THE SUM-of-SQUARES PROBLEM

and

DISSIPATIVE DISTRIBUTED SYSTEMS

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Linear differential distributed (n-d) systems

Let $R \in \mathbb{R}^{\bullet \times \mathtt{w}}[\xi_1, \cdots, \xi_{\mathtt{n}}]$, and consider

$$R\left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathtt{n}}}\right) w = 0. \quad (*)$$

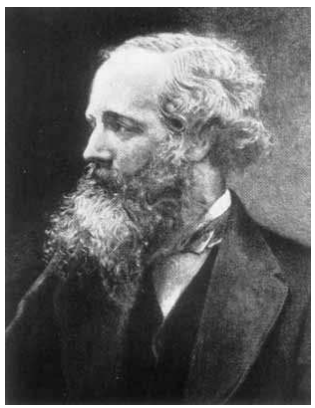
Define the associated 'behavior'

$$\mathfrak{B} = \{ w \in \mathfrak{C}^{\infty}(\mathbb{R}^{\mathtt{n}}, \mathbb{R}^{\mathtt{w}}) \mid (*) \text{ holds} \}.$$

<u>Notation</u> $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\mathtt{n}}$.

Example

Maxwell's equations: $\mathtt{n} = 4, \mathtt{w} = 10.$



$$
\begin{aligned}
\nabla \cdot \vec{E} &= \frac{1}{\varepsilon_0}\rho\,, \\
\nabla \times \vec{E} &= -\frac{\partial}{\partial t}\vec{B}, \\
\nabla \cdot \vec{B} &= 0\,, \\
c^2 \nabla \times \vec{B} &= \frac{1}{\varepsilon_0}\vec{j} + \frac{\partial}{\partial t}\vec{E}.
\end{aligned}
$$

QDF's

Use multi-index notation. Consider

$$w \mapsto \sum_{k,\ell} \left(\frac{d^k}{dx^k} w\right)^\top \Phi_{k,\ell} \left(\frac{d^\ell}{dx^\ell} w\right)$$

$\mathfrak{C}^\infty(\mathbb{R}^{\mathtt{n}}, \mathbb{R}^{\mathtt{w}}) \to \mathfrak{C}^\infty(\mathbb{R}^{\mathtt{n}}, \mathbb{R})$. $\Phi_{k,\ell} \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$; $\Phi_{k,\ell} = \Phi_{\ell,k}^\top$.

Introduce the $2\mathtt{n}$-variable polynomial matrix Φ

$$\Phi(\zeta, \eta) = \sum_{k,\ell} \Phi_{k,\ell} \zeta^k \eta^\ell.$$

Denote the QDF as Q_Φ.

Dissipative distributed systems

$\mathfrak{B} \in \mathfrak{L}_{\mathtt{n}}^{\mathtt{w}}$ **is**

dissipative **with respect to the 'supply rate'** Q_{Φ}

:$\Leftrightarrow$

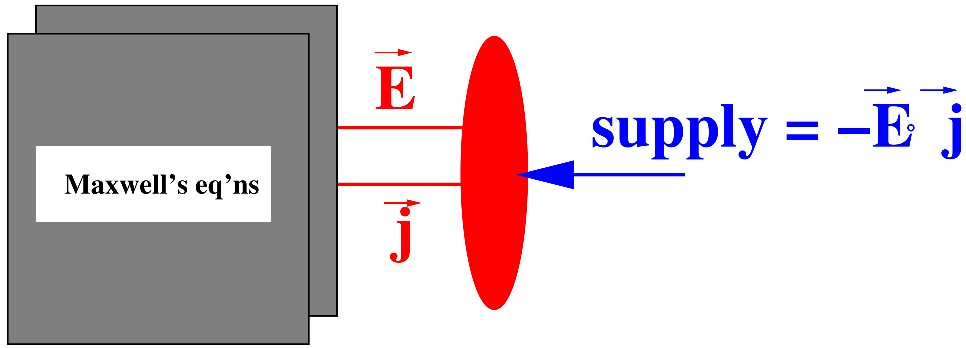
$$\int_{\mathbb{R}^{\mathtt{n}}} Q_{\Phi}(w)\ dx \geq 0$$

for all $w \in \mathfrak{B}$ **of compact support, i.e., for all** $w \in \mathfrak{B} \cap \mathfrak{D}$**.**

$\mathfrak{D} := \mathfrak{C}^{\infty}$ **and 'compact support'.**

if = holds **:**$\Leftrightarrow$ ***'conservative'*****.**

Example



$$\begin{aligned} \varepsilon_0 \frac{\partial}{\partial t} \nabla \cdot \vec{E} + \nabla \cdot \vec{j} &= 0, \\ \varepsilon_0 \frac{\partial^2}{\partial t^2} \vec{E} + \varepsilon_0 c^2 \nabla \times \nabla \times \vec{E} + \frac{\partial}{\partial t} \vec{j} &= 0. \end{aligned}$$

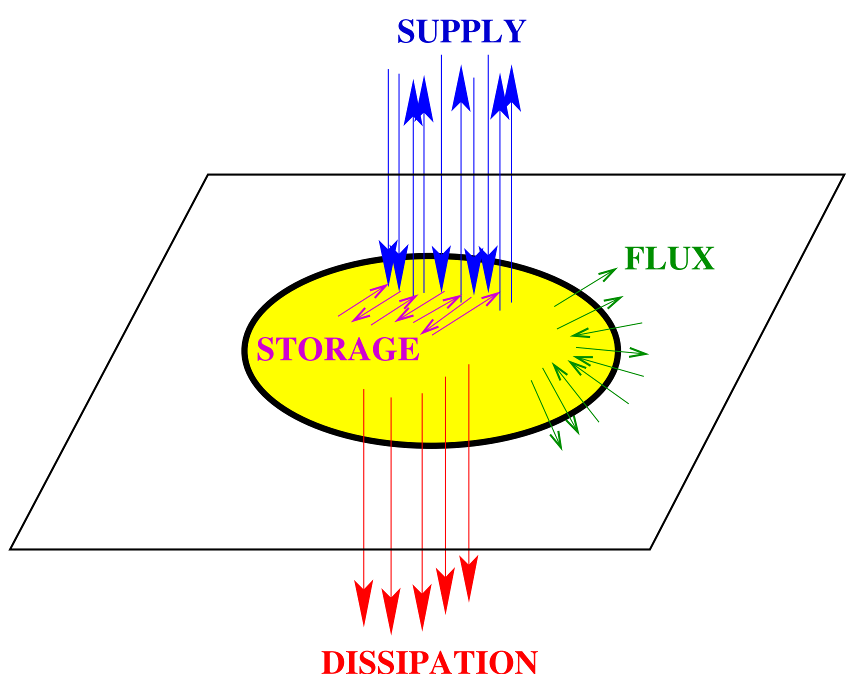
Conservative ⇔ for compact support sol'ns:

$$\iiiint_{\mathbb{R}^4} \vec{E}(x,y,z,t) \cdot \vec{j}(x,y,z,t)\, dx\, dy\, dz\, dt = 0$$

Local dissipation law

Can this be reinterpreted as: As the system evolves, some of the supply is locally stored, some locally dissipated, and some redistributed over space?
!! Invent *storage and flux*, locally defined in time and space, such that in every spatial domain there holds:

$$\frac{d}{dt}\ \text{Storage + Spatial flux} \leq \text{Supply.}$$



Reduction to $\mathfrak{C}^\infty$

$$R\left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathtt{n}}}\right) w = 0$$

kernel representation of $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\mathtt{n}}$.

Another representation: image representation

$$w = M\left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathtt{n}}}\right) \ell$$

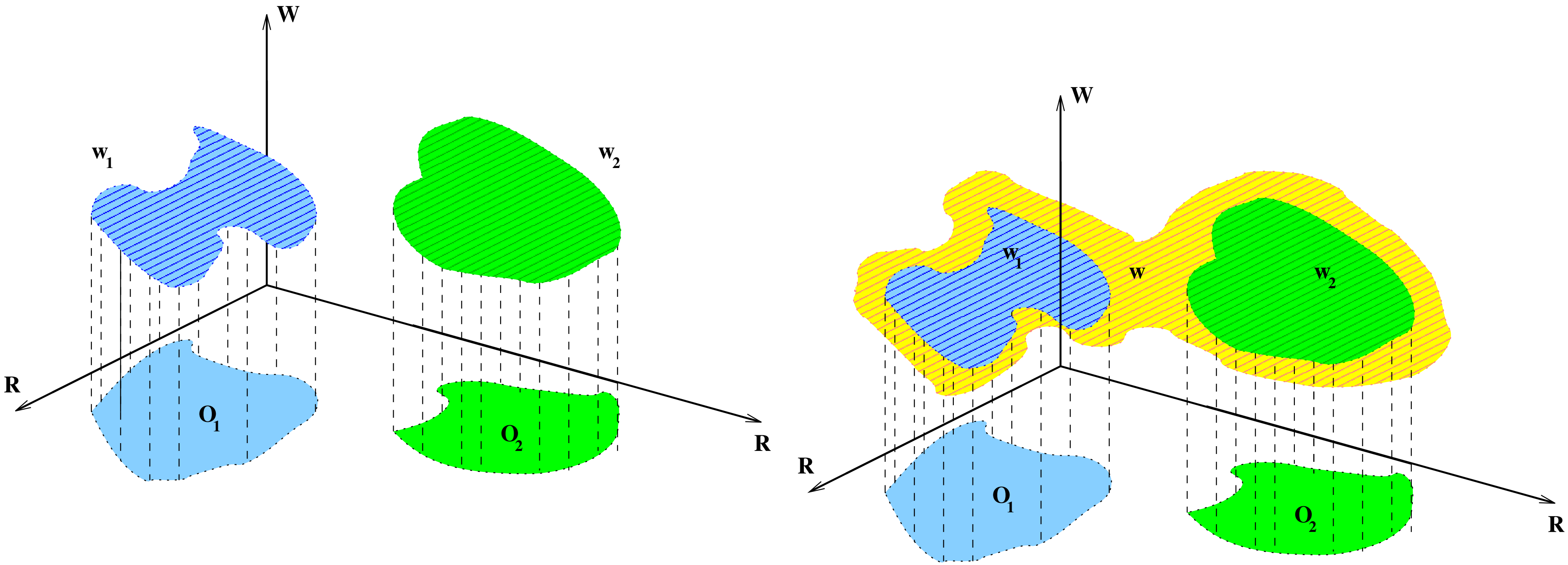
w replaced by ℓ, 'free'.

Elimination thm $\Rightarrow$ $\operatorname{im}\left(M\left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathtt{n}}}\right)\right) \in \mathfrak{L}^{\mathtt{w}}_{\mathtt{n}}$!

Do all $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\mathtt{n}}$ admit an image representation???

iff it is 'controllable'.

Controllability



w **'patches'** $w_1, w_2 \in \mathfrak{B}$.

$\forall\ w_1, w_2 \in \mathfrak{B}\ \exists\ w \in \mathfrak{B}$: **Controllability :⇔ 'patchability'.**

For controllable systems, the compact support trajectories are 'representative' of the whole behavior.

Are Maxwell's equations controllable ?

The following equations
in the *scalar potential* $\phi : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$ and
the *vector potential* $\vec{A} : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$
generate exactly the solutions to Maxwell's equations:

$$
\begin{aligned}
\vec{E} &= -\frac{\partial}{\partial t}\vec{A} - \nabla\phi, \\
\vec{B} &= \nabla \times \vec{A}, \\
\vec{j} &= \varepsilon_0 \frac{\partial^2}{\partial t^2}\vec{A} - \varepsilon_0 c^2 \nabla^2 \vec{A} + \varepsilon_0 c^2 \nabla \left(\nabla \cdot \vec{A}\right) + \varepsilon_0 \frac{\partial}{\partial t}\nabla\phi, \\
\rho &= -\varepsilon_0 \frac{\partial}{\partial t}\nabla \cdot \vec{A} - \varepsilon_0 \nabla^2 \phi.
\end{aligned}
$$

Proves controllability. Illustrates the interesting connection

controllability $\Leftrightarrow \exists$ potential!

Local dissipation law (stated for $\mathtt{n} = 4$)

<u>Thm</u>: $\mathtt{n} = 4 : x, y, z; t :$ space/time; $\mathfrak{B} \in \mathfrak{L}_4^{\mathtt{w}}$, controllable.

$$\iiiint_{\mathbb{R}^4} Q_\Phi(w)\; dx dy dz\, dt \geq 0 \qquad \text{for all } w \in \mathfrak{B} \cap \mathfrak{D}$$

$$\Updownarrow$$

$\exists$ an image representation $w = M\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \frac{\partial}{\partial t}\right)\ell$ of $\mathfrak{B}$,

and QDF's S, the *storage,* and $F_x, F_y, F_z,$ the *flux,*

such that the *local dissipation law*

$$\frac{\partial}{\partial t} S(\ell) + \frac{\partial}{\partial x} F_x(\ell) + \frac{\partial}{\partial y} F_y(\ell) + \frac{\partial}{\partial z} F_z(\ell) \leq Q_\Phi(w)$$

holds for all (w, ℓ) that satisfy $w = M\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \frac{\partial}{\partial t}\right)\ell$.

Energy stored in EM fields

Maxwell's equations are dissipative (in fact, conservative) with respect to $-\vec{E}\cdot\vec{j}$, the rate of energy supplied.

Introduce the ***stored energy density,*** S, and

the ***energy flux density*** **(the *Poynting vector*),** $\vec{F}$,

$$S\left(\vec{E},\vec{B}\right) := \frac{\varepsilon_0}{2}\vec{E}\cdot\vec{E} + \frac{\varepsilon_0 c^2}{2}\vec{B}\cdot\vec{B},$$

$$\vec{F}\left(\vec{E},\vec{B}\right) := \varepsilon_0 c^2 \vec{E}\times\vec{B}.$$

Local conservation law for Maxwell's equations:

$$\boxed{\frac{\partial}{\partial t}S\left(\vec{E},\vec{B}\right) + \nabla\cdot\vec{F}\left(\vec{E},\vec{B}\right) = -\vec{E}\cdot\vec{j}.}$$

Involves $\vec{B}$, unobservable from $\left(\vec{E},\ \vec{j}\right)$.

Local dissipation law (General case)

Thm: $\mathfrak{B} \in \mathfrak{L}_{\mathtt{n}}^{\mathtt{w}}$ controllable, is globally dissipative w.r.t. Q_Φ:

$$\int_{\mathbb{R}^{\mathtt{n}}} Q_\Phi(w)\, dx \geq 0 \qquad \text{for all } w \in \mathfrak{B} \cap \mathfrak{D}$$

$$\Updownarrow$$

$\exists$ an image representation $w = M(\frac{d}{dx})\ell$ of $\mathfrak{B}$,
and an $\mathtt{n}$-vector of QDF's Q_Φ such that the *local dissipation law*

$$\nabla \cdot Q_\Psi(\ell) \leq Q_\Phi(w)$$

holds for all (w, ℓ) that satisfy $w = M\left(\frac{d}{dx}\right)\ell$.

Idea of the proof

Using controllability and image representations, we may assume, WLOG: $\mathfrak{B} = \mathfrak{C}^{\infty}\left(\mathbb{R}^{\mathtt{n}}, \mathbb{R}^{\mathtt{w}}\right)$

To be shown

Global dissipation :$\Leftrightarrow$

$$\int_{\mathbb{R}^{\mathtt{n}}} Q_{\Phi}\left(w\right) \geq 0 \text{ for all } w \in \mathfrak{D}$$

$$\Updownarrow$$

$$\exists\ \Psi : \quad \nabla \cdot Q_{\Psi}\left(w\right) \leq Q_{\Phi}\left(w\right) \text{ for all } w \in \mathfrak{C}^{\infty}$$

$\Leftrightarrow$: Local dissipation

$$\int_{\mathbb{R}^n} Q_\Phi(w) \geq 0 \text{ for all } w \in \mathfrak{D}$$

$$\Updownarrow \quad \textbf{(Parseval)}$$

$$\Phi(-i\omega, i\omega) \geq 0 \text{ for all } \omega \in \mathbb{R}^n$$

$$\int_{\mathbb{R}^n} Q_\Phi(w) \geq 0 \text{ for all } w \in \mathfrak{D}$$

$\Updownarrow$ **(Parseval)**

$$\Phi(-i\omega, i\omega) \geq 0 \text{ for all } \omega \in \mathbb{R}^n$$

$\Updownarrow$ **(Factorization equation $\cong$ SOS)**

$$\exists\ D: \quad \Phi(-\xi, \xi) = D^\top(-\xi) D(\xi)$$

$$\int_{\mathbb{R}^n} Q_\Phi(w) \geq 0 \text{ for all } w \in \mathfrak{D}$$

$\Updownarrow$ **(Parseval)**

$$\Phi(-i\omega, i\omega) \geq 0 \text{ for all } \omega \in \mathbb{R}^n$$

$\Updownarrow$ **(Factorization equation $\cong$ SOS)**

$$\exists\ D: \quad \Phi(-\xi, \xi) = D^\top(-\xi) D(\xi)$$

$\Updownarrow$ **(easy)**

$$\exists\ \Psi: \quad (\zeta + \eta)^\top \Psi(\zeta, \eta) = \Phi(\zeta, \eta) - D^\top(\zeta) D(\eta)$$

$$\int_{\mathbb{R}^n} Q_\Phi(w) \geq 0 \text{ for all } w \in \mathfrak{D}$$

$\Updownarrow$ **(Parseval)**

$$\Phi(-i\omega, i\omega) \geq 0 \text{ for all } \omega \in \mathbb{R}^n$$

$\Updownarrow$ **(Factorization equation $\cong$ SOS)**

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$\Updownarrow$ **(easy)**

$$\exists\ \Psi: \quad (\zeta + \eta)^\top \Psi(\zeta, \eta) = \Phi(\zeta, \eta) - D^\top(\zeta) D(\eta)$$

$\Updownarrow$ **(clearly)**

$$\exists\ \Psi: \quad \nabla \cdot Q_\Psi(w) \leq Q_\Phi(w) \text{ for all } w \in \mathfrak{C}^\infty$$

Idea of the proof

Assuming factorizability, we indeed obtain:

Global dissipation :⇔

$$\int_{\mathbb{R}^{\mathrm{n}}} Q_{\Phi}(w) \geq 0 \text{ for all } w \in \mathfrak{D}$$

$$\Updownarrow$$

$$\exists \ \Psi : \quad \nabla \cdot Q_{\Psi}(w) \leq Q_{\Phi}(w) \text{ for all } w \in \mathfrak{C}^{\infty}$$

⇔: **Local dissipation**

However, for $\mathrm{n} > 1$, this factorization needs rational functions

The factorization equation

$$X^\top(-\xi)\,X(\xi) = Y(\xi) \qquad \textbf{(FE)}$$

with $Y \in \mathbb{R}^{\bullet\times\bullet}[\xi]$ **given, and** X **the unknown.** ***Solvable??***

$$\Leftrightarrow Y(\xi) = Y^\top(-\xi) \text{ and } Y(i\omega) \geq 0 \ \forall \omega \in \mathbb{R}^{\mathrm{n}}.$$

$\cong$ **the SOS problem**

$$X^\top(\xi)\,X(\xi) = Y(\xi) \qquad \textbf{(SOS)}$$

with $Y \in \mathbb{R}^{\bullet\times\bullet}[\xi]$ **given, and** X **the unknown.** ***Solvable??***

$$\Leftrightarrow Y(\xi) = Y^\top(\xi) \text{ and } Y(\alpha) \geq 0 \ \forall \alpha \in \mathbb{R}^{\mathrm{n}}.$$

Idea of the proof

$\Rightarrow$ **solvability of the factorization eq'n**

$$\Phi\left(-i\omega, i\omega\right) \geq 0 \text{ for all } \omega \in \mathbb{R}^{\mathrm{n}}$$

$$\Updownarrow$$

(Factorization equation)

$$\exists\ D: \quad \Phi\left(-\xi, \xi\right) = D^{\top}\left(-\xi\right) D\left(\xi\right)$$

over the rational functions, i.e., with D a matrix with elements in $\mathbb{R}\left(\xi_1, \cdots, \xi_{\mathrm{n}}\right)$.

The need to introduce rational functions in this factorization equation and an image representation of $\mathfrak{B}$ (to reduce the pbm to free variables) are the causes of the unavoidable presence of (possibly unobservable, i.e., 'hidden') latent variables in the local dissipation law.

???

$$X^\top(-\xi)\,X(\xi) = Y(\xi)$$

Can be made into an LMI by

$$Y(\xi) \rightsquigarrow \Phi(\zeta,\eta), \quad \Phi(-\xi,\xi) = Y(\xi)$$

and solving

$$\exists\,??\ \Psi : \qquad (\zeta+\eta)^\top\,\Psi(\zeta,\eta) \leq \Phi(\zeta,\eta)$$

For 1-d systems (ODE case), we know a great deal: Available storage, required supply reasoning gives a proof of factorizability. $\exists$ upper and lower bounds for Ψ. Yields low rank factorizations. Sol'n set convex, compact. State models: ARE, ARineq, LMI's.

Does any of this generalize to SOS via PDE's?

???

The nature and need of these hidden variables

Needed also e.g. in Lyapunov theory, etc.?

<u>Reference</u>: H. Pillai and JCW, Dissipative distributed systems,
***SIAM Journal on Control and Optimization*,**
Volume 40, pages 1406-1430, 2002.

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