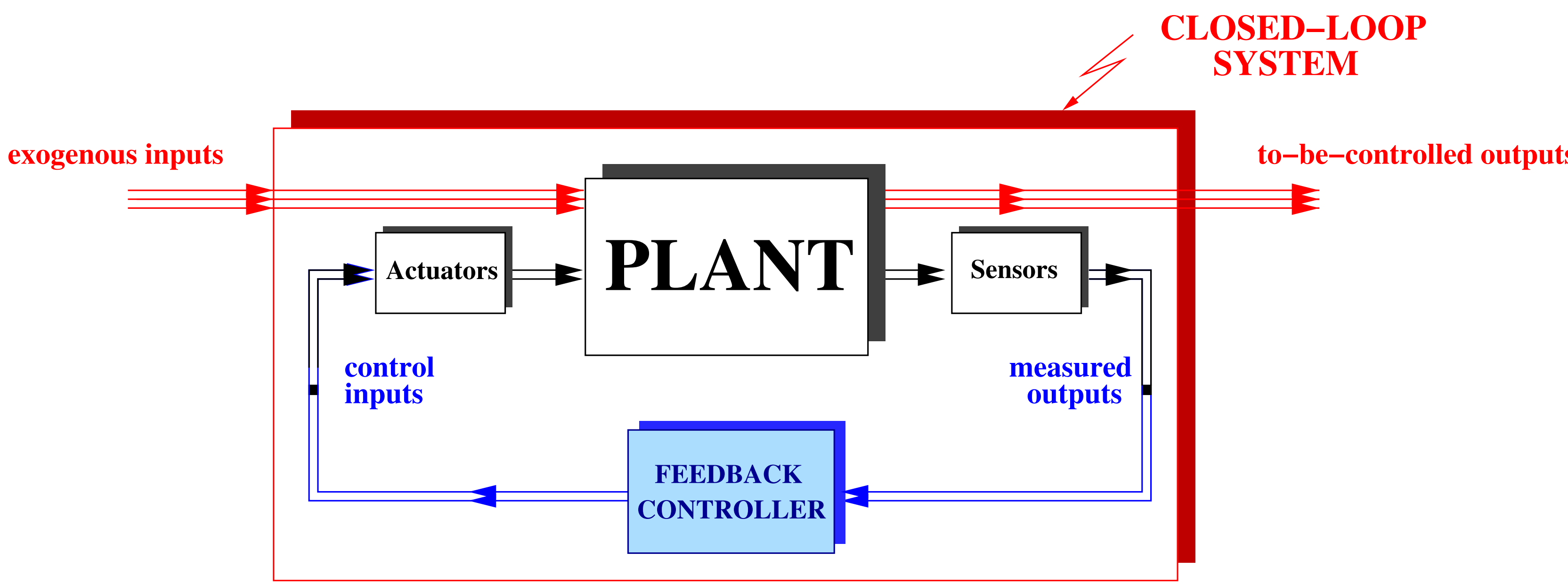


Read-out function:

$g(\mathrm{x}, \mathrm{u})$: output value with state x and input value u.

The input/state/output view turned out to be a very effective and fruitful paradigm

- for control (stabilization, robustness, ...)



The input/state/output view turned out to be
a very effective and fruitful paradigm

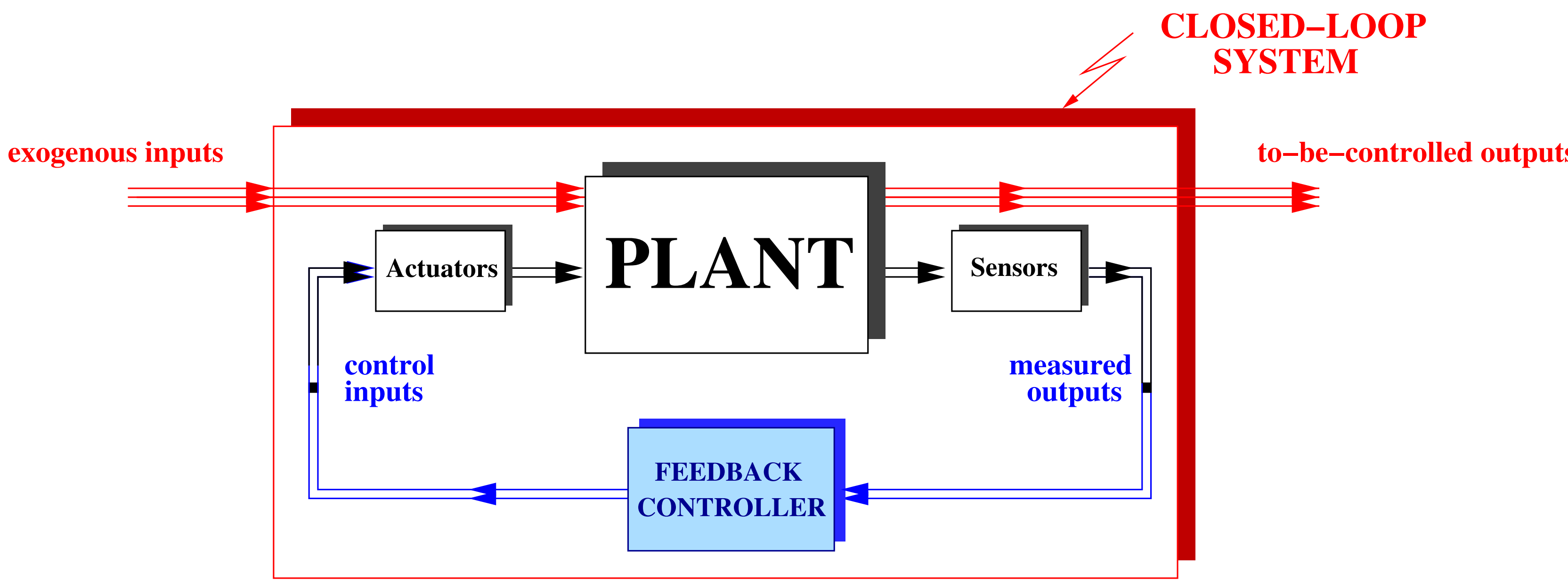
- for **control** (stabilization, robustness, ...)
- **prediction** of one signal from another, **filtering**
- understanding **system representations**
  (transfer f'n, input/state/output, etc.)
- model simplification, **reduction**
- **system ID:** models from data
- etc., etc., etc.

Let’s take a closer look at the i/o framework ...

in control

Difficulties with i/o

active control



versus **passive control**

Dampers, heat fins, pressure valves, ...

Controllers without sensors and actuators

Difficulties with i/o

active control versus **passive** control

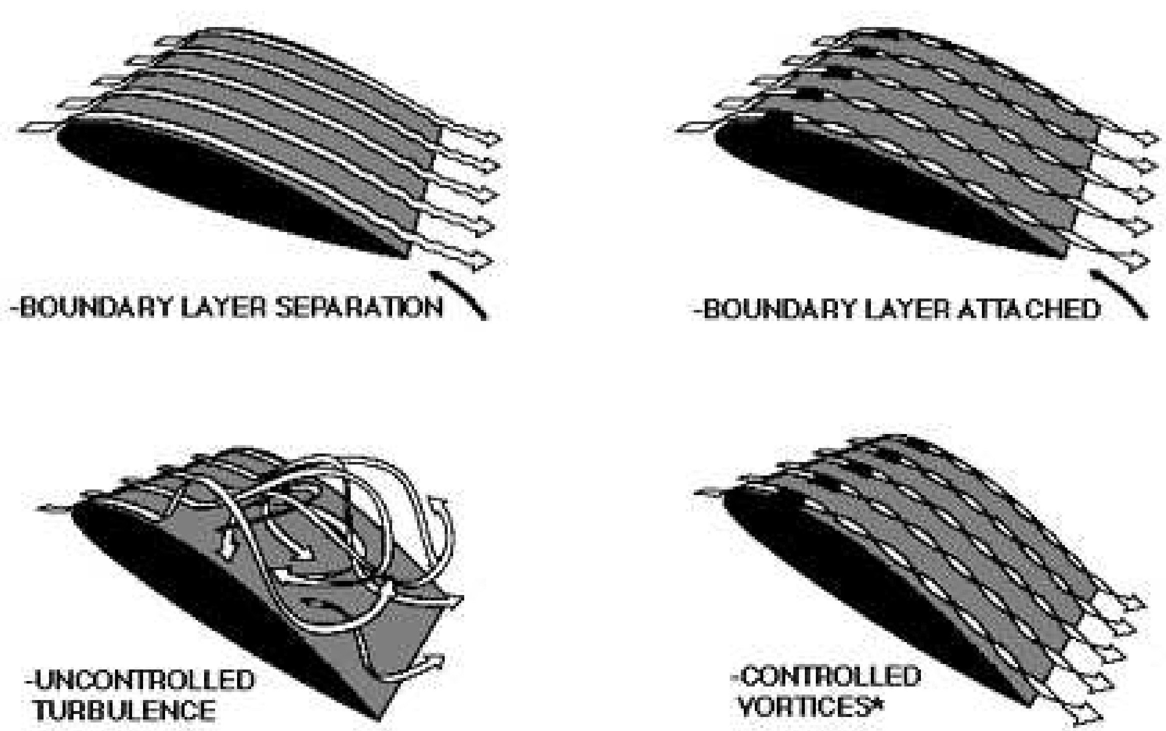
Controlling turbulence

for airplanes, sharks, dolphins, golf balls, bicycling helmets, etc.

Difficulties with i/o

active **control versus** **passive** **control**

Controlling turbulence



Difficulties with i/o

active control versus **passive** control

Controlling turbulence

Nagano 1998

Difficulties with i/o

active control versus **passive** control

Controlling turbulence

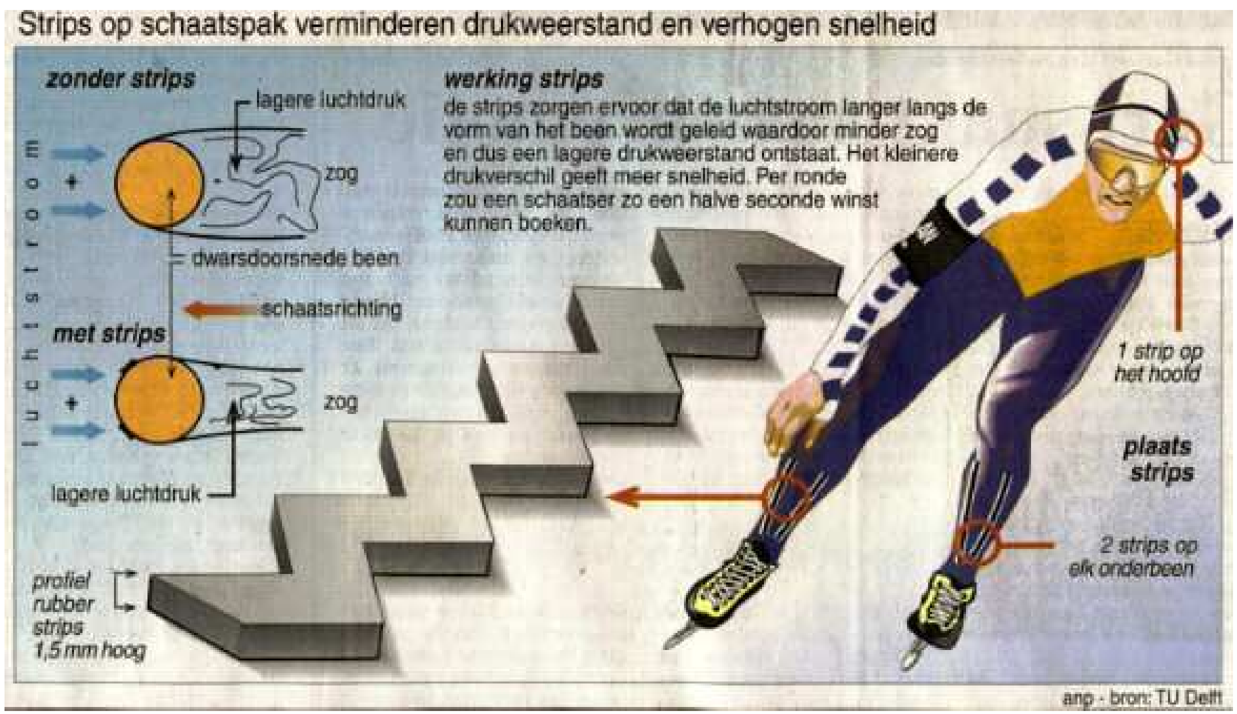
Nagano 1998

Difficulties with i/o

active control versus **passive** control

Controlling turbulence

Nagano 1998

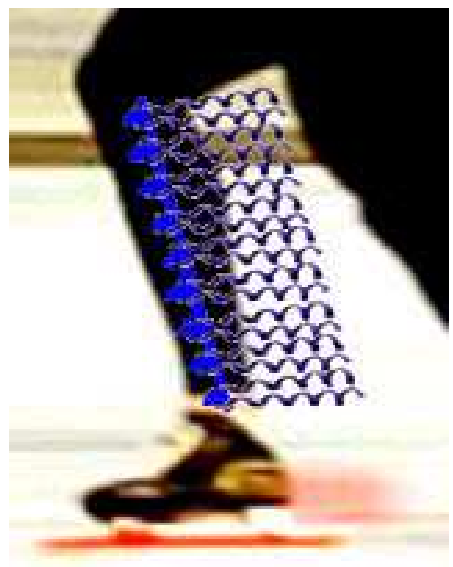


Difficulties with i/o

active control versus **passive** control

Controlling turbulence

Nagano 1998

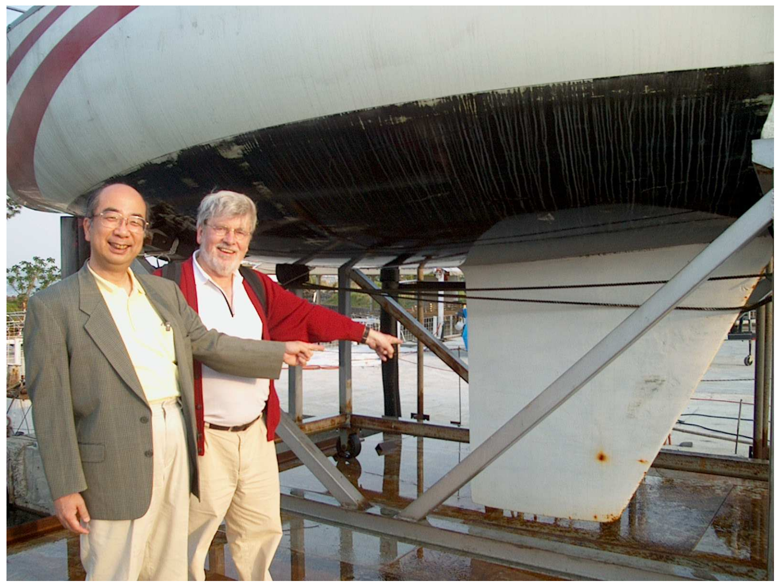


These are beautiful controllers! But, the only people not calling this "control", are the control engineers ...

Difficulties with i/o

active control versus **passive** control

Another example: the stabilizer of a ship



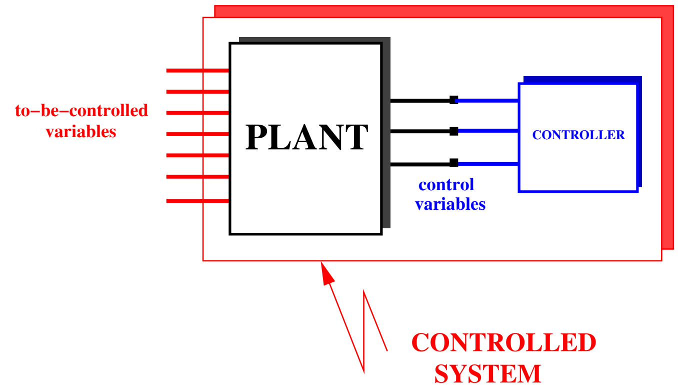
These are beautiful controllers! But, the only people not calling this ”stabilization”, are the control engineers ...

Btw, this interconnection is, but shouldn’t be, called ‘singular’

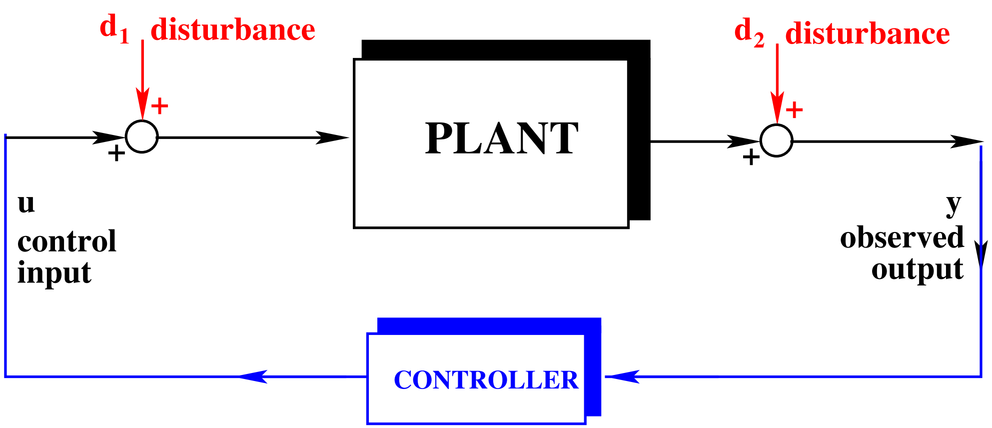
Difficulties with i/o

active control versus **passive** control

The appropriate figure is



With the 'classical' interconnection figure



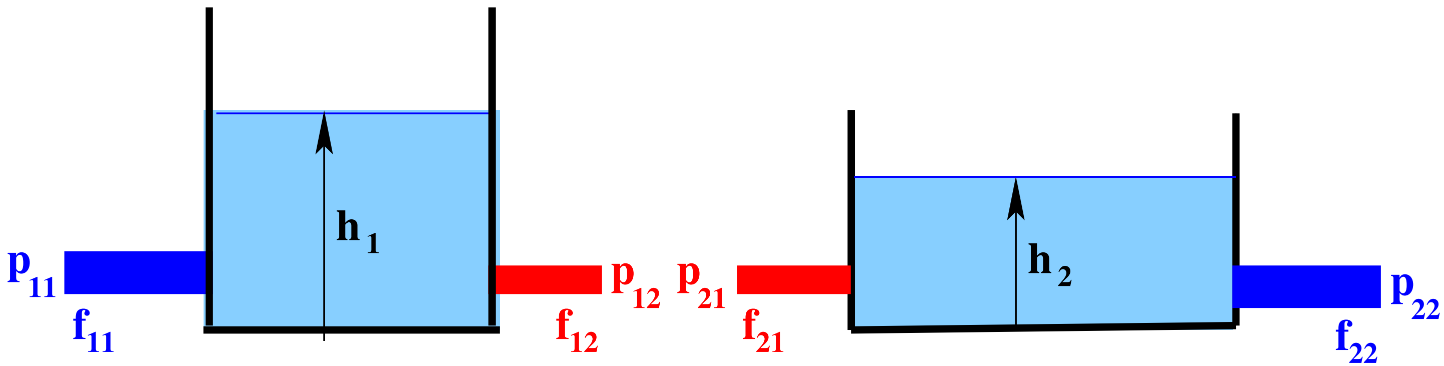
such controllers do not stabilize, because

dynamic order controlled system $<$ dynamic order plant $+$dynamic order contro

Let's take a closer look at the i/o framework ...

for interconnection

i/o and interconnection



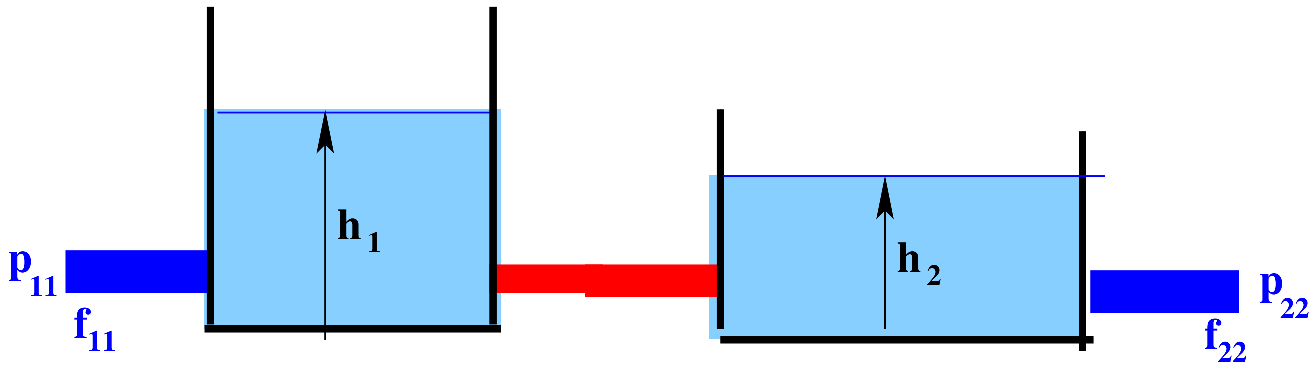
$$\frac{d}{dt}h_1 = F_1(h_1, p_{11}, p_{12}), f_{11} = H_{11}(h_1, p_{11}), f_{12} = H_{12}(h_1, p_{12})$$

$$\frac{d}{dt}h_2 = F_1(h_2, p_{21}, p_{22}), f_{21} = H_{21}(h_2, p_{21}), f_{22} = H_{22}(h_2, p_{22})$$

inputs: **the pressures** $p_{11}, p_{12}, p_{21}, p_{22}$

outputs: **the flows** $f_{11}, f_{12}, f_{21}, f_{22}$

i/o and interconnection



$$\frac{d}{dt}h_1 = F_1(h_1, p_{11}, p_{12}), f_{11} = H_{11}(h_1, p_{11}), f_{12} = H_{12}(h_1, p_{12})$$

$$\frac{d}{dt}h_2 = F_1(h_2, p_{21}, p_{22}), f_{21} = H_{21}(h_2, p_{21}), f_{22} = H_{22}(h_2, p_{22})$$

Interconnection: $p_{12} = p_{21}, f_{12} + f_{21} = 0$

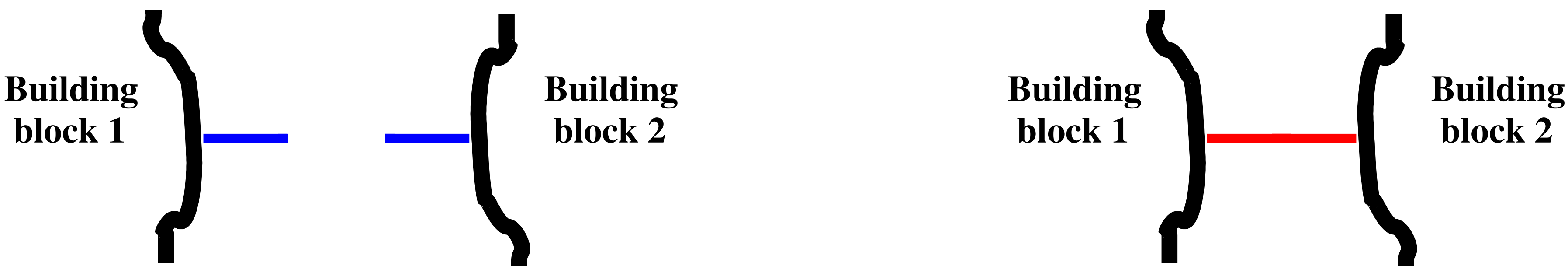
This identifies 2 inputs AND (NOT WITH) 2 outputs,

the sort of thing SIMULINK© forbids.

This situation is the rule, not the exception (in fluidics, mechanics,...)

Interconnection is not input-to-output assignment!

Sharing variables, not input-to-output assignment, is the basic mechanism by which systems interact.



Before interconnection:

the variables on the interconnected terminals are independent.

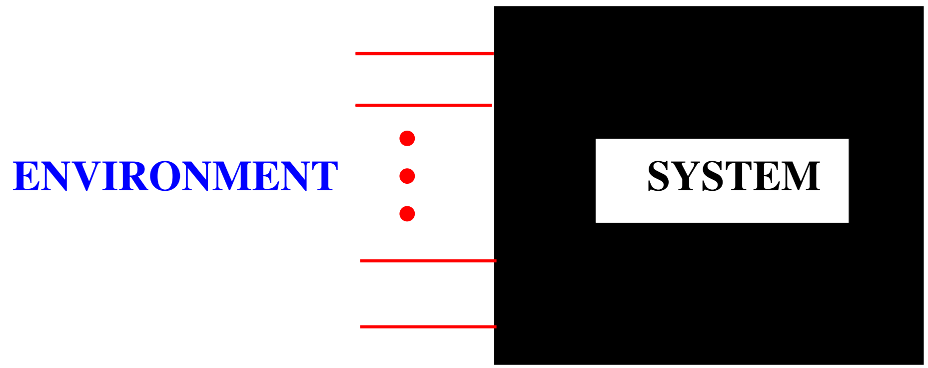
After interconnection: they are set equal.

Let’s take a closer look at the i/o framework ...

for modeling

i/o in modeling

Physical systems often interact with their environment through physical terminals

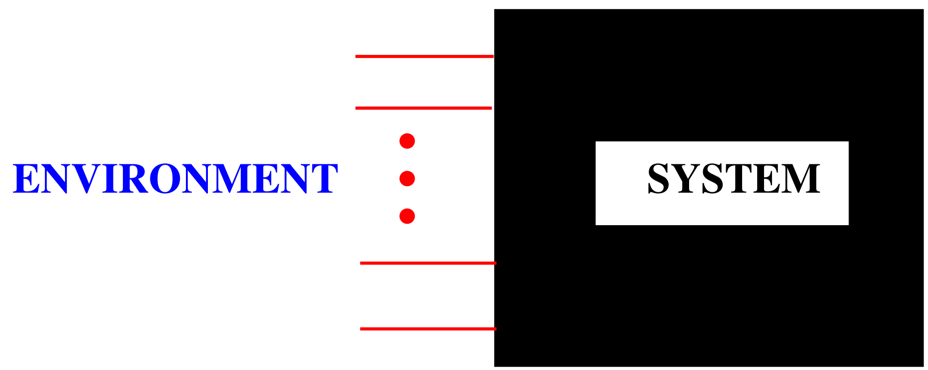


On each of these terminals many variables 'live':

- **voltage & current**
- **position & force**
- **pressure & flow**
- **price & demand**
- **angle & momentum**
- **etc. & etc.**

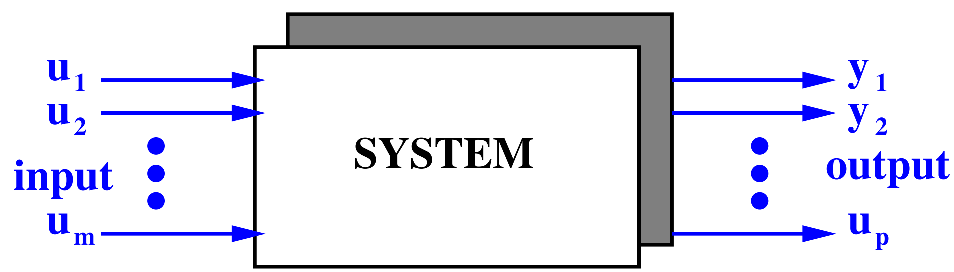
i/o in modeling

Physical systems often interact with their environment through **physical** terminals



Situation is NOT:

on one terminal there is an input, on another there is an output.



This picture is misleading, if superficially interpreted.

i/o in modeling

Physical systems often interact with their environment through physical terminals

The selection of what is an input and what is an output

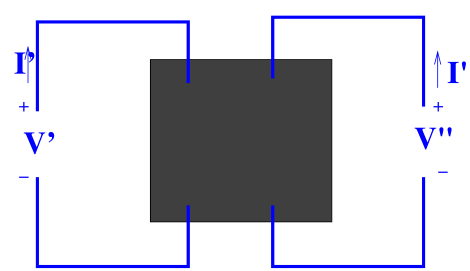
- most often does not need to be made
- if it made, it should be made after the modeling is done
- sometimes it cannot be made

i/o in modeling

Physical systems often interact with their environment through physical terminals

The selection of what is an input and what is an output

- **does not need to be made**
- **if it made, it should be made after the modeling is done**



voltage controlled?

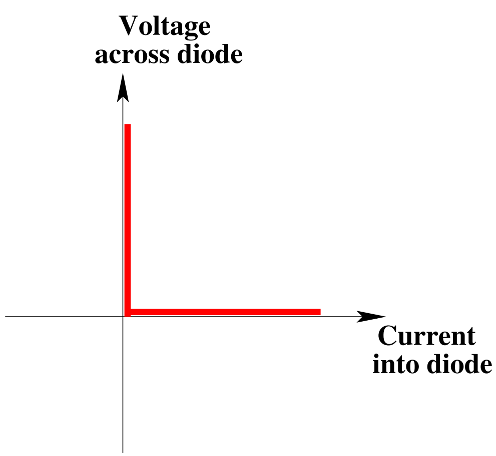
- **sometimes it cannot be made**

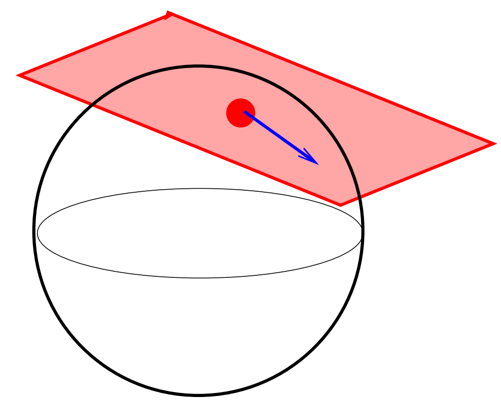
i/o in modeling

Physical systems often interact with their environment through physical terminals

The selection of what is an input and what is an output

- does not need to be made
- if it made, it should be made after the modeling is done
- sometimes it cannot be made





variables: (x, v) $\quad \frac{d}{dt}x = v$

tangent bundle of the sphere is not 'trivial'

Conclusion

The inability of the i/o framework to properly deal with

(i) interconnections

and

(ii) passive control

is lethal.

Just as the state, the input/output partition needs to be constructed from first principles models. Contrary to the state, such a partition may not be useful, or even possible

We need a better, more flexible, universal, simpler framework that properly deals with

open & connected.

General formalism

Generalities

What is a model? As a mathematical concept.
What is a dynamical system? What is the role of differential equations in thinking about dynamical models?

Generalities

Intuition

We have a ‘phenomenon’ that produces ‘outcomes’ (‘events’).
We wish to model the outcomes that can occur.

Before we model the phenomenon:
the outcomes are in a set, which we call the *universum*.

After we model the phenomenon:
the outcomes are declared (thought, believed)
to belong to the *behavior* of the model,
a subset of this universum.

This subset is what we consider the mathematical model.

Generalities

This way we arrive at the

Definition

A ***math. model*** is a subset $\mathfrak{B}$ of a universum $\mathfrak{U}$ of outcomes

$$\mathfrak{B} \subseteq \mathfrak{U}.$$

$\mathfrak{B}$ is called the ***behavior*** of the model.
For example, the ideal gas law states that the temperature T, pressure P, volume V, and quantity (number of moles) N of an ideal gas satisfy

$$\frac{PV}{NT} = R$$

with R a universal constant.

Generalities

So, before Boyle, Charles, and Avogadro got into the act, T, P, V and N may have seemed unrelated, yielding

$$\mathfrak{U} = \mathbb{R}_+^4.$$

The ideal gas law restricts the possibilities to

$$\mathfrak{B} = \{(T, P, V, N) \in \mathbb{R}_+^4 \mid PV/NT = R\}$$

Features

- Generality, applicability
- shows the role of model equations
- $\rightsquigarrow$ notion of equivalent models
- $\rightsquigarrow$ notion of more powerful model
- Structure, symmetries
- ...

We will only consider **deterministic** models.

Stochastic models: there is a map P (the 'probability')

$$\boxed{P : \mathcal{A} \to [0, 1]}$$

with $\mathcal{A}$ a 'σ-algebra' of subsets of $\mathfrak{U}$.

$P(\mathfrak{B}) =$ 'the degree of certainty (belief, plausibility, propensity, relative frequency) that outcomes are in $\mathfrak{B}$;

$\cong$ the degree of validity of $\mathfrak{B}$ as a model.

We will only consider **deterministic** models.

Stochastic models: there is a map P (the 'probability')

$$\boxed{P : \mathcal{A} \to [0, 1]}$$

with $\mathcal{A}$ a 'σ-algebra' of subsets of $\mathfrak{U}$.

Fuzzy models: there is a map μ (the 'membership function')

$$\boxed{\mu : \mathfrak{U} \to [0, 1]}$$

$\mu(x) =$ 'the extent to which $x \in \mathfrak{U}$ belongs to the model'.

We will only consider deterministic models.

Stochastic models: there is a map P (the 'probability')

$$\boxed{P : \mathcal{A} \to [0, 1]}$$

with $\mathcal{A}$ a 'σ-algebra' of subsets of $\mathfrak{U}$.

Determinism: $\mathcal{A} = \{\varnothing, \mathfrak{B}, \mathfrak{B}^{\text{complement}}, \mathfrak{U}\}, P(\mathfrak{B}) = 1$.

Fuzzy models: there is a map μ (the 'membership function')

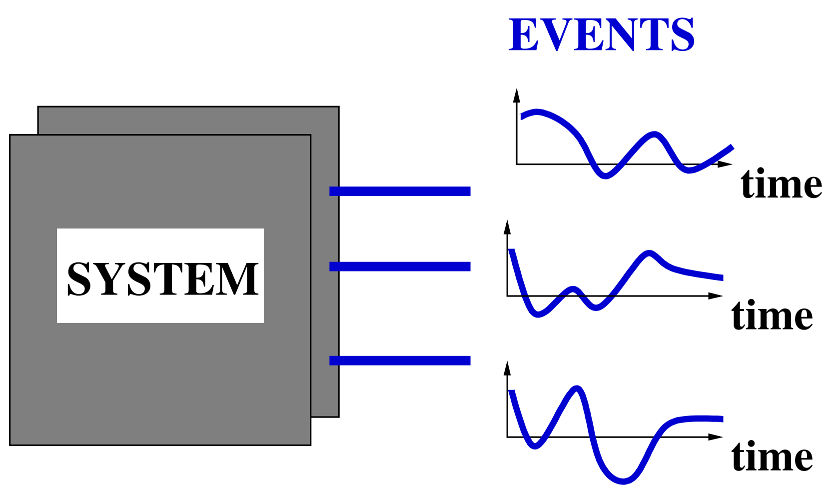
$$\boxed{\mu : \mathfrak{U} \to [0, 1]}$$

Determinism: μ is 'crisp':

$$\text{image}(\mu) = \{0, 1\}, \;\; \mathfrak{B} = \mu^{-1}(\{1\}) := \{x \in \mathfrak{U} \mid \mu(x) = 1\}$$

Dynamical systems

In dynamics, the outcomes are functions of time $\leadsto$



Which event trajectories are possible?

Dynamical systems

Definition

A dynamical system = $\boxed{\Sigma := (\mathbb{T}, \mathbb{W}, \mathfrak{B})}$

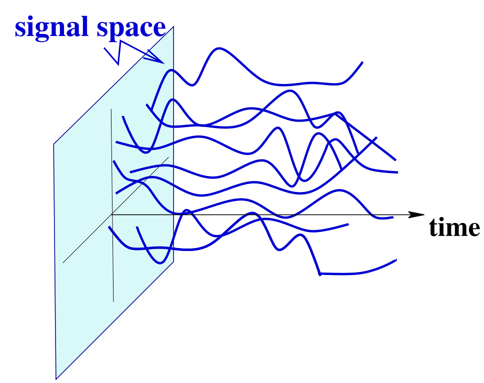
with $\mathbb{T} \subseteq \mathbb{R}$, the *time-axis* (= the relevant time instances),
$\mathbb{W}$, the *signal space*
(= where the variables take on their values),
$\boxed{\mathfrak{B} \subseteq \mathbb{W}^{\mathbb{T}}}$ ***the behavior*** **(= the admissible trajectories).**

Dynamical systems

Definition

A dynamical system = $\Sigma := (\mathbb{T}, \mathbb{W}, \mathfrak{B})$

with $\mathbb{T} \subseteq \mathbb{R}$, the *time-axis* (= the relevant time instances),
$\mathbb{W}$, the *signal space*
(= where the variables take on their values),
$\mathfrak{B} \subseteq \mathbb{W}^{\mathbb{T}}$ *the behavior* (= the admissible trajectories).



Totality of 'legal' trajectories =: the behavior

End of Part I

Part II: Linear Differential Systems

Dynamical systems

Definition

A dynamical system = $\Sigma := (\mathbb{T}, \mathbb{W}, \mathfrak{B})$

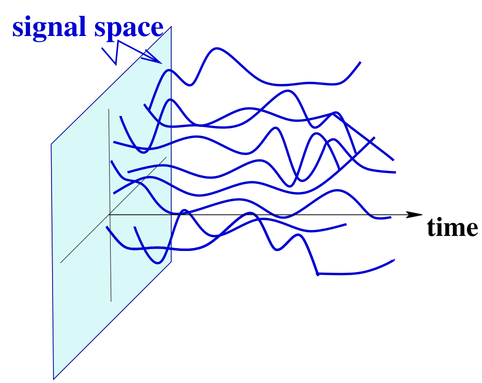
with $\mathbb{T} \subseteq \mathbb{R}$, the *time-axis* (= the relevant time instances),
$\mathbb{W}$, the *signal space*
(= where the variables take on their values),
$\mathfrak{B} \subseteq \mathbb{W}^{\mathbb{T}}$ *the behavior* (= the admissible trajectories).

Dynamical systems

Definition

A dynamical system = $\Sigma := (\mathbb{T}, \mathbb{W}, \mathfrak{B})$

with $\mathbb{T} \subseteq \mathbb{R}$, the *time-axis* (= the relevant time instances),
$\mathbb{W}$, the *signal space*
(= where the variables take on their values),
$\mathfrak{B} \subseteq \mathbb{W}^{\mathbb{T}}$ *the behavior* (= the admissible trajectories).



Totality of 'legal' trajectories =: the behavior

Dynamical systems

Definition

A dynamical system = $\Sigma := (\mathbb{T}, \mathbb{W}, \mathfrak{B})$

with $\mathbb{T} \subseteq \mathbb{R}$, the *time-axis* (= the relevant time instances),
$\mathbb{W}$, the *signal space*
(= where the variables take on their values),
$\mathfrak{B} \subseteq \mathbb{W}^{\mathbb{T}}$ *the behavior* (= the admissible trajectories).

For a trajectory ('an event') $w : \mathbb{T} \to \mathbb{W}$, we thus have:

$w \in \mathfrak{B}$: the model allows the trajectory w,
$w \notin \mathfrak{B}$: the model forbids the trajectory w.

Dynamical systems

Definition

A dynamical system = $\Sigma := (\mathbb{T}, \mathbb{W}, \mathfrak{B})$

with $\mathbb{T} \subseteq \mathbb{R}$, the *time-axis* (= the relevant time instances),
$\mathbb{W}$, the *signal space*
(= where the variables take on their values),
$\mathfrak{B} \subseteq \mathbb{W}^{\mathbb{T}}$ *the behavior* (= the admissible trajectories).

Usually,
$\mathbb{T} = \mathbb{R}$, or $[0, \infty)$, etc. (in continuous-time systems),
or $\mathbb{Z}$, or $\mathbb{N}$, etc. (in discrete-time systems).

Dynamical systems

Definition

A dynamical system = $\boxed{\Sigma := (\mathbb{T}, \mathbb{W}, \mathfrak{B})}$

with $\mathbb{T} \subseteq \mathbb{R}$**, the *time-axis* (= the relevant time instances),**
$\mathbb{W}$**, the *signal space***
(= where the variables take on their values),
$\boxed{\mathfrak{B} \subseteq \mathbb{W}^{\mathbb{T}}}$ ***the behavior*** **(= the admissible trajectories).**

Usually,

$\mathbb{W} \subseteq \mathbb{R}^{\mathtt{w}}$ **(in lumped systems),**
a function space
(in distributed systems, time a distinguished variable),
a finite set (in DES)' etc.

Dynamical systems

Definition

A dynamical system = $\boxed{\Sigma := (\mathbb{T}, \mathbb{W}, \mathfrak{B})}$

with $\mathbb{T} \subseteq \mathbb{R}$, the *time-axis* (= the relevant time instances),
$\mathbb{W}$, the *signal space*
(= where the variables take on their values),
$\boxed{\mathfrak{B} \subseteq \mathbb{W}^{\mathbb{T}}}$ ***the behavior*** **(= the admissible trajectories).**

Emphasis:

$\mathbb{T} = \mathbb{R}$,
$\mathbb{W} = \mathbb{R}^{\mathtt{w}}$,
$\mathfrak{B}$ = **solution set of system of (linear constant coefficient) ODE's, or difference eqn's, or PDE's. $\rightsquigarrow$ 'differential systems'.**

A series of examples

Examples

Let's put Kepler and Newton in this setting.

K1+K2+K3 obviously define a dynamical system $\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$

$\mathbb{T} = \mathbb{R}, \quad \mathbb{W} = \mathbb{R}^3,$

$\mathfrak{B} =$ **all** $w : \mathbb{R} \to \mathbb{R}^3$ **that satisfy Kepler's 3 laws.**

Nice example of a dynamical model 'without equations'.

Examples

Let's put Kepler and Newton in this setting.

K1+K2+K3 obviously define a dynamical system $\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$

$\mathbb{T} = \mathbb{R}, \quad \mathbb{W} = \mathbb{R}^3,$

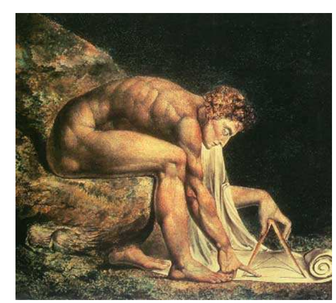
$\mathfrak{B} =$ **all** $w : \mathbb{R} \to \mathbb{R}^3$ **that satisfy Kepler's 3 laws.**

Nice example of a dynamical model 'without equations'.

Is it a differential system?

This question turned out to be of revolutionary importance...

$\Leftrightarrow$



Examples

Flows: $$\frac{d}{dt}x(t) = f(x(t)),$$

$\mathfrak{B} =$ all state trajectories.

Observed flows: $$\frac{d}{dt}x(t) = f(x(t)); \quad y(t) = h(x(t)),$$

$\mathfrak{B} =$ all possible output trajectories.

Note:

1. It may be impossible to express $\mathfrak{B}$ as the solutions of a differential equation involving only y.
2. The auxiliary (latent variable) nature of x.

Examples

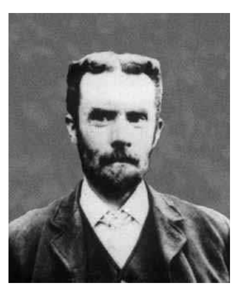
Input / output systems

$$f_1\left(y(t), \frac{d}{dt}y(t), \frac{d^2}{dt^2}y(t), \ldots, t\right) = f_2\left(u(t), \frac{d}{dt}u(t), \frac{d^2}{dt^2}u(t), \ldots, t\right)$$

$\mathbb{T} = \mathbb{R}$ **(time),**

$\mathbb{W} = \mathbb{U} \times \mathbb{Y}$ **(input $\times$ output signal spaces),**

$\mathfrak{B} =$ **all input / output pairs.**



Examples

Input / state / output systems

$$\frac{d}{dt}x(t) = f(x(t), u(t), t), \;\; y(t) = h(x(t), u(t), t)$$

What do we want to call the behavior?

the (u, y, x)'s, or the (u, y)'s?

Is the (u, y) behavior described by a differential eq'n?

Examples

Codes

$\mathfrak{C} \subseteq \mathbb{A}^{\mathbb{I}} =$ **the code;** **yields the system** $\Sigma = (\mathbb{I}, \mathbb{A}, \mathfrak{C})$.

Redundancy structure, error correction possibilities, etc., are visible in the code behavior $\mathfrak{C}$. **It is the central object of study.**

Formal languages

$\mathbb{A}$ $=$ **a (finite) alphabet,**

$\mathfrak{L} \subseteq \mathbb{A}^{*} =$ **the language** **= all 'legal' 'words'** $a_1 a_2 \cdots a_{\mathrm{k}} \cdots$

$\mathbb{A}^{*} =$ **all finite strings with symbols from** $\mathbb{A}$.

yields the system $\Sigma = (\mathbb{N}, \mathbb{A}, \mathfrak{L})$.

Examples: **All words appearing in the *Webster* dictionary**

All LaTeX documents.

Examples

Thermodynamics: a theory of open systems

Thermodynamics is the only theory of a general nature of which I am convinced that it will never be overthrown.

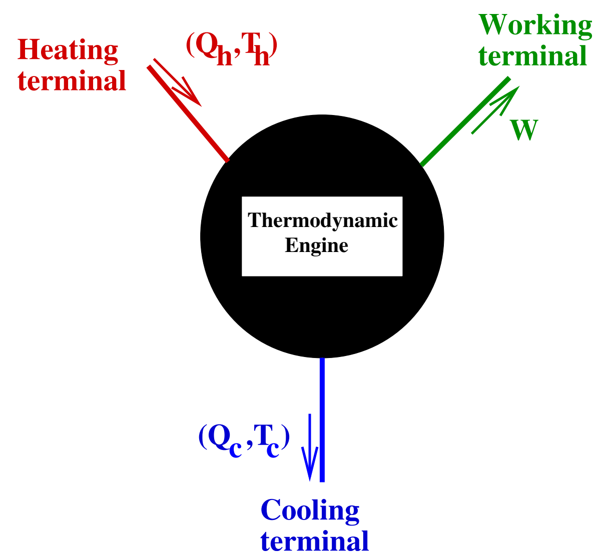
Albert Einstein

The law that entropy always increases – the second law of thermodynamics – holds, I think, the supreme position among the laws of nature.

Arthur Eddington

Examples

Thermodynamics: a theory of open systems



time-axis: $\mathbb{R}$

Q: Variables of interest? **A:** Q_h, T_h, Q_c, T_c, W

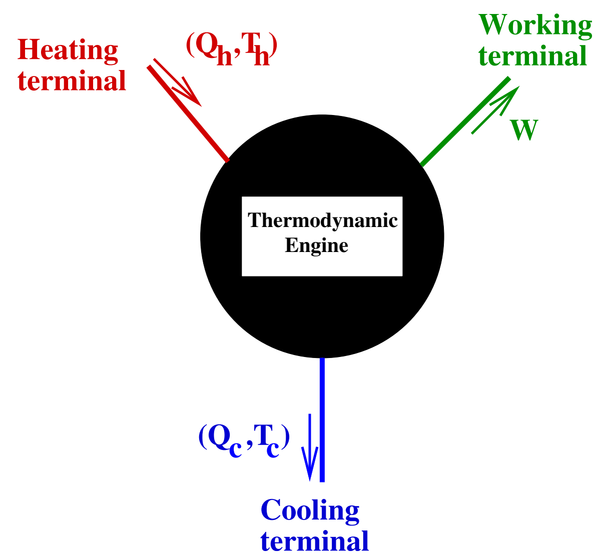
$\rightsquigarrow$ **signal space:** $\mathbb{W} = \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}$

Behavior $\mathfrak{B}$: a suitable family of trajectories.

But, there are some universal laws that restrict the $\mathfrak{B}$'s that are 'thermodynamic'.

Examples

Thermodynamics: a theory of open systems



<u>First and second law</u>:

$$\oint (Q_h - Q_c - W)\, dt = 0; \quad \oint \left(\frac{Q_h}{T_h} - \frac{Q_c}{T_c}\right) dt \le 0.$$

These laws deal with 'open' systems.

But <u>not</u> with input/output systems!

$\mathfrak{L}^\bullet$: Linear time-invariant differential systems

More structure

$$\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$$

is said to be **linear**

if $\mathbb{W}$ is a vector space, and $\mathfrak{B}$ a linear subspace of $\mathbb{W}^{\mathbb{T}}$.

More structure

$$\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$$

is said to be **time-invariant**

if $\mathbb{T} = \mathbb{R}, \mathbb{R}_+, \mathbb{Z},$ **or** $\mathbb{Z}_+$ **and if** $\mathfrak{B}$ **satisfies**

$$\sigma^t \mathfrak{B} \subseteq \mathfrak{B} \text{ for all } t \in \mathbb{T}.$$

σ^t **denotes the shift,** $\sigma^t f(t') := f(t' + t)$.

More structure

$$\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$$

is said to be **differential**

if $\mathbb{T} = \mathbb{R}$, or $\mathbb{R}_+$, etc., and if $\mathfrak{B}$ is the solution set of a (system of) ODE's.

a **difference system** if, etc.

More structure

$$\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$$

is said to be **symmetric**

w.r.t. the transformation group $\{T_g, g \in \mathfrak{G}\}$ on $\mathbb{W}^{\mathbb{T}}$

if $T_g \mathfrak{B} = \mathfrak{B}$ for all $g \in \mathfrak{G}$.

Examples:

1. time-invariance, time-reversibility
2. permutation symmetry, rotation symmetry, translation symmetry, Euclidean symmetry,
3. etc., etc.

$\mathfrak{L}^{\mathtt{w}}$

$R \in \mathbb{R}^{\bullet \times \mathtt{w}}[\xi]$ $\quad$ $R(\frac{d}{dt})w = 0$ **defines the** **linear, time-invariant, differential** **system:** $\Sigma = (\mathbb{R}, \mathbb{R}^{\mathtt{w}}, \mathfrak{B})$ **with**

$$\mathfrak{B} = \{w \in \mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \mid R(\tfrac{d}{dt})w = 0\}.$$

$\mathfrak{L}^{\mathtt{w}}$

$R \in \mathbb{R}^{\bullet \times \mathtt{w}}[\xi]$ $\quad$ $R(\frac{d}{dt})w = 0$ defines the

linear, time-invariant, differential system: $\Sigma = (\mathbb{R}, \mathbb{R}^{\mathtt{w}}, \mathfrak{B})$ with

$$\mathfrak{B} = \{w \in \mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \mid R(\tfrac{d}{dt})w = 0\}.$$

NOTATION

$\mathfrak{L}^{\bullet}$: all such systems (with any - finite - number of variables)

$\mathfrak{L}^{\mathtt{w}}$: with $\mathtt{w}$ variables

$\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ (no ambiguity regarding $\mathbb{T}$, $\mathbb{W}$)

$\mathfrak{L}^{\mathtt{w}}$

$R \in \mathbb{R}^{\bullet \times \mathtt{w}}[\xi]$ $\quad$ $R(\frac{d}{dt})w = 0$ defines the

linear, time-invariant, differential system: $\Sigma = (\mathbb{R}, \mathbb{R}^{\mathtt{w}}, \mathfrak{B})$ with

$$\mathfrak{B} = \{w \in \mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \mid R(\tfrac{d}{dt})w = 0\}.$$

NOMENCLATURE

Elements of $\mathfrak{L}^{\bullet}$: *linear differential systems*

$R(\frac{d}{dt})w = 0$: a *kernel representation* of the

corresponding $\Sigma \in \mathfrak{L}^{\bullet}$ or $\mathfrak{B} \in \mathfrak{L}^{\bullet}$

Overview

Starting from this vantage point, a rich theory has been developed

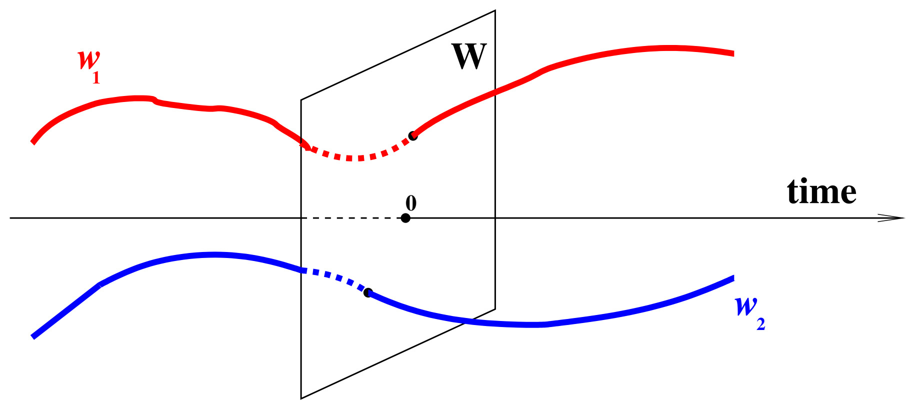
1. **Modeling by tearing, zooming, and linking**
2. **Controllability and stabilizability**
3. **Control by interconnection:**

   from stabilization to LQ and $\mathcal{H}_\infty$-control
4. **Observability, observers and the like**
5. **SYSID, the MPUM, subspace ID**
6. **System representations**
7. **PDE's**
8. **etc., etc., ...**

Controllability

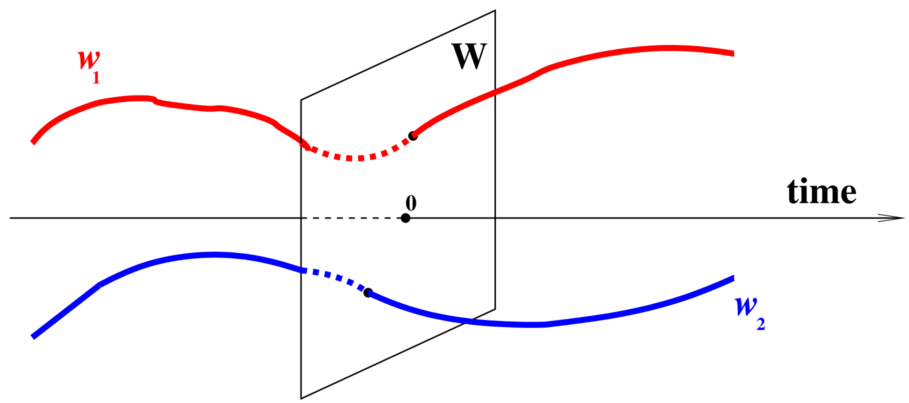
Controllability

Take any two trajectories $w_1, w_2 \in \mathfrak{B}$.

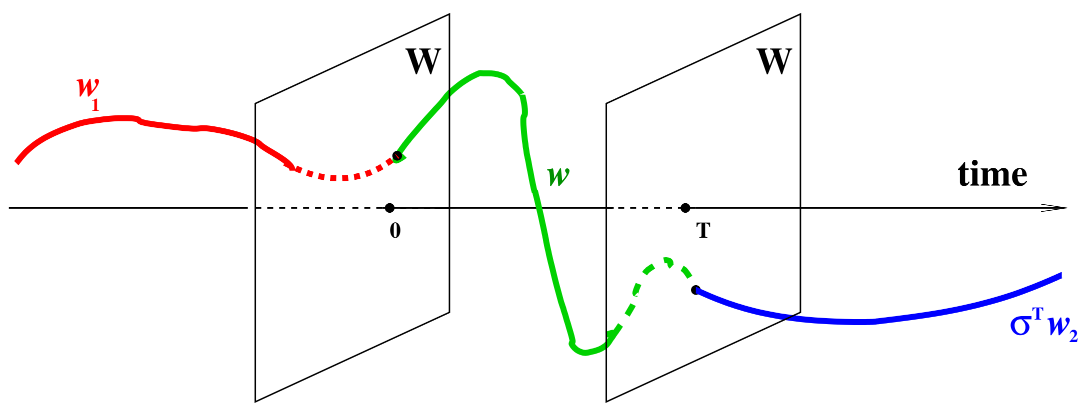


Controllability

Take any two trajectories $w_1, w_2 \in \mathfrak{B}$.



Controllability:



Controllability

The time-invariant system $\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$ is said to be

controllable

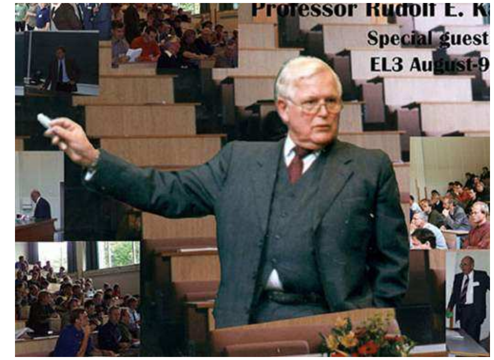
if for all $w_1, w_2 \in \mathfrak{B}$ there exists $w \in \mathfrak{B}$ and $T \geq 0$ such that

$$w(t) = \begin{cases} w_1(t) & t < 0 \\ w_2(t - T) & t \geq T \end{cases}$$

Controllability :⇔

legal trajectories must be 'patch-able', 'concatenable'.

State Controllability

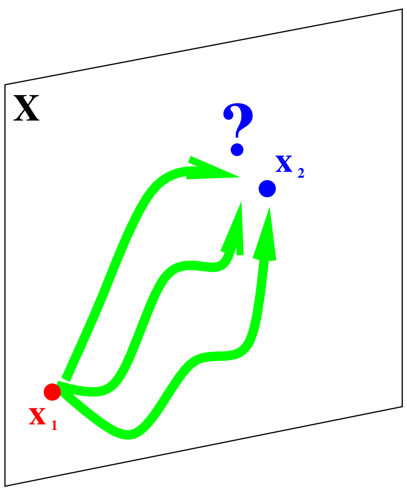


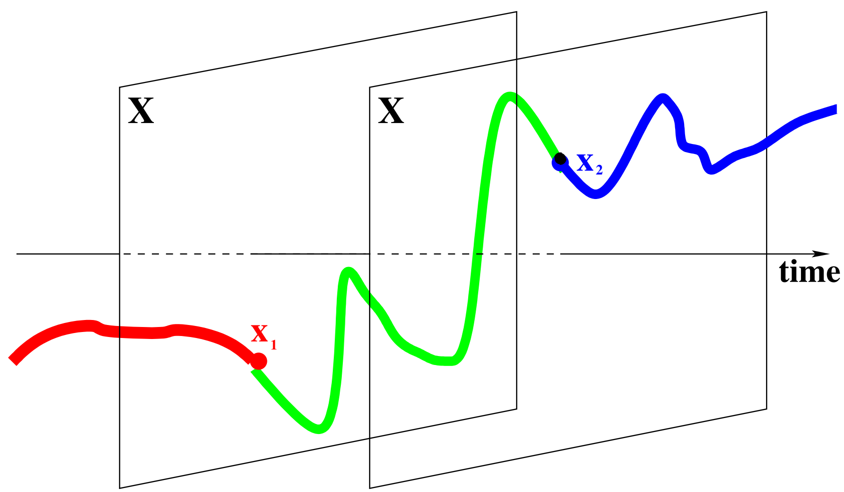
Special case: classical Kalman definitions for

$$\frac{d}{dt}x = f\left(x, u\right).$$

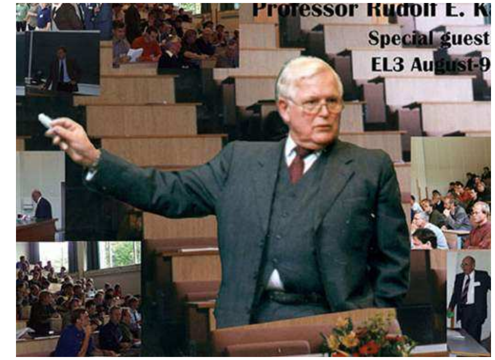
controllability: variables = state or (input, state)

This is a special case of our controllability:





State Controllability



Special case: classical Kalman definitions for

$$\frac{d}{dt}x = f\,(x, u)\,.$$

controllability: variables = state or (input, state)

If a system is not (state) controllable, why is it?

Insufficient influence of the control?

Or bad choice of the state?

Or not properly editing the equations?

Kalman's definition addresses a rather special situation.

Tests

Given a system representation, derive algorithms in terms of the parameters for controllability.

Consider the system $\mathfrak{B} \in \mathfrak{L}^\bullet$ defined by

$$R\left(\frac{d}{dt}\right) w = 0.$$

Under what conditions on $R \in \mathbb{R}^{\bullet \times \mathtt{w}}[\xi]$ does it define a controllable system?

Theorem: $R\left(\frac{d}{dt}\right) w = 0$ defines a controllable system

$$\Leftrightarrow$$

$\text{rank}\,(R(\lambda)) =$ constant over $\lambda \in \mathbb{C}$.

Tests

Notes:

- If $R\left(\frac{d}{dt}\right) w = 0$ has R of full row rank, then

  controllability $\Leftrightarrow$ $R(\lambda)$ is of full row rank $\forall\ \lambda \in \mathbb{C}$.

  Equivalently, R is right-invertible as a polynomial matrix ($\Leftrightarrow$ 'left prime').

Tests

<u>Notes</u>:

- $\frac{d}{dt}x = Ax + Bu, w = x$ or (x, u) is controllable iff

$$\mathrm{rank}\,([A - \lambda I \quad B]) = \dim\,(x) \quad \forall\, \lambda \in \mathbb{C}.$$

Popov-Belevich-Hautus test for controllability.

Of course,

$$\Leftrightarrow \mathrm{rank}\left(\begin{bmatrix} B & AB & \cdots & A^{\dim(x)-1}B \end{bmatrix}\right) = \dim\,(x)\,.$$

Tests

Notes:

- When is

$$p\left(\frac{d}{dt}\right)w_1 = q\left(\frac{d}{dt}\right)w_2$$

controllable? $p, q \in \mathbb{R}[\xi]$, not both zero.

Controllable $\Leftrightarrow$ rank$([p(\lambda) \quad -q(\lambda)]) = 1 \; \forall \lambda \in \mathbb{C}$.

Iff p and q are co-prime. No common factors!

Testable via Sylvester matrix, etc.

Generalizable.

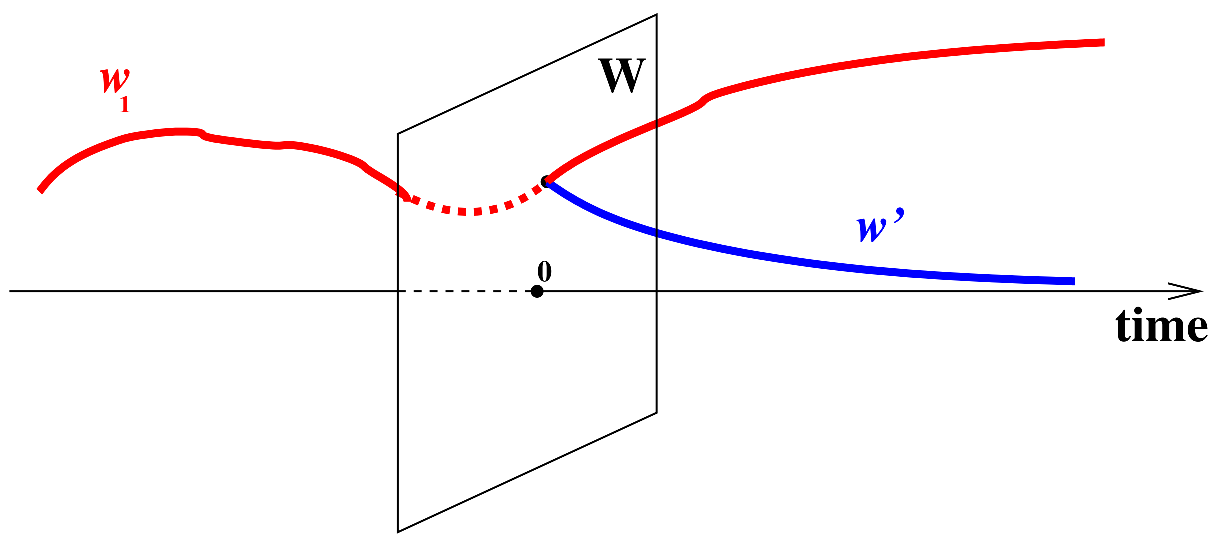
Stabilizability

The system $\Sigma = (\mathbb{T}, \mathbb{R}^{\mathtt{w}}, \mathfrak{B})$ is said to be **stabilizable** if, for all $w \in \mathfrak{B}$, there exists $w' \in \mathfrak{B}$ such that

$$w(t) = w'(t) \text{ for } t < 0 \quad \text{and} \quad w'(t) \underset{t\to\infty}{\longrightarrow} 0.$$

Stabilizability :⇔

legal trajectories can be steered to a desired point.



Stabilizability

Consider the system defined by

$$R\left(\frac{d}{dt}\right)w = 0.$$

Under which conditions on $R \in \mathbb{R}^{\bullet \times \mathtt{w}}[\xi]$ does it define a stabilizable system?

<u>Theorem</u>: $R\left(\frac{d}{dt}\right)w = 0$ defines a stabilizable system

$$\Leftrightarrow$$

$\mathrm{rank}\,(R\,(\lambda)) =$ **constant over** $\{\lambda \in \mathbb{C} \mid \mathrm{Real}\,(\lambda) \geq 0\}$.

Image representations

Representations of $\mathfrak{L}^\bullet$: $\quad R\left(\frac{d}{dt}\right) w = 0$

called a *'kernel' representation*. Sol'n set $\in \mathfrak{L}^\bullet$, by definition.

$$R\left(\frac{d}{dt}\right) w = M\left(\frac{d}{dt}\right) \ell$$

called a *'latent variable' representation* of the behavior of the w-variables.

'Elimination th'm' $\Rightarrow \in \mathfrak{L}^\bullet$.

Image representations

Representations of $\mathfrak{L}^\bullet$: $\quad R\left(\frac{d}{dt}\right) w = 0$

called a *'kernel' representation*. Sol'n set $\in \mathfrak{L}^\bullet$, by definition.

$$R\left(\frac{d}{dt}\right) w = M\left(\frac{d}{dt}\right) \ell$$

called a *'latent variable' representation* of the behavior of the w-variables.

'Elimination th'm' $\Rightarrow \in \mathfrak{L}^\bullet$.

Missing link:

$$w = M\left(\frac{d}{dt}\right) \ell$$

called an *'image' representation* of $\mathfrak{B} = \operatorname{im}\left(M\left(\frac{d}{dt}\right)\right)$.

Elimination theorem $\Rightarrow$ every image is also a kernel.

¿¿ Which kernels are also images ??

Controllability!

Image representations

Theorem: (Controllability and image representations):

The following are equivalent for $\mathfrak{B} \in \mathfrak{L}^\bullet$:

1. $\mathfrak{B}$ is controllable
2. $\mathfrak{B}$ admits an image representation

$$w = M\left(\frac{d}{dt}\right)\ell$$

3. etc., etc.

Numerical test

- **Image representation leads to an effective numerical test.**
- $\exists$ **similar results & algorithms for time-varying systems.**
- $\exists$ **partial results for nonlinear systems.**

Controllable part

The *'controllable part'* of $\mathfrak{B} \in \mathfrak{L}^\bullet$ can be defined in many equivalent ways. Most expedient:

$$\mathfrak{B}_{\text{controllable}} := \text{ largest controllable } \mathfrak{B}' \in \mathfrak{L}^{\mathtt{w}}, \mathfrak{B}' \subseteq \mathfrak{B}$$

Two systems

$$P_1(\frac{d}{dt})w_1 = Q_1(\frac{d}{dt})w_2 \qquad P_2(\frac{d}{dt})w_1 = Q_2(\frac{d}{dt})w_2$$

have the same controllable part iff they have the same transfer function

$$P_1^{-1}Q_1 =: G_1 = G_2 := P_2^{-1}Q_2$$

Transfer function: determines the controllable part only.

Limited description. Limitation of tf. f'n manipulations.

Polynomial representations

Representations with $\mathbb{R}[\xi]$-matrices of $\mathfrak{B} \in \mathfrak{L}^\bullet$

1. $R\left(\frac{d}{dt}\right) w = 0$ by definition
2. WLOG: R full row rank
3. R left prime over $\mathbb{R}[\xi]$ $(\exists S : RS = I)$ $\Leftrightarrow$ $\mathfrak{B}$ controllable
4. $w = M\left(\frac{d}{dt}\right) \ell$ $\Leftrightarrow$ $\mathfrak{B}$ controllable
5. if controllable,

   WLOG: M right prime over $\mathbb{R}[\xi]$ $(\exists N : NM = I)$

   'observable image representation': $\exists N : \ell = N(\frac{d}{dt})w$.

Representations with rational functions

Let $G \in \mathbb{R}^{\bullet \times \mathtt{w}}[\xi]$.

What does $G(\frac{d}{dt})w = 0$ mean?

Representations with rational functions

Let $G \in \mathbb{R}^{\bullet \times \mathtt{w}}[\xi]$.

What does $G(\frac{d}{dt})w = 0$ mean?

Joint work with

Yutaka Yamamoto

Representations with rational functions

The behavior defined by $G(\frac{d}{dt})w = 0$ is defined as that of

$$Q(\frac{d}{dt})w = 0$$

$G = P^{-1}Q$ a left co-prime factorization over $\mathbb{R}[\xi]$ of G

Representations with $\mathbb{R}(\xi)$-matrices of $\mathfrak{B} \in \mathfrak{L}^\bullet$.

1. WLOG, with G (strictly) proper, etc.
2. G left prime over ring of stable rational f'ns $\Leftrightarrow$ $\mathfrak{B}$ stabilizable
3. $w = G(\frac{d}{dt})\ell \quad \Leftrightarrow \mathfrak{B}$ controllable
4. if controllable, WLOG: G right prime over stable rational f'ns

   'observable im. repr'on': $\exists F$ stable rational : $\ell = F(\frac{d}{dt})w$.

PDE's

PDE's

Much of the theory also holds for PDE's.

$\mathbb{T} = \mathbb{R}^{\mathtt{n}}$, **the set of independent variables, often** $\mathtt{n} = 4$,
$\mathbb{W} = \mathbb{R}^{\mathtt{w}}$, **the set of dependent variables,**
$\mathfrak{B} =$ **sol'ns of a linear constant coefficient system of PDE's.**

PDE's

Much of the theory also holds for PDE's.

$\mathbb{T} = \mathbb{R}^{\mathtt{n}}$, the set of independent variables, often $\mathtt{n} = 4$,
$\mathbb{W} = \mathbb{R}^{\mathtt{w}}$, the set of dependent variables,
$\mathfrak{B}$ = sol'ns of a linear constant coefficient system of PDE's.

Let $R \in \mathbb{R}^{\bullet \times \mathtt{w}}[\xi_1, \cdots, \xi_{\mathtt{n}}]$, and consider

$$R\left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathtt{n}}}\right) w = 0. \qquad (*)$$

Define the associated behavior

$$\mathfrak{B} = \{ w \in \mathfrak{C}^{\infty}(\mathbb{R}^{\mathtt{n}}, \mathbb{R}^{\mathtt{w}}) \mid (*) \text{ holds} \}.$$

Notation for $\mathtt{n}$-D linear differential systems:

$$(\mathbb{R}^{\mathtt{n}}, \mathbb{R}^{\mathtt{w}}, \mathfrak{B}) \in \mathfrak{L}^{\mathtt{w}}_{\mathtt{n}}, \quad \text{or } \mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\mathtt{n}}.$$

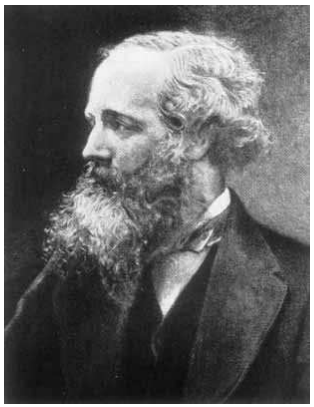
Example

Maxwell's eq'ns, diffusion eq'n, wave eq'n, . . .

$$
\begin{aligned}
\nabla \cdot \vec{E} &= \frac{1}{\varepsilon_0}\rho\,, \\
\nabla \times \vec{E} &= -\frac{\partial}{\partial t}\vec{B}, \\
\nabla \cdot \vec{B} &= 0\,, \\
c^2\nabla \times \vec{B} &= \frac{1}{\varepsilon_0}\vec{j} + \frac{\partial}{\partial t}\vec{E}.
\end{aligned}
$$

Example

Maxwell's eq'ns, diffusion eq'n, wave eq'n, . . .



$$\begin{aligned}
\nabla \cdot \vec{E} &= \frac{1}{\varepsilon_0}\rho\,, \\
\nabla \times \vec{E} &= -\frac{\partial}{\partial t}\vec{B}, \\
\nabla \cdot \vec{B} &= 0\,, \\
c^2\nabla \times \vec{B} &= \frac{1}{\varepsilon_0}\vec{j} + \frac{\partial}{\partial t}\vec{E}.
\end{aligned}$$

$\mathbb{T} = \mathbb{R} \times \mathbb{R}^3$ (time and space) $\mathtt{n} = 4$,

$w = \left(\vec{E}, \vec{B}, \vec{j}, \rho\right)$

(electric field, magnetic field, current density, charge density),

$\mathbb{W} = \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}$, $\mathtt{w} = 10$,

$\mathfrak{B}$ = set of solutions to these PDE's.

Note: 10 variables, 8 equations! $\Rightarrow \exists$ free variables. 'open' system.

Submodule theorem

$R \in \mathbb{R}^{\bullet\times\bullet}[\xi_1, \cdots, \xi_{\mathtt{n}}]$ defines $\mathfrak{B} = \ker\left(R\left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathtt{n}}}\right)\right)$, but not vice-versa.

¿¿ $\exists$ 'intrinsic' characterization of $\mathfrak{B} \in \mathfrak{L}_{\mathtt{n}}^{\mathtt{w}}$??

Is there a mathematical 'object' that characterizes a $\mathfrak{B} \in \mathfrak{L}_{\mathtt{n}}^{\mathtt{w}}$?

Define the ***annihilators*** of $\mathfrak{B} \in \mathfrak{L}_{\mathtt{n}}^{\mathtt{w}}$ by

$$\mathfrak{N}_{\mathfrak{B}} := \{n \in \mathbb{R}^{\mathtt{w}}[\xi_1, \cdots, \xi_{\mathtt{n}}] \mid n^\top\left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathtt{n}}}\right)\mathfrak{B} = 0\}.$$

Proposition:

$\mathfrak{N}_{\mathfrak{B}}$ is a $\mathbb{R}[\xi_1, \cdots, \xi_{\mathtt{n}}]$ sub-module of $\mathbb{R}^{\mathtt{w}}[\xi_1, \cdots, \xi_{\mathtt{n}}]$.

Submodule theorem

$R \in \mathbb{R}^{\bullet \times \bullet}[\xi_1, \cdots, \xi_n]$ defines $\mathfrak{B} = \ker\left(R\left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}\right)\right)$,
but not vice-versa.

¿¿ $\exists$ ‘intrinsic’ characterization of $\mathfrak{B} \in \mathfrak{L}_n^{\mathtt{w}}$??

Is there a mathematical ‘object’ that characterizes a $\mathfrak{B} \in \mathfrak{L}_n^{\mathtt{w}}$?

Define the ***annihilators*** of $\mathfrak{B} \in \mathfrak{L}_n^{\mathtt{w}}$ by

$$\mathfrak{N}_{\mathfrak{B}} := \{n \in \mathbb{R}^{\mathtt{w}}[\xi_1, \cdots, \xi_n] \mid n^\top\left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}\right)\mathfrak{B} = 0\}.$$

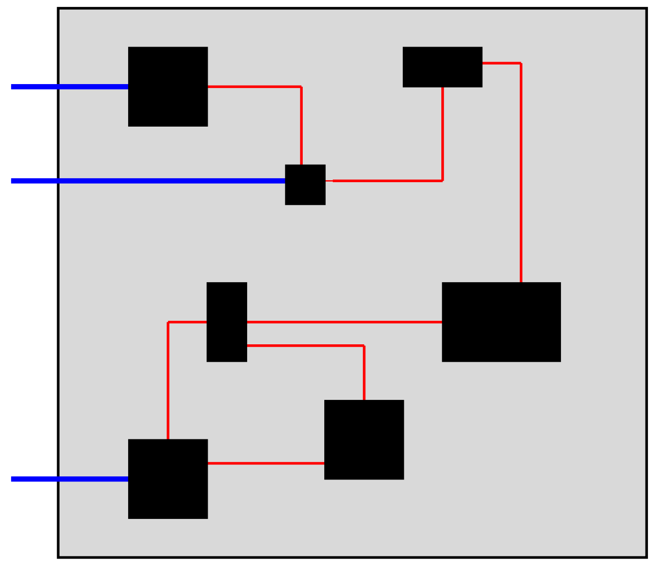
Proposition:

$\mathfrak{N}_{\mathfrak{B}}$ is a $\mathbb{R}[\xi_1, \cdots, \xi_n]$ sub-module of $\mathbb{R}^{\mathtt{w}}[\xi_1, \cdots, \xi_n]$.

Theorem: $\mathfrak{L}_n^{\mathtt{w}} \overset{\text{bijective}}{\longleftrightarrow}$ submodules of $\mathbb{R}^{\mathtt{w}}[\xi_1, \cdots, \xi_n]$

Elimination theorem

Motivation: In many problems, we want to eliminate variables. For example, first principle modeling



$\rightsquigarrow$ model containing both variables the model aims at ('manifest' variables), and auxiliary variables introduced in the modeling process ('latent' variables).

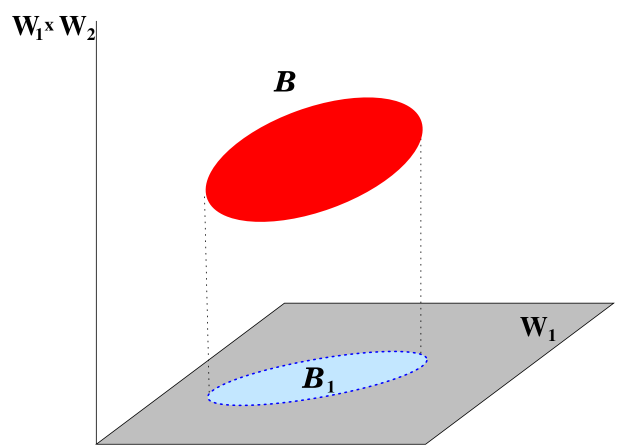
¿ Can these latent variables be eliminated from the equations ?

Elimination theorem

This leads to the following important question, first in polynomial matrix language. Consider

$$R_1(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n})w_1 = R_2(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n})w_2.$$

Obviously, the behavior of the (w_1, w_2)'s is described by a system of PDE's. ¿ Is the behavior of the w_1's alone also ?



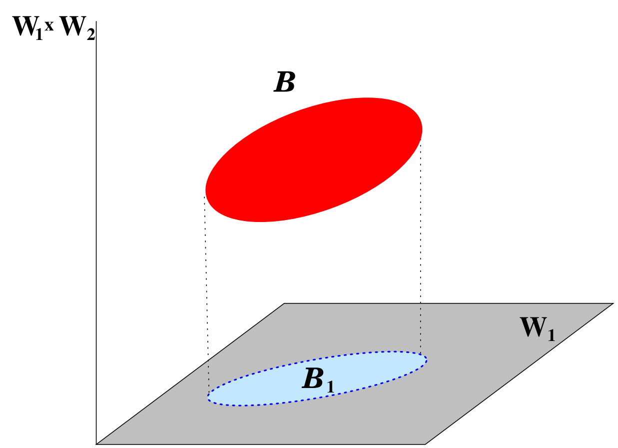
Elimination theorem

In the language of behaviors:

Let $\mathfrak{B} \in \mathfrak{L}_{\mathtt{n}}^{\mathtt{w}_1+\mathtt{w}_2}$**. Define**

$$\mathfrak{B}_1 = \{w_1 \in \mathfrak{C}^\infty(\mathbb{R}^{\mathtt{n}}, \mathbb{R}^{\mathtt{w}_1}) \mid \ \exists\ w_2 \text{ such that } (w_1, w_2) \in \mathfrak{B}\}.$$

Does this 'projection' $\mathfrak{B}_1$ belong to $\mathfrak{L}_{\mathtt{n}}^{\mathtt{w}_1}$?



Theorem: It does!

$\mathfrak{L}^\bullet$ is closed under projection !!

The Fundamental Principle

Proof: 'Fundamental principle'. Consider

$$F(x) = y$$

Given: $F : \mathbb{X} \to \mathbb{Y}$, $y \in \mathbb{Y}$; **Unknown:** $x \in \mathbb{X}$.

¿ Does there exists a sol'n x ?

Examples:

1.

2.

3.

The Fundamental Principle

Proof: 'Fundamental principle'. Consider

$$F(x) = y$$

Given: $F : \mathbb{X} \to \mathbb{Y}$, $y \in \mathbb{Y}$; **Unknown:** $x \in \mathbb{X}$.

¿ Does there exists a sol'n x ?

Examples:

1. $F \in \mathbb{R}^{n_1 \times n_2}$, $y \in \mathbb{R}^{n_2}$, $x \in \mathbb{R}^{n_1}$
2.
3.

The Fundamental Principle

Proof: 'Fundamental principle'. Consider

$$F(x) = y$$

Given: $F : \mathbb{X} \rightarrow \mathbb{Y}, \quad y \in \mathbb{Y}$; **Unknown:** $x \in \mathbb{X}$.

¿ **Does there exists a sol'n x ?**

Examples:

1.

2. **ODE's:**

$$F(\frac{d}{dt})x = y$$

with $F \in \mathbb{R}^{n_1 \times n_2}[\xi], y \in \mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^{n_2}), x \in \mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^{n_1})$.

Or over distributions, $y \in \mathfrak{D}'(\mathbb{R}, \mathbb{R}^{n_2}), x \in \mathfrak{D}'(\mathbb{R}, \mathbb{R}^{n_1})$.

The Fundamental Principle

Proof: ‘Fundamental principle’. Consider

$$F(x) = y$$

Given: $F : \mathbb{X} \to \mathbb{Y}$, $y \in \mathbb{Y}$; **Unknown:** $x \in \mathbb{X}$.

¿ Does there exists a sol’n x ?

Examples:

1.

2.

3. **PDE’s:**

$$F\left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}\right) x = y$$

$F \in \mathbb{R}^{n_1 \times n_2}[\xi_1, \cdots, \xi_n]$, $y \in \mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R}^{n_2})$, $x \in \mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R}^{n_1})$, **or over distributions.**

The Fundamental Principle for PDE's

$$F\left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathrm{n}}}\right)x = y$$

Given: $F \in \mathbb{R}^{\mathrm{n}_1 \times \mathrm{n}_2}[\xi_1, \cdots, \xi_{\mathrm{n}}]$, $y \in \mathfrak{C}^{\infty}(\mathbb{R}^{\mathrm{n}}, \mathbb{R}^{\mathrm{n}_2})$,
Unknown: $x \in \mathfrak{C}^{\infty}(\mathbb{R}^{\mathrm{n}}, \mathbb{R}^{\mathrm{n}_1})$.

¿ Does there exists a sol'n x ?

Obvious necessary condition:

$$(n \in \mathbb{R}^{n_1}[\xi_1, \cdots, \xi_{\mathrm{n}}]) \wedge (\mathrm{n}^{\top}(\xi_1, \cdots, \xi_{\mathrm{n}})F(\xi_1, \cdots, \xi_{\mathrm{n}}) = 0)$$
$$\Rightarrow \quad n^{\top}\left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathrm{n}}}\right)y = 0.$$

Theorem (Fundamental principle): This is a n.a.s.c.

The Fundamental Principle for PDE's

$$F(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n})x = y$$

<u>Given</u>: $F \in \mathbb{R}^{n_1 \times n_2}[\xi_1, \cdots, \xi_n]$, $y \in \mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R}^{n_2})$,
<u>Unknown</u>: $x \in \mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R}^{n_1})$.

¿ Does there exists a sol'n x ?

<u>Theorem</u> (Fundamental principle): This is a n.a.s.c.

Since the n's form a (finitely generated) $\mathbb{R}[\xi_1, \cdots, \xi_n]$-module, this is a finite condition!

<u>Example</u>:

Take $0 \neq F \in \mathbb{R}[\xi_1, \cdots, \xi_n]$. PDE $F(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n})x = y$.
Always solvable!

The elimination theorem

There exist effective algorithms for $\boxed{(R_1, R_2) \mapsto R}$.

$\rightsquigarrow$ Computer algebra, Gröbner bases.

It follows from all this that $\mathfrak{L}_{\mathrm{n}}^{\bullet}$ has very nice properties. In particular, it is closed under:

- **Intersection:** $(\mathfrak{B}_1, \mathfrak{B}_2 \in \mathfrak{L}_{\mathrm{n}}^{\mathrm{w}}) \Rightarrow (\mathfrak{B}_1 \cap \mathfrak{B}_2 \in \mathfrak{L}^{\mathrm{w}})$
- **Addition:** $(\mathfrak{B}_1, \mathfrak{B}_2 \in \mathfrak{L}_{\mathrm{n}}^{\mathrm{w}}) \Rightarrow (\mathfrak{B}_1 + \mathfrak{B}_2 \in \mathfrak{L}_{\mathrm{n}}^{\mathrm{w}})$
- **Projection:** $(\mathfrak{B} \in \mathfrak{L}_{\mathrm{n}}^{\mathrm{w}_1+\mathrm{w}_2}) \Rightarrow (\Pi_{w_1}\mathfrak{B} \in \mathfrak{L}_{n}^{\mathrm{w}_1})$ Π_{w_1} : projection
- **Action of a linear differential operator:**

$$(\mathfrak{B} \in \mathfrak{L}_{n}^{\mathrm{w}_1}, P \in \mathbb{R}^{\mathrm{w}_2 \times \mathrm{w}_1}[\xi_1, \cdots, \xi_{\mathrm{n}}]) \Rightarrow (P(\tfrac{d}{dt})\mathfrak{B} \in \mathfrak{L}_{\mathrm{n}}^{\mathrm{w}_2}).$$

- **Inverse image of a linear differential operator:**

$$(\mathfrak{B} \in \mathfrak{L}_{\mathrm{n}}^{\mathrm{w}_2}, P \in \mathbb{R}^{\mathrm{w}_2 \times \mathrm{w}_1}[\xi_1, \cdots, \xi_{\mathrm{n}}]) \Rightarrow (P(\tfrac{d}{dt}))^{-1}\mathfrak{B} \in \mathfrak{L}_{\mathrm{n}}^{\mathrm{w}_1}.$$

Elimination theorem

Which PDE's describe $(\rho, \vec{E}, \vec{j})$ in Maxwell's equations ?

Eliminate $\vec{B}$ from Maxwell's equations $\rightsquigarrow$

$$
\begin{aligned}
\nabla \cdot \vec{E} &= \frac{1}{\varepsilon_0}\rho\,, \\
\varepsilon_0 \frac{\partial}{\partial t} \nabla \cdot \vec{E} + \nabla \cdot \vec{j} &= 0, \\
\varepsilon_0 \frac{\partial^2}{\partial t^2} \vec{E} + \varepsilon_0 c^2 \nabla \times \nabla \times \vec{E} + \frac{\partial}{\partial t} \vec{j} &= 0.
\end{aligned}
$$

$$R\left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}\right) w = 0$$

is called a kernel representation of the associated $\mathfrak{B} \in \mathfrak{L}_n^w$.

Another representation: image representation

$$w = M\left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}\right) \ell.$$

'Elimination' thm $\Rightarrow$ $\mathrm{im}\left(M\left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}\right)\right) \in \mathfrak{L}_n^w$!

$$R\left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathrm{n}}}\right) w = 0$$

is called a kernel representation of the associated $\mathfrak{B} \in \mathfrak{L}_{\mathrm{n}}^{\mathtt{w}}$.

Another representation: image representation

$$w = M\left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathrm{n}}}\right) \ell.$$

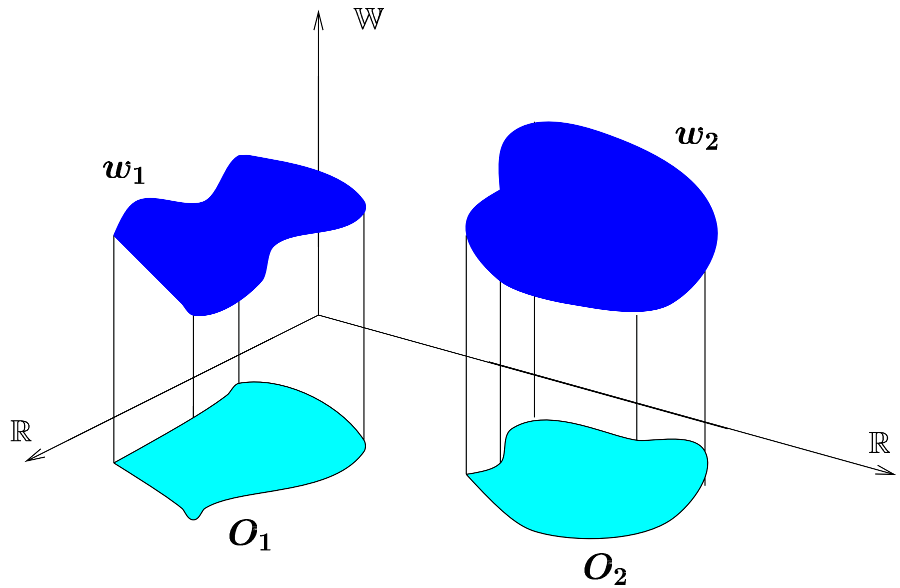
'Elimination' thm $\quad \Rightarrow \quad \operatorname{im}\left(M\left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathrm{n}}}\right)\right) \in \mathfrak{L}_{\mathrm{n}}^{\mathtt{w}}$!

Which linear diff. systems admit an image representation???

$\mathfrak{B} \in \mathfrak{L}_{\mathrm{n}}^{\mathtt{w}}$ admits an image representation iff it is 'controllable'.

Controllability for PDE's

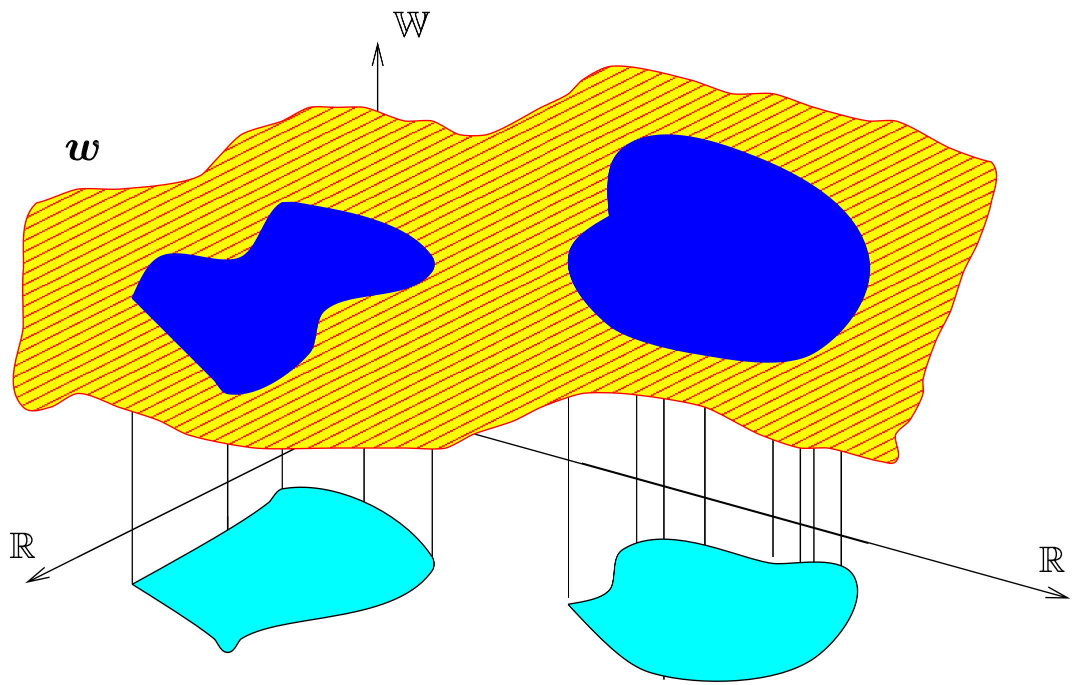
Controllability def'n in pictures:



$$w_1, w_2 \in \mathfrak{B}.$$

Controllability for PDE's

$\exists\, w \in \mathfrak{B}$ **'patches'** $w_1, w_2 \in \mathfrak{B}$.



Controllability :⇔ 'patch-ability'.

Are Maxwell's equations controllable ?

The following equations in the *scalar potential* $\phi : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$ and the *vector potential* $\vec{A} : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$, generate exactly the solutions to Maxwell's equations:

$$
\begin{aligned}
\vec{E} &= -\frac{\partial}{\partial t}\vec{A} - \nabla\phi, \\
\vec{B} &= \nabla \times \vec{A}, \\
\vec{j} &= \varepsilon_0 \frac{\partial^2}{\partial t^2}\vec{A} - \varepsilon_0 c^2 \nabla^2 \vec{A} + \varepsilon_0 c^2 \nabla\left(\nabla \cdot \vec{A}\right) + \varepsilon_0 \frac{\partial}{\partial t}\nabla\phi, \\
\rho &= -\varepsilon_0 \frac{\partial}{\partial t}\nabla \cdot \vec{A} - \varepsilon_0 \nabla^2 \phi.
\end{aligned}
$$

Proves controllability. Illustrates the interesting connection

controllability $\Leftrightarrow$ $\exists$ potential!

Conclusion

The flexibility and generality of the behavioral approach in modeling, for system representations, for passive control, dealing with PDE's, etc. is evident.

Exemplified by the notion of controllability.

Conclusion

The flexibility and generality of the behavioral approach in modeling, for system representations, for passive control, dealing with PDE's, etc. is evident.

Exemplified by the notion of controllability.

Nature and Nature's laws lay hid in night
God said, 'Let Newton be' and all was light

Mathematical Systems Theory lay bound by might
Ratio said, 'Let Behaviors be' and all was right

Details & copies of the lecture frames are available from/at

`Jan.Willems@esat.kuleuven.be`

`http://www.esat.kuleuven.be/~jwillems`

Details & copies of the lecture frames are available from/at

Jan.Willems@esat.kuleuven.be

http://www.esat.kuleuven.be/∼jwillems

Thank you

Thank you

Thank you

Thank you

Thank you

Thank you

Thank you

Thank you