Question

Let $P \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}[\xi]$**. Assume** $P(\xi) = P^\top(-\xi)$**. Consider**

$$\int_{-\infty}^{+\infty} \left[w(t)^\top \;\; P(\tfrac{d}{dt})w(t)\right] \;\; dt$$

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- **Under which conditions on** P **is this integral** ≥ 0
  for 'blips' := short duration compact support w**'s ?**

OPTIMALITY w.r.t. BLIPS

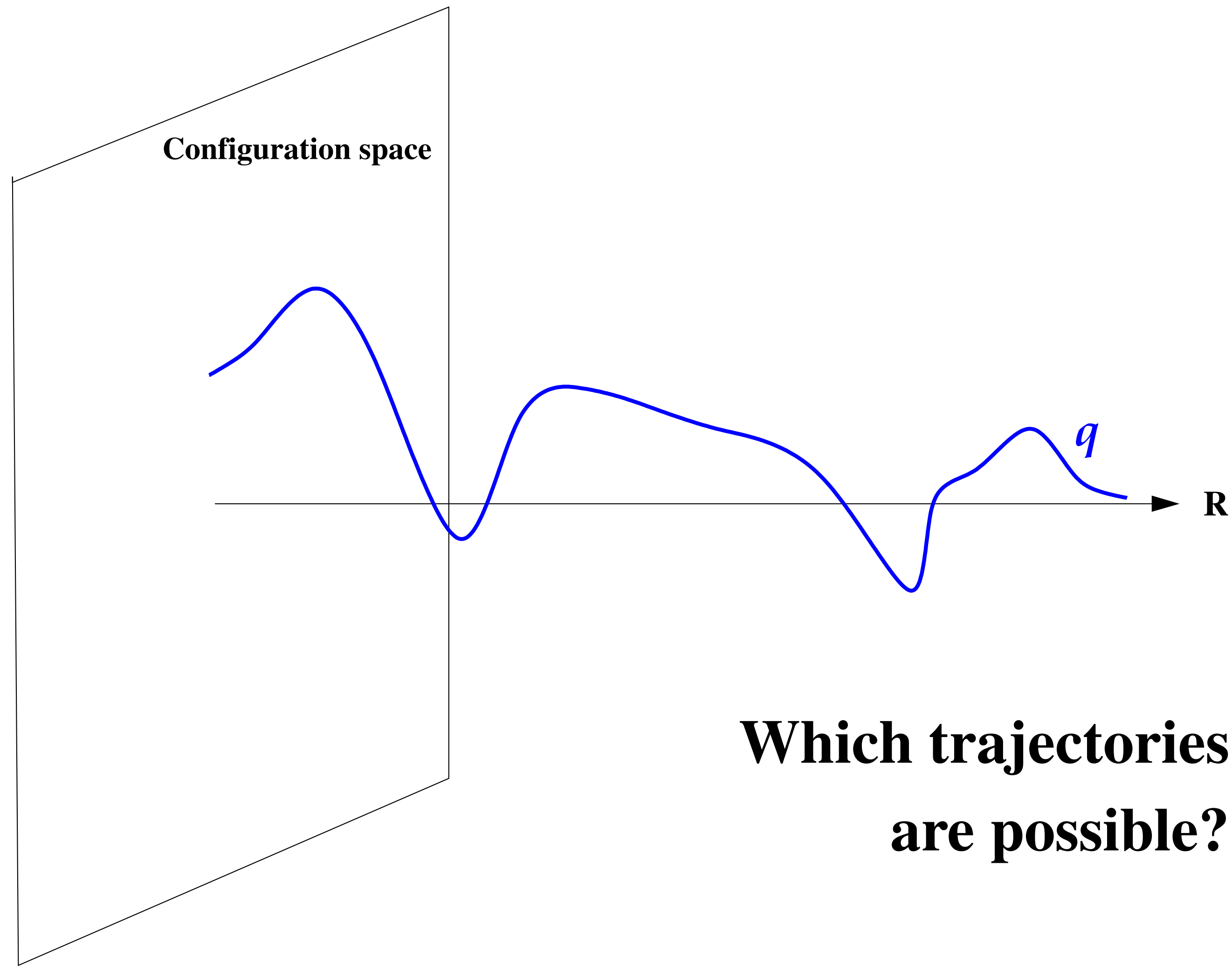
Jan C. Willems
K.U. Leuven, Belgium

ECC-CDC 2005 **Sevilla, December 12, 2005**

Joint paper with

Maria Elena Valcher
Universitá di Padova

Motivation: variational principles of mechanics



Which trajectories are possible?

Motivation: variational principles of mechanics

Which trajectories are possible?

Configuration space: $\mathbb{R}^{\mathtt{n}}$

$$\begin{array}{lrcl}
\textit{kinetic energy} & (\mathrm{q},\dot{\mathrm{q}}) \in \mathbb{R}^{\mathtt{n}} \times \mathbb{R}^{\mathtt{n}} & \mapsto & K(\mathrm{q},\dot{\mathrm{q}}) \in \mathbb{R} \\
\textit{potential energy} & \mathrm{q} \in \mathbb{R}^{\mathtt{n}} & \mapsto & P(\mathrm{q}) \in \mathbb{R} \\
\textit{`Lagrangian'} & (\mathrm{q},\dot{\mathrm{q}}) \in \mathbb{R}^{\mathtt{n}} \times \mathbb{R}^{\mathtt{n}} & \mapsto & L(\mathrm{q},\dot{\mathrm{q}}) = K(\mathrm{q},\dot{\mathrm{q}}) - P(\mathrm{q})
\end{array}$$

Motivation: variational principles of mechanics

Which trajectories are possible?

Configuration space: $\mathbb{R}^n$

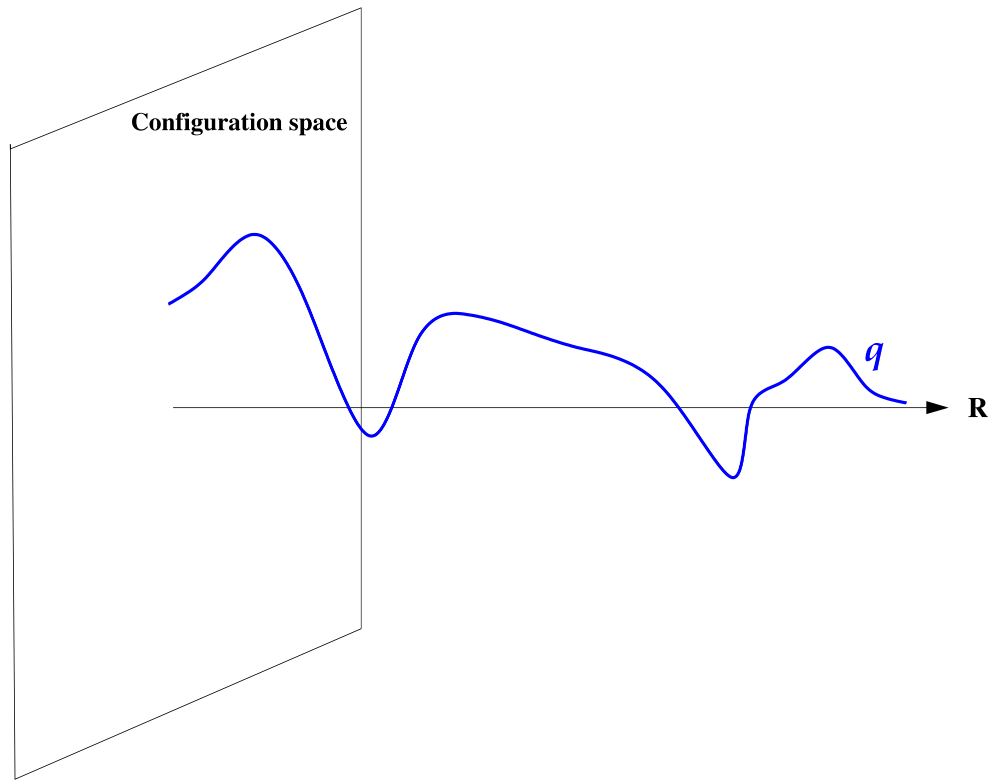
kinetic energy	$(q,\dot{q}) \in \mathbb{R}^n \times \mathbb{R}^n$	$\mapsto$	$K(q,\dot{q}) \in \mathbb{R}$
potential energy	$q \in \mathbb{R}^n$	$\mapsto$	$P(q) \in \mathbb{R}$
'Lagrangian'	$(q,\dot{q}) \in \mathbb{R}^n \times \mathbb{R}^n$	$\mapsto$	$L(q,\dot{q}) = K(q,\dot{q}) - P(q)$

Variational principles: $\rightsquigarrow$ **The possible trajectories are those that minimize the 'action integral'**

$$\int_{-\infty}^{+\infty} L\left(q(t), \frac{dq}{dt}(t)\right)\ dt$$

Motivation: variational principles of mechanics

¿¿¿ What does this 'minimization' mean ??? When does



'minimize'

$$\int_{-\infty}^{+\infty} L\left(q(t), \frac{dq}{dt}(t)\right) \; dt \;\textbf{?}$$

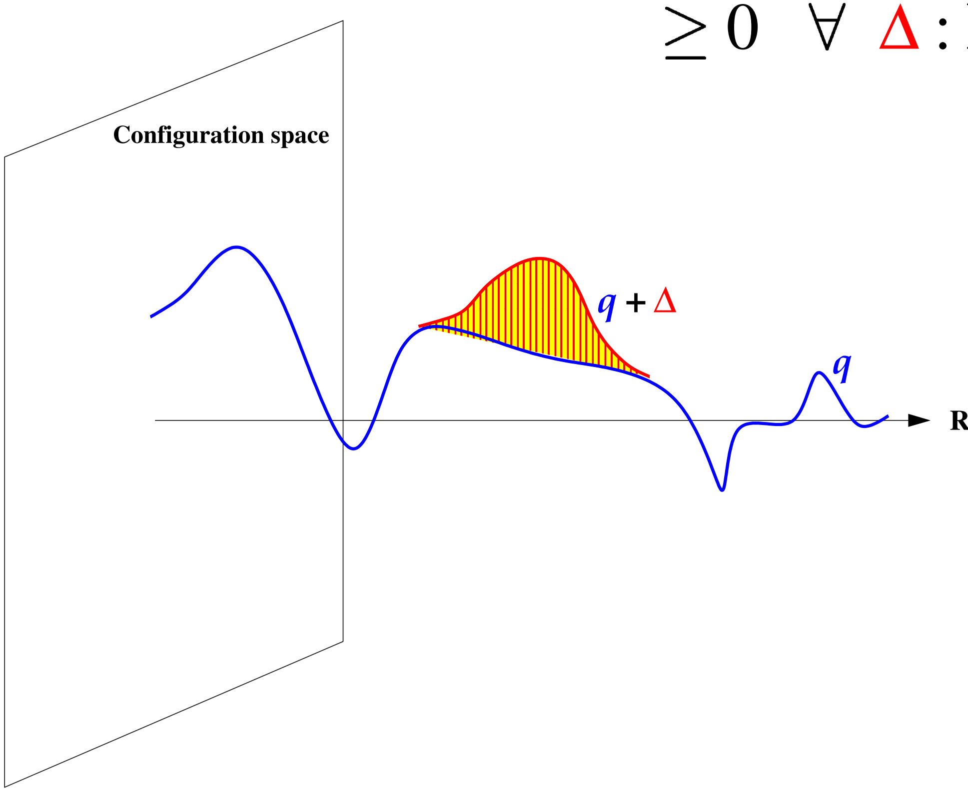
Optimality

Definition: $q : \mathbb{R} \to \mathbb{R}^n$ ***minimizes*** **the action integral** $:\Leftrightarrow$

$$\partial_A(q,\Delta) :=$$

$$\int_{-\infty}^{+\infty} \left[L\left(q(t) + \Delta(t), \frac{dq}{dt}(t) + \frac{d\Delta}{dt}(t) \right) - L\left(q(t), \frac{dq}{dt}(t) \right) \right] dt$$

$\geq 0 \quad \forall \ \Delta : \mathbb{R} \to \mathbb{R}^n$ **of compact support**



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$\geq 0 \quad \forall \ \Delta : \mathbb{R} \to \mathbb{R}^n$ **of compact support**

Necessary conditions:

1. **'Euler-Lagrange equations':**

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}} \left(q(t), \frac{dq}{dt}(t) \right) - \frac{\partial L}{\partial q} \left(q(t), \frac{dq}{dt}(t) \right) = 0$$

2. **The second variation integral is ≥ 0 over all compact support domains.**

Example

Configuration space $\mathbb{R}$; Lagrangian $\frac{1}{2}(M\dot{\mathrm{q}}^2 - K\mathrm{q}^2)$.

$$\partial_A(q,\Delta) = \int_{-\infty}^{+\infty} -\Delta\,(K\frac{d^2q}{dt^2} + Mq)\,dt + \frac{1}{2}\int_{-\infty}^{+\infty} \left[M(\frac{d\Delta}{dt})^2 - K\Delta^2\right]\,dt.$$

Euler-Lagrange:

$$M\frac{d^2q}{dt^2} + Kq = 0$$

Every Euler-Lagrange solution satisfies $\partial_A(q,\Delta) \geq 0$

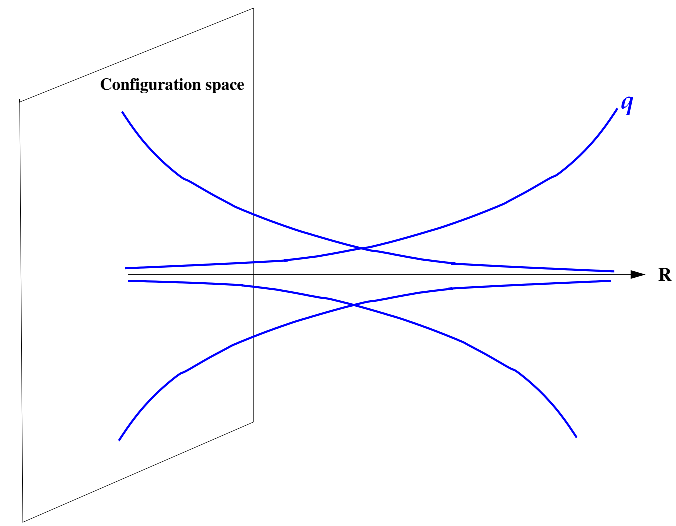
$$\Leftrightarrow \quad \int_{-\infty}^{+\infty} \left[M(\frac{d\Delta}{dt})^2 - K\Delta^2\right]\,dt \geq 0$$

$\forall\ \Delta$ **of compact support. Non-negativity if and only if $K < 0$.**

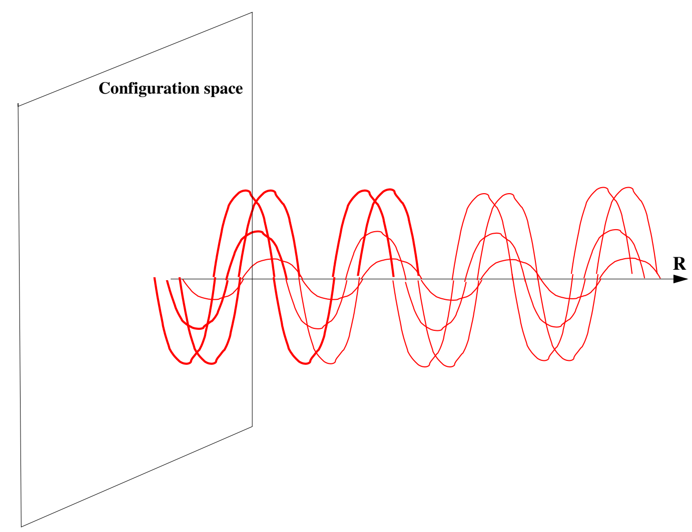
<u>Note</u>: This is an integral of the sort announced in the beginning.

Example

The hyperbolic flow $\dfrac{d^2q}{dt^2} - q = 0$ **does consist of minima w.r.t.** $\frac{1}{2}\left(\dot{q}^2 + q^2\right)$



The oscillator $\dfrac{d^2q}{dt^2} + q = 0$ **does not consists of minima w.r.t.** $\frac{1}{2}\left(\dot{q}^2 - q^2\right)$**.**



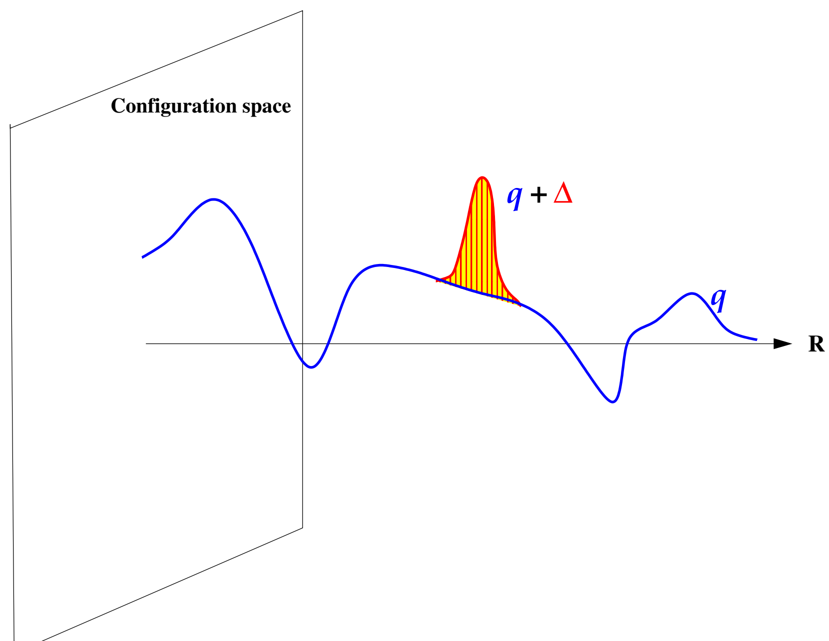
Optimality w.r.t. blips

$q : \mathbb{R} \to \mathbb{R}^n$ is ***minimizes*** **the action integral w.r.t. blips** $:\Leftrightarrow$

$$\partial_A(q, \Delta) :=$$

$$\int_{-\infty}^{+\infty} \left[L\left(q(t) + \Delta(t), \frac{dq}{dt}(t) + \frac{d\Delta}{dt}(t) \right) - L\left(q(t), \frac{dq}{dt}(t) \right) \right] dt$$

$\geq 0 \quad \forall \ \Delta : \mathbb{R} \to \mathbb{R}^n$ **of sufficiently short compact support**



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Necessary conditions:

1. **Euler-Lagrange:**

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\mathrm{q}}} \left(q(t), \frac{dq}{dt}(t) \right) - \frac{\partial L}{\partial \mathrm{q}} \left(q(t), \frac{dq}{dt}(t) \right) = 0$$

2. **The second variation integral is** ≥ 0 **over all 'short' compact support domains.**

The LQ question

$\rightsquigarrow$ **Let** $P \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}[\xi]$**. Assume** $P(\xi) = P^\top(-\xi)$**. Consider**

$$\int_{-\infty}^{+\infty} \left[w(t)^\top \; P(\tfrac{d}{dt})w(t)\right] \; dt$$

- **Under which conditions on** P **is this integral** ≥ 0
  for all $w \in \mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{n}})$ **of compact support ?**

- **Under which conditions on** P **is this integral** ≥ 0
  for 'blips' := short duration compact support w**'s ?**

  precisely, when does there exist $\varepsilon > 0$ **such that**
  $w \in \mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{n}})$ **and** $|\text{support}(w)| \leq \varepsilon$
  $\Rightarrow$ **this integral is** ≥ 0 **?**

Answer

- $$\int_{-\infty}^{+\infty} \left[w(t)^\top \ P(\frac{d}{dt}) w(t) \right] \ dt \geq 0$$

for all $w \in \mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{n}})$ **of compact support**

$$\Updownarrow$$

$$P(i\omega) = P^\top(-i\omega)^\top \geq 0 \ \forall \ \ \omega \in \mathbb{R}$$

Answer

- $$\int_{-\infty}^{+\infty} \left[w(t)^\top \; P(\frac{d}{dt}) w(t) \right] \; dt \geq 0$$

  for all $w \in \mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{n}})$ **of compact support**

  $$\Updownarrow$$

  $$P(i\omega) = P^\top(-i\omega)^\top \geq 0 \;\; \forall \;\; \omega \in \mathbb{R}$$

- $$\int_{-\infty}^{+\infty} \left[w(t)^\top \; P(\frac{d}{dt}) w(t) \right] \; dt \geq 0 \quad \textbf{for all blips}$$

  $$\Updownarrow$$

  $P(i\omega) = P^\top(-i\omega)^\top \geq 0$ **for all** $\omega \in \mathbb{R}$ **sufficiently large**

Classical: compact support LQ non-negativity

The following are equivalent:

1.

$$\int_{-\infty}^{+\infty} \left[w(t)^\top \;\; P(\frac{d}{dt}) w(t) \right] \;\; dt \geq 0$$

for all $w \in \mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{n}})$ **of compact support**

2.

$$P(i\omega) = P^\top(-i\omega)^\top \geq 0 \;\; \forall \;\; \omega \in \mathbb{R}$$

3. **there exists** $F \in \mathbb{R}^{\bullet \times \mathtt{n}}[\xi]$ **such that**

$$P(\xi) = F^\top(-\xi) F(\xi) \rightsquigarrow \int_{-\infty}^{+\infty} |F(\frac{d}{dt}) w(t)|^2 \;\; dt$$

4. **etc., etc.**

Main result: LQ non-negativity for blips

The following are equivalent:

1.
$$\int_{-\infty}^{+\infty} \left[w(t)^\top \ P(\frac{d}{dt}) w(t) \right] \ dt \geq 0$$

for all blips $\in \mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{n}})$

2. $P(i\omega) = P^\top(-i\omega)^\top \geq 0 \ \forall \ \ \omega \in \mathbb{R}$ **sufficiently large**

3. there exists a unimodular $U \in \mathbb{R}^{\mathtt{n} \times \mathtt{n}}[\xi]$ **such that** $P'(\xi) := U^\top(-\xi) P(\xi) U(\xi)$ **equals**

$$P' = \begin{bmatrix} Q & 0 \\ 0 & 0 \end{bmatrix}$$

Main result: LQ non-negativity for blips

The following are equivalent:

1.
2.
3. **with Q of the form**

$$Q(\xi) = \Delta(-\xi) Q_{\text{leading}} \Delta(\xi) + Q'(\xi),$$

with

$$\Delta(\xi) = \text{diag}(\xi^{\mathtt{n}_{1,1}}, \xi^{\mathtt{n}_{2,2}}, \cdots, \xi^{\mathtt{n}_{\dim(Q),\dim(Q)}}),$$

$$Q_{\text{leading}} = Q_{\text{leading}}^{\top} > 0,$$

degree of the $(\mathtt{k}, \mathtt{l})$-th element, $\mathtt{k} \neq \mathtt{l}$, of $Q' < \mathtt{n}_{\mathtt{k},\mathtt{k}} + \mathtt{n}_{\mathtt{l},\mathtt{l}}$.

Main result: LQ non-negativity for blips

The following are equivalent:

1.

2.

3.

$$Q(\xi) = \begin{bmatrix} \star & \star & \star & \star & \star \\ \star & \xi^{2\mathtt{n}_1} + \cdots & \star & \xi^{\mathtt{n}_1+\mathtt{n}_2-1} + \cdots & \star \\ \star & \star & \star & \star & \star \\ \star & (-\xi)^{\mathtt{n}_1+\mathtt{n}_2-1} + \cdots & \star & \xi^{2\mathtt{n}_2} + \cdots & \star \\ \star & \star & \star & \star & \star \end{bmatrix}$$

A corollary

Configuration space: $\mathbb{R}^{\mathtt{n}}$

kinetic energy	$\dot{\mathrm{q}} \in \mathbb{R}^{\mathtt{n}}$	$\mapsto$	$\dot{\mathrm{q}}^\top K \dot{\mathrm{q}} \in \mathbb{R}, K + K^\top > 0$
potential energy	$\mathrm{q} \in \mathbb{R}^{\mathtt{n}}$	$\mapsto$	$\mathrm{q}^\top P \mathrm{q} \in \mathbb{R}, P = P^\top$
'Lagrangian'	$(\mathrm{q}, \dot{\mathrm{q}}) \in \mathbb{R}^{\mathtt{n}} \times \mathbb{R}^{\mathtt{n}}$	$\mapsto$	$L(\mathrm{q}, \dot{\mathrm{q}}) = \dot{\mathrm{q}}^\top K \dot{\mathrm{q}} - \mathrm{q}^\top P \mathrm{q}$

$\rightsquigarrow$ **traj'ies: those that minimize, for blips, the action integral**

$$\int_{-\infty}^{+\infty} L\left(q(t), \frac{dq}{dt}(t)\right) dt$$

Equivalently, those that satisfy 'Euler-Lagrange':

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\mathrm{q}}}\left(q(t), \frac{dq}{dt}(t)\right) - \frac{\partial L}{\partial \mathrm{q}}\left(q(t), \frac{dq}{dt}(t)\right) = 0$$

Beyond LQ

It is known that if the *'Lagrangian'*

$$L(\mathrm{q}, \dot{\mathrm{q}})$$

is strictly convex in $\dot{\mathrm{q}}$, then a trajectory minimizes the action integral for 'blips' (small & short variations, suitably defined)

$$\int_{-\infty}^{+\infty} L\left(q(t), \frac{dq}{dt}(t)\right) dt$$

$\Leftrightarrow$ it satisfies 'Euler-Lagrange':

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\mathrm{q}}}\left(q(t), \frac{dq}{dt}(t)\right) - \frac{\partial L}{\partial \mathrm{q}}\left(q(t), \frac{dq}{dt}(t)\right) = 0$$

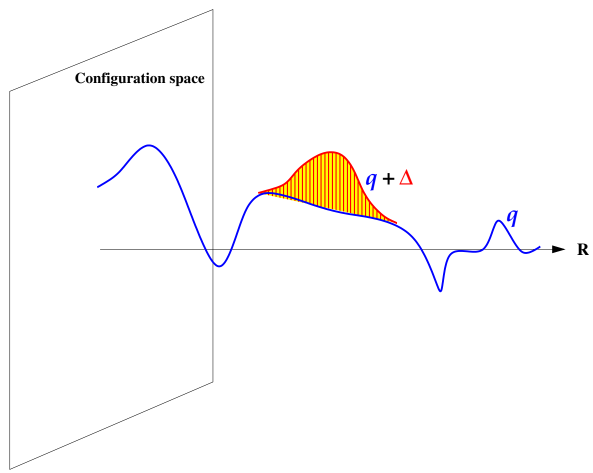
High order generalizations ...

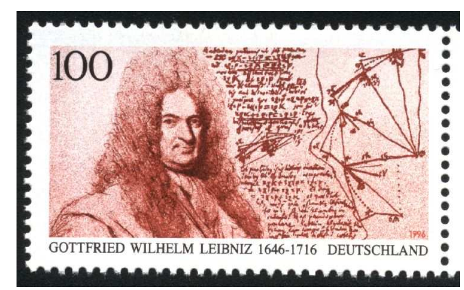
The morale of the story

The variational principles of mechanics, apparently first formulated by Maupertuis, play an important role in discussions around Leibniz' dictum that

Ours is the best of all possible worlds

Pierre de Maupertuis (1698-1778)





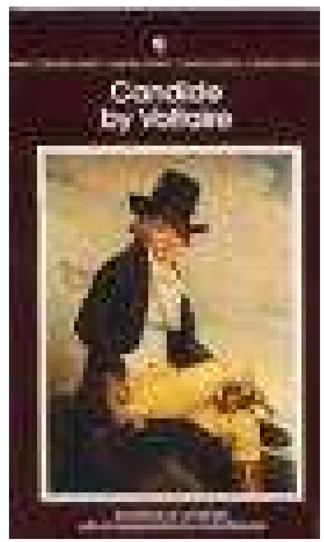
Gottfried Wilhelm Leibniz (1646-1716)

The morale of the story

Ours is the best of all possible worlds

Ridiculed by Voltaire in Candide (1759) with Dr. Pangloss

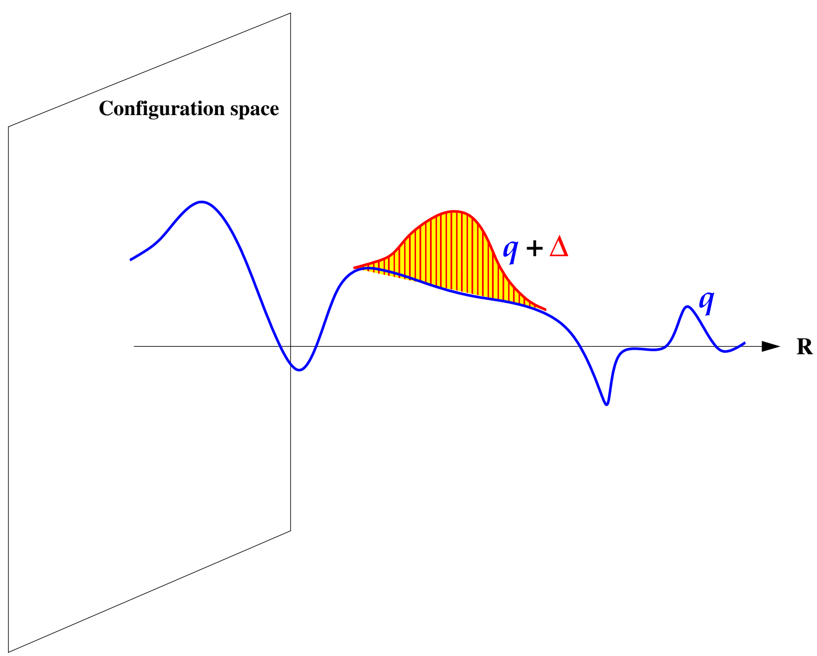
François-Marie Voltaire (1694-1778)

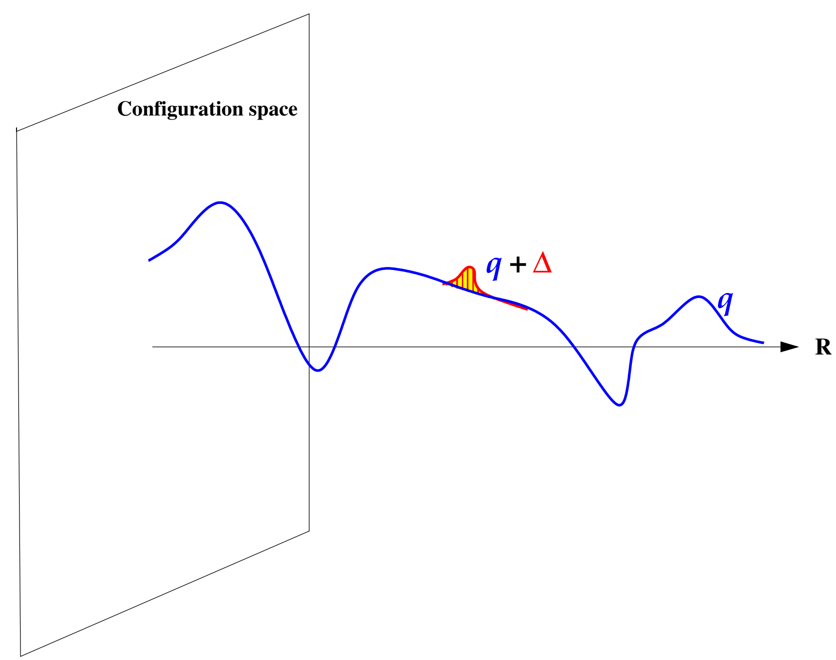


Once one dismisses
The rest of possible worlds
One finds that this is
The best of all possible worlds

The morale of the story

Ours is the best of all possible worlds





Mechanics does not deal with 'global' optimality, but, at best, with optimality w.r.t. short & small variations: 'blipjes'

Conclusion

$$\int_{-\infty}^{+\infty} \left[w(t)^\top \;\; P(\frac{d}{dt}) w(t) \;\; dt \right] \geq 0 \quad \textbf{for all blips}$$

$$\Updownarrow$$

$$P(i\omega) = P^\top(-i\omega)^\top \geq 0 \;\; \textbf{for all} \;\; \omega \in \mathbb{R} \;\; \textbf{sufficiently large}$$

Details & copies of the lecture frames are available from/at

`Jan.Willems@esat.kuleuven.be`

`http://www.esat.kuleuven.be/~jwillems`

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Thank you

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