LQ CONTROL and QDF's

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I also draw on work by

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and co-workers.

Introduction and outline

Introduction

Basic merits of the **behavioral approach**:

- Language for modeling, independent of system representation, adapted to first principles modeling, no signal flow graphs
- Interconnection $\mapsto$ sharing variables
- General, simple notions as controllability, observability, state, etc.
- Versatile theory of LTI systems: diff. eq'n models, state models, transfer function models, etc.

Introduction

In the context of **control**:

- Control = interconnection
- Control = finding a good subbehavior
- Obtain controlled behavior first,

  $\rightsquigarrow$ synthesis, 'implementation' problem;

  regularity, feedback, etc.
- LQ and $\mathcal{H}_\infty$ theory via **quadratic differential forms**

Outline

Today: general introduction to LQ control using QDF's

- QDF's and their positivity
- A couple of preliminaries on behaviors
- Stationarity, optimality of quadratic integrals w.r.t. compact support variations
- w.r.t. one-sided variations
- Outline of the LQ trajectory optimization problem

QDF's

QDF's

Let $\Phi_{k,\ell} \in \mathbb{R}^{\mathtt{w}_1 \times \mathtt{w}_2}$, $\quad k, \ell = 0, 1, 2, \ldots, N$
and $w_i \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}_i})$, $\quad i = 1, 2.$

The map

$$\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}_1}) \times \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}_2}) \to \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R})$$

defined by

$$(w_1, w_2) \mapsto \sum_{k,\ell=0}^{N} \left(\frac{d^k}{dt^k} w_1\right)^\top \Phi_{k,\ell} \left(\frac{d^\ell}{dt^\ell} w_2\right)$$

is called a **bilinear differential form (BDF)**.

QDF's

Let $\Phi_{k,\ell} \in \mathbb{R}^{\mathtt{w} \times \mathtt{w}}$, $\quad k, \ell = 0, 1, 2, \ldots, N$

and $w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$.

The map $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \rightarrow \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R})$ defined by

$$w \mapsto \sum_{k,\ell=0}^{N} \left(\frac{d^k}{dt^k} w\right)^\top \Phi_{k,\ell} \left(\frac{d^\ell}{dt^\ell} w\right)$$

is called a quadratic differential form (QDF).

QDF's

Compact notation and parametrization $\rightsquigarrow$ $\mathbb{R}[\zeta,\eta]$ **and matrices of real polynomials in two indeterminates:**

$$\Phi(\zeta,\eta) = \sum_{k,\ell=0}^{N} \Phi_{k,\ell}\zeta^k\eta^\ell$$

with $\Phi_{k,\ell} \in \mathbb{R}^{\mathtt{w}_1\times\mathtt{w}_2}$**. In 1↔1 relation with the BDF**

$$L_\Phi : \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}_1}) \times \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}_2}) \to \mathfrak{C}^\infty(\mathbb{R},\mathbb{R})$$

$$L_\Phi(w_1,w_2) := \sum_{k,\ell=0}^{N}\left(\frac{d^k}{dt^k}w_1\right)^\top \Phi_{k,\ell}\left(\frac{d^\ell}{dt^\ell}w_2\right)$$

called the BDF L_Φ **induced by** $\Phi(\zeta,\eta)$**.**

QDF's

Compact notation and parametrization $\rightsquigarrow$ $\mathbb{R}[\zeta, \eta]$

and matrices of real polynomials in two indeterminates:

$$\Phi(\zeta, \eta) = \sum_{k,\ell=0}^{N} \Phi_{k,\ell}\zeta^k\eta^\ell$$

with $\Phi_{k,\ell} \in \mathbb{R}^{\mathtt{w}_1 \times \mathtt{w}_2}$. With $\mathtt{w}_1 = \mathtt{w}_2 = \mathtt{w}$ $\rightsquigarrow$

$$Q_\Phi : \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \rightarrow \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R})$$

$$L_\Phi(w, w) = Q_\Phi(w) = \sum_{k,\ell=0}^{N} (\tfrac{d^k}{dt^k} w)^\top \Phi_{k,\ell} (\tfrac{d^\ell}{dt^\ell} w)$$

called the QDF Q_Φ induced by $\Phi(\zeta, \eta)$

QDF's

With $\mathtt{w}_1 = \mathtt{w}_2 = \mathtt{w} \leadsto$

$$Q_\Phi : \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \to \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R})$$

$$L_\Phi(w, w) = Q_\Phi(w) = \sum_{k,\ell=0}^{N} (\tfrac{d^k}{dt^k} w)^\top \Phi_{k,\ell} (\tfrac{d^\ell}{dt^\ell} w)$$

called the QDF Q_Φ induced by $\Phi(\zeta, \eta)$

WLOG $\Phi_{k,\ell} = \Phi_{\ell,k}^\top$ i.e. $\Phi = \Phi^\star$, $\Phi^\star(\zeta, \eta) := \Phi(\eta, \zeta)^\top$

symmetry

symmetric 2-var. pol. matrices are in $1 \leftrightarrow 1$ relation with QDF's

- **Total energy for oscillator**

$$M\frac{d^2}{dt^2}w + Kw = 0$$

equals

$$Q_\Phi(w) = \frac{1}{2}M(\frac{d}{dt}w)^2 + \frac{1}{2}Kw^2.$$

$\rightsquigarrow$

$$\Phi(\zeta,\eta) = \frac{1}{2}M\zeta\eta + \frac{1}{2}K$$

QDF's

- $Q_\Phi(w_1, w_2) = w_2 \frac{d}{dt} w_1$

  ¿Polynomial matrix for Q_Φ?

$$w_2\left(\frac{d}{dt}w_1\right) = \frac{1}{2}\begin{bmatrix}\frac{d}{dt}w_1 & \frac{d}{dt}w_2\end{bmatrix}\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\begin{bmatrix}w_1\\ w_2\end{bmatrix}$$

$$+\frac{1}{2}\begin{bmatrix}w_1 & w_2\end{bmatrix}\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}\begin{bmatrix}\frac{d}{dt}w_1\\ \frac{d}{dt}w_2\end{bmatrix}$$

Therefore $\quad \Phi(\zeta, \eta) = \frac{1}{2}\begin{bmatrix}0 & \zeta\\ \eta & 0\end{bmatrix}$

Positivity of QDF's

Q_Φ (or Φ) is (pointwise) **non-negative** : $\Leftrightarrow$

$$Q_\Phi(w)(t) \geq 0 \quad \forall\, w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}}),$$ and $t \in \mathbb{R}$.

Q_Φ is **average non-negative** : $\Leftrightarrow$

$$\int_{-\infty}^{+\infty} Q_\Phi(w)\, dt \geq 0 \quad \forall\, w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$$ of compact support.

Q_Φ is **half-line non-negative** : $\Leftrightarrow$

$$\int_{-\infty}^{0} Q_\Phi(w)\, dt \geq 0 \quad \forall\, w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$$ of compact support.

$\exists$ positivity, strict positivity, ... analogues.

Positivity of QDF's

Q_Φ is (pointwise) non-negative $\Leftrightarrow$

$$\mathrm{Mat}(\Phi) \geq 0.$$

$$\mathrm{Mat}(\Phi) := \begin{bmatrix} \Phi_{0,0} & \Phi_{0,1} & \Phi_{0,2} & \cdots \\ \Phi_{1,0} & \Phi_{1,1} & \Phi_{1,2} & \cdots \\ \Phi_{2,0} & \Phi_{2,1} & \Phi_{2,2} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

$\Leftrightarrow \exists\ D \in \mathbb{R}^{\mathtt{w}\times\bullet}[\xi] :$ $\Phi(\zeta,\eta) = D^\top(\zeta)D(\eta)$

Positivity of QDF's

Q_Φ is **average non-negative**

$\Leftrightarrow$

$$\Phi(-i\omega, i\omega) \geq 0 \quad \forall\, \omega \in \mathbb{R}.$$

$\Leftrightarrow$ **(LMI)** $\exists$ a **'storage function'** $\Psi = \Psi^\star \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta, \eta]$:

$$\Phi(\zeta, \eta) + (\zeta + \eta)\Psi(\zeta, \eta) \geq 0$$

i.e. $Q_\Phi(w) + \frac{d}{dt} Q_\Psi(w) \geq 0 \;\; \forall\, w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$.

$\Leftrightarrow$ $\exists$ a **factorization** (is also an **LMI**):

$$\Phi(-\xi, \xi) = F^\top(-\xi) F(\xi)$$

Positivity of QDF's

Roughly:

Q_Φ is **half-line non-negative**

$\Leftrightarrow$ **(LMI)** $\exists \ \Psi = \Psi^\star \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$:

$$-\Psi \geq 0$$

$$\Phi(\zeta,\eta) + (\zeta+\eta)\Psi(\zeta,\eta) \geq 0$$

$\Leftrightarrow$

1. $\Phi(-i\omega, i\omega) \geq 0 \quad \forall\, \omega \in \mathbb{R}$
2. A certain Pick matrix is > 0

Preliminaries: Behaviors

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Introduction

Get the physics right.

After that, it is all mathematics.

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Get the physics right.

After that, it is all mathematics.

Physics does not have signal flow graphs...

Interconnection appears by sharing variables...

Use a mathematical (graph) structure in circuit theory that supports more than just 2-terminal elements.

LTI systems $\mathfrak{L}^{\mathtt{w}}$

The *behavior* $\mathfrak{B} \subseteq (\mathbb{R}^{\mathtt{w}})^{\mathbb{R}}$ belongs to $\mathfrak{L}^{\mathtt{w}}$: $\Leftrightarrow$

$\exists$ a polynomial matrix $R \in \mathbb{R}^{\bullet \times \mathtt{w}}[\xi]$ such that

$$\mathfrak{B} = \{w \in \mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \mid R\left(\frac{d}{dt}\right) w = 0\}$$

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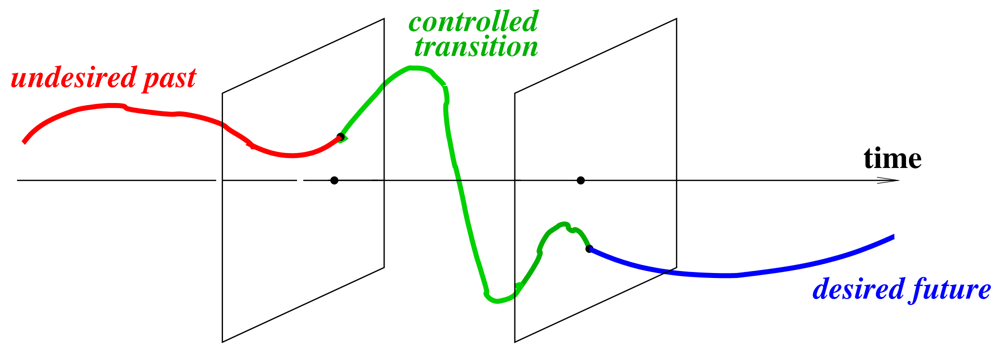
$\mathfrak{B}$ is said to be **controllable** : $\Leftrightarrow$

the trajectories in $\mathfrak{B}$ are ‘patch-able’, ‘concatenable’.



$\mathfrak{B}$ is said to be controllable $:\Leftrightarrow$

the trajectories in $\mathfrak{B}$ are 'patch-able', 'concatenable'.

$\Leftrightarrow$ $R(\lambda)$ has the same rank for all $\lambda \in \mathbb{C}$.

$\Leftrightarrow$ $\mathfrak{B}$ is an image: $\exists \ M \in \mathbb{R}^{\mathtt{w} \times \bullet}[\xi]$ such that

$$\mathfrak{B} = \operatorname{im}\left(M\left(\frac{d}{dt}\right)\right).$$

LTI systems $\mathfrak{L}^{\mathtt{w}}$

$\mathfrak{B}$ is said to be **controllable** :$\Leftrightarrow$

the trajectories in $\mathfrak{B}$ are 'patch-able', 'concatenable'.

$\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$:$\Leftrightarrow$

$$R\left(\frac{d}{dt}\right) w = 0$$

$\exists$ *kernel representation*

iff $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ and controllable

$$w = M\left(\frac{d}{dt}\right) \ell$$

$\exists$ *image representation*

LTI systems $\mathfrak{L}^{\mathtt{w}}$

$\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ is said to be **autonomous** $:\Leftrightarrow$

$$w_1, w_2 \in \mathfrak{B}, w_1(t) = w_2(t) \text{ for } t < 0 \;\Rightarrow\; w_1 = w_2.$$

$\Leftrightarrow \mathfrak{B}$ is finite dimensional

$\Leftrightarrow \exists$ kernel repr. $R\left(\frac{d}{dt}\right) w = 0$, with R square, $\det(R) \neq 0$.

$\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ is said to be stable $:\Leftrightarrow$

$$w \in \mathfrak{B} \;\Rightarrow\; w(t) \rightarrow 0 \text{ for } t \rightarrow \infty.$$

$\Leftrightarrow \exists$ kernel repr. $R\left(\frac{d}{dt}\right) w = 0$,

with R square and $\det(R)$ a Hurwitz polynomial.

stable $\Rightarrow$ autonomous.

LTI systems $\mathfrak{L}^{\mathtt{w}}$

After you get used to $w \in \mathfrak{B}$, the rest is easy.

Problem formulation

Aims

Object of study: for $\Phi = \Phi^\star \in \mathbb{R}^{w \times w}[\zeta, \eta]$

the stationarity, minimality, etc. of the integral:

$$\int_{-\infty}^{+\infty} Q_\Phi(w)\, dt$$

viewed as a map from $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^w)$ to $\mathbb{R}$

Aims

Object of study: for $\Phi = \Phi^\star \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$

the **stationarity, minimality, etc.** of the integral:

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viewed as a map from $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$ to $\mathbb{R}$

1. Char. the **stationary** traj. w.r.t. **compact support variations**
2. the **local minima** w.r.t. compact support variations
3. stationary trajectories w.r.t. **one-sided variations**
4. local minima w.r.t. **one-sided variations**
5. Minimize with **initial and terminal conditions** on w.

Level of generality

$$\int_{-\infty}^{+\infty} Q_\Phi(w)\, dt$$

for $w \in \mathfrak{B}$, $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$, controllable, $w = M\left(\frac{d}{dt}\right)\ell$

$$\rightsquigarrow \quad \int_{-\infty}^{+\infty} Q_{\Phi'}(\ell)\, dt$$

$\ell \in \mathfrak{C}^\infty$, with $\Phi'(\zeta,\eta) = M^\top(\zeta)\Phi(\zeta,\eta)M(\eta)$.

Solve for stationary, minimizing ℓ, return to $w = M\left(\frac{d}{dt}\right)\ell$.

Level of generality

$$\int_{-\infty}^{+\infty} w^\top G w \, dt$$

with $G \in \mathbb{R}^{\mathtt{w} \times \mathtt{w}}(\xi)$ a rational weighting function.

$$\rightsquigarrow \quad G = P^{-1} Q, \qquad P, Q \in \mathbb{R}^{\bullet \times \bullet}[\xi]$$

$$P\left(\frac{d}{dt}\right) w = Q\left(\frac{d}{dt}\right) v \rightsquigarrow w' = \begin{bmatrix} w \\ v \end{bmatrix}$$

Level of generality

$$\int_{-\infty}^{+\infty} (G_1 w)^\top (G_2 w)\ dt$$

with $G_1, G_2 \in \mathbb{R}^{\bullet \times \bullet}(\xi)$ rational **weighting functions**, with

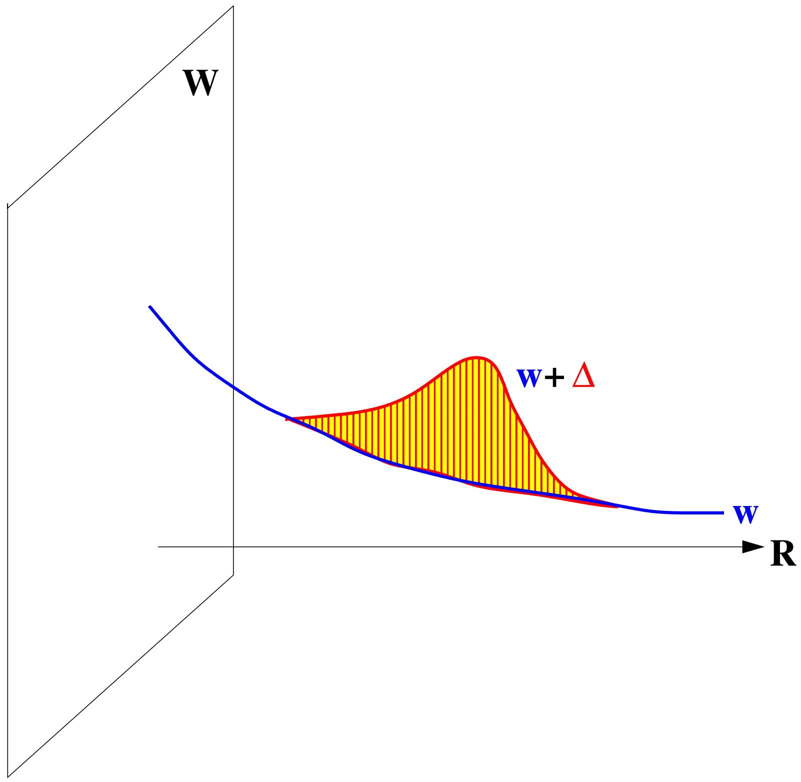
$$R_1\left(\frac{d}{dt}\right) w_1 = R_2\left(\frac{d}{dt}\right) w_2$$

e.g. $$\frac{d}{dt}x = Ax + Bw_1, \ \ w_2 = Cx + Dw_2$$

- **quadratic** functionals
- **rational** weightings
- variables related by **linear differential systems**

Compact support variations

Compact support variations



What do we mean by w 'stationary', a 'local minimum'?

Compact support variations

Note that $Q_\Phi\,(w+\Delta) - Q_\Phi\,(w) = 2L_\Phi(w,\Delta) + Q_\Phi(\Delta)$

$w \in \mathfrak{C}^\infty\,(\mathbb{R},\mathbb{R}^{\mathtt{W}})$ is said to be stationary for $\int Q_\Phi$ w.r.t. compact support variations if

$$\int_{-\infty}^{+\infty} \left(Q_\Phi(w+\Delta) - Q_\Phi(w)\right)\,dt = \int_{-\infty}^{+\infty} Q_\Phi(\Delta)\,dt$$

for all $\Delta \in \mathfrak{C}^\infty\,(\mathbb{R},\mathbb{R}^{\mathtt{W}})$ of compact support. i.e.

$$\int_{-\infty}^{+\infty} L_\Phi(w,\Delta)\,dt = 0 \quad \forall\Delta$$

Compact support variations

$w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$ is said to be **stationary** for $\int Q_\Phi$ w.r.t. compact support variations if

$$\int_{-\infty}^{+\infty} \left(Q_\Phi(w + \Delta) - Q_\Phi(w) \right) \, dt = \int_{-\infty}^{+\infty} Q_\Phi(\Delta) \, dt$$

for all $\Delta \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$ of compact support. And a **local minimum** if

$$\int_{-\infty}^{+\infty} \left(Q_\Phi(w + \Delta) - Q_\Phi(w) \right) \, dt \geq 0$$

for all $\Delta \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$ of compact support.

$\Leftrightarrow$ stationarity and $\int_{-\infty}^{+\infty} Q_\Phi(\Delta) \, dt \geq 0.$

Compact support variations

Theorem:

1. Stationary $\in \mathfrak{L}^{\mathtt{w}}$
2. w is stationary $\Leftrightarrow$ $\Phi(-\frac{d}{dt}, \frac{d}{dt})w = 0$
3. Either no local minima, or stationary $\Rightarrow$ local minimum
4. Local minima $\Leftrightarrow$
   (i) $\Phi(-\frac{d}{dt}, \frac{d}{dt})w = 0$
   (ii) Q_Φ is average non-negative,
   i.e. $\Phi(-i\omega, i\omega) \geq 0 \quad \forall \omega \in \mathbb{R}$

If $\det(\Phi(-\xi, \xi)) \neq 0$, stationary is an autonomous system.

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If $\det(\Phi(-\xi, \xi)) \neq 0$, stationary is an autonomous system.

Of crucial importance in mechanics

Examples

$$\int_{-\infty}^{+\infty} \left(w^2 - \left(\frac{d}{dt} w\right)^2 \right) dt \;\rightsquigarrow\; \Phi(\zeta, \eta) = 1 - \zeta\eta \rightsquigarrow$$

$$w + \frac{d^2}{dt^2} w = 0$$ harmonic oscillator

local minimum? $\rightsquigarrow \; 1 - \omega^2$ **NO!**

Examples

$$\int_{-\infty}^{+\infty}\left(w^2-\left(\tfrac{d}{dt}w\right)^2\right)\,dt \;\rightsquigarrow\; \Phi(\zeta,\eta)=1-\zeta\eta \rightsquigarrow$$

$$w+\frac{d^2}{dt^2}w=0$$

harmonic oscillator

local minimum? $\rightsquigarrow\; 1-\omega^2$ **NO!**

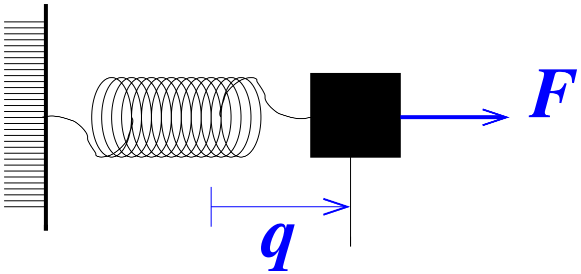
$$\int_{-\infty}^{+\infty}\left(w^2+\left(\tfrac{d}{dt}w\right)^2\right)\,dt \rightsquigarrow \Phi(\zeta,\eta)=1+\zeta\eta \rightsquigarrow$$

$$w-\frac{d^2}{dt^2}w=0$$

hyperbolic flow

local minimum? $\rightsquigarrow\; 1+\omega^2$ **YES!**

Examples



$$Kq + D\frac{d}{dt}q + M\frac{d^2}{dt^2}q = F$$

Energy absorbed $$\int_{-\infty}^{+\infty} F\left(\frac{d}{dt}q\right)\, dt$$

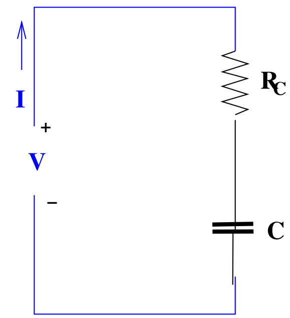
$$\rightsquigarrow \left[\frac{q}{F}\right] = \left[\frac{1}{\left(K + D\frac{d}{dt} + M\frac{d^2}{dt^2}\right)}\right] w$$

$$\rightsquigarrow \quad \frac{1}{2}\left(K + D\zeta + M\zeta^2\right)\eta + \frac{1}{2}\left(K + D\eta + M\eta^2\right)\zeta \quad \rightsquigarrow$$

$$D\frac{d^2}{dt^2}q = 0$$ **for** $D > 0 \rightsquigarrow$ $q = \alpha + \beta t$ **local minima**

Stat. traj. include (strictly) the dissipation-free ones $q(t)$ **= constant.**

Examples



Energy absorbed $\int_{-\infty}^{+\infty} V\, I\, dt$

$$\rightsquigarrow \begin{bmatrix} V \\ I \end{bmatrix} = \begin{bmatrix} 1 + R_C C \frac{d}{dt} \\ C \frac{d}{dt} \end{bmatrix} V_c$$

$$\rightsquigarrow \quad \tfrac{1}{2}\,(1 + R_C C \zeta)\, C\eta + \tfrac{1}{2}\,(1 + R_C C\eta)\, C\zeta \quad \rightsquigarrow$$

$$-R_C C^2 \frac{d^2}{dt^2} q = 0 \;\rightsquigarrow\; V(t) = \alpha + \beta t, I(t) = \beta'$$ local minima

The stat. traj. include (strictly) the dissipation-free traj. $I(t) = 0$.

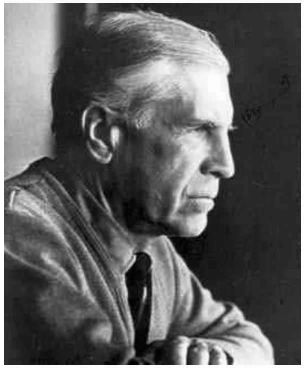
State representation

The stationary trajectories in other representations. E.g.

$$\frac{d}{dt}x = Ax + Bu \qquad \int_{-\infty}^{+\infty} (u^\top Ru + x^\top Qx)\, dt$$

$$\rightsquigarrow \qquad \frac{d}{dt}\begin{bmatrix} x \\ \lambda \end{bmatrix} = H \begin{bmatrix} x \\ \lambda \end{bmatrix}, \quad u = -R^{-1}B^\top \lambda$$

$$H = \begin{bmatrix} A & -BR^{-1}B^\top \\ -Q & -A^\top \end{bmatrix}$$



Local minima $\Rightarrow R \geq 0.$

Stability

Stationarity, local minimality have no bearing on stability.

In fact, if they are 'time-symmetric' notions, while stability is not.

In mechanics, there is no stability.

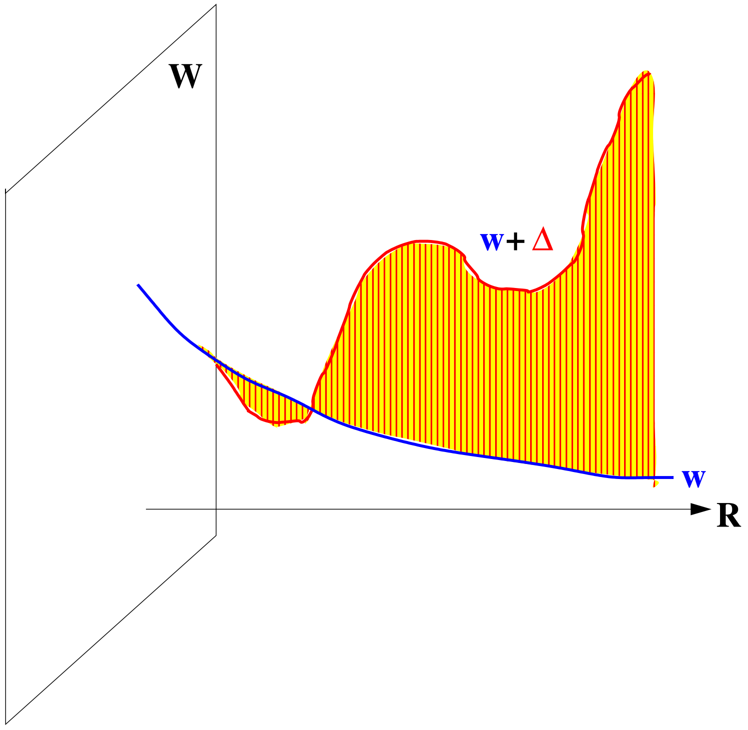
In control applications, stability can be imposed

or, as we now aim at,

enforced by imposing stronger requirements on Q_Φ.

One-sided variations

One-sided variations



When is w a local minimum?

One-sided variations

$w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$ **is said to be a**

local minimum w.r.t. one-sided variations **:** $\Leftrightarrow$

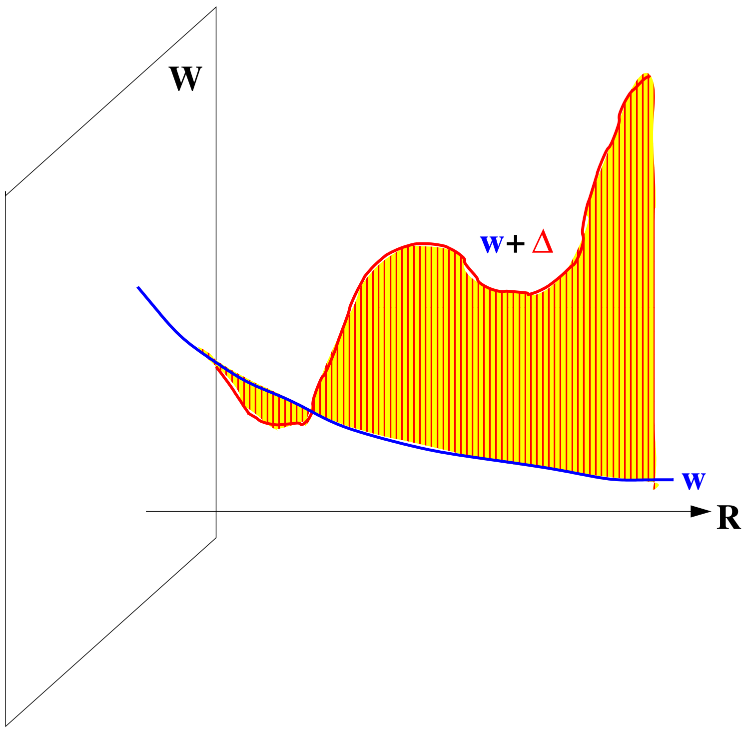
$$\int_{-\infty}^{+\infty} (Q_\Phi(w + \Delta) - Q_\Phi(w)) \; dt \geq 0$$

for all $\Delta \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$ **with support on a half line** $[t, +\infty)$ **for some** $t \in \mathbb{R}$**.**

One-sided variations

$$\int_{-\infty}^{+\infty} \left(Q_\Phi(w+\Delta) - Q_\Phi(w)\right)\, dt \geq 0$$

for all $\Delta \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$ with support on a half line



Main new result

Theorem:

w is a local minimum w.r.t. one-sided variations

$\Rightarrow$ local minimum w.r.t. compact support variations, i.e.

$$\Phi(-\frac{d}{dt}, \frac{d}{dt})w = 0 \quad \textbf{and} \quad \Phi(-i\omega, i\omega) \geq 0 \ \forall \omega \in \mathbb{R}.$$

Assume > 0.

Main new result

Theorem:

w is then a local minimum w.r.t. one-sided variations $\Leftrightarrow$ (roughly)

1. $\Phi(-\frac{d}{dt}, \frac{d}{dt})w = 0$
2. Q_Φ is half-line nonnegative
3. $w(t) \to 0$ for $t \to \infty$

Stability is a consequence of minimality, shows importance in control.

Algorithmic problem: Extract the stable trajectories from $\Phi(-\frac{d}{dt}, \frac{d}{dt})w = 0. \rightsquigarrow$ 'spectral factorization'.

Main new result

Computation:

$$\Phi(-i\omega, i\omega) > 0 \;\forall \omega \in \mathbb{R}$$

$$\Leftrightarrow \;\exists\; H \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi] \text{ with } \det(H) \text{ Hurwitz :}$$

$$\Phi(-\xi, \xi) = H^\top(-\xi)H(\xi)$$

Hence $H(\frac{d}{dt})w = 0$ **gives the one-sided optimal trajectories.**

Computation much more tricky: 'SOS', spectral factorization, ARE, LMI's with extremality conditions.

Conclusion:

Optimality w.r.t. compact support has nothing to do with stability.

Optimality w.r.t. one-sided variations delivers exactly stability.

Controller implementation

Implementation

How does this relate to the classical view of control, where an input trajectory selection, or a feedback law selection are the aim?

Implementation

Start with $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$, criterion like QDF Q_Φ.

Obtain a (stationary, or optimal w.r.t. compact support or one-sided variations) controlled behavior

$$\mathfrak{K} \in \mathfrak{L}^{\mathtt{w}}, \qquad \mathfrak{K} \subseteq \mathfrak{B}$$

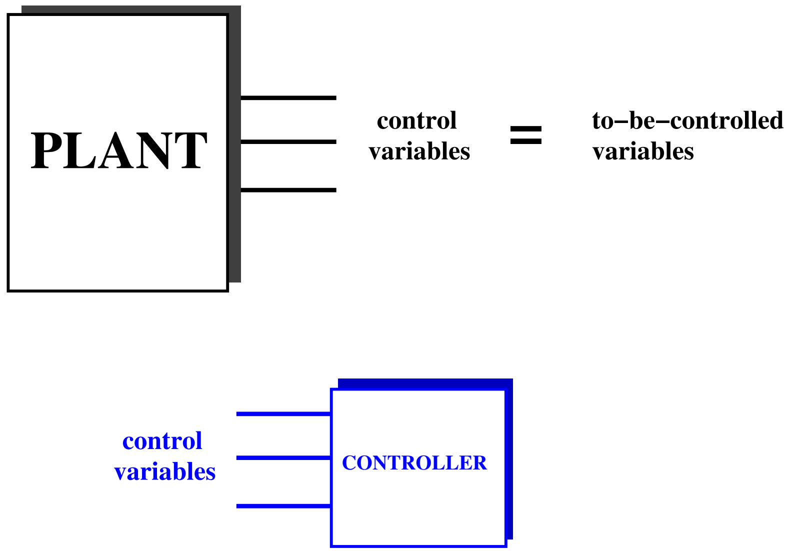
Find a controller $\mathfrak{C} \in \mathfrak{L}^{\mathtt{w}}$ such that

$$\mathfrak{K} = \mathfrak{B} \cap \mathfrak{C}$$

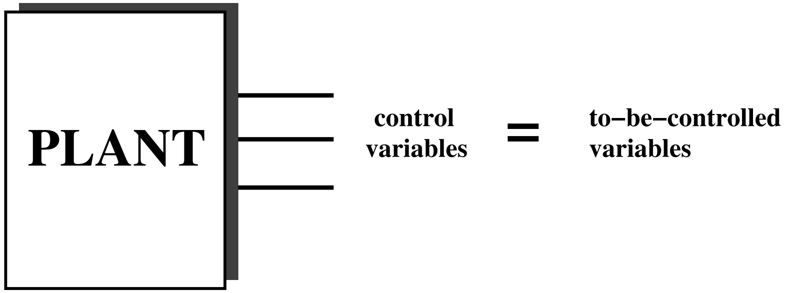
regular interconnection, or feedback...

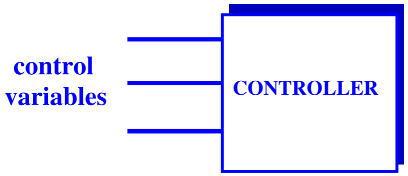
cfr. the work of Trentelman c.s.

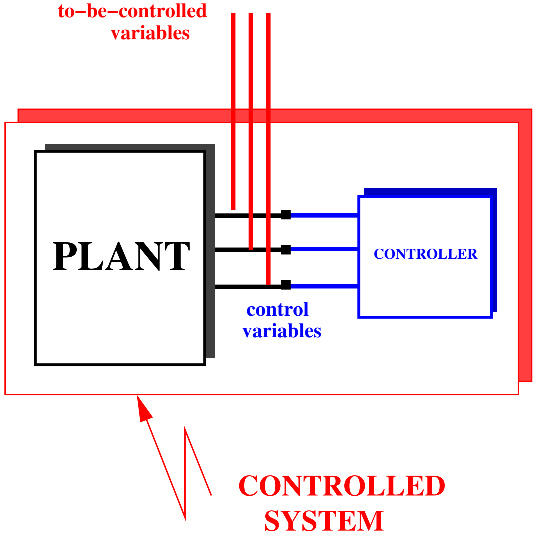
Implementation



Implementation







Implementation

$$\mathfrak{K} = \mathfrak{B} \cap \mathfrak{C}$$

Given a first principles representation of the plant $\mathfrak{B}$ and Φ,

end up with a adapted representation of the controller $\mathfrak{C}$.

LQ optimal trajectories

Optimal trajectories

Given $\Phi \in \mathbb{R}^{\mathtt{w} \times \mathtt{w}}[\zeta, \eta]$ and $I \in \mathbb{R}^{\bullet \times \mathtt{w}}[\xi]$, $a \in \mathbb{R}^{\bullet}$:

! Minimize or infimize $\int_0^{+\infty} Q_\Phi(w)\, dt$ **subject to**

$$I(\tfrac{d}{dt})w(0) = a$$

and possibly conditions on w and its derivatives as $t \to +\infty$.

Optimal trajectories

Given $\Phi \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ and $I \in \mathbb{R}^{\bullet\times\mathtt{w}}[\xi]$, $a \in \mathbb{R}^{\bullet}$:

! **Minimize or infimize** $\int_0^{+\infty} Q_\Phi(w)\, dt$ subject to

$$I(\tfrac{d}{dt})w(0) = a$$

and possibly conditions on w and its derivatives as $t \to +\infty$.

1. When is infimum $> +\infty$?
2. When is the infimum a minimum ?
3. When is the minimum unique ?
4. Given $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ when is there an optimal continuation ?
5. etc.

Optimal trajectories

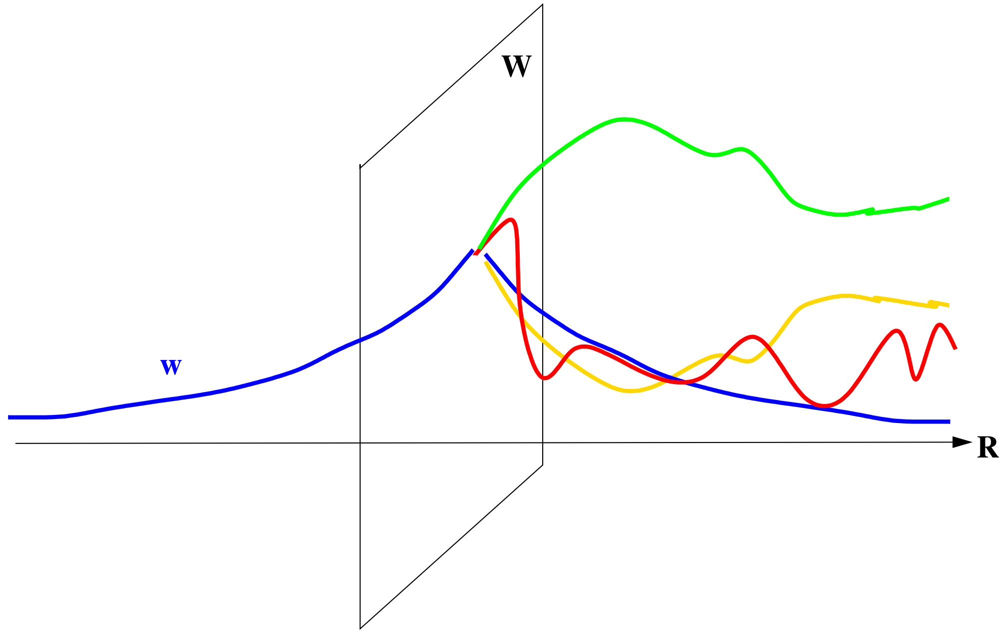
These problems are dealt with in complete generality in the paper.

Example: Given $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$, controllable.

$$\text{! Minimize} \int_0^{+\infty} ||w||^2 \, dt$$

subject to $w \in \mathfrak{B}$ and $w|_{(-\infty,0]}$ given.

Optimal trajectories



Find continuation that minimizes $\int_0^{+\infty} ||w||^2 \, dt$.

How to compute it? Is it unique? Is it stable?

Optimal trajectories

Assume $\mathfrak{B}$ given in observable image representation

$$w = M\left(\frac{d}{dt}\right)\ell.$$

Find the stable stationary trajectories for

$$Q_\Phi \quad \text{with} \quad \Phi(\zeta,\eta) = M^\top(\zeta)M(\eta)$$

via spectral factorization of

$$M^\top(-\xi)M(\xi) = H^\top(-\xi)H(\xi) \quad \det(H) \text{ Hurwitz}.$$

Optimal trajectories

Finally 'match' the required initial conditions for

$$H(\frac{d}{dt})w^* = 0$$

with those of the given $w \in \mathfrak{B}$.

Existence, uniqueness, stability ...

Summary

Highligths

- QDF's and their positivity.
  The role of $\mathbb{R}[\zeta, \eta]$.
- Stationarity w.r.t. compact support variations: readily
- local minimality $\Leftarrow$ average positivity.
- w.r.t. one-sided variations $\Leftarrow$ half-line positivity. Extracts stable stationary traj.
- Outline of the LQ trajectory optimization problem

Thank you

Thank you

Thank you

Thank you

Thank you

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