HIDDEN VARIABLES
in
DISSIPATIVE SYSTEMS

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Theme

Theme

Is the storage function of a dissipative system (necessarily, or without loss of generality) an ‘observable’ function of the external system variables?

Not really!

Preliminaries

The *behavior* $\mathfrak{B}$ belongs to $\mathfrak{L}^{\mathtt{w}}$ $:\Leftrightarrow$
$\exists$ a polynomial matrix $R \in \mathbb{R}^{\bullet \times \mathtt{w}}[\xi]$ such that

$$\mathfrak{B} = \{w \in \mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \mid R(\tfrac{d}{dt})w = 0\}$$

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$\mathfrak{B}$ is controllable :⇔
the trajectories are 'patch-able', 'concatenable'.

⇔ $R(\lambda)$ has the same rank for all $\lambda \in \mathbb{C}$.

LTI systems, QDF's

The quadratic map acting on $w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$,

$$w \mapsto \sum_{k,\ell} (\frac{d^k}{dt^k} w)^\top \Phi_{k,\ell} (\frac{d^\ell}{dt^\ell} w)$$

is called a *quadratic differential form* (QDF).

$\Phi_{k,\ell} \in \mathbb{R}^{\mathtt{w} \times \mathtt{w}}$; WLOG: $\Phi_{k,\ell} = \Phi_{\ell,k}^\top$.

Introduce the 2-variable polynomial matrix Φ

$$\Phi(\zeta, \eta) = \sum_{k,\ell} \Phi_{k,\ell} \zeta^k \eta^\ell .$$

Denote the QDF as Q_Φ.

Latent variables, observability

More often that not, a **behavior** is specified through **auxiliary** variables

state variables, interconnection variables, potentials, image representations, internal variables, ...

$\leadsto$ the model

$$R(\frac{d}{dt})w = M(\frac{d}{dt})\ell$$

ℓ's: 'latent', w's: 'manifest'

Latent variables, observability

$$\mathfrak{B} = \{w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \mid \exists\, \ell \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\dim(\ell)})$$

$$\text{such that} \quad R(\frac{d}{dt})w = M(\frac{d}{dt})\ell\}$$

actually belongs to $\mathfrak{L}^{\mathtt{w}}$.

$\rightsquigarrow$ many possible representations of $\mathfrak{B}$

Latent variables, observability

We call these latent variables ***observable*** if

$$(w, \ell_1), (w, \ell_2) \;\Rightarrow\; \ell_1 = \ell_2$$

i.e. $\exists$ a representation

$$R'(\tfrac{d}{dt})w = 0, \quad \ell = M'(\tfrac{d}{dt})w$$

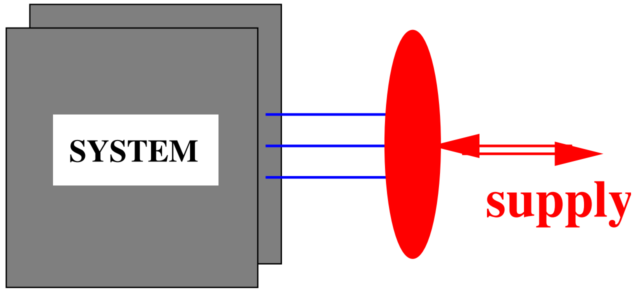
for

$$R(\frac{d}{dt})w = M(\frac{d}{dt})\ell$$

Dissipative systems

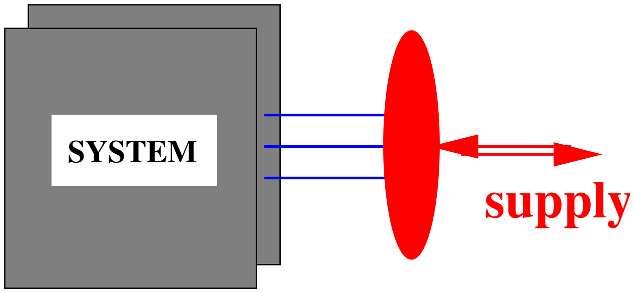
Dissipative systems

Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ be a behavior and
Q_Φ be a QDF, with $Q_\Phi(w)$ called the supply rate



¿¿ When do we want to call $\mathfrak{B}$ dissipative w.r.t. Q_Φ ??

Dissipative systems

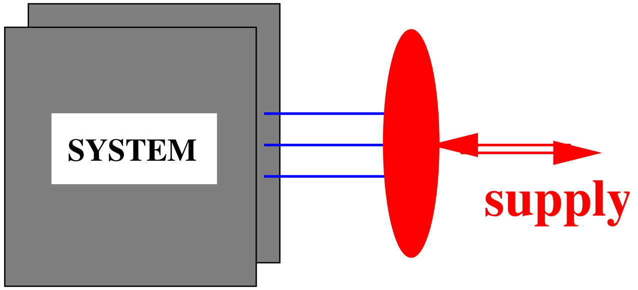


Classical choices:

$$\exists\, V \text{ such that } \quad \frac{d}{dt}V(x) \leq s = Q_\Phi(w)$$

$$\frac{d}{dt}Q_\Psi(w) \leq s = Q_\Phi(w) \quad \forall w \in \mathfrak{B}$$

Dissipative systems



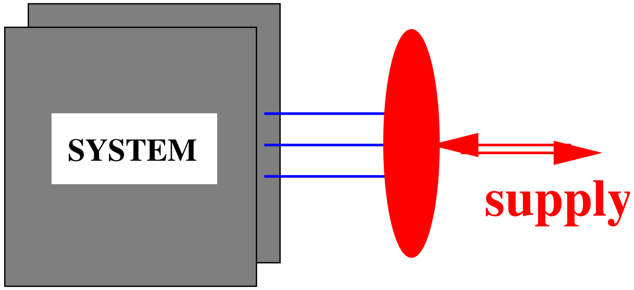
$\mathfrak{B}$ is **dissipative** w.r.t. Q_Φ $:\Leftrightarrow$

$\exists$ a latent variable repr'ion $R(\frac{d}{dt})w = M(\frac{d}{dt})\ell$
and a two-variable polynomial matrix

Ψ such that $\frac{d}{dt}Q_\Psi(\ell) \leq Q_\Phi(w)$

Q_Ψ $=:$ the ***storage function***, may depend on ℓ!

Dissipative systems



Note: Observability $\Rightarrow \ \exists$

$$\Psi' \text{ such that } \frac{d}{dt} Q_{\Psi'}(w) \ \leq \ Q_{\Phi}(w) \quad \forall\, w \in \mathfrak{B}$$

Call such a storage function **observable.**

¿¿ Does it suffice to look at *observable* storage f'ns ??

Controllable systems have observable storage f'ns

Dissipativity and Controllability

Assume $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$, controllable. Equivalent:

- **Dissipativity:**

  $\exists\ \Psi$ such that $\frac{d}{dt} Q_\Psi(\ell) \leq Q_\Phi(w)$

- **with an observable storage f'n:**

  $\exists\ \Psi'$ such that $\frac{d}{dt} Q_{\Psi'}(w) \leq Q_\Phi(w)$

- **Global dissipativity:**

  $\int_{-\infty}^{+\infty} Q_\Phi(w)\, dt \geq 0$

  for all $w \in \mathfrak{B}$ of compact support

- $\cdots$

General problem

Assume $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$, not controllable, and a QDF Q_Φ.

N&SC's for dissipativity are not known!

Is this a relevant problem?

Example

Consider $\frac{d}{dt}x = Ax, w_1 = Cx,\ \ w_2$ free,

$Q_\Phi(w_1, w_2) = w_1^\top w_2.$ Dissipative?

Non-observable state representation:

$$\frac{d}{dt}x = Ax, w_1 = Cx, \frac{d}{dt}z = -A^\top z + Cw_2.$$

Then

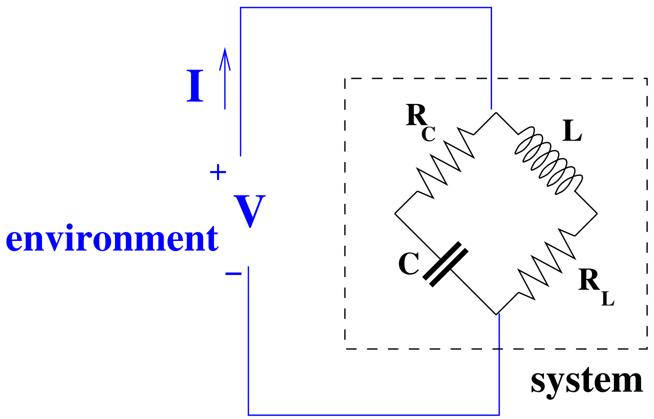
$$\frac{d}{dt}x^\top z = w_1^\top w_2$$

$\rightsquigarrow$ dissipativeness.

But there does not exist an observable storage f'n.

Relevance in circuit theory

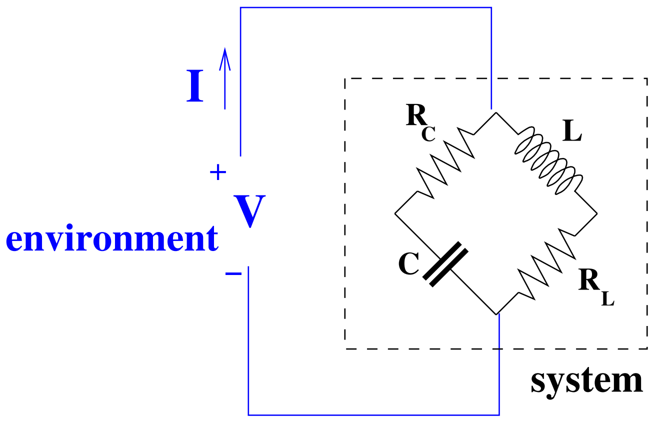
Relation with electrical circuit synthesis



Latent variable repr'ion for the port behavior (V, I):

$$
\begin{aligned}
R_L I_L + L \frac{d}{dt} I_L &= V, \\
V_C + R_C C \frac{d}{dt} V_C &= V, \\
\frac{V - V_C}{R_C} + I_L &= I.
\end{aligned}
$$

Relation with electrical circuit synthesis

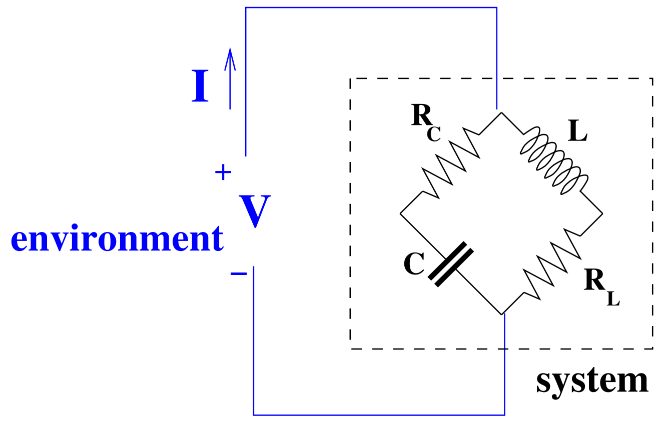


After elimination, we obtain, in the case
$R_L = 1, R_C = 1, L = 1, C = 1,$

$$(1 + \frac{d}{dt})V = (1 + \frac{d}{dt})I.$$

Not controllable, but really real.

Relation with electrical circuit synthesis

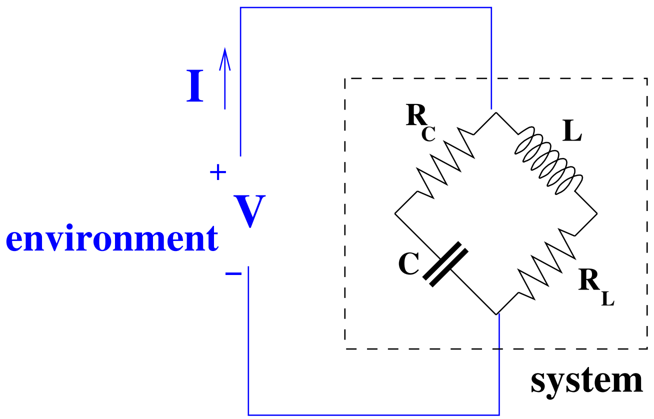


Storage f'n: the internally stored energy,

$$\frac{1}{2}C{V_C}^2 + \frac{1}{2}L{I_L}^2$$

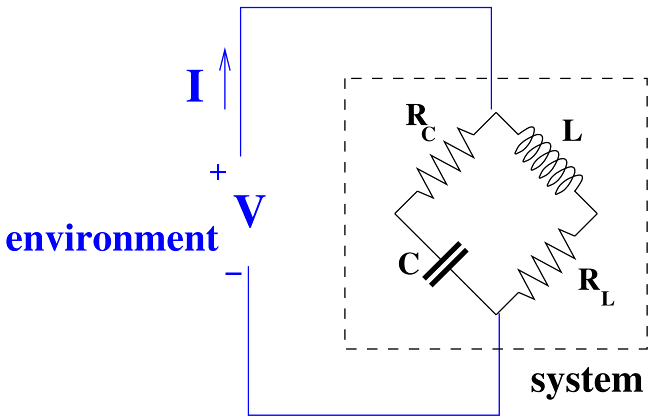
When $R_L = 1, R_C = 1, L = 1, C = 1,$ **not observable.**

Relation with electrical circuit synthesis



There are first order state repr. with an observable storage function. ⇒ can be synthesized with one capacitor, positive resistors, gyrators, and transformers.

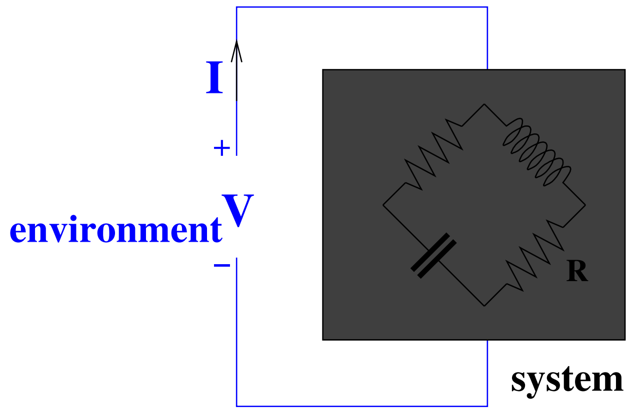
Relation with electrical circuit synthesis



However, no reciprocal first order realizations (without gyrators) that use *only one* reactive element (a capacitor or an inductor), positive resistors, and transformers.

i.e. no ‘observable’ reciprocal realizations.

Relation with electrical circuit synthesis



Problem 1:

Which behaviors $\mathfrak{B}$ are realizable as the port behavior of a circuit containing a finite number of passive resistors, capacitors, inductors, (gyrators,) and transformers?

Relation with electrical circuit synthesis

Which behaviors $\mathfrak{B}$ are realizable as the port behavior of a circuit containing a finite number of passive resistors, capacitors, inductors, (gyrators,) and transformers?

Necessary: $\mathfrak{B} \in \mathfrak{L}^2$, single input / single output, and transfer f'n positive real.

In the controllable case, this is Brune's well-known 1931 n.a.s.c. (driving point impedance synthesis).

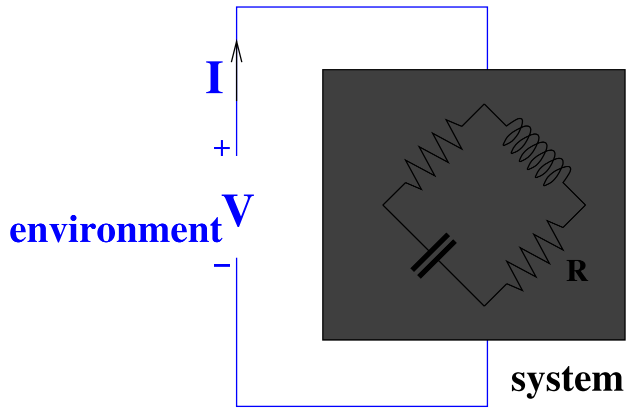
Relation with electrical circuit synthesis

Which behaviors $\mathfrak{B}$ are realizable as the port behavior of a circuit containing a finite number of passive resistors, capacitors, inductors, (gyrators,) and transformers?

Necessary: $\mathfrak{B} \in \mathfrak{L}^2$, **single input / single output,**
and transfer f'n positive real.

$\Leftrightarrow$ $\mathfrak{B}$ **is dissipative w.r.t.** VI**, with a** ≥ 0 **storage function (but in general unobservable), and therefore it is not clear at the moment what this says in terms of** $\mathfrak{B}$ **when** $\mathfrak{B}$ **is not controllable.**

Relation with electrical circuit synthesis



Problem 2:

Is it possible to realize a controllable single input / single output system with a rational, p.r. transfer f'n by means of a finite number of passive resistors, capacitors, and inductors, but without transformers?

Relation with electrical circuit synthesis

Is it possible to realize a controllable single input / single output system with a rational, p.r. transfer f'n by means of a finite number of passive resistors, capacitors, and inductors, but without transformers?

The Bott-Duffin (1949) synthesis procedure realizes a non-controllable system that has the correct impedance (i.e., the correct controllable part), but not necessarily the correct behavior. There are standard synthesis procedures known that do realize the correct behavior, but they need transformers.

Summary

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- We need non-observable storage f'ns in the context of non-controllable systems
- For PDE's (e.g. Maxwell's eq's) we need non-obs. storage f'ns even for controllable systems
- Several open problems:
  When is a non-controllable system dissipative?
  What non-controllable systems are realizable?
  Faithful transformerless synthesis of controllable systems
- Relevance of non-controllable, non-observable systems for physical systems theory

Reference

JCW, Hidden variables in dissipative systems
Proceedings CD rom

The slides of this presentation and the manuscript are available from my webpage

`http://www.esat.kuleuven.ac.be/∼jwillems`

Thank you

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