A TUTORIAL INTRODUCTION TO QUADRATIC DIFFERENTIAL FORMS

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THEORY

- Introduction
- Basic definitions:
  bilinear/quadratic differential forms (BDF's, QDF's)
- Two-variable polynomial matrices
- Calculus of BDF's, QDF's

Introduction

Given: a linear differential system, with variables w

Often necessary to study **functionals** of w and its derivatives $\frac{d^j}{dt^j}w$, for example in

- **Lyapunov functions** for high-order diff. eq'ns;
- **Performance criteria** in control and filtering problems;
- Modeling **physical quantities/properties**,

  as power, energy; dissipativity, conservation laws;

Of special interest *quadratic* and *bilinear* functionals.

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Of special interest *quadratic* and *bilinear* functionals.

Could reduce to 1-st order eq'ns and **constant** functionals;

but why not address such issues in the original representation?

Example: Lyapunov stability

Consider trajectories $(u, y) \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^2)$ described by

$$p(\frac{d}{dt})y = q(\frac{d}{dt})u$$

Lyapunov stability: assume $u = 0$; ¿$\lim_{t\to\infty} y(t) = 0$?

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Check if there exists a quadratic functional

$$Q(y) = \sum_{k,\ell} Q_{k,\ell}(\frac{d^k}{dt^k}y)(\frac{d^\ell}{dt^\ell}y)$$

with $Q(y)(t) \geq 0$ and $\frac{d}{dt}Q(y)(t) < 0$

along solutions of $p(\frac{d}{dt})y = 0 \ldots$

Why cast this into state form (nontrivial for multivariable case!)?

Bilinear differential forms

Let $\Phi_{k,\ell} \in \mathbb{R}^{\mathtt{w}_1 \times \mathtt{w}_2}$, $k, \ell = 0, 1, 2, \ldots, L$ and $w_i \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}_i})$.

The functional

$L_\Phi : \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}_1}) \times \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}_2}) \to \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R})$ defined by

$$L_\Phi(w_1, w_2) := \sum_{k,\ell=0}^{L} \left(\frac{d^k}{dt^k} w_1\right)^\top \Phi_{k,\ell} \left(\frac{d^\ell}{dt^\ell} w_2\right)$$

is called a **bilinear differential form (BDF)**.

Quadratic differential forms

Let $\Phi_{k,\ell} \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}, k, \ell = 0, 1, 2, \ldots, L$ and $w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$.

The functional $Q_\Phi : \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \to \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R})$ defined by

$$Q_\Phi(w) := \sum_{k,\ell=0}^{L} (\tfrac{d^k}{dt^k} w)^\top \Phi_{k,\ell} (\tfrac{d^\ell}{dt^\ell} w)$$

is called a **quadratic differential form (QDF)**.

Example

QDF: Total energy in spring-mass system

$$M\frac{d^2}{dt^2}w + Kw = 0$$

$$E_{\mathrm{tot}}(t) = \tfrac{1}{2}M(\tfrac{d}{dt}w(t))^2 + \tfrac{1}{2}Kw(t)^2$$

$$E_{\mathrm{tot}}(t) = \begin{bmatrix} w(t) & \frac{d}{dt}w(t) \end{bmatrix} \begin{bmatrix} \frac{1}{2}K & 0 \\ 0 & \frac{1}{2}M \end{bmatrix} \begin{bmatrix} w(t) \\ \frac{d}{dt}w(t) \end{bmatrix}$$

Two-variable polynomial matrices

Entries are polynomials with real coefficients in two indeterminates:

$$\Phi(\zeta,\eta) = \sum_{k,\ell=0}^{L} \Phi_{k,\ell}\zeta^k\eta^\ell$$

with $\Phi_{k,\ell} \in \mathbb{R}^{\mathtt{w}_1 \times \mathtt{w}_2}$.

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with $\Phi_{k,\ell} \in \mathbb{R}^{\mathtt{w}_1 \times \mathtt{w}_2}$. In 1↔1 relation with the BDF L_Φ

$\mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}_1}) \times \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}_2}) \to \mathfrak{C}^\infty(\mathbb{R},\mathbb{R})$

$$L_\Phi(w_1,w_2) := \sum_{k,\ell=0}^{L} \left(\frac{d^k}{dt^k}w_1\right)^\top \Phi_{k,\ell}\left(\frac{d^\ell}{dt^\ell}w_2\right)$$

the **bilinear differential form** L_Φ **(BDF) induced by** $\Phi(\zeta,\eta)$

Two-variable polynomial matrices and QDF's

Let $\mathtt{w}_1 = \mathtt{w}_2 = \mathtt{w}$ in

$$\Phi(\zeta, \eta) = \sum_{k,\ell=0}^{L} \Phi_{k,\ell} \zeta^k \eta^\ell$$

The QDF $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \rightarrow \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R})$

$$L_\Phi(w, w) = Q_\Phi(w) = \sum_{k,\ell=0}^{L} \left(\frac{d^k}{dt^k} w\right)^\top \Phi_{k,\ell} \frac{d^\ell}{dt^\ell} w$$

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WLOG $\Phi_{k,\ell} = \Phi_{\ell,k}^\top$ i.e. $\Phi(\zeta,\eta) = \Phi(\eta,\zeta)^\top$ (symmetry)

$1 \leftrightarrow 1$ relation with QDF's

Examples

- Total energy for oscillator $M\frac{d^2}{dt^2}w + Kw = 0$ induced by

$$\Phi(\zeta,\eta) = \frac{1}{2}M\zeta\eta + \frac{1}{2}K$$

since $Q_\Phi(w) = \frac{1}{2}M(\frac{d}{dt}w)^2 + \frac{1}{2}Kw^2$.

Examples

- $Q_\Phi(w_1, w_2) = w_2 \frac{d}{dt} w_1$

  ¿Polynomial matrix for Q_Φ?

$$w_2\left(\frac{d}{dt}w_1\right) = \frac{1}{2}\begin{bmatrix}\frac{d}{dt}w_1 & \frac{d}{dt}w_2\end{bmatrix}\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\begin{bmatrix}w_1\\ w_2\end{bmatrix}$$

$$+\frac{1}{2}\begin{bmatrix}w_1 & w_2\end{bmatrix}\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}\begin{bmatrix}\frac{d}{dt}w_1\\ \frac{d}{dt}w_2\end{bmatrix}$$

Therefore $\Phi(\zeta, \eta) = \frac{1}{2}\begin{bmatrix}0 & \zeta\\ \eta & 0\end{bmatrix}$

The calculus of QDF's

1. Basics of linear differential systems
2. Differentiation
3. Integration
4. QDF's along behaviors
5. Positivity

Linear differential systems

$$R_0 + R_1 \frac{d}{dt} w + R_2 \frac{d^2}{dt^2} w + \ldots + R_L \frac{d^L}{dt^L} w = 0$$

$R_i \in \mathbb{R}^{\mathtt{g} \times \mathtt{w}}$, $i = 0, \ldots, L$. **Associated one-variable polynomial matrix:**

$$R(\xi) = R_0 + R_1 \xi + \ldots + R_L \xi^L \in \mathbb{R}^{\mathtt{g} \times \mathtt{w}}[\xi]$$

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$R(\frac{d}{dt}) w = 0$ **kernel representation**

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More than a representation issue:

- $\exists$ ***calculus of representations*;**
- ***Time-domain* properties $\longleftrightarrow$ *algebraic* properties**

Linear differential systems

Often in order to model the behavior of w (**‘manifest’** variable), we need to consider the ℓ (**’latent’** variable) as well:

$$R(\tfrac{d}{dt})w = M(\tfrac{d}{dt})\ell$$

latent variable repr’on

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'State' variable is special latent variable ('Markovian')

1-st order representation is **consequence** of state **property**!

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'State' variable is special latent variable ('Markovian')

1-st order representation is consequence of state property!

Observability of ℓ from w: $(w = 0) \Rightarrow (\ell = 0)$

The calculus of QDF's: differentiation

Consider Q_Φ induced by $\Phi \in \mathbb{R}_s^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$

The derivative of Q_Φ is

$$\frac{d}{dt}Q_\Phi : \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}}) \to \mathfrak{C}^\infty(\mathbb{R},\mathbb{R})$$

$$\left(\frac{d}{dt}Q_\Phi\right)(w) := \frac{d}{dt}Q_\Phi(w)$$

Also a QDF!

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¿Which matrix in $\mathbb{R}_s^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ induces $\frac{d}{dt}Q_\Phi$?

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¿Which matrix in $\mathbb{R}_s^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ induces $\frac{d}{dt}Q_\Phi$?

The matrix $\overset{\bullet}{\Phi}(\zeta,\eta) := (\zeta+\eta)\Phi(\zeta,\eta)$!

Calculus of QDF's: integration

Consider compact-support $\mathfrak{C}^\infty$-trajectories (denoted $\mathfrak{D}(\mathbb{R}, \mathbb{R}^\bullet)$), let

$$L_\Phi : \mathfrak{D}(\mathbb{R}, \mathbb{R}^{w_1}) \times \mathfrak{D}(\mathbb{R}, \mathbb{R}^{w_2}) \to \mathfrak{D}(\mathbb{R}, \mathbb{R})$$

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Integral of L_Φ defined

$$\int L_\Phi : \mathfrak{D}(\mathbb{R}, \mathbb{R}^{\mathtt{w}_1}) \times \mathfrak{D}(\mathbb{R}, \mathbb{R}^{\mathtt{w}_2}) \to \mathbb{R}$$

$$\int L_\Phi(w_1, w_2) := \int_{-\infty}^{t} L_\Phi(w_1, w_2) dt$$

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Is it a BDF? Not always, but when? Analogous question for QDF's.

'Path independence' (cfr. Brockett's work in the 1960's)

Integration

¿Given Q_Φ, does there exist a $\Psi(\zeta,\eta)$ such that $\frac{d}{dt}Q_\Psi = Q_\Phi$?

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Theorem: Let $\Phi \in \mathbb{R}_s^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$. The following are equivalent:

1. there exists $\Psi \in \mathbb{R}_s^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ such that $\Phi(\zeta,\eta) = (\zeta+\eta)\Psi(\zeta,\eta)$, equivalently, $\frac{d}{dt}Q_\Psi = Q_\Phi$;
2. $\Phi(-\xi,\xi) = 0$.

QDF's along behaviors: example

Often need to evaluate QDF's on $w \in \mathfrak{B}$ ("along $\mathfrak{B}$")

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Example: Mass-spring system

$$\mathfrak{B} = \{w \mid M\frac{d^2}{dt^2}w + Kw = 0\}$$

Total energy $\rightsquigarrow$ $\Phi(\zeta,\eta) = \frac{1}{2}M\zeta\eta + \frac{1}{2}K$

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$\frac{d}{dt}Q_\Phi(w) = 0$ for all $w \in \mathfrak{B}$ expressed as

$$\begin{aligned}(\zeta+\eta)\Phi(\zeta,\eta) &= (\zeta+\eta)(\frac{1}{2}M\zeta\eta + \frac{1}{2}K)\\ &= \frac{1}{2}\underbrace{(M\zeta^2+K)}_{=0 \text{ if evaluated on } w\in\mathfrak{B}}\eta + \frac{1}{2}\zeta\underbrace{(M\eta^2+K)}_{=0 \text{ if evaluated on } w\in\mathfrak{B}}\end{aligned}$$

QDF's which are zero along behaviors

Q_Φ is zero on $\mathfrak{B}$, written $Q_\Phi \overset{\mathfrak{B}}{=} 0$, if $Q_\Phi(w) = 0$ for all $w \in \mathfrak{B}$

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Theorem: Let $\mathfrak{B} = \ker R(\frac{d}{dt})$. Then $Q_\Phi \overset{\mathfrak{B}}{=} 0$ if and only if there exists $F \in \mathbb{R}^{\bullet\times\bullet}[\zeta,\eta]$ such that

$$\Phi(\zeta,\eta) = R(\zeta)^\top F(\zeta,\eta) + F(\eta,\zeta)^\top R(\eta)$$

Equivalence of QDF's along behaviors

$Q_\Phi \stackrel{\mathfrak{B}}{=} Q_\Psi$ iff exists $F \in \mathbb{R}^{\cdot\times\cdot}[\zeta,\eta]$ such that

$$\Phi(\zeta,\eta) - \Psi(\zeta,\eta) = R(\zeta)^\top F(\zeta,\eta) + F(\eta,\zeta)^\top R(\eta)$$

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Example: $\zeta^3\eta^3 + 1 \stackrel{\mathfrak{B}}{=} \zeta\eta + 1$ when $\mathfrak{B} = \ker\left(\frac{d^2}{dt^2} + 1\right)$.

Indeed,

$$\frac{d^2}{dt^2}w + w = 0 \Longrightarrow \frac{d^3}{dt^3}w = -\frac{d}{dt}w$$

In two-variable polynomial terms:

$$\zeta^3\eta^3 + 1 = (\zeta\eta + 1) + (\zeta^2 + 1)(\zeta\eta^3) + (\zeta^3\eta)(\eta^2 + 1)$$

Positivity of QDF's

$\Phi \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ is **nonnegative** (written $\Phi \geq 0$) if $Q_\Phi(w) \geq 0$ for all $w \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}})$.

$\Phi \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ is **positive** (written $\Phi > 0$) if $\Phi \geq 0$ and $(Q_\Phi(w) = 0) \Rightarrow (w = 0)$.

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$$\Phi \geq 0 \Leftrightarrow \text{ exists } D \in \mathbb{R}^{\bullet\times\mathtt{w}}[\xi] : \Phi(\zeta,\eta) = D^\top(\zeta)D(\eta)$$

$$\Phi > 0 \Leftrightarrow \text{ exists } D \in \mathbb{R}^{\bullet\times\mathtt{w}}[\xi] : \Phi(\zeta,\eta) = D^\top(\zeta)D(\eta)$$

and rank $D(\lambda) = \mathtt{w}\ \forall \lambda \in \mathbb{C}$

Positivity of QDF's along behaviors

$\Phi \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ is **nonnegative along** $\mathfrak{B}$ (written $\Phi \overset{\mathfrak{B}}{\geq} 0$) if $Q_\Phi(w) \geq 0$ for all $w \in \mathfrak{B}$.

$\Phi \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ is **positive along** $\mathfrak{B}$ (written $\Phi \overset{\mathfrak{B}}{>} 0$) if $\Phi \overset{\mathfrak{B}}{\geq} 0$ and $(Q_\Phi(w) = 0) \Rightarrow (w = 0)$.

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$$\Phi \overset{\mathfrak{B}}{\geq} 0 \Leftrightarrow \exists \Phi' \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta] \text{ s.t. } \Phi(\zeta,\eta) \overset{\mathfrak{B}}{=} \Phi'(\zeta,\eta) \text{ and } \Phi' \geq 0$$

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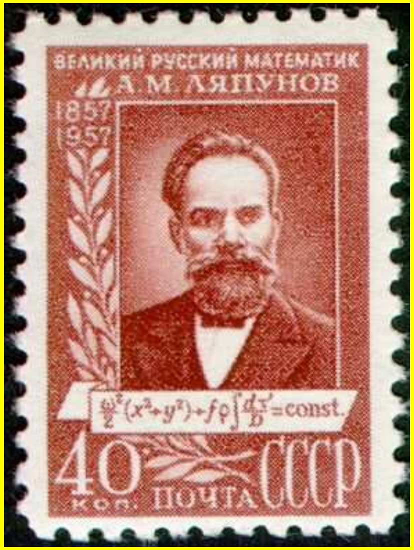
$$\Phi'(\zeta,\eta) = D^\top(\zeta)D(\eta) \text{ and rank } \begin{bmatrix} D(\lambda) \\ R(\lambda) \end{bmatrix} = \mathtt{w} \quad \forall \lambda \in \mathbb{C}$$

APPLICATIONS

- Lyapunov theory
- The construction of storage functions
- ...

Is there

a Lyapunov theory for systems described by high order differential equations?



cfr. early work by Fuhrmann.

An example

Consider the mechanical system

$$Kw + D\frac{d}{dt}w + M\frac{d^2}{dt^2}w = 0$$

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The stored energy equals

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The dissipation equals

$$\frac{d}{dt}E(w, \frac{d}{dt}w) \stackrel{\mathfrak{B}}{=} -(\frac{d}{dt}w)^\top D(\frac{d}{dt}w)$$

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Conclude stability if e.g.

$$K = K^\top \geq 0, M = M^\top \geq 0, D + D^\top \geq 0$$

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asymptotic stability if e.g.

$$K = K^\top > 0, M = M^\top > 0, D + D^\top > 0$$

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The **dissipation equals**

$$\frac{d}{dt}E(w, \frac{d}{dt}w) \overset{\mathfrak{B}}{=} -(\frac{d}{dt}w)^\top D(\frac{d}{dt}w)$$

1. No need to put the system in **state** form.
2. Draw conclusions directly from **polynomial matrix calculus**.

An example

Consider the mechanical system

$$Kw + D\frac{d}{dt}w + M\frac{d^2}{dt^2}w = 0 \qquad \rightsquigarrow \quad R(\xi) = K + D\xi + M\xi^2.$$

The stored energy equals

$$E(w, \frac{d}{dt}w) = \frac{1}{2}w^\top Kw + \frac{1}{2}(\frac{d}{dt}w)^\top M(\frac{d}{dt}w) \qquad \rightsquigarrow \quad \frac{1}{2}M + \frac{1}{2}K\zeta\eta$$

The dissipation equals

$$\frac{d}{dt}E(w, \frac{d}{dt}w) \overset{\mathfrak{B}}{=} -(\frac{d}{dt}w)^\top D(\frac{d}{dt}w) \qquad \rightsquigarrow \quad \frac{1}{2}(D + D^\top)\zeta\eta$$

Which $R(\xi), V(\zeta, \eta), \dot{V}_{\mathfrak{B}}(\zeta, \eta)$ lead to stability?

Lyapunov theorem

Given: $\mathfrak{B}$, $\mathtt{w}$ variables, autonomous. Is $\mathfrak{B}$ stable?

¿ $\lim_{t\to\infty} w(t) = 0$ for all $w \in \mathfrak{B}$?

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Given: $\mathfrak{B}$, $\mathtt{w}$ variables, autonomous. Is $\mathfrak{B}$ stable?

¿ $\lim_{t\to\infty} w(t) = 0$ for all $w \in \mathfrak{B}$?

Theorem: $\mathfrak{B}$ is stable $\Leftrightarrow$ there exists

$\Phi \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ such that

$$Q_\Phi \overset{\mathfrak{B}}{\geq} 0 \qquad \text{and} \qquad Q_{\overset{\bullet}{\Phi}} \overset{\mathfrak{B}}{<} 0.$$

Recall $$\overset{\bullet}{\Phi}(\zeta,\eta) := (\zeta+\eta)\Phi(\zeta,\eta)$$

The general theory teaches us how to verify $\mathfrak{B}$-positivity.

Construction of Lyapunov functions

Recall the construction for first order representations

$$\frac{d}{dt}x = Ax, \qquad A \text{ Hurwitz.}$$

Take $Q = Q^\top < 0$ and solve the Lyapunov eq'n

$$A^\top P + PA = Q$$

for $P = P^\top > 0$.

Lyapunov function is $x^\top P x$, its derivative is $x^\top Q x$.

This completely generalizes to high order differential equations.

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- Choose $\Psi \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$
- Solve the *polynomial Lyapunov equation* in $X \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi]$

$$R(-\xi)^\top X(\xi) + X(-\xi)^\top R(\xi) = \Psi(-\xi,\xi)$$

- Define $\Phi(\zeta,\eta) = \frac{\Psi(\zeta,\eta) - R(\zeta)^\top X(\eta) - X(\zeta)^\top R(\eta)}{\zeta+\eta}$

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Then $\quad Q_{\overset{\bullet}{\Phi}} \overset{\mathfrak{B}}{=} Q_\Psi \quad$ **i.e.** $\quad \frac{d}{dt} Q_\Phi \overset{\mathfrak{B}}{=} Q_\Psi$

and $\quad Q_\Phi \overset{\mathfrak{B}}{>} 0 \quad$ **if** $\quad \mathfrak{B}$ **is stable and** $Q_\Psi \overset{\mathfrak{B}}{<} 0$

Example

$\mathfrak{B} \;\cong\; w + \frac{d}{dt}w + \frac{d^2}{dt^2}w = 0 \;\rightsquigarrow R(\xi) = 1 + \xi + \xi^2$

$\Psi(\zeta,\eta) = -2\zeta\eta, \leq 0 : Q_\Psi(w) = -2(\frac{d}{dt}w)^2$; negative on $\mathfrak{B}$.

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The *polynomial Lyapunov equation* becomes

$$(x_0 - x_1\xi)(1 + \xi + \xi^2) + (1 - \xi + \xi^2)(x_0 + x_1\xi) = -2\xi^2$$

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$$\Phi(\zeta,\eta) = \frac{x(\zeta)r(\eta) + r(\zeta)x(\eta)}{\zeta + \eta} = \frac{\zeta(1 + \eta + \eta^2) + \eta(1 + \zeta + \zeta^2)}{\zeta + \eta} = 1 + \zeta\eta$$

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$\rightsquigarrow$ L.f. $Q_\Phi(w) = w^2 + (\frac{d}{dt}w)^2$, derivative: $\frac{d}{dt}Q_\Phi = Q_\Psi(w) = -2(\frac{d}{dt}w)^2$.

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This construction theorem leads to Lyapunov proofs of

the Hurwitz criterion, and the Kharitonov theorem.

Dissipative systems

both the supply rate and the storage function in linear system theory lead to QDF's.

Dissipative systems

Definition: $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ is said to be dissipative

w.r.t. the supply rate Q_Φ with storage function Q_Ψ if the

dissipation inequality

$$Q_{\overset{\bullet}{\Psi}}(\ell) = \frac{d}{dt} Q_\Psi(\ell) \leq Q_\Phi(w)$$

holds for all $(w, \ell) \in \mathfrak{B}_{\text{full}}$, a latent variable repr. of $\mathfrak{B}$.

If equality holds: 'conservative'.

Dissipative systems

If the storage function acts on w, i.e., if

$$Q_{\overset{\bullet}{\Psi}}(w) = \frac{d}{dt} Q_{\Psi}(w) \leq Q_{\Phi}(w)$$

for all $w \in \mathfrak{B}$, then we call the storage function observable.

We consider only observable storage functions and dissipation rates.

Dissipative systems

$$Q_{\overset{\bullet}{\Psi}}(w) - Q_{\Phi}(w) = -||D(\tfrac{d}{dt})(w)||^2$$

Defines the **dissipation rate** D.

Dissipative systems

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Defines the dissipation rate D.

Central problem: Given R and Φ, construct $\Psi \leftrightarrow D$.

Existence of storage f'ns

Theorem: Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$, controllable, Q_Φ a QDF, the supply rate.

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$$\int_{-\infty}^{+\infty} Q_\Phi(w)\, dt \geq 0$$

for all $w \in \mathfrak{B}$ of compact support.

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$$Q_{\overset{\bullet}{\Psi}} \overset{\mathfrak{B}}{\leq} Q_\Phi$$

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2. **Dissipativity**

3. **Dissipativity** : $\exists \;\; \Psi, D$ such that

$$Q_{\overset{\bullet}{\Psi}}(\zeta,\eta) \overset{\mathfrak{B}}{=} Q_\Phi(\zeta,\eta) + D^\top(\zeta)D(\eta)$$

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2. **Dissipativity**

3. $$M^\top(-i\omega)\Phi(-i\omega,\omega)M(i\omega) \geq 0$$

   for all $\omega \in \mathbb{R}$, with $w = M(\frac{d}{dt})\ell$ any image repr. of $\mathfrak{B}$.

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4. **Dissipation function** : $\exists\ F$ such that

$$M^\top(-\xi)\Phi(-\xi,\xi)M(\xi) = F^\top(-\xi)F(\xi)$$

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4. Dissipation function
5. Other representations, adapted conditions ...

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Non-negative storage f'ns

Theorem: Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$, controllable, Q_Φ a QDF, the supply rate.

The following are equivalent:

1. 'half-line dissipativity'

$$\int_{-\infty}^{0} Q_\Phi(w)\, dt \geq 0$$

for all $w \in \mathfrak{B}$ of compact support.

Non-negative storage f'ns

Theorem: Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$, **controllable**, Q_Φ a QDF, the supply rate.

The following are equivalent:

1. 'half-line dissipativity'
2. Dissipativity with a non-negative storage function

   $\exists\ \Psi$ such that

$$Q_\Psi \overset{\mathfrak{B}}{\geq} 0 \quad \text{and} \quad Q_{\overset{\bullet}{\Psi}} \overset{\mathfrak{B}}{\leq} Q_\Phi .$$

Non-negative storage f'ns

Theorem: Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$, controllable, Q_Φ a QDF, the supply rate.

The following are equivalent:

1. 'half-line dissipativity'
2. Dissipativity with a non-negative storage function
3. A Pick matrix condition on $M^\top(-\xi)\Phi(-\xi,\xi)M(\xi)$ with $w = M(\frac{d}{dt})\ell$ any image representation of $\mathfrak{B}$.

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Theorem: Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$, controllable, Q_Φ a QDF, the supply rate. The following are equivalent:

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Storage functions

Remarks:

1. If there exists a storage function, there exists one that is a QDF.

Every observable storage f'n is a memoryless state f'n!

Storage functions

Algorithmic issues.

2. The set of observable storage functions is

convex, compact, and attains its maximum and minimum:

$$Q_{\Psi_{\text{available}}} \overset{\mathfrak{B}}{\leq} Q_{\Psi} \overset{\mathfrak{B}}{\leq} Q_{\Psi_{\text{required}}}$$

$$Q_{\Psi_{\text{available}}}(w)(0) := \text{supremum}\{-\int_0^{\infty} Q_{\Phi}(\hat{w})\,dt\}$$

$$Q_{\Psi_{\text{required}}}(w)(0) := \text{infimum}\{\int_{-\infty}^{0} Q_{\Phi}(\hat{w})\,dt\}$$

with the sup and inf over all $\hat{w}$ such that the concatenations,

$$\hat{w} \wedge_0 w,\ w \wedge_0 \hat{w} \in \mathfrak{B}.$$

Storage functions

Algorithmic issues.

3. The condition: Given $R(\frac{d}{dt})w = 0$ and Φ, ¿ $\exists$ Ψ such that

$$Q_{\overset{\bullet}{\Psi}} \overset{\mathfrak{B}}{\leq} Q_\Phi$$

is actually an **LMI**.

Most easily seen by going to image representation:

$\cong$ given Φ ¿ $\exists$ Ψ such that

$$(\zeta + \eta)\Psi(\zeta, \eta) \leq \Phi(\zeta, \eta).$$

Obviously an LMI in the coefficients of Ψ.

Storage functions

Algorithmic issues.

4. We can also compute the dissipation rate first: Given Φ,

¿ $\exists\, \Delta$ such that

$$\Delta + \Delta^\top \geq 0 \quad \text{and} \quad \Phi(-\xi, \xi) = \begin{bmatrix} I \\ -I\xi \\ \vdots \\ (-1)^{\mathrm{n}}\xi^{\mathrm{n}} \end{bmatrix}^\top \Delta \begin{bmatrix} I \\ I\xi \\ \vdots \\ \xi^{\mathrm{n}} \end{bmatrix}$$

Obviously an LMI in the coefficients of Δ.

there is much more ...

... many more applications, many more to be expected from various areas:

- **B/QDF's for distributed systems (Pillai e.a) ;**
- **SOS (Parrilo)**
- **Representation-free H_∞ control- and filtering (Trentelman, Belur)**
- **LQ-control for higher-order systems (Valcher)**
- **Balancing and model reduction**
- **Bilinear- and quadratic *difference* forms (*discrete-time*) (Fujii & Kaneko)**

Conclusion

State systems $\Leftrightarrow$ quadratic functionals

High order linear differential eq'ns $\Leftrightarrow$ QDF's

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High order linear differential eq'ns $\Leftrightarrow$ QDF's

Stay with the original, parsimonious, model

No need to put things in state form...

Thank you

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