

THE SUM-of-SQUARES PROBLEM and DISSIPATIVE DISTRIBUTED SYSTEMS

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Linear differential distributed (n-d) systems

Let
$$R \in \mathbb{R}^{ullet imes imes}[\xi_1,\cdots,\xi_{ ext{n}}],$$
 and consider

$$R\left(rac{\partial}{\partial x_1},\cdots,rac{\partial}{\partial x_n}
ight) oldsymbol{w}=0.$$
 (*)

Define the associated 'behavior'

$$\mathfrak{B}=\{oldsymbol{w}\in\mathfrak{C}^{\infty}\left(\mathbb{R}^{\mathrm{n}},\mathbb{R}^{\mathrm{w}}\right)\mid(*)\ \mathsf{holds}\ \}.$$

Notation $\mathfrak{B} \in \mathfrak{L}_{n}^{w}$.

Example

Maxwell's equations: n = 4, w = 10.



$$egin{array}{lll}
abla \cdot ec{E} &=& rac{1}{arepsilon_0}
ho \,, \
abla imes ec{E} &=& -rac{\partial}{\partial t}ec{B} \,, \
abla \cdot ec{B} &=& 0 \,, \
abla \cdot ec{B} &=& rac{1}{arepsilon_0}ec{j} + rac{\partial}{\partial t}ec{E} \,. \end{array}$$

QDF's

Use multi-index notation. Consider

$$w\mapsto \sum_{k,\ell} \left(rac{d^k}{dx^k}w
ight)^ op \Phi_{k,\ell} \left(rac{d^\ell}{dx^\ell}w
ight)$$

$$\mathfrak{C}^{\infty}\left(\mathbb{R}^{\mathrm{n}},\mathbb{R}^{\mathtt{w}}
ight)
ightarrow\mathfrak{C}^{\infty}\left(\mathbb{R}^{\mathrm{n}},\mathbb{R}
ight)$$
. $\Phi_{k,\ell}\in\mathbb{R}^{\mathtt{w} imes\mathtt{w}};\;\Phi_{k,\ell}=\Phi_{\ell,k}^{ op}.$

Introduce the 2n-variable polynomial matrix Φ

$$\Phi(\zeta,\eta) = \sum_{k,\ell} \Phi_{k,\ell} \zeta^k \eta^\ell.$$

Denote the QDF as Q_{Φ} .

Dissipative distributed systems

$$\mathfrak{B}\in\mathfrak{L}_n^{\scriptscriptstyle{W}}$$
 is

dissipative with respect to the 'supply rate' Q_{Φ}

:⇔

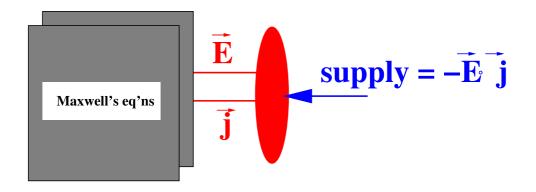
$$\int_{\mathbb{R}^{ ext{n}}}Q_{\Phi}\left(w
ight)\;dx\geq0$$

for all $w \in \mathfrak{B}$ of compact support, i.e., for all $w \in \mathfrak{B} \cap \mathfrak{D}$.

 $\mathfrak{D}:=\mathfrak{C}^{\infty}$ and 'compact support'.

if = holds :⇔ 'conservative'.

Example



$$egin{array}{lll} arepsilon_0 rac{\partial}{\partial t}
abla \cdot ec{m{E}} \, + \,
abla \cdot ec{m{j}} &= 0, \ &arepsilon_0 rac{\partial^2}{\partial t^2} ec{m{E}} + arepsilon_0 c^2
abla imes
abla imes
abla imes ec{m{E}} \, + \, rac{\partial}{\partial t} ec{m{j}} &= 0. \end{array}$$

Conservative ⇔ for compact support sol'ns:

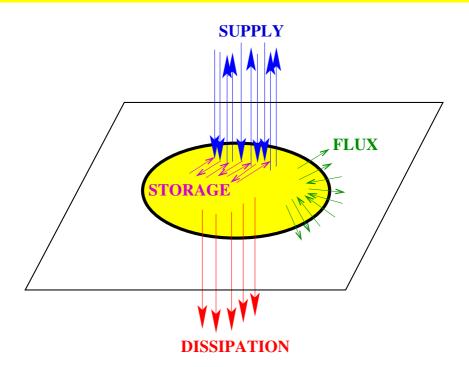
$$\iiint_{\mathbb{R}^4} ec{E}(x,y,z,t) \cdot ec{j}(x,y,z,t) \; dx \, dy \, dz \; dt = 0$$

Local dissipation law

Can this be reinterpreted as: As the system evolves, some of the supply is locally stored, some locally dissipated, and some redistributed over space?

!! Invent storage and flux, locally defined in time and space, such that in every spatial domain there holds:

 $\frac{d}{dt}$ Storage + Spatial flux \leq Supply.



Reduction to \mathfrak{C}^{∞}

$$R\left(rac{\partial}{\partial x_1},\cdots,rac{\partial}{\partial x_{
m n}}
ight) {f w}=0$$

kernel representation of $\mathfrak{B}\in\mathfrak{L}_{\mathrm{n}}^{\mathtt{w}}$.

Another representation: image representation

$$oldsymbol{w} = M\left(rac{\partial}{\partial x_1},\cdots,rac{\partial}{\partial x_{ ext{n}}}
ight)oldsymbol{\ell}$$

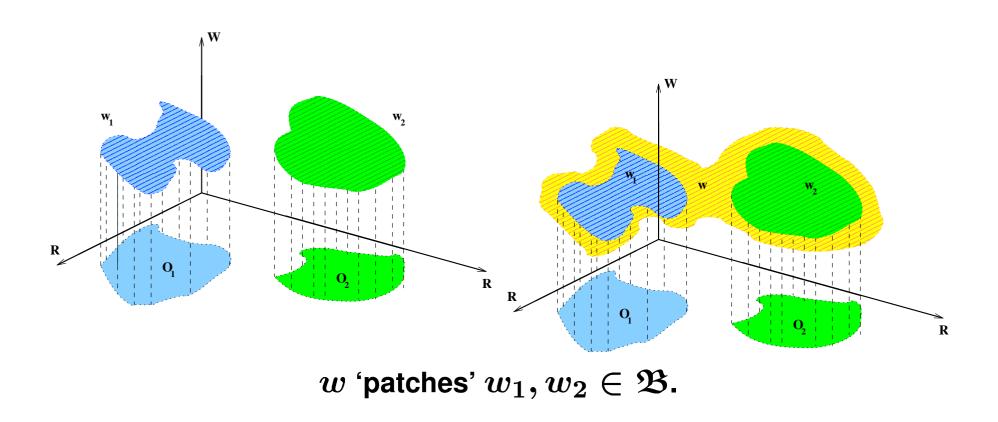
w replaced by ℓ , 'free'.

Elimination thm
$$\Rightarrow$$
 $\operatorname{im}\left(M\left(rac{\partial}{\partial x_1},\cdots,rac{\partial}{\partial x_n}
ight)
ight)\in\mathfrak{L}^{\scriptscriptstyle{W}}_{\scriptscriptstyle{\mathrm{n}}}$!

Do all $\mathfrak{B} \in \mathfrak{L}_n^{\mathtt{W}}$ admit an image representation???

iff it is 'controllable'.

Controllability



 $orall \; w_1, w_2 \in \mathfrak{B} \; \exists \; w \in \mathfrak{B} \; : \; extstyle extstyl$

For controllable systems, the compact support trajectories are 'representative' of the whole behavior.

Are Maxwell's equations controllable?

The following equations

in the scalar potential $\phi: \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$ and the vector potential $\vec{A}: \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$ generate exactly the solutions to Maxwell's equations:

$$egin{array}{lll} ec{E} &=& -rac{\partial}{\partial t} ec{A} -
abla \phi, \ ec{B} &=&
abla imes ec{A}, \ ec{j} &=& arepsilon_0 rac{\partial^2}{\partial t^2} ec{A} - arepsilon_0 c^2
abla^2 ec{A} + arepsilon_0 c^2
abla \left(
abla \cdot ec{A} \right) + arepsilon_0 rac{\partial}{\partial t}
abla \phi, \
ho &=& -arepsilon_0 rac{\partial}{\partial t}
abla \cdot ec{A} - arepsilon_0
abla^2 \phi. \end{array}$$

Proves controllability. Illustrates the interesting connection

controllability ⇔ ∃ potential!

Local dissipation law (stated for n = 4)

Thm: n = 4:x,y,z;t: space/time; $\mathfrak{B} \in \mathfrak{L}_4^{\mathtt{W}}$, controllable.

$$\iiint_{\mathbb{R}^{4}}Q_{\Phi}\left(w
ight)\,dxdydz\,dt\geq0\qquad ext{for all }w\in\mathfrak{B}\cap\mathfrak{D}$$



 \exists an image representation $w=M\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z},\frac{\partial}{\partial t}\right)$ ℓ of \mathfrak{B} , and QDF's S, the storage, and F_x,F_y,F_z , the flux, such that the local dissipation law

$$\left| rac{\partial}{\partial t} S\left(oldsymbol{\ell}
ight) + rac{\partial}{\partial x} F_x\left(oldsymbol{\ell}
ight) + rac{\partial}{\partial y} F_y\left(oldsymbol{\ell}
ight) + rac{\partial}{\partial z} F_z\left(oldsymbol{\ell}
ight) \leq Q_\Phi\left(w
ight)
ight|$$

holds for all (w, ℓ) that satisfy $w = M\left(rac{\partial}{\partial x}, rac{\partial}{\partial y}, rac{\partial}{\partial z}, rac{\partial}{\partial t}
ight) \ell$.

Energy stored in EM fields

Maxwell's equations are dissipative (in fact, conservative) with respect to $-\vec{E} \cdot \vec{j}$, the rate of energy supplied.

Introduce the stored energy density, S, and the energy flux density (the Poynting vector), \vec{F} ,

$$S\left(ec{m{E}}, ec{m{B}}
ight) := rac{arepsilon_0}{2} ec{m{E}} \cdot ec{m{E}} + rac{arepsilon_0 c^2}{2} ec{m{B}} \cdot ec{m{B}},$$

$$ec{F}\left(ec{E},ec{B}
ight):=arepsilon_{0}c^{2}ec{E} imesec{B}.$$

Local conservation law for Maxwell's equations:

$$\left[rac{\partial}{\partial t}S\left(ec{m{E}},ec{m{B}}
ight)+m{
abla}\cdotec{m{F}}\left(ec{m{E}},ec{m{B}}
ight)=-ec{m{E}}\cdotec{m{j}}.
ight]$$

Involves \vec{B} , unobservable from $\left(\vec{E},\,\vec{j}\right)$.

Local dissipation law (General case)

<u>Thm</u>: $\mathfrak{B} \in \mathfrak{L}^{\scriptscriptstyle{\mathbb{W}}}_{\scriptscriptstyle{\mathrm{n}}}$ controllable, is globally dissipative w.r.t. Q_{Φ} :

$$\int_{\mathbb{R}^{\mathrm{n}}} \, Q_{\Phi} \left(w
ight) \, dx \geq 0$$
 for all $w \in \mathfrak{B} \cap \mathfrak{D}$

1

 \exists an image representation $m{w}=M(rac{d}{dx})m{\ell}$ of ${\mathfrak B},$ and an ${
m n}$ -vector of QDF's Q_Φ such that the *local dissipation law*

$$abla \cdot Q_{\Psi}\left(\ell
ight) \leq Q_{\Phi}\left(w
ight)$$

holds for all $({m w}, {m \ell})$ that satisfy ${m w} = M\left(rac{d}{dx}
ight){m \ell}$.

Idea of the proof

Using controllability and image representations, we may assume, WLOG: $\mathfrak{B}=\mathfrak{C}^{\infty}\left(\mathbb{R}^{n},\mathbb{R}^{\mathtt{W}}\right)$

To be shown

Global dissipation :⇔

$$\int_{\mathbb{R}^{ ext{n}}}Q_{\Phi}\left(w
ight)\geq0 ext{ for all }w\in\mathfrak{D}$$



$$\exists \; \Psi: \;\; oldsymbol{
abla} \cdot Q_{\Psi}\left(w
ight) \leq Q_{\Phi}\left(w
ight) \; ext{for all } w \in \mathfrak{C}^{\infty}$$

⇔: Local dissipation

$$\int_{\mathbb{R}^{ ext{n}}} oldsymbol{Q}_{\Phi}\left(oldsymbol{w}
ight) \geq 0 ext{ for all } oldsymbol{w} \in \mathfrak{D}$$
 \updownarrow (Parseval) $\Phi\left(-i\omega,i\omega
ight) \geq 0 ext{ for all } \omega \in \mathbb{R}^{ ext{n}}$

$$\int_{\mathbb{R}^n} Q_\Phi\left(w
ight) \geq 0 ext{ for all } w \in \mathfrak{D}$$
 \Leftrightarrow (Parseval) $\Phi\left(-i\omega,i\omega
ight) \geq 0 ext{ for all } \omega \in \mathbb{R}^n$ \Leftrightarrow (Factorization equation \cong SOS) $\Phi\left(-i\omega,i\omega
ight) = \Phi\left(-i\omega,i\omega
ight) = \Phi\left(-i\omega,i\omega
ight) = 0 ext{ for all } w \in \mathbb{R}^n$

$$\int_{\mathbb{R}^n} Q_\Phi\left(w
ight) \geq 0 ext{ for all } w \in \mathfrak{D}$$
 \updownarrow (Parseval) $\Phi\left(-i\omega,i\omega
ight) \geq 0 ext{ for all } \omega \in \mathbb{R}^n$ \updownarrow (Factorization equation \cong SOS) $\exists \ D: \quad \Phi\left(-\xi,\xi
ight) = D^\top\left(-\xi
ight) D\left(\xi
ight)$ \updownarrow (easy)

$$\exists \; \Psi : \quad \left(\zeta + \eta
ight)^{ op} \Psi\left(\zeta, \eta
ight) = \Phi\left(\zeta, \eta
ight) - D^{ op}\left(\zeta
ight) D\left(\eta
ight)$$

$$\int_{\mathbb{R}^n} Q_\Phi\left(w\right) \geq 0 \text{ for all } w \in \mathfrak{D}$$

$$\updownarrow \quad (\mathsf{Parseval})$$

$$\Phi\left(-i\omega, i\omega\right) \geq 0 \text{ for all } \omega \in \mathbb{R}^n$$

$$\updownarrow \quad (\mathsf{Factorization equation} \cong \mathsf{SOS})$$

$$\exists \ D: \quad \Phi\left(-\xi, \xi\right) = D^\top \left(-\xi\right) D\left(\xi\right)$$

$$\updownarrow \quad (\mathsf{easy})$$

$$\exists \ \Psi: \quad (\zeta + \eta)^\top \Psi\left(\zeta, \eta\right) = \Phi\left(\zeta, \eta\right) - D^\top \left(\zeta\right) D\left(\eta\right)$$

$$\updownarrow \quad (\mathsf{clearly})$$

$$\exists \ \Psi: \quad \nabla \cdot Q_\Psi\left(w\right) \leq Q_\Phi\left(w\right) \text{ for all } w \in \mathfrak{C}^\infty$$

Idea of the proof

Assuming factorizability, we indeed obtain:

Global dissipation :⇔

$$\int_{\mathbb{R}^{ ext{n}}}Q_{\Phi}\left(w
ight)\geq0 ext{ for all }w\in\mathfrak{D}$$

$$\exists \; \Psi: \;\; oldsymbol{
abla} \cdot Q_{\Psi}\left(w
ight) \leq Q_{\Phi}\left(w
ight) \; ext{for all } w \in \mathfrak{C}^{\infty}$$

⇔: Local dissipation

However, for n > 1, this factorization needs rational functions

The factorization equation

$$X^{ op}\left(-\xi\right)X\left(\xi
ight)=Y\left(\xi
ight)$$
 (FE)

with $Y \in \mathbb{R}^{ullet imes ullet}[\xi]$ given, and X the unknown. Solvable??

$$\Leftrightarrow Y(\xi) = Y^{ op}(-\xi) \;\; ext{and} \;\; Y(i\omega) \geq 0 \;\; orall \omega \in \mathbb{R}^{ ext{n}}.$$

 \cong the SOS problem

$$X^{\top}\left(\xi\right)X\left(\xi\right)=Y\left(\xi\right)$$
 (SOS)

with $Y \in \mathbb{R}^{ullet imes ullet}[\xi]$ given, and X the unknown. Solvable??

$$\Leftrightarrow Y(\xi) = Y^ op(\xi)$$
 and $Y(lpha) \geq 0 \,\, orall lpha \in \mathbb{R}^{ ext{n}}$.

Idea of the proof

⇒ solvability of the factorization eq'n

$$\Phi\left(-i\omega,i\omega
ight)\geq0$$
 for all $\omega\in\mathbb{R}^{ ext{n}}$



(Factorization equation)

$$\exists \ D: \ \ \Phi\left(-\xi,\xi
ight) = D^{ op}\left(-\xi
ight)D\left(\xi
ight)$$

over the rational functions, i.e., with $oldsymbol{D}$ a matrix with elements in $\mathbb{R}\left(\xi_1,\cdots,\xi_n\right)$.

The need to introduce rational functions in this factorization equation and an image representation of \mathfrak{B} (to reduce the pbm to free variables) are the causes of the unavoidable presence of (possibly unobservable, i.e., 'hidden') latent variables in the local dissipation law.

$$X^{\top}\left(-\xi\right)X\left(\xi\right)=Y\left(\xi\right)$$

Can be made into an LMI by

$$Y(\xi) \leadsto \Phi(\zeta, \eta), \quad \Phi(-\xi, \xi) = Y(\xi)$$

and solving

$$\exists \ ?? \ \Psi : \quad \left(\zeta + \eta\right)^{\top} \Psi \left(\zeta, \eta\right) \leq \Phi \left(\zeta, \eta\right)$$

For 1-d systems (ODE case), we know a great deal: Available storage, required supply reasoning gives a proof of factorizability. \exists upper and lower bounds for Ψ . Yields low rank factorizations. Sol'n set convex, compact. State models: ARE, ARineq, LMI's.

Does any of this generalize to SOS via PDE's?

???

The nature and need of these hidden variables

Needed also e.g. in Lyapunov theory, etc.?

Reference: H. Pillai and JCW, Dissipative distributed systems, *SIAM Journal on Control and Optimization*, Volume 40, pages 1406-1430, 2002.

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