SYSTEMS THEORY
and
PHYSICAL MODELS

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Problematique

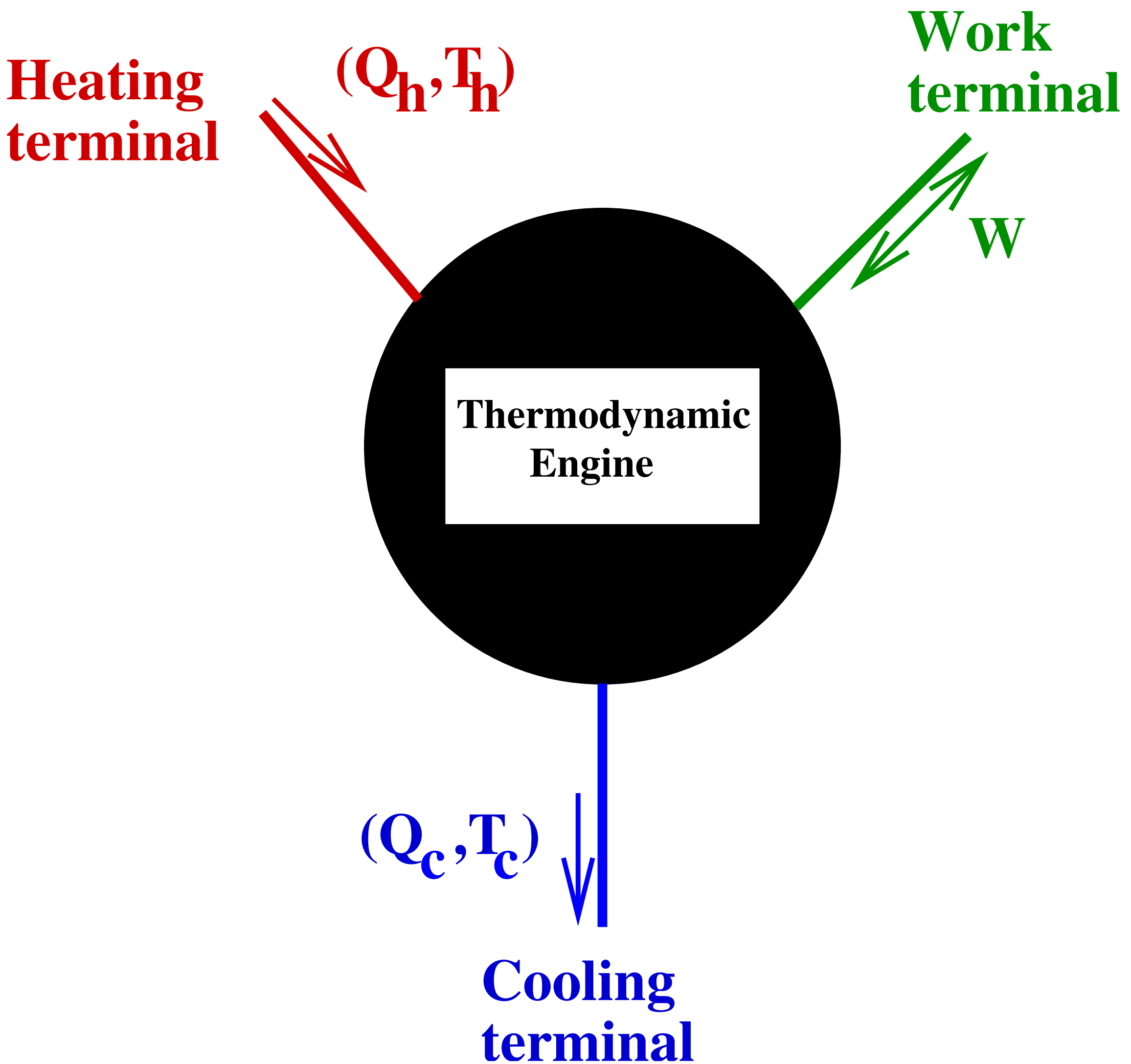
Develop a suitable *mathematical* framework for discussing dynamical systems

aimed at modeling, analysis, and synthesis.

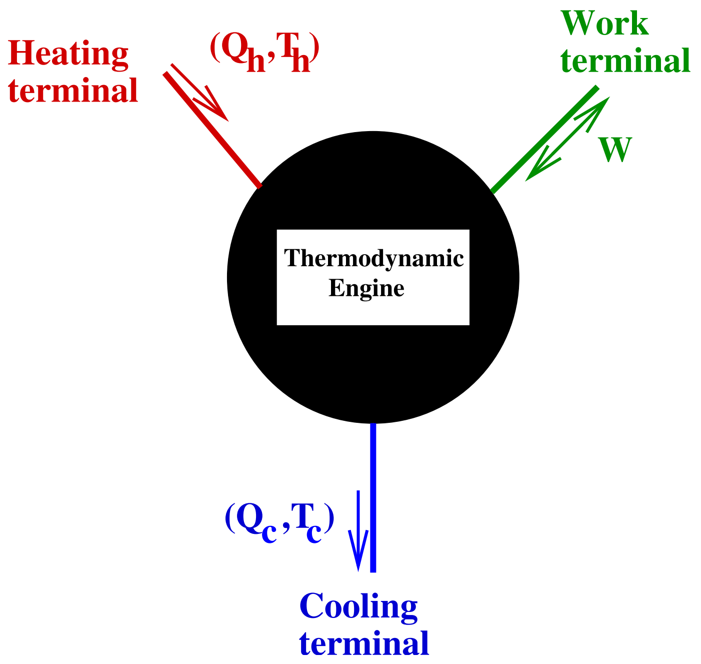
$\leadsto$ **control, signal processing, system id., . . .**

$\leadsto$ **engineering systems, economics, physics, . . .**

Paradigmatic examples - thermodynamics



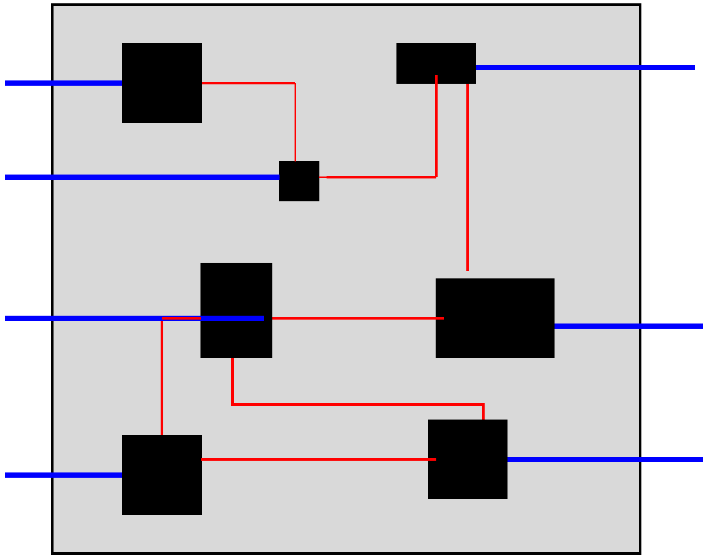
Paradigmatic examples - thermodynamics



Express the first and second law $\rightsquigarrow$ **?**

Paradigmatic examples - modeling

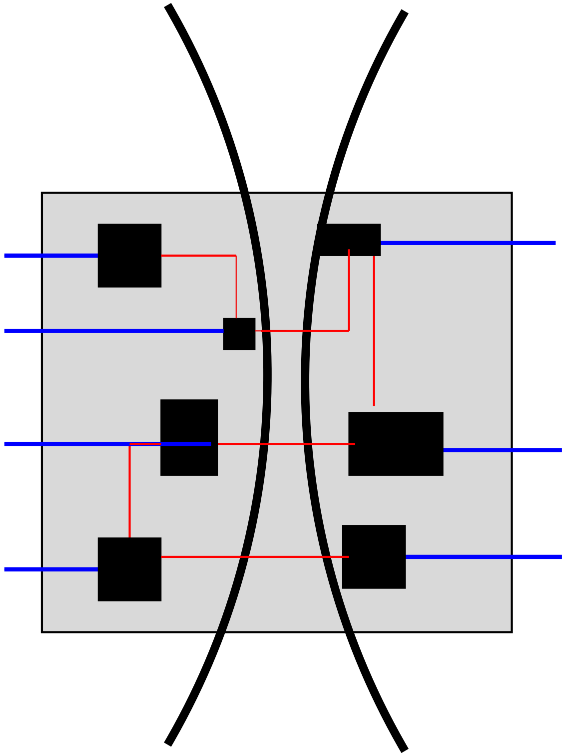
¡ Model this interconnected system !



Tearing, zooming & linking

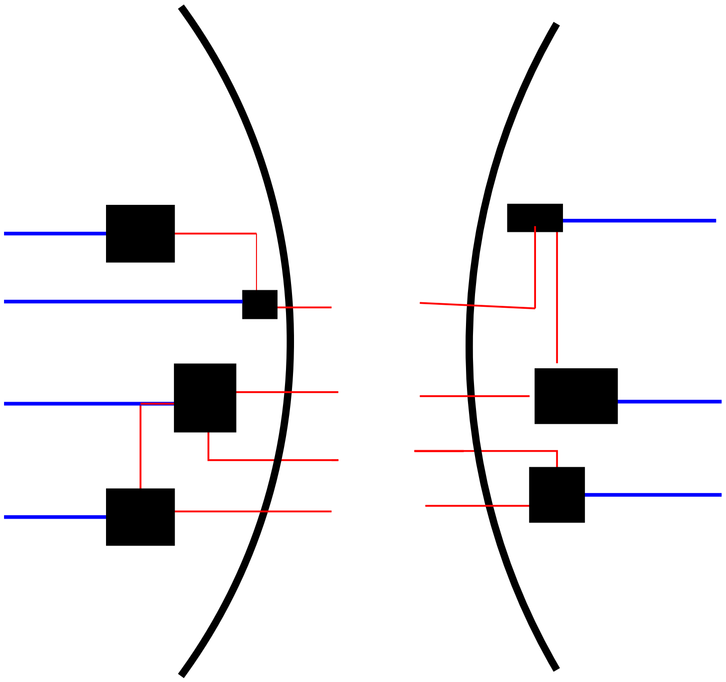
Paradigmatic examples - modeling

Tearing:



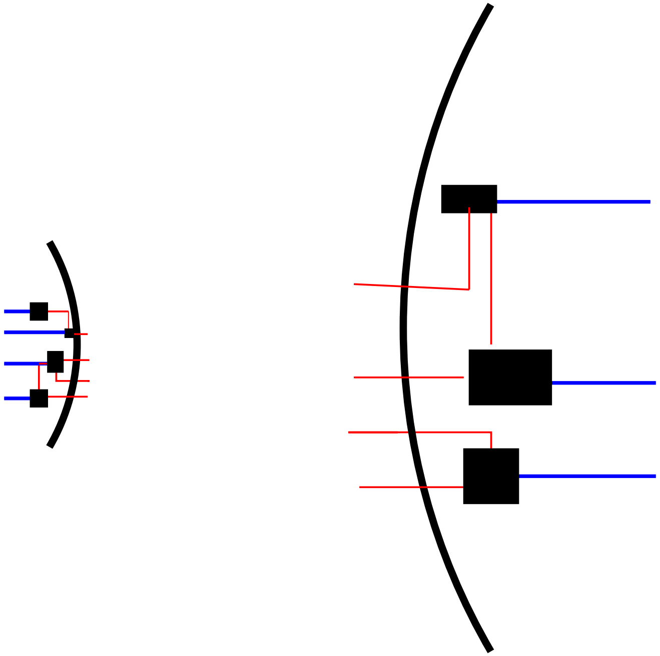
Paradigmatic examples - modeling

Tearing:



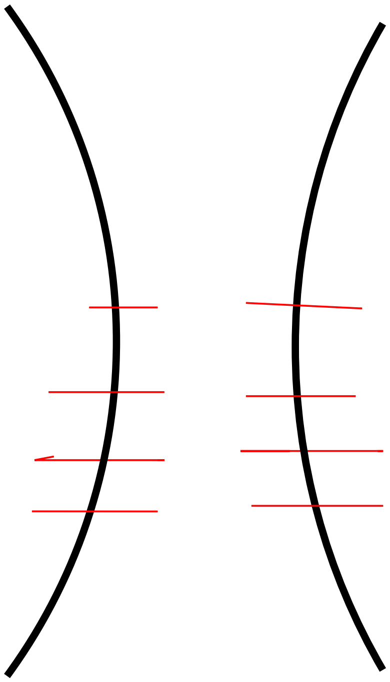
Paradigmatic examples - modeling

Zooming:



Paradigmatic examples - modeling

Linking:



Paradigmatic examples - modeling

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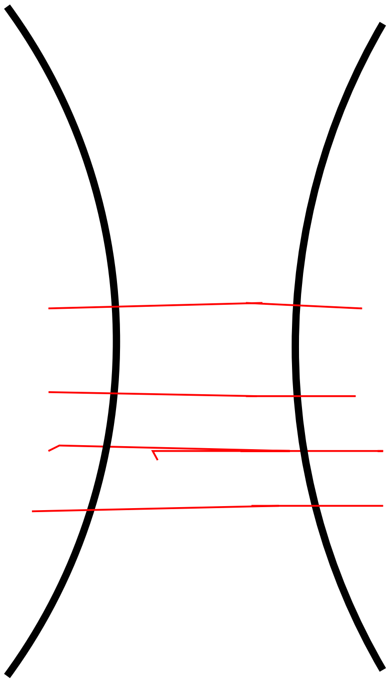

THEMES

1. Closed systems
2. Input/output ;
   input/state/output systems
3. Beyond causality: behavioral systems

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1. Closed systems
2. Input/output ;
   input/state/output systems
3. Beyond causality: behavioral systems

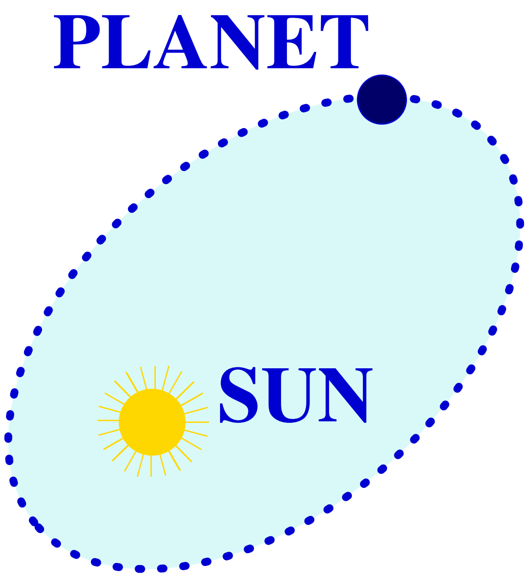
- Submodule thm
- Elimination thm
- Controllability and image representation thm

How it all began ...

How does it move?

Kepler's laws

Johannes Kepler (1571-1630)

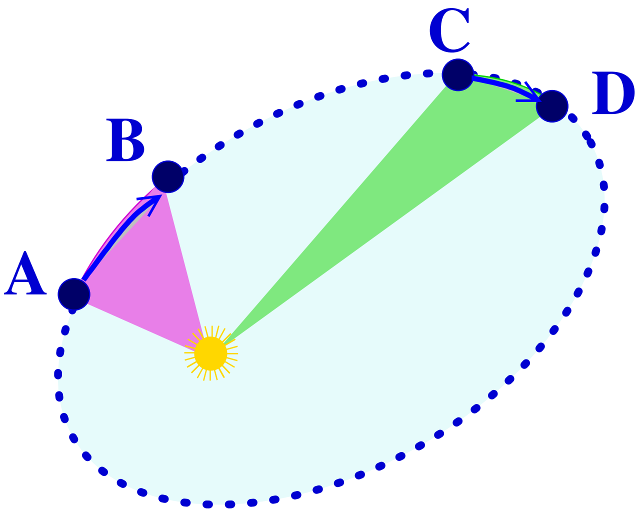


Kepler's first law:

Ellipse, sun in focus

Kepler’s laws

Johannes Kepler (1571-1630)

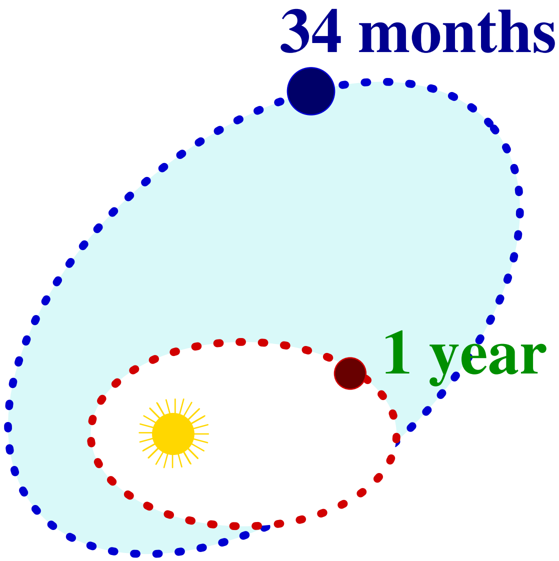


Kepler’s second law:

= areas in = times

Kepler's laws

Johannes Kepler (1571-1630)

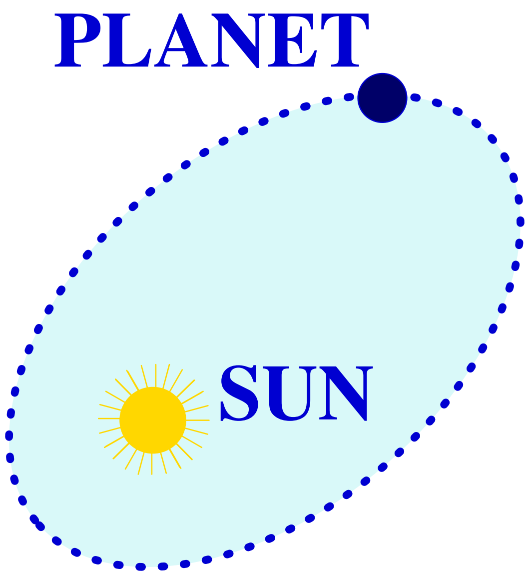


Kepler's third law:

$$(\text{period})^2 \cong (\text{diameter})^3$$

The state of the planet

What determines the orbit uniquely?



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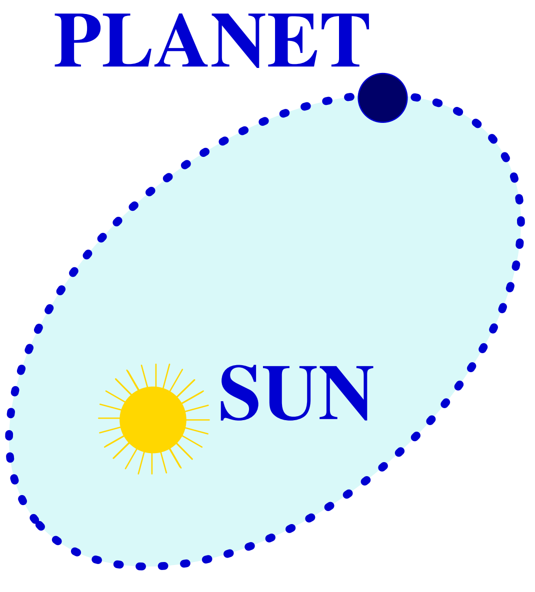
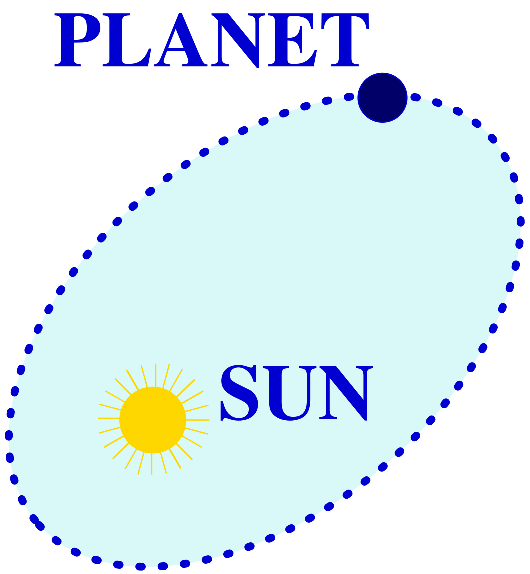


The position?

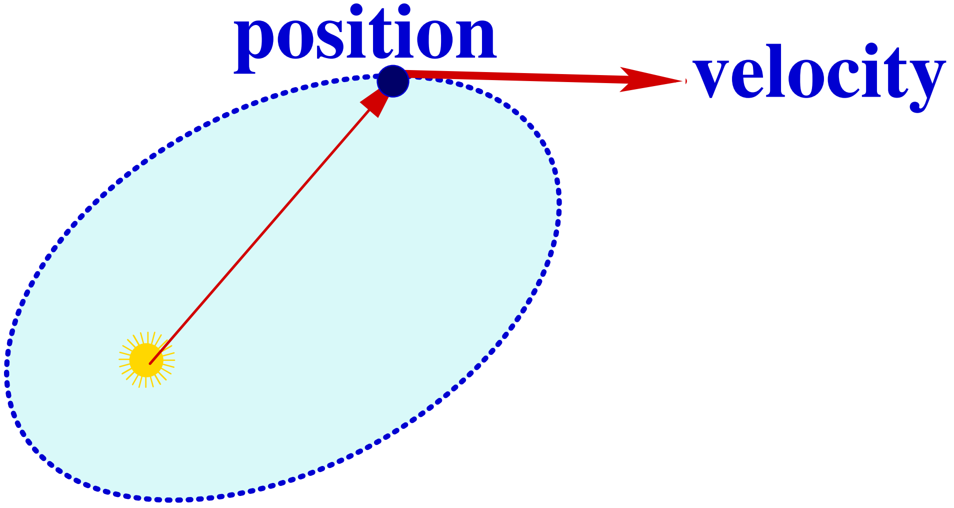
The state of the planet

What determines the orbit uniquely?



The position and the direction of motion?

The state of the planet



The state = the position & the velocity

The equation of the planet

Consequence:

acceleration = function of position and velocity

$$\frac{d^2}{dt^2}w(t) = A(w(t), \frac{d}{dt}w(t))$$

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acceleration = function of position and velocity

$$\frac{d^2}{dt^2}w(t) = A(w(t), \frac{d}{dt}w(t))$$

$\rightsquigarrow$ via calculus and calculation

$$\frac{d^2}{dt^2}w(t) + \frac{1_{w(t)}}{|w(t)|^2} = 0$$

Newton's laws

$$F'(t) = m\frac{d^2}{dt^2}w(t)$$ (2-nd law)

$$F''(t) = m\frac{1_{w(t)}}{|w(t)|^2}$$ (gravity)

$$F'(t) + F''(t) = 0$$ (3-rd law)

$$\Downarrow$$

$$\frac{d^2}{dt^2}w(t) + \frac{1_{w(t)}}{|w(t)|^2} = 0$$

Kepler's laws

K.1, K.2, & K.3

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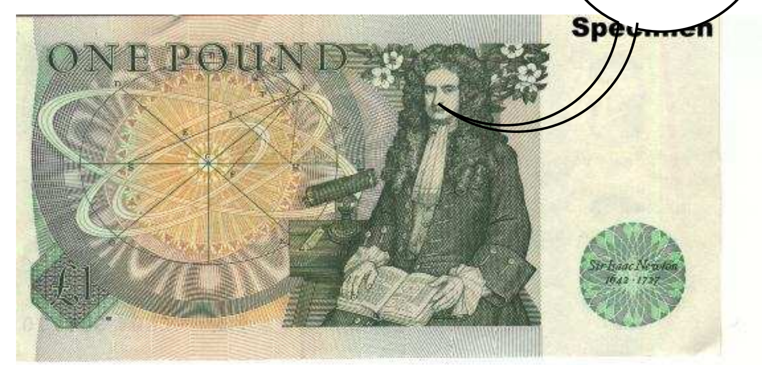
Hypotheses
non
fingo



Kepler's laws

K.1, K.2, & K.3

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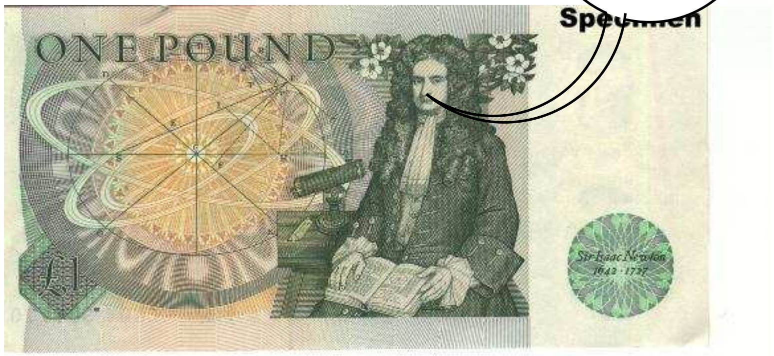

Hypotheses non fingo

$$F = m\frac{d^2}{dt^2}w$$

The paradigm of closed systems

‘Axiomatization’

K.1, K.2, & K.3

K.1, K.2, & K.3

$$\rightsquigarrow \quad \frac{d^2}{dt^2}w(t) + \frac{1_{w(t)}}{|\frac{d}{dt}w(t)|^2} = 0$$

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$$\rightsquigarrow \quad \boxed{\frac{d}{dt} x = f(x)}$$

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$\rightsquigarrow$ **‘dynamical systems’, flows**

K.1, K.2, & K.3

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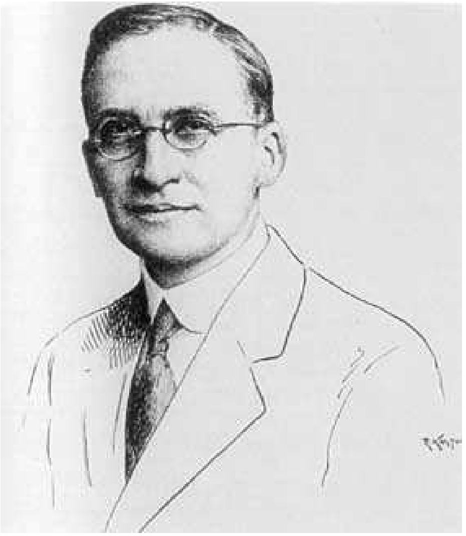
$\rightsquigarrow$ ‘dynamical systems’, flows

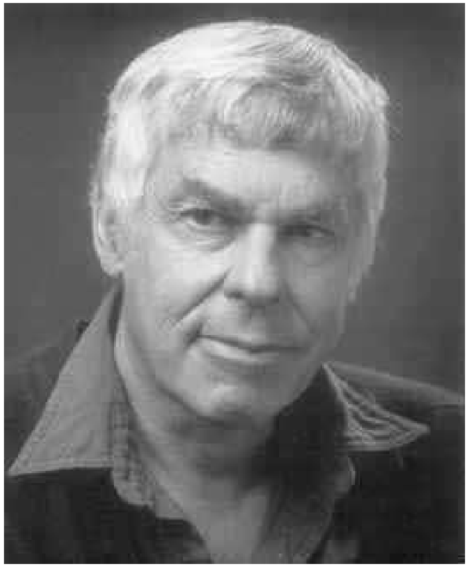
$\rightsquigarrow$ closed systems as paradigm of dynamics

‘Axiomatization’

Henri Poincaré (1854-1912)



George Birkhoff (1884-1944)



Stephen Smale (1930-   )

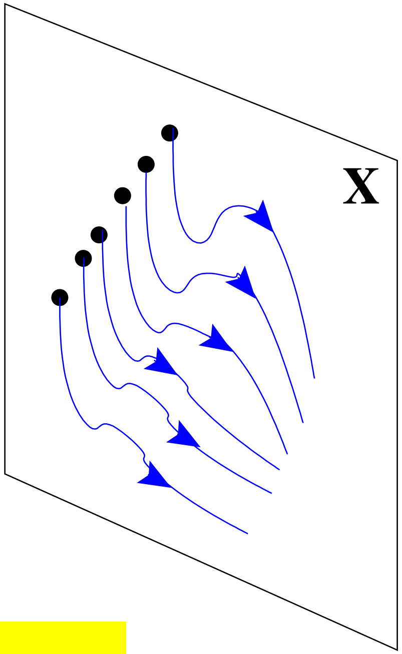
‘Axiomatization’

A *dynamical system* is defined by

a **state space** X and

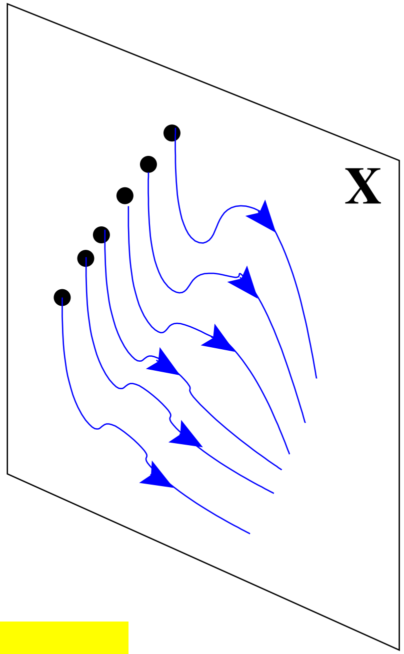
a **state transition function**

ϕ : $\cdots$ such that $\cdots$



$\phi(t, x)$ = state at time t starting from state x

'Axiomatization'



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$\phi(t, x)$ = state at time t starting from state x

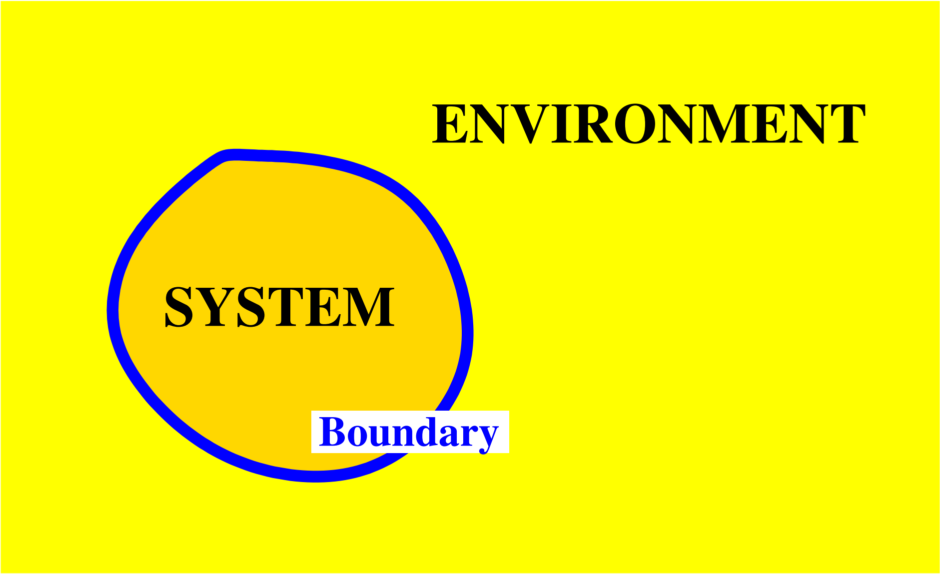
How could they forget about Newton's second law,

about Maxwell's eq'ns, about thermodynamics,

about tearing & zooming & linking, ...?

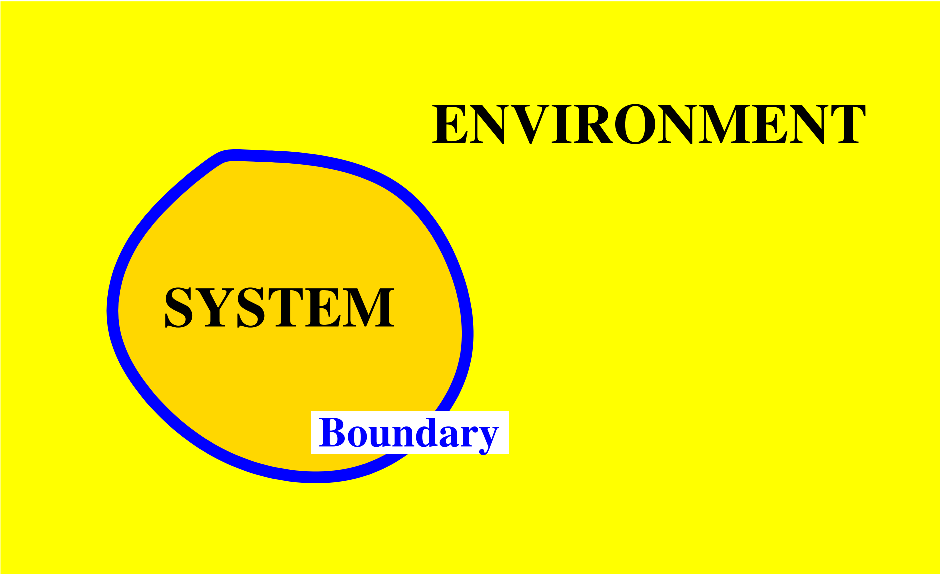
‘Axiomatization’

Reply: assume ‘fixed boundary conditions’



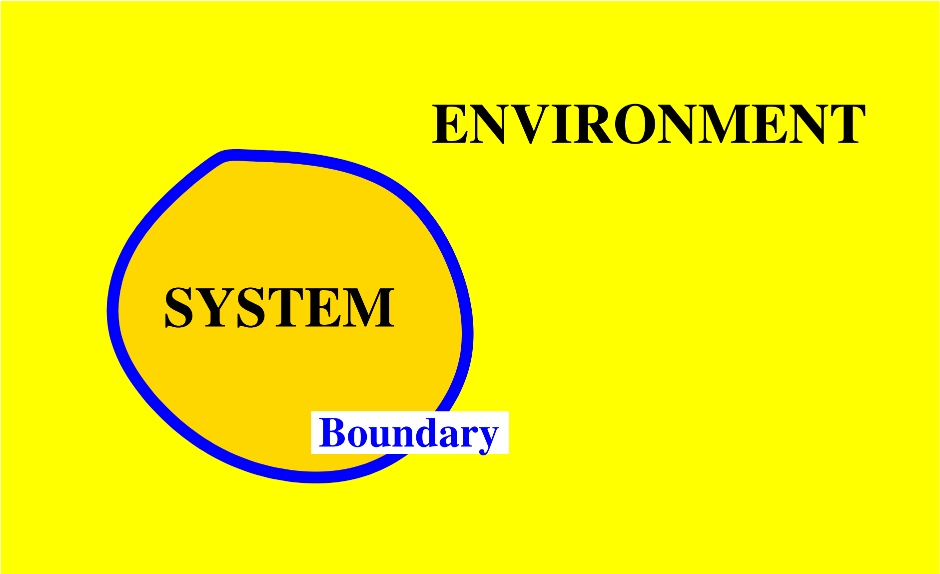
⇝ **an absurd situation: to model a system,**
we have to model also the environment!

‘Axiomatization’



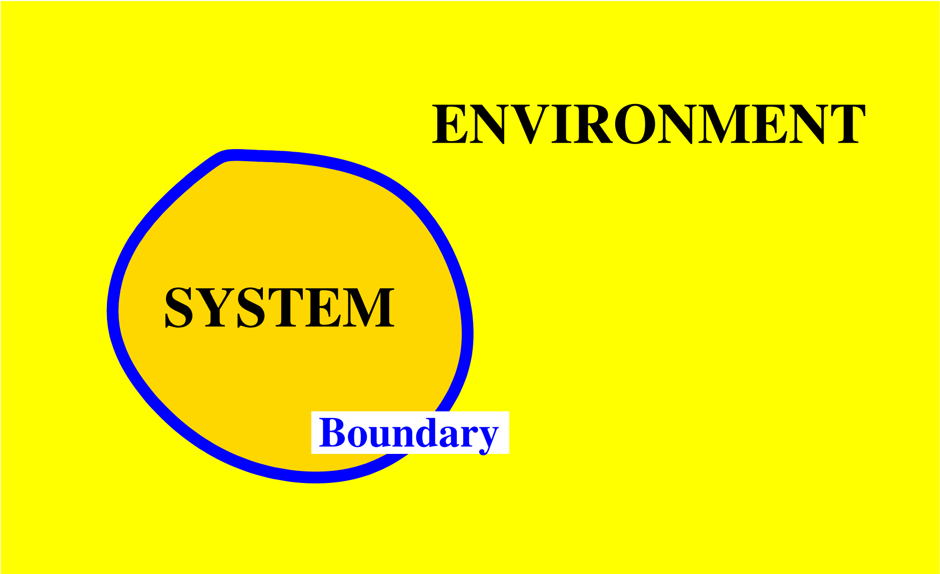
Chaos theory, cellular automata, sync, etc.,
‘function’ in this framework ...

‘Axiomatization’



Chaos: not a property of the physical laws, but just as much of what the system is interconnected to.

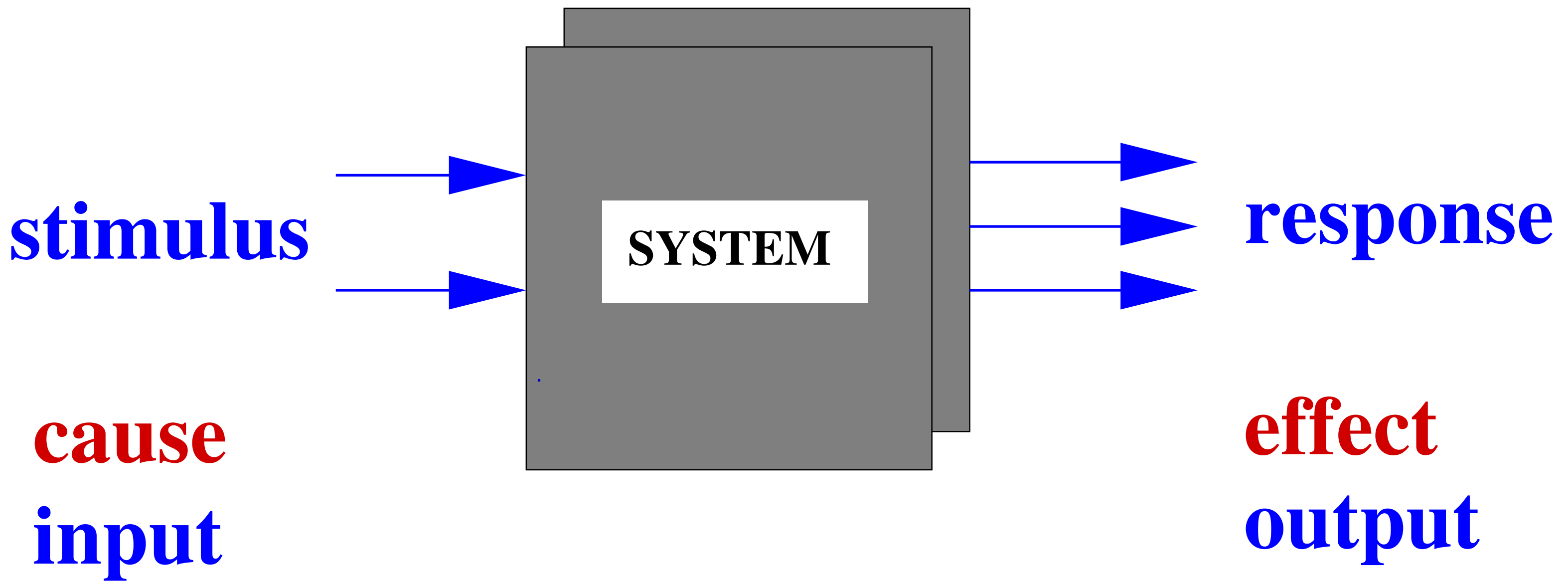
‘Axiomatization’

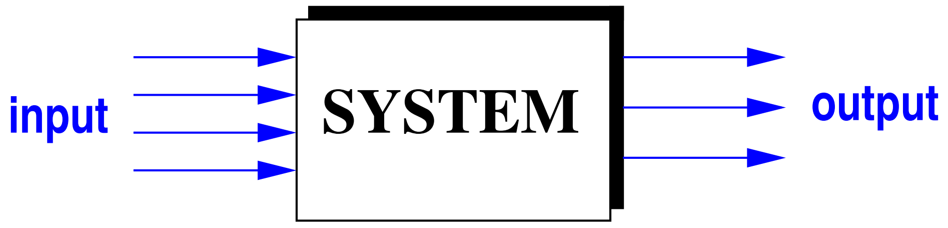


Turbulence may not be a property of Navier-Stokes, but just as much of the boundary conditions.

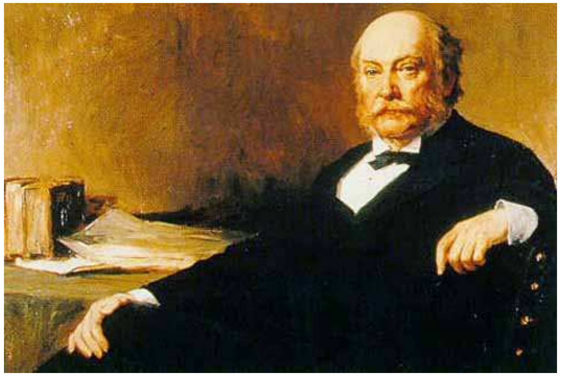
Meanwhile, in engineering, ...

Input/output systems

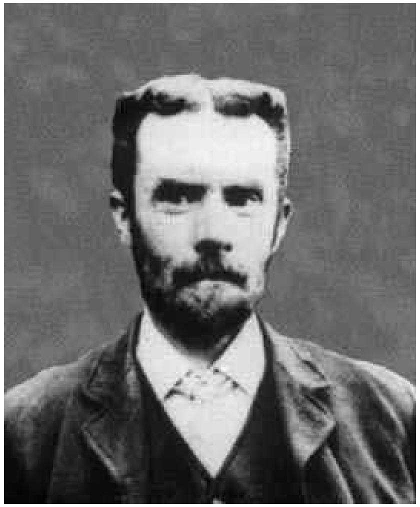




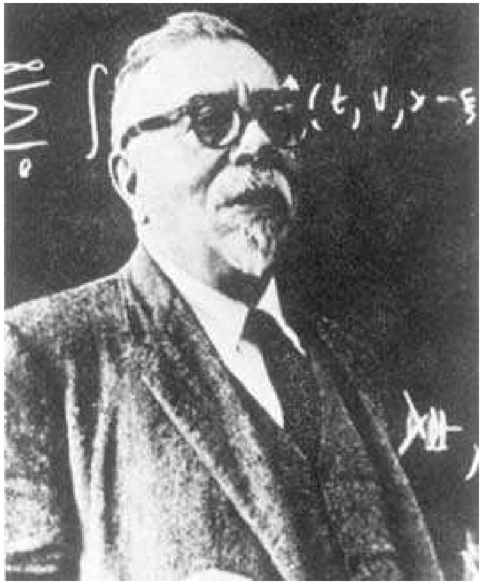
The originators



Lord Rayleigh (1842-1919)

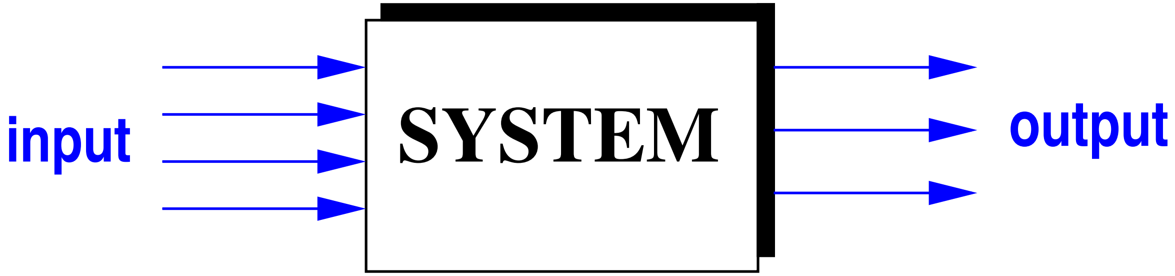


Oliver Heaviside (1850-1925)



Norbert Wiener (1894-1964)

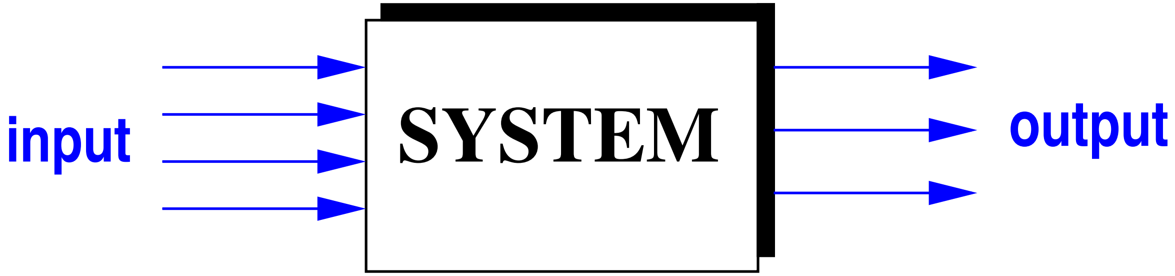
Mathematical description



$$y(t) = \int_{0 \text{ or } -\infty}^{t} H(t-t')u(t')\,dt'$$

$$y(t) = H_0(t) + \int_{-\infty}^{t} H_1(t-t')u(t')\,dt' + \int_{-\infty}^{t}\int_{-\infty}^{t'} H_2(t-t', t'-t'')u(t')u(t'')\,dt'dt'' + \cdots$$

Mathematical description



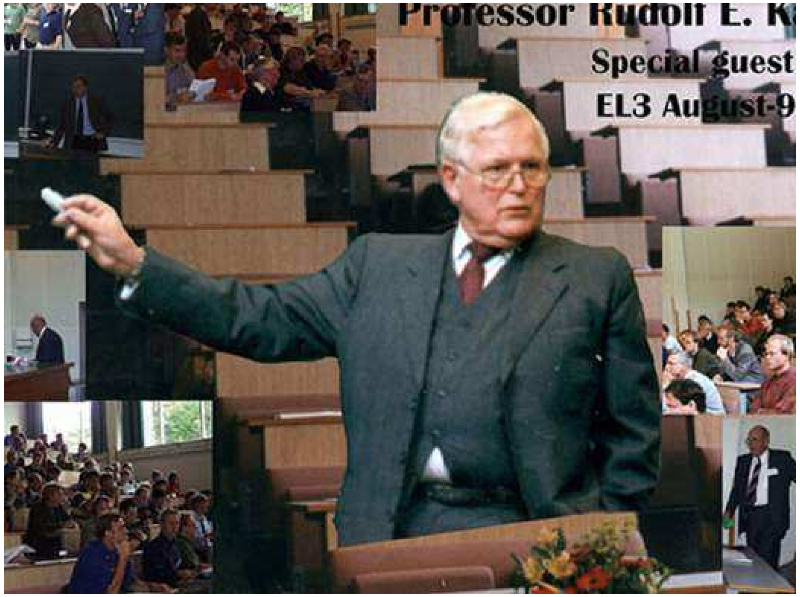
$$y(t) = \int_{0 \text{ or } -\infty}^{t} H(t-t')u(t')\,dt'$$

These models fail to deal with ‘initial conditions’.

A physical system is SELDOMLY an i/o map

Input/state/output systems

$$\rightsquigarrow \quad \frac{d}{dt}x = f(x,u), \; y = g(x,u)$$

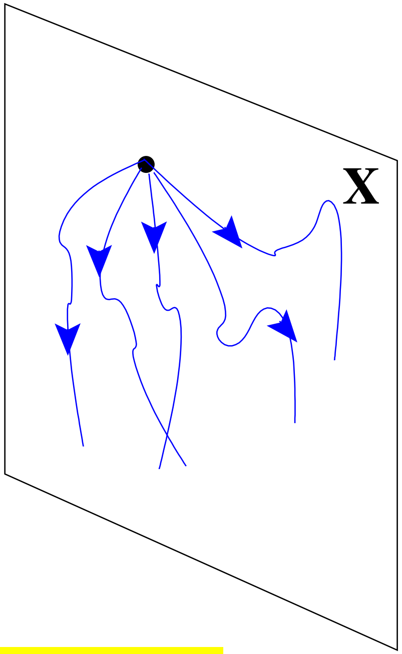


Rudolf Kalman (1930-)

‘Axiomatization’

State transition function:

$\phi(t, \mathrm{x}, u)$: state at time t from x using input u.

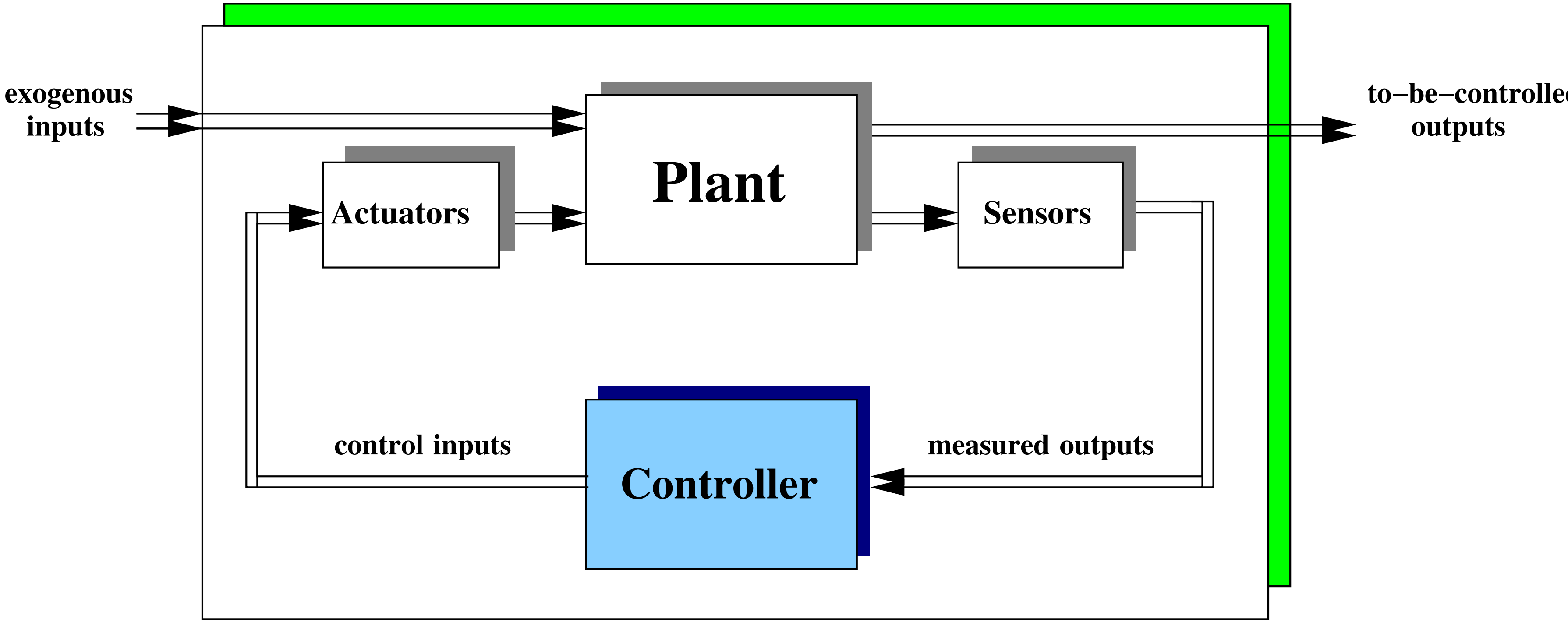


Read-out function:

$g(\mathrm{x}, \mathrm{u})$: output with state x and input value u.

The input/state/output view turned out to be a very effective and fruitful paradigm

- for control (stabilization, robustness, ...)



The input/state/output view turned out to be a very effective and fruitful paradigm

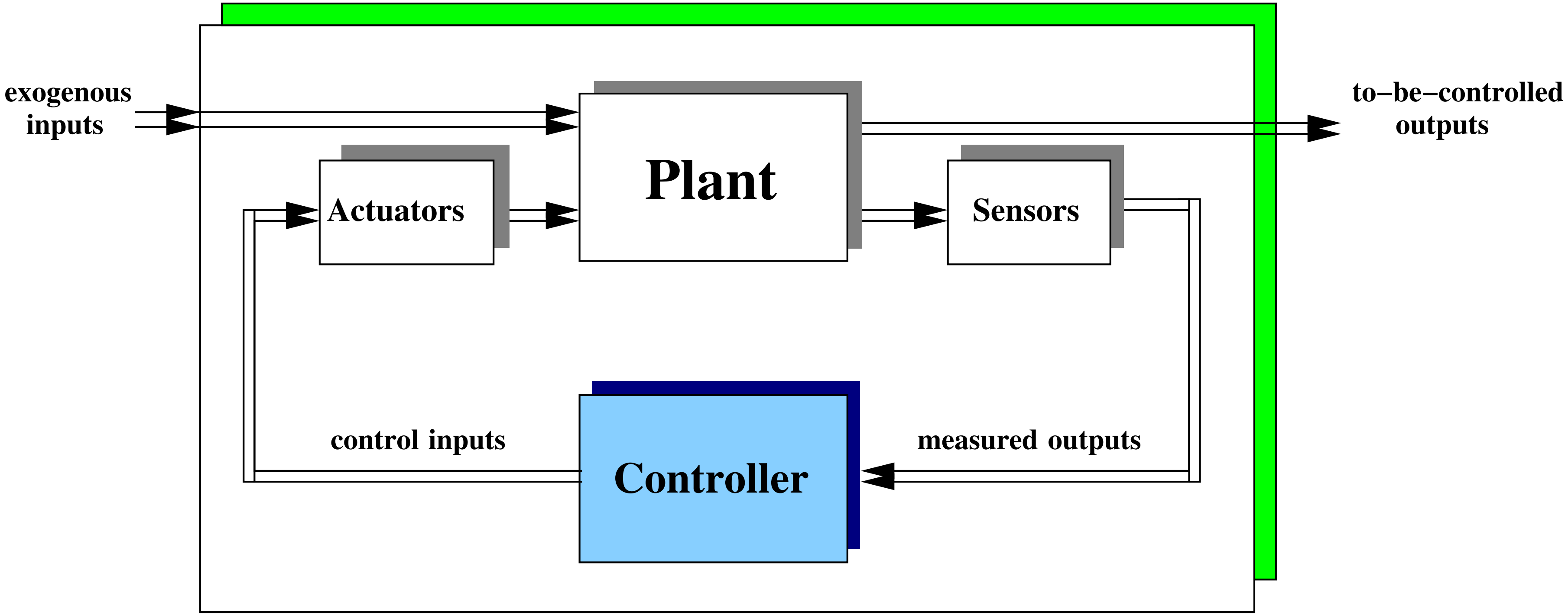
- for **control** (stabilization, robustness, ...)
- **prediction** of one signal from another
- **system ID:** models from data
- understanding **system representations** (state, transfer f'n, etc.)
- etc., etc., etc.

Let's take a close look at the i/o framework ...

in control (only)

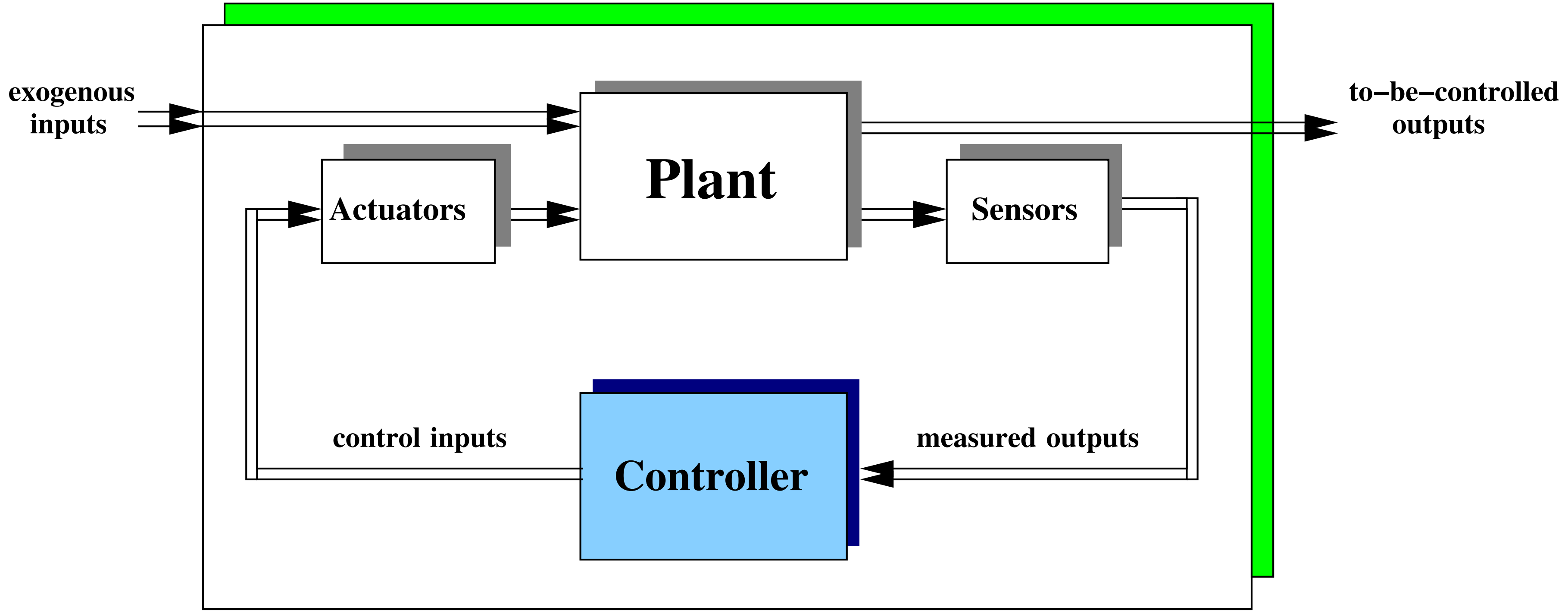
Difficulties with i/o

active control



Difficulties with i/o

active control



versus **passive control**

Difficulties with i/o

active control versus passive control

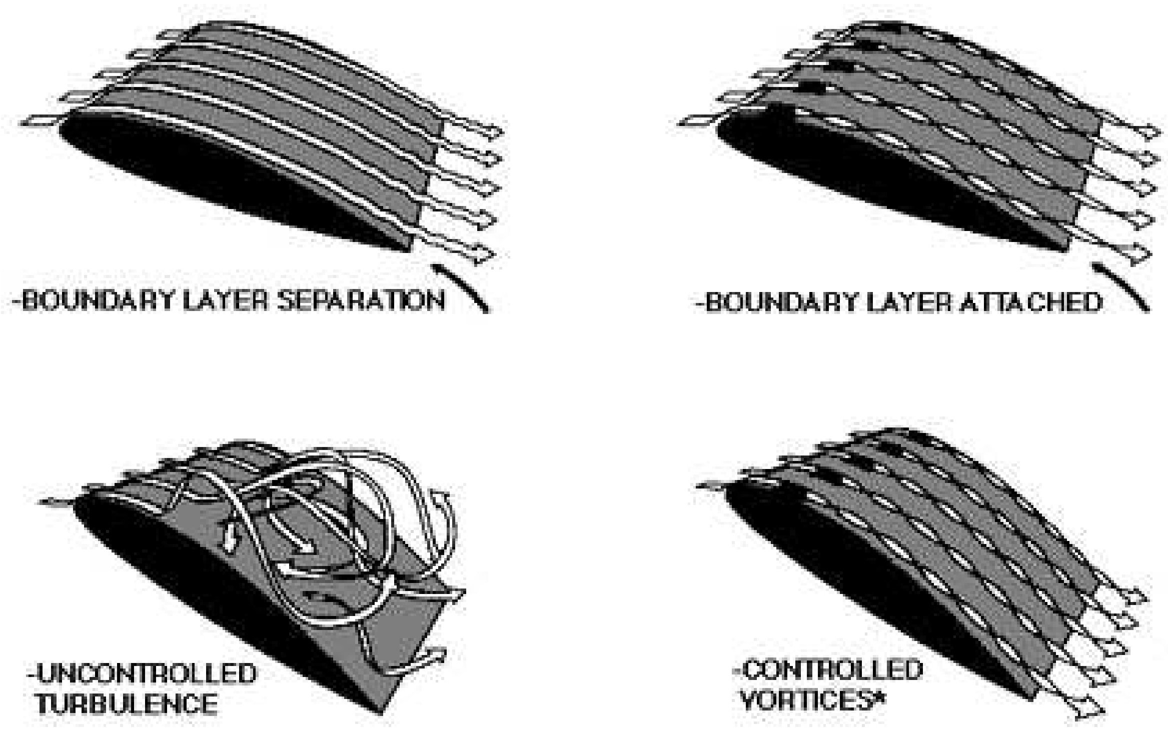
Controlling turbulence

for airplanes, sharks, dolphins, golf balls, bicycling helmets, etc.

Difficulties with i/o

active control versus **passive control**

Controlling turbulence



Difficulties with i/o

active control   versus **passive control**

Controlling turbulence

Nagano 1998

Difficulties with i/o

active control versus passive control

Controlling turbulence

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Difficulties with i/o

active control versus passive control

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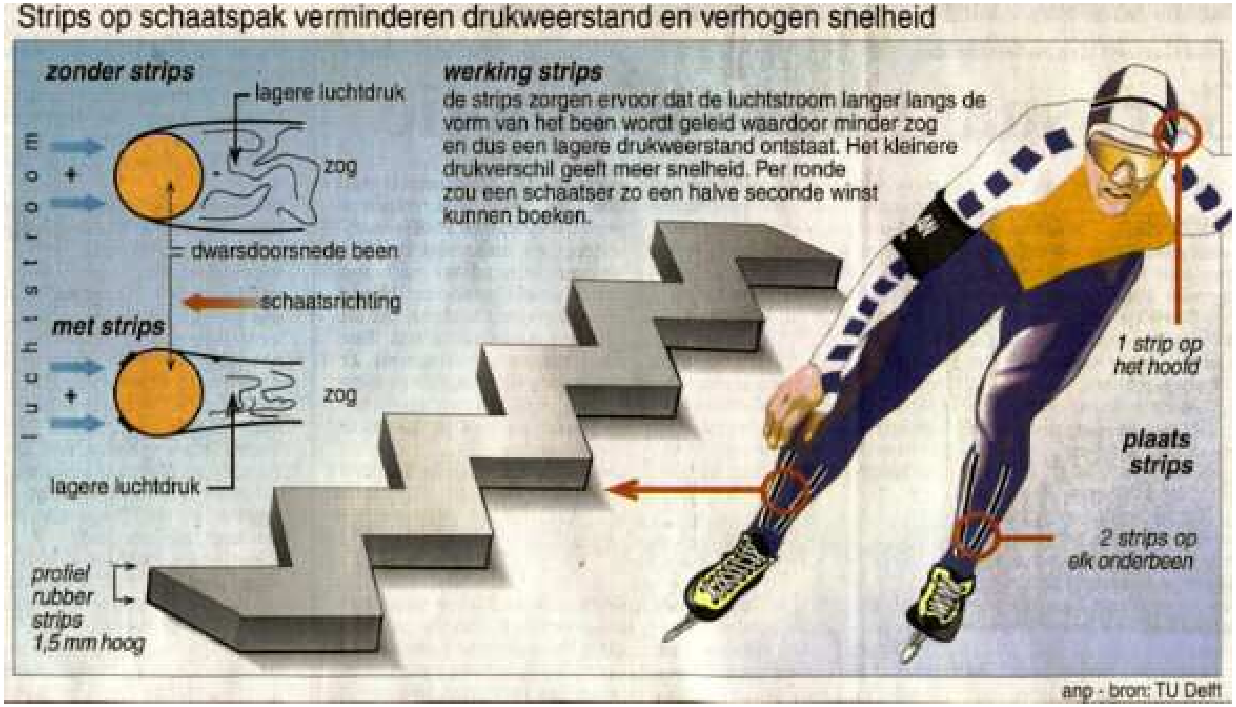


Difficulties with i/o

active control versus passive control

Nagano 1998

These are beautiful controllers!

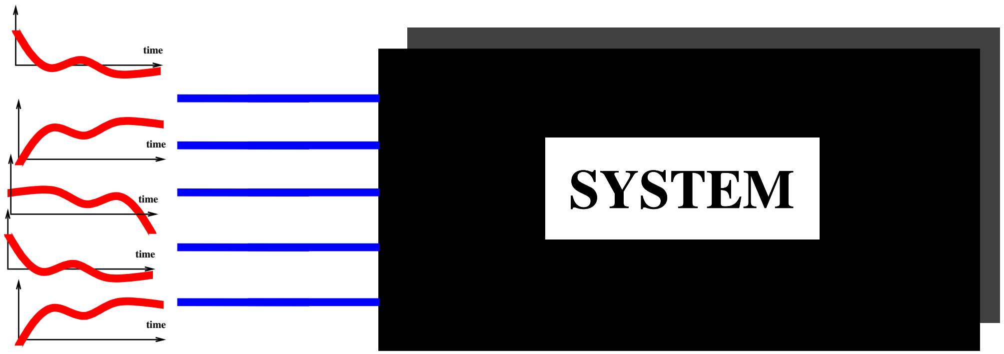
active control versus passive control

Nagano 1998

These are beautiful controllers! But, the only people not calling this "control", are the control engineers ...

At last, a consistent framework ...

The behavior



Which event trajectories are possible?

The behavior =

all trajectories of the system variables which, according to the mathematical model, are **possible**.

Definition: A **system** Σ is defined as

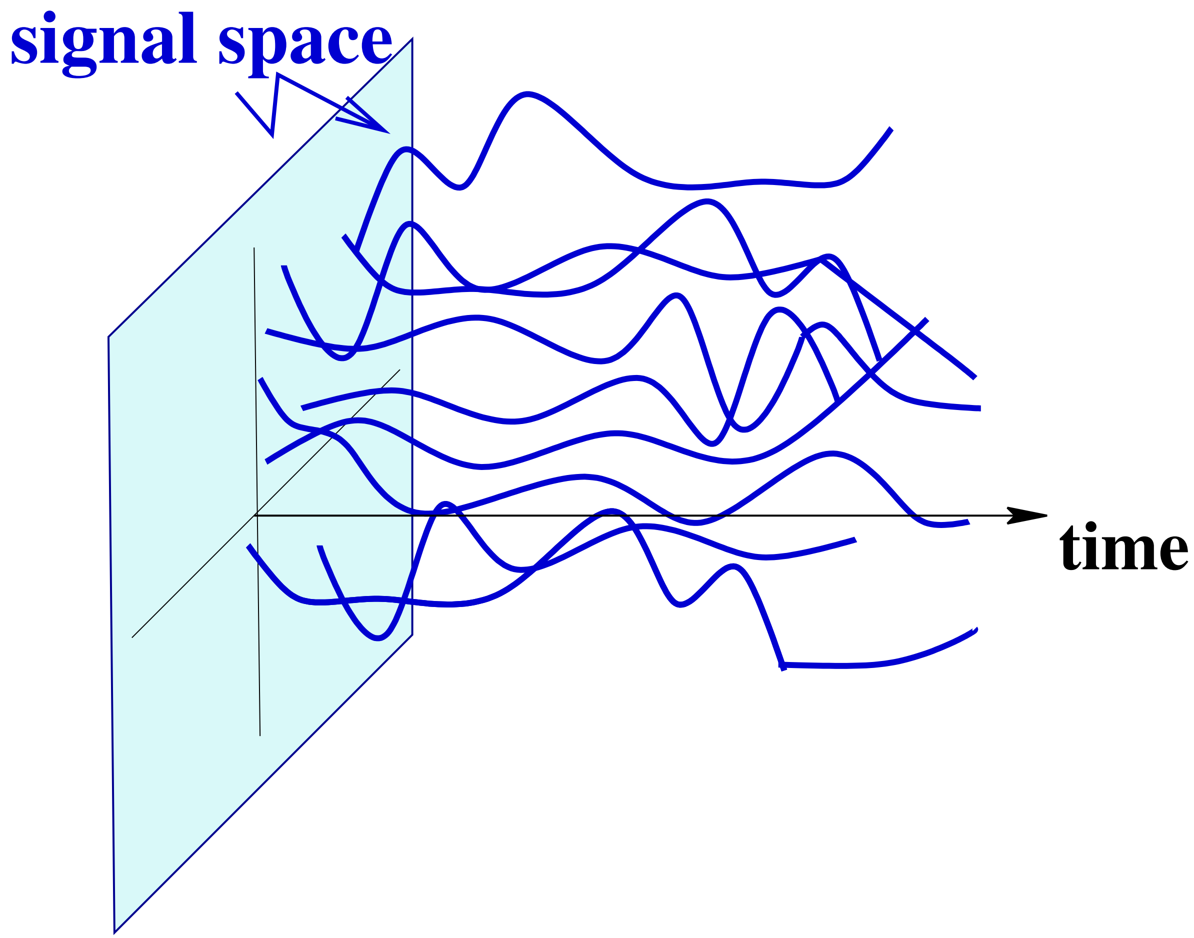
$$\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$$

with $\mathbb{T}$ = the set of **independent variables**

$\mathbb{W}$ = the **signal space**

$\mathfrak{B} \subseteq \mathbb{W}^{\mathbb{T}}$ the **behavior**.

The behavior



Totality of ‘legal’ trajectories =: the behavior

Linear shift-invariant differential systems

There exists an extensive theory for these systems

$$\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$$

$\mathbb{T} = \mathbb{R}^{\mathtt{n}}$, the set of independent variables,

$\mathbb{W} = \mathbb{R}^{\mathtt{w}}$, the set of dependent variables,

$\mathfrak{B} =$ sol'ns lin. const. coeff. system ODE's or PDE's.

Linear shift-invariant differential systems

There exists an extensive theory for these systems

$$\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$$

$\mathbb{T} = \mathbb{R}^{\mathtt{n}}$, **the set of independent variables,**
often $\mathtt{n} = 1$ **(dynamical systems)**
or $\mathtt{n} = 4$ **(distributed systems),**

$\mathbb{W} = \mathbb{R}^{\mathtt{w}}$, **the set of dependent variables,**

$\mathfrak{B}$ = **sol'ns lin. const. coeff. system ODE's or PDE's.**

Linear shift-invariant differential systems

Let $R \in \mathbb{R}^{\bullet \times \mathtt{w}}[\xi_1, \cdots, \xi_{\mathtt{n}}]$, and consider

$$R(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathtt{n}}})w = 0. \quad (*)$$

Linear shift-invariant differential systems

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$$R\left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathtt{n}}}\right) w = 0. \quad (*)$$

Define the associated behavior

$$\mathfrak{B} = \{ w \in \mathfrak{C}^{\infty}(\mathbb{R}^{\mathtt{n}}, \mathbb{R}^{\mathtt{w}}) \mid (*) \text{ holds } \}.$$

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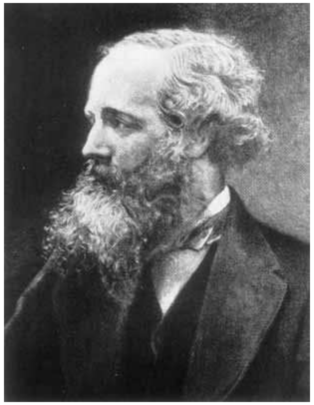
Notation for $\mathtt{n}$-D linear differential systems:

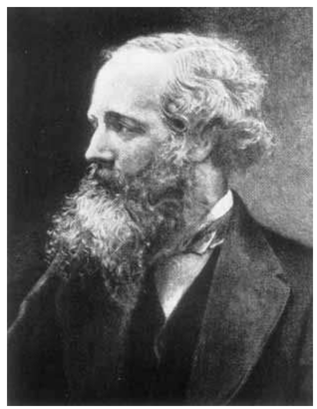
$$(\mathbb{R}^{\mathtt{n}}, \mathbb{R}^{\mathtt{w}}, \mathfrak{B}) \in \mathfrak{L}^{\mathtt{w}}_{\mathtt{n}}, \quad \text{or} \quad \mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\mathtt{n}}.$$

Examples: Maxwell's eq'ns, diff. eq'n, wave eq'n, . . .



$$
\begin{aligned}
\nabla \cdot \vec{E} &= \frac{1}{\varepsilon_0}\rho\,, \\
\nabla \times \vec{E} &= -\frac{\partial}{\partial t}\vec{B}, \\
\nabla \cdot \vec{B} &= 0\,, \\
c^2 \nabla \times \vec{B} &= \frac{1}{\varepsilon_0}\vec{j} + \frac{\partial}{\partial t}\vec{E}.
\end{aligned}
$$

Examples: Maxwell’s eq’ns, diff. eq’n, wave eq’n, . . .



$$\begin{aligned}
\nabla \cdot \vec{E} &= \frac{1}{\varepsilon_0}\rho\,, \\
\nabla \times \vec{E} &= -\frac{\partial}{\partial t}\vec{B}, \\
\nabla \cdot \vec{B} &= 0\,, \\
c^2\nabla \times \vec{B} &= \frac{1}{\varepsilon_0}\vec{j} + \frac{\partial}{\partial t}\vec{E}.
\end{aligned}$$

$\mathbb{T} = \mathbb{R} \times \mathbb{R}^3$ (time and space) $\mathtt{n} = 4$,

$w = (\vec{E}, \vec{B}, \vec{j}, \rho)$ (electric field, magnetic field,

current density, charge density),

$\mathbb{W} = \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}$, $\mathtt{w} = 10$,

$\mathfrak{B} \in \mathfrak{L}_4^{10}$ = the set of solutions to these PDE’s.

Note: 10 variables, 8 equations! $\Rightarrow \exists$ free variables.

Submodule theorem

$R \in \mathbb{R}^{\bullet\times\bullet}[\xi_1, \cdots, \xi_n]$ **defines**
$\mathfrak{B} = \ker(R(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}))$, **but not vice-versa.**

¿¿ $\exists$ 'intrinsic' characterization of $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\mathtt{n}}$??

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Define the ***annihilators*** of $\mathfrak{B} \in \mathfrak{L}_{\mathtt{n}}^{\mathtt{w}}$ by

$$\mathfrak{N}_{\mathfrak{B}} := \{n \in \mathbb{R}^{\mathtt{w}}[\xi_1, \cdots, \xi_{\mathtt{n}}] \mid n^\top(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathtt{n}}})\mathfrak{B} = 0\}.$$

$\mathfrak{N}_{\mathfrak{B}}$ is a $\mathbb{R}[\xi_1, \cdots, \xi_{\mathtt{n}}]$-submodule of $\mathbb{R}^{\mathtt{w}}[\xi_1, \cdots, \xi_{\mathtt{n}}]$.

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Thm. 1: $\mathfrak{L}^{\mathtt{w}}_{\mathtt{n}} \overset{1:1}{\longleftrightarrow}$ submodules of $\mathbb{R}^{\mathtt{w}}[\xi_1, \cdots, \xi_{\mathtt{n}}]$

Assume $\mathfrak{B} \in \mathfrak{L}_{\mathtt{n}}^{\mathtt{w}_1+\mathtt{w}_2}$, and define

$$\mathfrak{B}_1 := \{w_1 \mid \exists\, w_2 \text{ such that } (w_1, w_2) \in \mathfrak{B}\}$$

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Elimination theorem

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Thm. 2 (‘Elimination’ thm): **It does!**

Which PDE's describe $(\rho, \vec{E}, \vec{j})$ in Maxwell's equations ?

Eliminate $\vec{B}$ from Maxwell's equations $\rightsquigarrow$

$$\begin{aligned} \nabla \cdot \vec{E} &= \frac{1}{\varepsilon_0}\rho\,, \\ \varepsilon_0 \frac{\partial}{\partial t}\nabla \cdot \vec{E} + \nabla \cdot \vec{j} &= 0, \\ \varepsilon_0 \frac{\partial^2}{\partial t^2}\vec{E} + \varepsilon_0 c^2 \nabla \times \nabla \times \vec{E} + \frac{\partial}{\partial t}\vec{j} &= 0. \end{aligned}$$

Controllability

$$\boxed{R(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathtt{n}}})w = 0}$$

is a ‘kernel representation’ of the associated $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\mathtt{n}}$.

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Another representation: image representation

$$w = M\left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathrm{n}}}\right) \ell.$$

‘Elimination’ thm $\quad \Rightarrow \quad \mathrm{im}\left(M\left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathrm{n}}}\right)\right) \in \mathfrak{L}_{\mathtt{n}}^{\mathtt{w}}$!

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Which systems admit an image representation???

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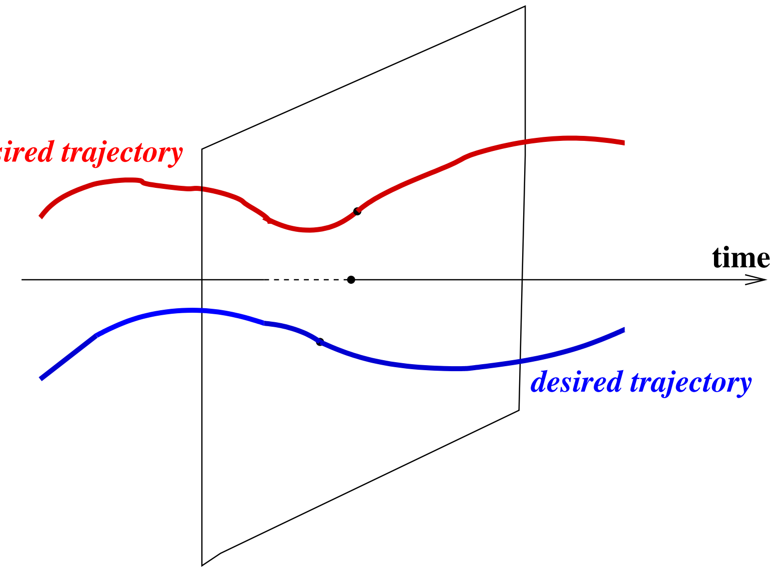
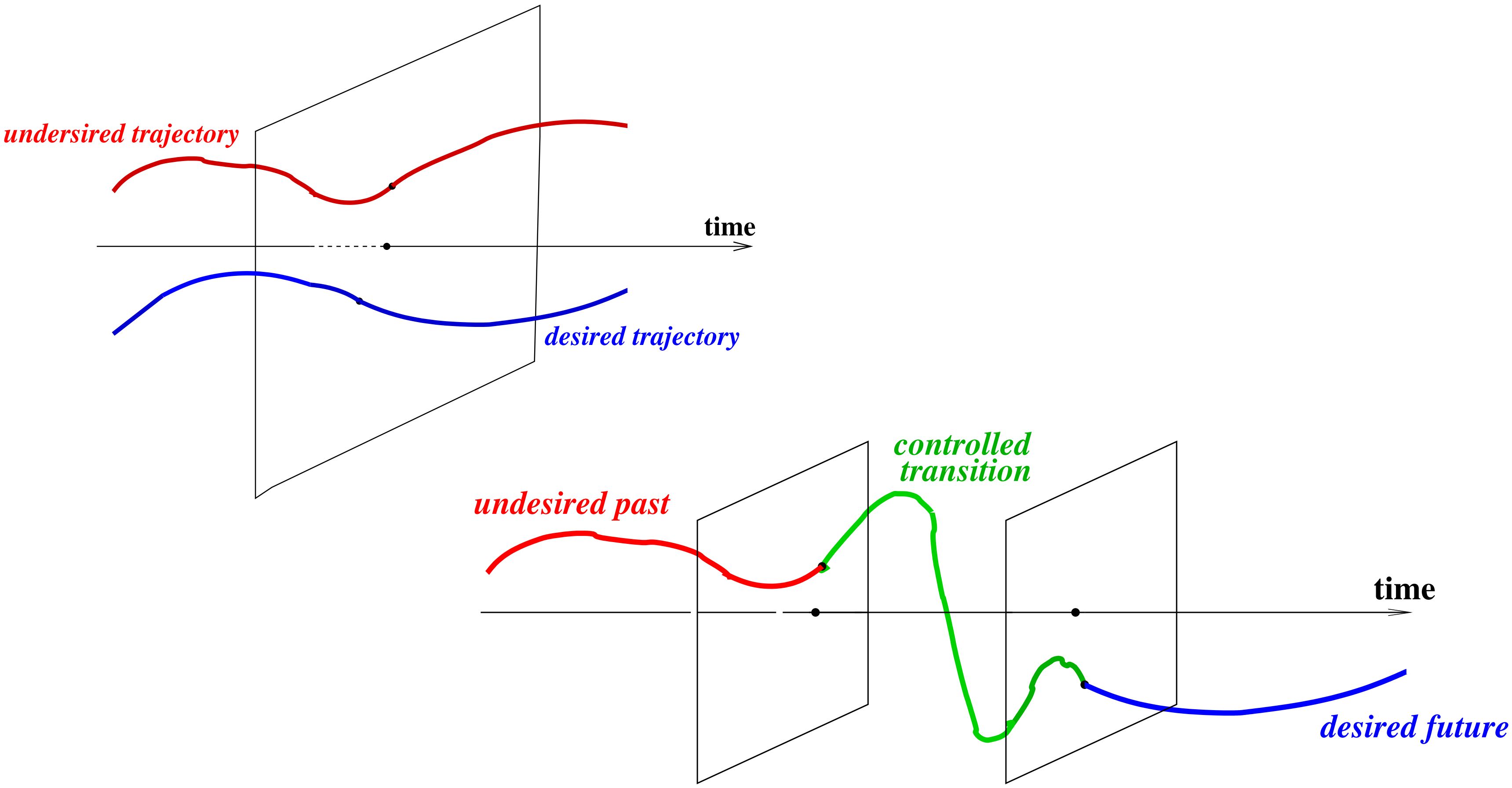
Which systems admit an image representation???

Thm. 3: $\mathfrak{B}$ admits image repr. $\Leftrightarrow$ it is 'controllable'.

Controllability def'n in pictures:



Controllability def'n in pictures:



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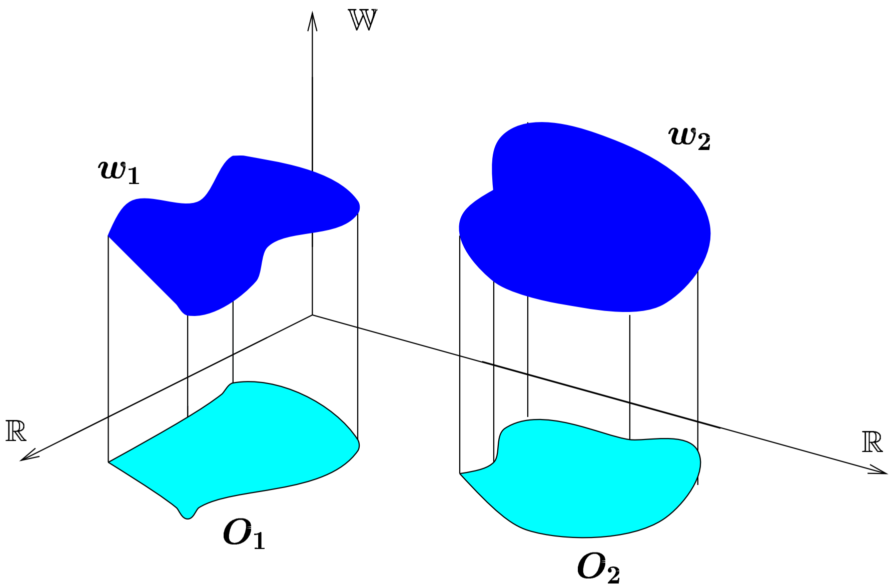
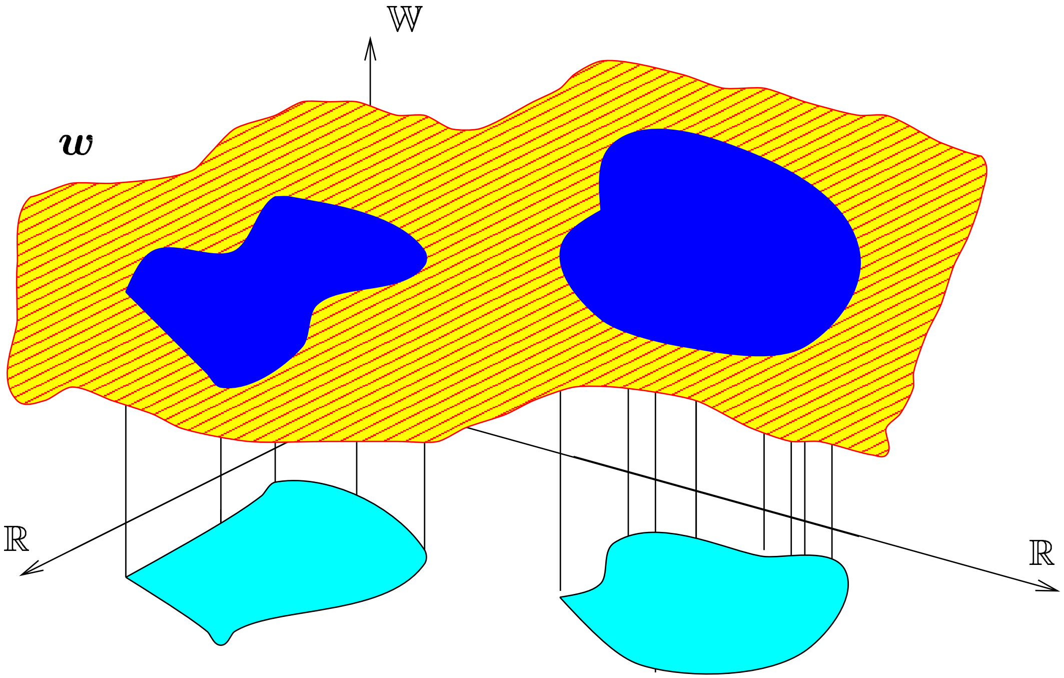


$$w_1, w_2 \in \mathfrak{B}.$$

Controllability for PDE's

$w \in \mathfrak{B}$ **'patches'** $w_1, w_2 \in \mathfrak{B}$.



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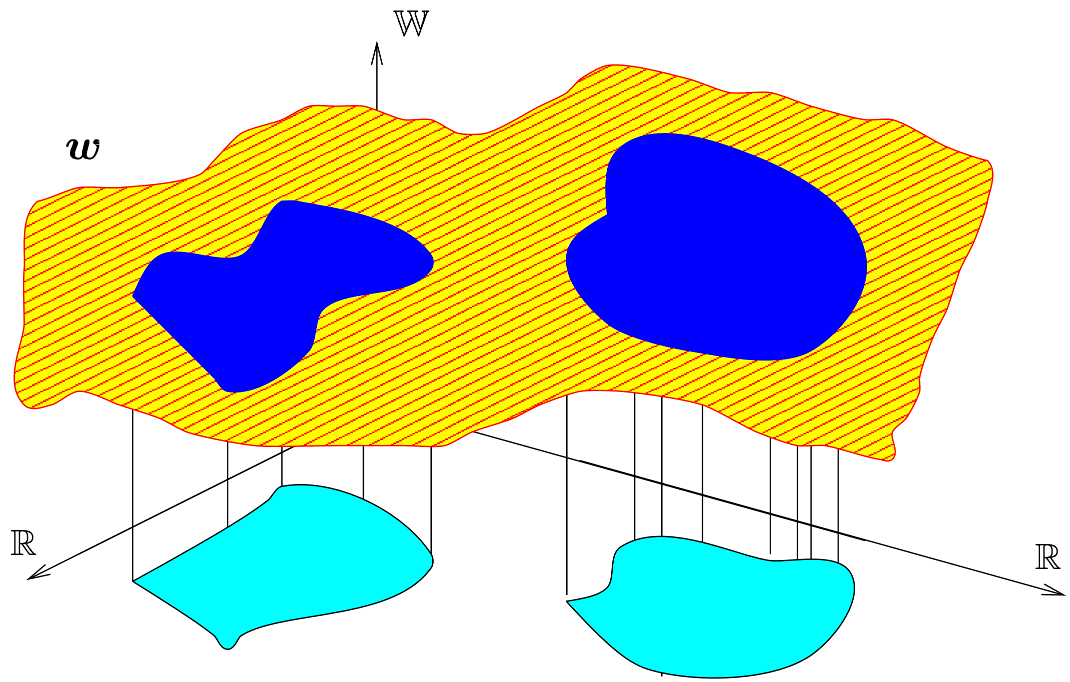

Controllability :⇔ 'patch-ability'.

Are Maxwell's equations controllable?

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The following equations in the *scalar potential* ϕ and the *vector potential* $\vec{A}$, generate exactly the solutions to Maxwell's equations:

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\begin{aligned}
\vec{E} &= -\frac{\partial}{\partial t}\vec{A} - \nabla\phi, \\
\vec{B} &= \nabla \times \vec{A}, \\
\vec{j} &= \varepsilon_0 \frac{\partial^2}{\partial t^2}\vec{A} - \varepsilon_0 c^2 \nabla^2 \vec{A} + \varepsilon_0 c^2 \nabla(\nabla\cdot\vec{A}) + \varepsilon_0 \frac{\partial}{\partial t}\nabla\phi, \\
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proves controllability. Illustrates the connection

controllability $\Leftrightarrow$ $\exists$ potential !

Observability of the image representation

$$w = M(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n})\ell$$

is defined as: ℓ can be deduced from w,

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Not all controllable systems admit an observable image repr'ion. For $n = 1$, they do. For $n > 1$, exceptionally.

The latent variable in an image repr'ion may be 'hidden'.

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Example: **Maxwell's equations do not allow a potential representation that is observable.**

Sources

Dynamical systems (ODE's): $\sim$ 1985

2-D discrete set of ind. variables: Rocha $\sim$ 1990
Differential-delay systems:
Glüsing-Luerssen, Rocha, Zampieri, Vettori
PDE's: Shankar, Pillai, Oberst, Zerz $\sim$ 1995

Generalization from $\mathbb{R}$, $\mathbb{C}$ to quaternions:
Pereira, Vettori $\sim$ 2004

Thank you

Thank you

Thank you

Thank you

Thank you

Thank you

Thank you

Details & copies of the lecture frames are available from/at

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