DISSIPATIVE DISTRIBUTED SYSTEMS

Jan C. Willems

K.U. Leuven, Belgium

University of California at Santa Barbara **December 4, 2003**

A dissipative system absorbs supply, 'globally', over time + space.

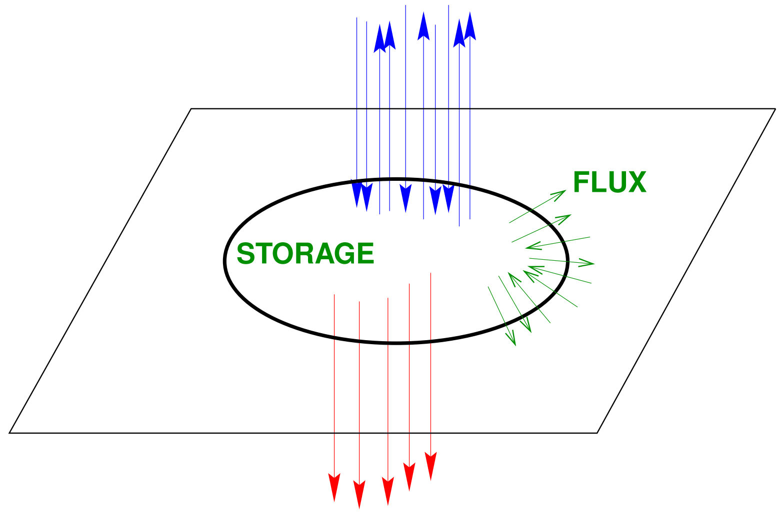
¿¿ Can this be expressed 'locally', as

rate of change in storage + spatial flux $\leq$ supply rate

A dissipative system absorbs supply, 'globally', over time + space.

¿¿ Can this be expressed 'locally', as

rate of change in storage + spatial flux $\leq$ supply rate



rate of change in storage + spatial flux

= supply rate + (non-negative) dissipation rate ??

OUTLINE

- **Lyapunov theory and dissipative dynamical systems**

- **Linear differential systems**
- **Dissipative distributed systems**
- **Local dissipation law**
- **The factorization equation**

LYAPUNOV FUNCTIONS

Consider the classical dynamical system, the *'flow'*

$$\Sigma : \quad \frac{d}{dt}x = f(x)$$

with $x \in \mathbb{X} = \mathbb{R}^{\mathrm{n}}$, the *state space,* and $f : \mathbb{X} \rightarrow \mathbb{X}$.

Denote the set of solutions $x : \mathbb{R} \rightarrow \mathbb{X}$ by $\mathfrak{B}$, the *'behavior'.*

LYAPUNOV FUNCTIONS

Consider the classical dynamical system, the *'flow'*

$$\Sigma : \quad \frac{d}{dt}x = f(x)$$

with $x \in \mathbb{X} = \mathbb{R}^{\mathrm{n}}$, the *state space,* and $f : \mathbb{X} \to \mathbb{X}$.

Denote the set of solutions $x : \mathbb{R} \to \mathbb{X}$ by $\mathfrak{B}$, the *'behavior'.*

$$V : \mathbb{X} \to \mathbb{R}$$

is said to be a Lyapunov function for Σ if along $x \in \mathfrak{B}$

$$\frac{d}{dt} V(x(\cdot)) \leq 0$$

LYAPUNOV FUNCTIONS

Consider the classical dynamical system, the ***'flow'***

$$\Sigma : \quad \frac{d}{dt}x = f(x)$$

with $x \in \mathbb{X} = \mathbb{R}^{\mathrm{n}}$, **the** ***state space,*** **and** $f : \mathbb{X} \to \mathbb{X}$.

Denote the set of solutions $x : \mathbb{R} \to \mathbb{X}$ **by** $\mathfrak{B}$, **the** ***'behavior'.***

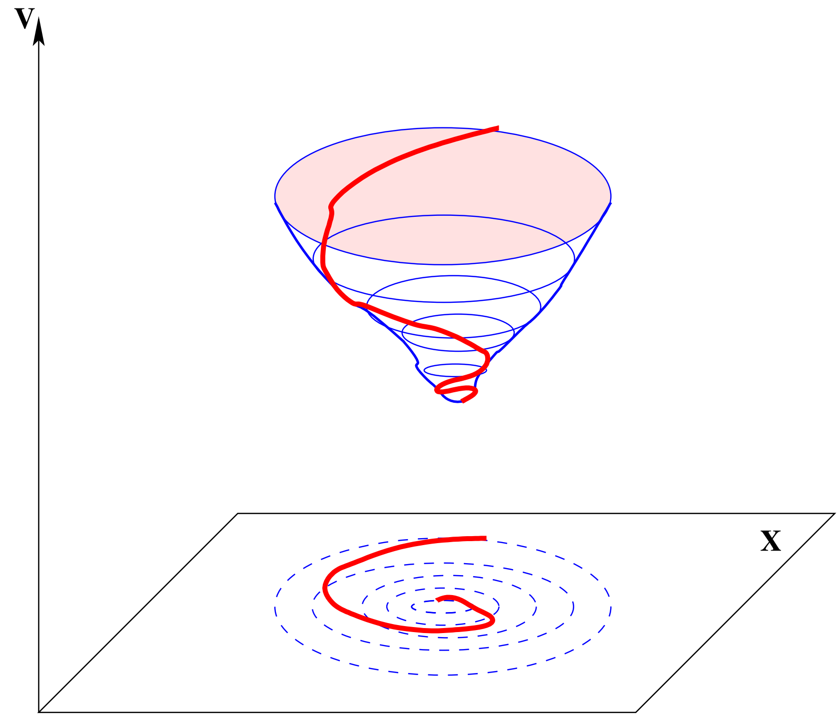
$$V : \mathbb{X} \to \mathbb{R}$$

is said to be a <u>Lyapunov function</u> **for** Σ **if along** $x \in \mathfrak{B}$

$$\frac{d}{dt} V(x(\cdot)) \leq 0$$

Equivalently, if $\overset{\bullet}{V}^{\Sigma} := \nabla V \cdot f \leq 0.$

Typical Lyapunov 'theorem':



$$V(x) > 0 \text{ and } \overset{\bullet}{V}^{\Sigma}(x) < 0 \text{ for } 0 \neq x \in \mathbb{X}$$

$$\Rightarrow$$

$\forall\ x \in \mathfrak{B},$ there holds $x(t) \to 0$ for $t \to \infty$ **'global stability'**

Lyapunov functions play a remarkably central role in the field.

Lyapunov functions play a remarkably central role in the field.

Aleksandr Mikhailovich Lyapunov (1857-1918)

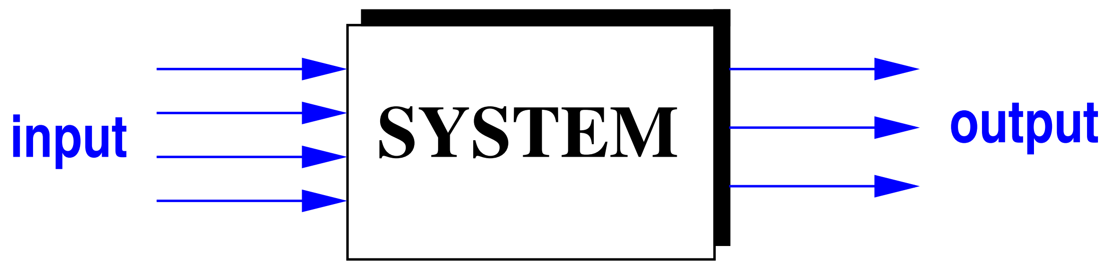
Introduced Lyapunov's 'second method' in his Ph.D. thesis (1899).

OPEN SYSTEMS

‘Open’ systems are a much more appropriate starting point for the study of dynamics.

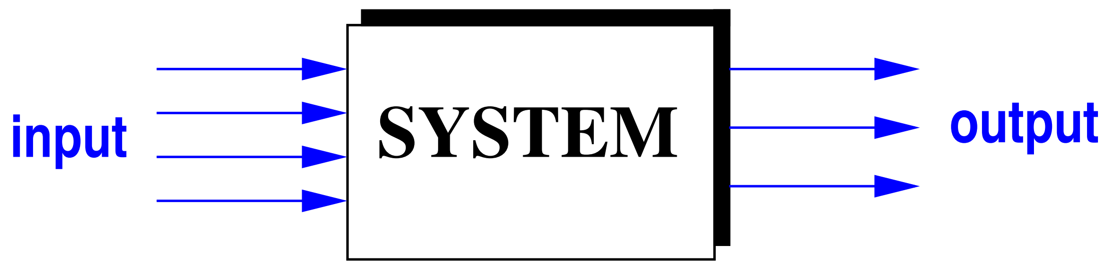
OPEN SYSTEMS

'Open' systems are a much more appropriate starting point for the study of dynamics. For example,



OPEN SYSTEMS

'Open' systems are a much more appropriate starting point for the study of dynamics. For example,



$\rightsquigarrow$ the **'dynamical system'**

$$\Sigma : \quad \frac{d}{dt}x = f(x,u), \quad y = h(x,u).$$

$u \in \mathbb{U} = \mathbb{R}^{\mathtt{m}}, y \in \mathbb{Y} = \mathbb{R}^{\mathtt{p}}, x \in \mathbb{X} = \mathbb{R}^{\mathtt{n}}$: **input, output, state**.

Behavior $\mathfrak{B} =$ all sol'ns $(u, y, x) : \mathbb{R} \to \mathbb{U} \times \mathbb{Y} \times \mathbb{X}$.

DISSIPATIVE DYNAMICAL SYSTEMS

Let $\boxed{s : \mathbb{U} \times \mathbb{Y} \rightarrow \mathbb{R}}$ **be a function, called the *supply rate*.**

Σ **is said to be *dissipative* w.r.t. the supply rate s if $\exists$**

$$\boxed{V : \mathbb{X} \rightarrow \mathbb{R},}$$

called the *storage function*, such that

$$\boxed{\boxed{\frac{d}{dt} V(x(\cdot)) \leq s(u(\cdot), y(\cdot))}}$$

along input/output/state trajectories ($\forall$ $(u(\cdot), y(\cdot), x(\cdot)) \in \mathfrak{B}$).

DISSIPATIVE DYNAMICAL SYSTEMS

Let $s : \mathbb{U} \times \mathbb{Y} \to \mathbb{R}$ **be a function, called the *supply rate*.**

Σ is said to be *dissipative* w.r.t. the supply rate s if $\exists$

$$V : \mathbb{X} \to \mathbb{R},$$

called the *storage function*, such that

$$\frac{d}{dt} V(x(\cdot)) \leq s(u(\cdot), y(\cdot))$$

along input/output/state trajectories ($\forall$ $(u(\cdot), y(\cdot), x(\cdot)) \in \mathfrak{B}$).

This inequality is called the *dissipation inequality.*

Equivalent to

$$\overset{\bullet}{V}^{\Sigma}(x,u) := \nabla V(x) \cdot f(x,u) \leq s(u, h(x,u))$$

for all $(u,x) \in \mathbb{U} \times \mathbb{X}$.

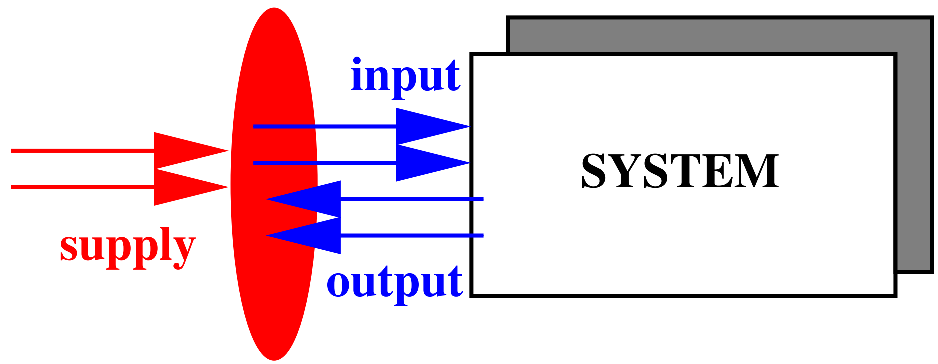
Equivalent to

$$\overset{\bullet}{V}^{\Sigma}(x,u) := \nabla V(x) \cdot f(x,u) \leq s(u, h(x,u))$$

for all $(u,x) \in \mathbb{U} \times \mathbb{X}$.

If equality holds: ‘conservative’ system.

$s(u, y)$ models something like the power delivered to the system when the input value is u and output value is y.



$V(x)$ then models the internally stored energy.

Dissipativity $:\Leftrightarrow$

rate of increase of internal energy $\leq$ power delivered.

Special case: 'closed' system: $s = 0$ then

dissipativeness $\leftrightarrow$ V is a Lyapunov function.

Dissipativity is the natural generalization to open systems of Lyapunov theory.

Special case: 'closed' system: $s = 0$ then

dissipativeness $\leftrightarrow$ V is a Lyapunov function.

Dissipativity is the natural generalization to open systems of Lyapunov theory.

Stability for closed systems $\simeq$ Dissipativity for open systems.

THE CONSTRUCTION OF STORAGE FUNCTIONS

Basic question:

Given (a representation of) Σ, the dynamics,

and given s, the supply rate,

is the system dissipative w.r.t. s, i.e.,

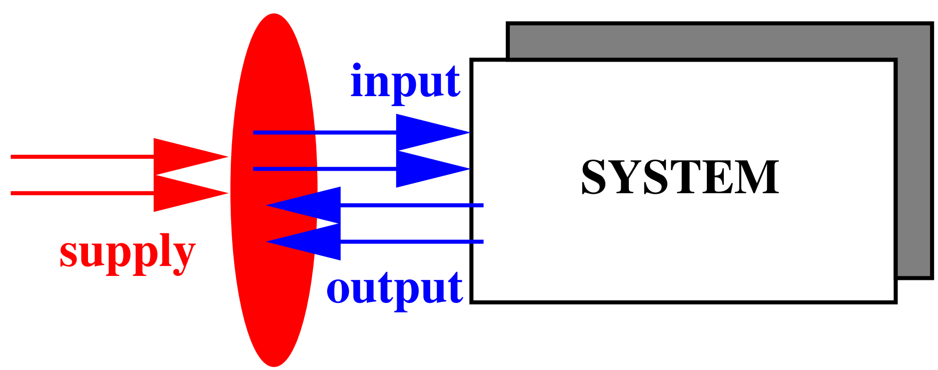
does there exist a storage function V such that

the dissipation inequality holds?

THE CONSTRUCTION OF STORAGE FUNCTIONS

Basic question:

Given (a representation of) Σ, the dynamics,
and given s, the supply rate,
is the system dissipative w.r.t. s, i.e.,
does there exist a storage function V such that
the dissipation inequality holds?



Assume s 'power', known dynamics, what is the internal stored energy?

The construction of storage f'ns is very well understood, particularly for finite dimensional linear systems and quadratic supply rates.

The construction of storage f'ns is very well understood, particularly for finite dimensional linear systems and quadratic supply rates.

Leads to the KYP-lemma, LMI's, ARIneq, ARE, semi-definite programming, spectral factorization, Lyapunov functions, $\mathcal{H}_\infty$ and robust control, positive and bounded real functions, electrical circuit synthesis, stochastic realization theory.

The construction of storage f'ns is very well understood, particularly for finite dimensional linear systems and quadratic supply rates.

Leads to the KYP-lemma, LMI's, ARIneq, ARE, semi-definite programming, spectral factorization, Lyapunov functions, $\mathcal{H}_\infty$ and robust control, positive and bounded real functions, electrical circuit synthesis, stochastic realization theory.

The storage function V is in general far from unique. There are two 'canonical' storage functions: the available storage and the required supply.

For conservative systems, V is unique.

Dissipative systems and storage f’ns play a remarkably central role in the field.

Dissipative systems and storage f'ns play a remarkably central role in the field.

The construction of storage functions
is the question which we shall discuss today
for systems described by PDE's.

OUTLINE

- **Lyapunov theory and dissipative dynamical systems**
- **Linear differential systems: Systems described by liner constant coefficient PDE's**
- **Dissipative distributed systems**
- **Local dissipation law**
- **The factorization equation**

PDE's: polynomial notation

Consider, for example, the PDE:

$$\begin{aligned} w_1(x_1,x_2) + \frac{\partial^2}{\partial x_2^2} w_1(x_1,x_2) + \frac{\partial}{\partial x_1} w_2(x_1,x_2) &= 0 \\ w_2(x_1,x_2) + \frac{\partial^3}{\partial x_2^3} w_1(x_1,x_2) + \frac{\partial^4}{\partial x_1^4} w_2(x_1,x_2) &= 0 \end{aligned}$$

PDE's: polynomial notation

Consider, for example, the PDE:

$$\begin{aligned} w_1(x_1,x_2) + \frac{\partial^2}{\partial x_2^2} w_1(x_1,x_2) + \frac{\partial}{\partial x_1} w_2(x_1,x_2) &= 0 \\ w_2(x_1,x_2) + \frac{\partial^3}{\partial x_2^3} w_1(x_1,x_2) + \frac{\partial^4}{\partial x_1^4} w_2(x_1,x_2) &= 0 \end{aligned}$$

$$\Updownarrow$$

Notation:

$$\xi_1 \leftrightarrow \frac{\partial}{\partial x_1,}\ \xi_2 \leftrightarrow \frac{\partial}{\partial x_2}, w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}, \quad R(\xi_1,\xi_2) = \begin{bmatrix} 1+\xi_2^2 & \xi_1 \\ \xi_2^3 & 1+\xi_1^4 \end{bmatrix}$$

PDE's: polynomial notation

Consider, for example, the PDE:

$$\begin{aligned} w_1(x_1,x_2) + \frac{\partial^2}{\partial x_2^2} w_1(x_1,x_2) + \frac{\partial}{\partial x_1} w_2(x_1,x_2) &= 0 \\ w_2(x_1,x_2) + \frac{\partial^3}{\partial x_2^3} w_1(x_1,x_2) + \frac{\partial^4}{\partial x_1^4} w_2(x_1,x_2) &= 0 \end{aligned}$$

$$\Updownarrow$$

Notation:

$$\xi_1 \leftrightarrow \frac{\partial}{\partial x_1,}\ \xi_2 \leftrightarrow \frac{\partial}{\partial x_2}, w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix},\ \ R(\xi_1,\xi_2) = \begin{bmatrix} 1+\xi_2^2 & \xi_1 \\ \xi_2^3 & 1+\xi_1^4 \end{bmatrix}$$

$$\boxed{R(\tfrac{\partial}{\partial x_1}, \tfrac{\partial}{\partial x_2})w = 0.}$$

LINEAR DIFFERENTIAL SYSTEMS

$\mathbb{T} = \mathbb{R}^{\mathtt{n}}$, **the set of independent variables, typically** $\mathtt{n} = 4$,
$\mathbb{W} = \mathbb{R}^{\mathtt{w}}$, **the set of dependent variables,**
$\mathfrak{B} =$ **the solutions of a linear constant coefficient system of PDE's.**

LINEAR DIFFERENTIAL SYSTEMS

$\mathbb{T} = \mathbb{R}^{\mathtt{n}}$, **the set of independent variables, typically** $\mathtt{n} = 4$,
$\mathbb{W} = \mathbb{R}^{\mathtt{w}}$, **the set of dependent variables,**
$\mathfrak{B} =$ **the solutions of a linear constant coefficient system of PDE's.**

Let $R \in \mathbb{R}^{\bullet \times \mathtt{w}}[\xi_1, \cdots, \xi_{\mathtt{n}}]$, **and consider**

$$R\left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathtt{n}}}\right) w = 0. \quad (*)$$

Define the associated behavior

$$\mathfrak{B} = \{ w \in \mathfrak{C}^{\infty}(\mathbb{R}^{\mathtt{n}}, \mathbb{R}^{\mathtt{w}}) \mid (*) \text{ holds } \}.$$

LINEAR DIFFERENTIAL SYSTEMS

$\mathbb{T} = \mathbb{R}^{\mathrm{n}}$, the set of independent variables, typically $\mathrm{n} = 4$,
$\mathbb{W} = \mathbb{R}^{\mathtt{w}}$, the set of dependent variables,
$\mathfrak{B} =$ the solutions of a linear constant coefficient system of PDE's.

Let $R \in \mathbb{R}^{\bullet \times \mathtt{w}}[\xi_1, \cdots, \xi_{\mathrm{n}}]$, and consider

$$R(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathrm{n}}})w = 0. \quad (*)$$

Define the associated behavior

$$\mathfrak{B} = \{w \in \mathfrak{C}^{\infty}(\mathbb{R}^{\mathrm{n}}, \mathbb{R}^{\mathtt{w}}) \mid (*) \text{ holds } \}.$$

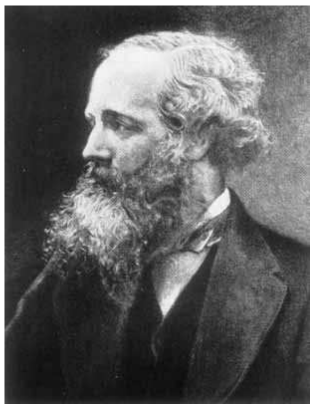
Notation for n-D linear differential systems:

$$(\mathbb{R}^{\mathrm{n}}, \mathbb{R}^{\mathtt{w}}, \mathfrak{B}) \in \mathfrak{L}^{\mathtt{w}}_{\mathrm{n}}, \quad \text{or } \mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\mathrm{n}}.$$

Examples: Maxwell's eq'ns, diffusion eq'n, wave eq'n, . . .

$$
\begin{aligned}
\nabla \cdot \vec{E} &= \frac{1}{\varepsilon_0}\rho\,, \\
\nabla \times \vec{E} &= -\frac{\partial}{\partial t}\vec{B}, \\
\nabla \cdot \vec{B} &= 0\,, \\
c^2 \nabla \times \vec{B} &= \frac{1}{\varepsilon_0}\vec{j} + \frac{\partial}{\partial t}\vec{E}.
\end{aligned}
$$

Examples: Maxwell's eq'ns, diffusion eq'n, wave eq'n, . . .



$$
\begin{aligned}
\nabla \cdot \vec{E} &= \frac{1}{\varepsilon_0}\rho\,, \\
\nabla \times \vec{E} &= -\frac{\partial}{\partial t}\vec{B}, \\
\nabla \cdot \vec{B} &= 0\,, \\
c^2 \nabla \times \vec{B} &= \frac{1}{\varepsilon_0}\vec{j} + \frac{\partial}{\partial t}\vec{E}.
\end{aligned}
$$

$\mathbb{T} = \mathbb{R} \times \mathbb{R}^3$ (time and space) $\mathtt{n} = 4$,

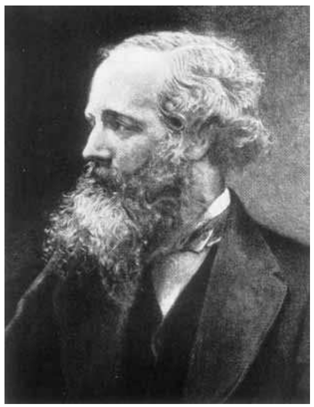
$w = (\vec{E}, \vec{B}, \vec{j}, \rho)$

(electric field, magnetic field, current density, charge density),

$\mathbb{W} = \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}$, $\mathtt{w} = 10$,

$\mathfrak{B}$ = set of solutions to these PDE's.

Examples: Maxwell's eq'ns, diffusion eq'n, wave eq'n, . . .



$$
\begin{aligned}
\nabla \cdot \vec{E} &= \frac{1}{\varepsilon_0}\rho\,, \\
\nabla \times \vec{E} &= -\frac{\partial}{\partial t}\vec{B}, \\
\nabla \cdot \vec{B} &= 0\,, \\
c^2 \nabla \times \vec{B} &= \frac{1}{\varepsilon_0}\vec{j} + \frac{\partial}{\partial t}\vec{E}.
\end{aligned}
$$

$\mathbb{T} = \mathbb{R} \times \mathbb{R}^3$ (time and space) $\mathtt{n} = 4$,

$w = (\vec{E}, \vec{B}, \vec{j}, \rho)$

(electric field, magnetic field, current density, charge density),

$\mathbb{W} = \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}$, $\mathtt{w} = 10$,

$\mathfrak{B}$ = set of solutions to these PDE's.

Note: 10 variables, 8 equations! $\Rightarrow \exists$ free variables.

$$\boxed{R\left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathrm{n}}}\right)w = 0}$$

is called a kernel representation of the associated $\mathfrak{B} \in \mathfrak{L}_{\mathtt{n}}^{\mathtt{w}}$.

$$\boxed{R(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n})w = 0}$$

is called a kernel representation of the associated $\mathfrak{B} \in \mathfrak{L}_n^w$.

Another representation: image representation

$$\boxed{w = M(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n})\ell.}$$

'Elimination' thm $\quad \Rightarrow \quad \mathrm{im}(M(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n})) \in \mathfrak{L}_n^w$ **!**

$$R(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathtt{n}}})w = 0$$

is called a kernel representation of the associated $\mathfrak{B} \in \mathfrak{L}_{\mathtt{n}}^{\mathtt{w}}$.

Another representation: image representation

$$w = M(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathtt{n}}})\ell.$$

'Elimination' thm $\quad \Rightarrow \quad \mathrm{im}(M(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathtt{n}}})) \in \mathfrak{L}_{\mathtt{n}}^{\mathtt{w}}$!

Do all linear differential systems admit an image representation???

$$R(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n})w = 0$$

is called a kernel representation of the associated $\mathfrak{B} \in \mathfrak{L}_n^w$.

Another representation: image representation

$$w = M(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n})\ell.$$

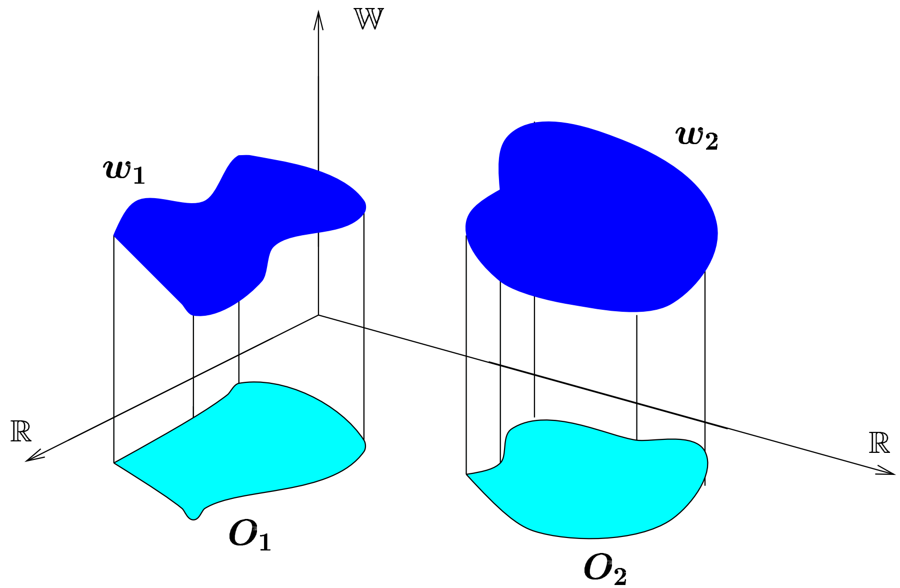
'Elimination' thm $\Rightarrow$ $\mathrm{im}(M(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n})) \in \mathfrak{L}_n^w$!

Do all linear differential systems admit an image representation???

$\mathfrak{B} \in \mathfrak{L}_n^w$ admits an image representation iff it is 'controllable'.

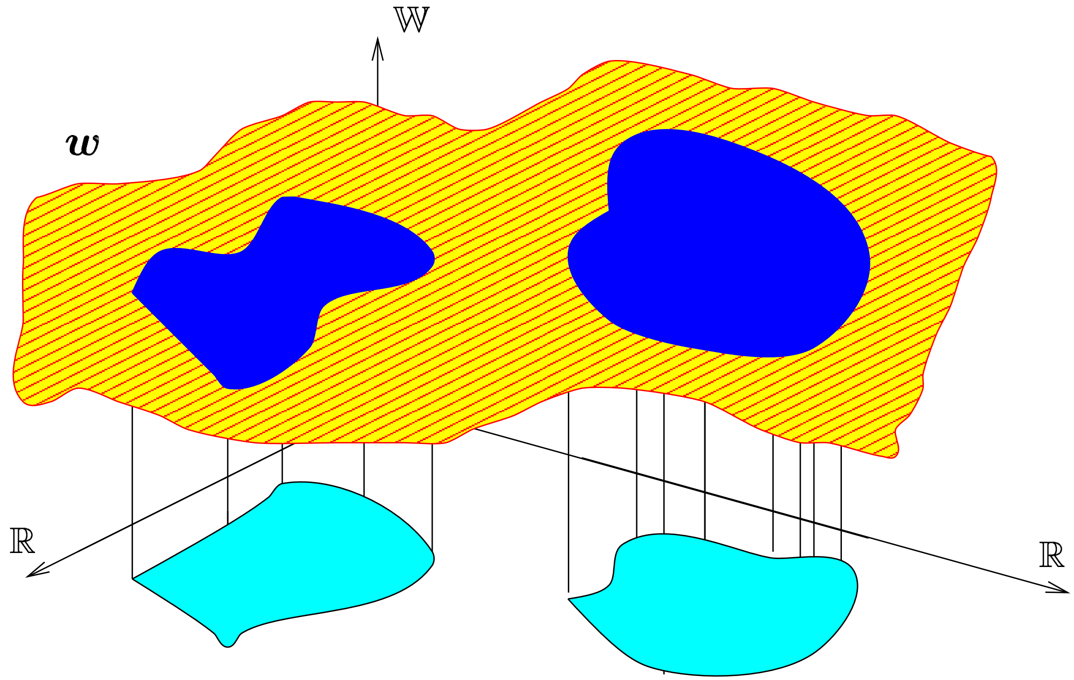
CONTROLLABILITY

Controllability def'n in pictures:

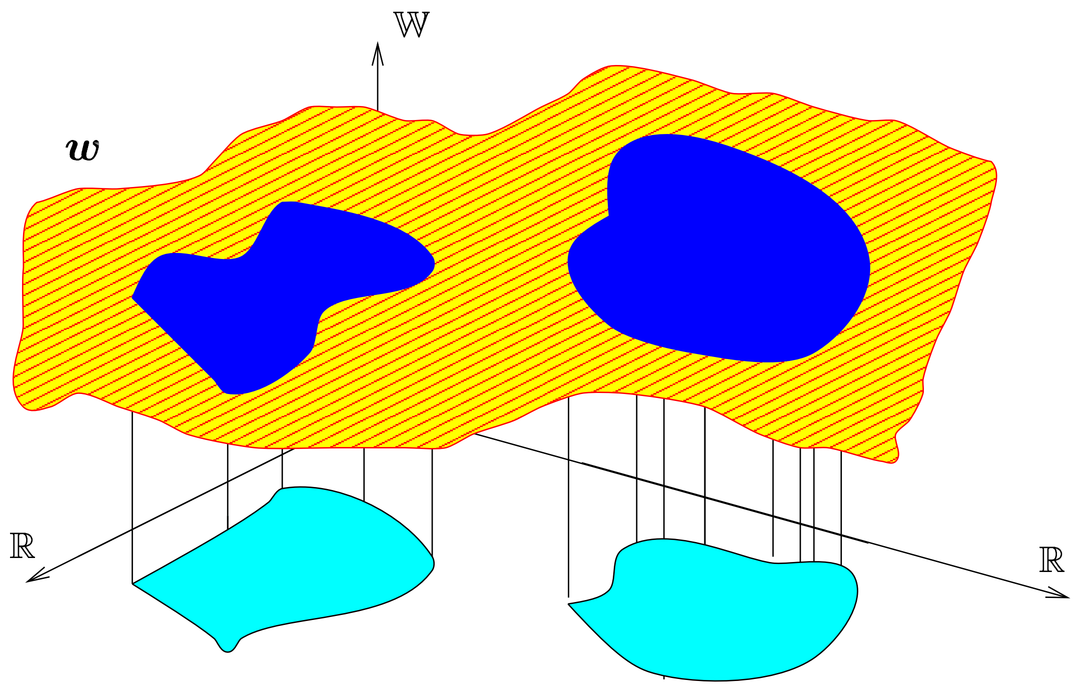


$$w_1, w_2 \in \mathfrak{B}.$$

$w \in \mathfrak{B}$ 'patches' $w_1, w_2 \in \mathfrak{B}$.



$w \in \mathfrak{B}$ **'patches'** $w_1, w_2 \in \mathfrak{B}$.



Controllability :⇔ 'patch-ability'.

Are Maxwell's equations controllable ?

Are Maxwell's equations controllable ?

The following equations in the ***scalar potential*** $\phi : \mathbb{R} \times \mathbb{R}^3 \rightarrow \mathbb{R}$ and the ***vector potential*** $\vec{A} : \mathbb{R} \times \mathbb{R}^3 \rightarrow \mathbb{R}^3$, generate exactly the solutions to Maxwell's equations:

$$
\begin{aligned}
\vec{E} &= -\frac{\partial}{\partial t}\vec{A} - \nabla \phi, \\
\vec{B} &= \nabla \times \vec{A}, \\
\vec{j} &= \varepsilon_0 \frac{\partial^2}{\partial t^2}\vec{A} - \varepsilon_0 c^2 \nabla^2 \vec{A} + \varepsilon_0 c^2 \nabla(\nabla \cdot \vec{A}) + \varepsilon_0 \frac{\partial}{\partial t}\nabla \phi, \\
\rho &= -\varepsilon_0 \frac{\partial}{\partial t}\nabla \cdot \vec{A} - \varepsilon_0 \nabla^2 \phi.
\end{aligned}
$$

Proves controllability.

Are Maxwell's equations controllable ?

The following equations in the *scalar potential* $\phi : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$ and the *vector potential* $\vec{A} : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$, generate exactly the solutions to Maxwell's equations:

$$
\begin{aligned}
\vec{E} &= -\frac{\partial}{\partial t}\vec{A} - \nabla \phi, \\
\vec{B} &= \nabla \times \vec{A}, \\
\vec{j} &= \varepsilon_0 \frac{\partial^2}{\partial t^2}\vec{A} - \varepsilon_0 c^2 \nabla^2 \vec{A} + \varepsilon_0 c^2 \nabla(\nabla \cdot \vec{A}) + \varepsilon_0 \frac{\partial}{\partial t}\nabla \phi, \\
\rho &= -\varepsilon_0 \frac{\partial}{\partial t}\nabla \cdot \vec{A} - \varepsilon_0 \nabla^2 \phi.
\end{aligned}
$$

Proves controllability. Illustrates the interesting connection

controllability $\Leftrightarrow$ $\exists$ potential!

OBSERVABILITY

Observability **of the image representation**

$$w = M(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n})\ell$$

is defined as: ℓ **can be deduced from** w**,**

i.e., $M(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n})$ **should be injective.**

OBSERVABILITY

Observability **of the image representation**

$$w = M(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathrm{n}}})\ell$$

is defined as: ℓ **can be deduced from** w**,**

i.e., $M(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathrm{n}}})$ **should be injective.**

Not all controllable systems admit an observable image representation. For $\mathrm{n} = 1$**, they do. For** $\mathrm{n} > 1$**, exceptionally so. The latent variable in an image representation** ℓ **may be 'hidden'.**

OBSERVABILITY

Observability **of the image representation**

$$w = M(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathrm{n}}})\ell$$

is defined as: ℓ **can be deduced from** w**,**

i.e., $M(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathrm{n}}})$ **should be injective.**

Not all controllable systems admit an observable image representation. For $\mathrm{n} = 1$**, they do. For** $\mathrm{n} > 1$**, exceptionally so. The latent variable in an image representation** ℓ **may be 'hidden'.**

Example: Maxwell's equations do not allow a potential representation that is observable.

OUTLINE

- **Lyapunov theory and dissipative dynamical systems**
- **Linear differential systems**

- **Dissipative distributed systems**

- **Local dissipation law**
- **The factorization equation**

NOTATION

Multi-index notation:

$$x = (x_1, \ldots, x_n),$$
$$k = (k_1, \ldots, k_n), \ell = (\ell_1, \ldots, \ell_n),$$
$$\xi = (\xi_1, \cdots, \xi_n), \zeta = (\zeta_1, \ldots, \zeta_n), \eta = (\eta_1, \ldots, \eta_n),$$
$$\frac{d}{dx} = (\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}), \frac{d^k}{dx^k} = (\frac{\partial^{k_1}}{\partial x_1^{k_1}}, \ldots, \frac{\partial^{k_n}}{\partial x_n^{k_n}}),$$
$$dx = dx_1 dx_2 \ldots dx_n,$$
$$R(\frac{d}{dx})w = 0 \quad \text{for} \quad R(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n})w = 0,$$
$$w = M(\frac{d}{dx})\ell \quad \text{for} \quad w = M(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n})\ell,$$

etc.

NOTATION

Multi-index notation:

$$x = (x_1, \ldots, x_n),$$
$$k = (k_1, \ldots, k_n), \ell = (\ell_1, \ldots, \ell_n),$$
$$\xi = (\xi_1, \cdots, \xi_n), \zeta = (\zeta_1, \ldots, \zeta_n), \eta = (\eta_1, \ldots, \eta_n),$$
$$\frac{d}{dx} = (\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}), \frac{d^k}{dx^k} = (\frac{\partial^{k_1}}{\partial x_1^{k_1}}, \ldots, \frac{\partial^{k_n}}{\partial x_n^{k_n}}),$$
$$dx = dx_1 dx_2 \ldots dx_n,$$
$$R(\tfrac{d}{dx})w = 0 \quad \text{for} \quad R(\tfrac{\partial}{\partial x_1}, \cdots, \tfrac{\partial}{\partial x_n})w = 0,$$
$$w = M(\tfrac{d}{dx})\ell \quad \text{for} \quad w = M(\tfrac{\partial}{\partial x_1}, \cdots, \tfrac{\partial}{\partial x_n})\ell,$$

etc.

$$\nabla \cdot := \frac{\partial}{\partial x_1} + \cdots + \frac{\partial}{\partial x_n}.$$

NOTATION

Multi-index notation:

$$x = (x_1, \ldots, x_{\mathrm{n}}),$$
$$k = (k_1, \ldots, k_{\mathrm{n}}), \ell = (\ell_1, \ldots, \ell_{\mathrm{n}}),$$
$$\xi = (\xi_1, \cdots, \xi_{\mathrm{n}}), \zeta = (\zeta_1, \ldots, \zeta_{\mathrm{n}}), \eta = (\eta_1, \ldots, \eta_{\mathrm{n}}),$$
$$\frac{d}{dx} = (\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_{\mathrm{n}}}), \frac{d^k}{dx^k} = (\frac{\partial^{k_1}}{\partial x_1^{k_1}}, \ldots, \frac{\partial^{k_{\mathrm{n}}}}{\partial x_{\mathrm{n}}^{k_{\mathrm{n}}}}),$$
$$dx = dx_1 dx_2 \ldots dx_{\mathrm{n}},$$
$$R(\frac{d}{dx})w = 0 \quad \text{for} \quad R(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathrm{n}}})w = 0,$$
$$w = M(\frac{d}{dx})\ell \quad \text{for} \quad w = M(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathrm{n}}})\ell,$$

etc.

For simplicity of notation, and for concreteness, we often take $\mathrm{n} = 4$, independent variables, t, time and x, y, z, space.

NOTATION

Multi-index notation:

$$x = (x_1, \dots, x_n),$$
$$k = (k_1, \dots, k_n), \ell = (\ell_1, \dots, \ell_n),$$
$$\xi = (\xi_1, \cdots, \xi_n), \zeta = (\zeta_1, \dots, \zeta_n), \eta = (\eta_1, \dots, \eta_n),$$
$$\frac{d}{dx} = (\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}), \frac{d^k}{dx^k} = (\frac{\partial^{k_1}}{\partial x_1^{k_1}}, \dots, \frac{\partial^{k_n}}{\partial x_n^{k_n}}),$$
$$dx = dx_1 dx_2 \dots dx_n,$$
$R(\frac{d}{dx})w = 0$ for $R(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n})w = 0,$

$w = M(\frac{d}{dx})\ell$ for $w = M(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n})\ell,$

etc.

$\nabla \cdot := \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z},$ ‘spatial flux’

QDF's

The quadratic map acting on $w : \mathbb{R}^{\mathtt{n}} \longrightarrow \mathbb{R}^{\mathtt{w}}$ and its derivatives, defined by

$$w \mapsto \sum_{k,\ell} \left(\frac{d^k}{dx^k} w\right)^\top \Phi_{k,\ell} \left(\frac{d^\ell}{dx^\ell} w\right)$$

is called ***quadratic differential form*** (QDF) on $\mathfrak{C}^\infty(\mathbb{R}^{\mathtt{n}}, \mathbb{R}^{\mathtt{w}})$.
$\Phi_{k,\ell} \in \mathbb{R}^{\mathtt{w} \times \mathtt{w}}$;

QDF's

The quadratic map acting on $w : \mathbb{R}^{\mathtt{n}} \longrightarrow \mathbb{R}^{\mathtt{w}}$ and its derivatives, defined by

$$w \mapsto \sum_{k,\ell} \left(\frac{d^k}{dx^k} w\right)^\top \Phi_{k,\ell} \left(\frac{d^\ell}{dx^\ell} w\right)$$

is called *quadratic differential form* (QDF) on $\mathfrak{C}^\infty(\mathbb{R}^{\mathtt{n}}, \mathbb{R}^{\mathtt{w}})$.
$\Phi_{k,\ell} \in \mathbb{R}^{\mathtt{w} \times \mathtt{w}}$; **WLOG:** $\Phi_{k,\ell} = \Phi_{\ell,k}^\top$.

QDF's

The quadratic map acting on $w : \mathbb{R}^{\mathtt{n}} \to \mathbb{R}^{\mathtt{w}}$ and its derivatives, defined by

$$w \mapsto \sum_{k,\ell} \left(\frac{d^k}{dx^k} w\right)^\top \Phi_{k,\ell} \left(\frac{d^\ell}{dx^\ell} w\right)$$

is called ***quadratic differential form*** (QDF) on $\mathfrak{C}^\infty(\mathbb{R}^{\mathtt{n}}, \mathbb{R}^{\mathtt{w}})$.
$\Phi_{k,\ell} \in \mathbb{R}^{\mathtt{w} \times \mathtt{w}}$; WLOG: $\Phi_{k,\ell} = \Phi_{\ell,k}^\top$.

Introduce the $2\mathtt{n}$-variable polynomial matrix Φ

$$\Phi(\zeta, \eta) = \sum_{k,\ell} \Phi_{k,\ell} \zeta^k \eta^\ell.$$

Denote the QDF as Q_Φ. QDF's are parametrized by $\mathbb{R}^{\bullet \times \bullet}[\zeta, \eta]$.

DISSIPATIVE DISTRIBUTED SYSTEMS

We consider only controllable linear differential systems and QDF's for supply rates.

DISSIPATIVE DISTRIBUTED SYSTEMS

We consider only controllable linear differential systems and QDF's for supply rates.

Definition: $\mathfrak{B} \in \mathfrak{L}_{\mathtt{n}}^{\mathtt{w}}$, controllable, is said to be ***dissipative*** with respect to the supply rate Q_{Φ} (a QDF) if

$$\boxed{\int_{\mathbb{R}^{\mathtt{n}}} Q_{\Phi}(w)\, dx \geq 0}$$

for all $w \in \mathfrak{B}$ of compact support, i.e., for all $w \in \mathfrak{B} \cap \mathfrak{D}$.

$\mathfrak{D} := \mathfrak{C}^{\infty}$ and 'compact support'.

Assume $\mathtt{n} = 4$:
independent variables $x, y, z; t$: space and time.

Assume $\mathtt{n} = 4$:
independent variables $x, y, z; t$: space and time.

Idea: $Q_\Phi(w)(x, y, z; t)\ dxdydz\ dt$:

'energy' supplied to the system
in the space-cube $[x, x + dx] \times [y, y + dy] \times [z, z + dz]$
during the time-interval $[t, t + dt]$.

Assume $\mathtt{n} = 4$**:**
independent variables $x, y, z; t$ **:** **space and time.**

Idea: $Q_\Phi(w)(x, y, z; t)\ dxdydz\ \ dt$:

'energy' supplied to the system
in the space-cube $[x, x + dx] \times [y, y + dy] \times [z, z + dz]$
during the time-interval $[t, t + dt]$ **.**

Dissipativity $:\Leftrightarrow$

$$\int_{\mathbb{R}} \left(\int_{\mathbb{R}^3} Q_\Phi(w)(x, y, z,\ t)\ dxdydz \right) dt \geq 0 \quad \text{for all } w \in \mathfrak{B} \cap \mathfrak{D}.$$

A dissipative system absorbs net energy.

Example: **Maxwell's eq'ns:**

dissipative (in fact, conservative) w.r.t. the QDF $-\vec{E}\cdot\vec{j}$.

Example: **Maxwell's eq'ns:**

dissipative (in fact, conservative) w.r.t. the QDF $-\vec{E}\cdot\vec{j}$.

In other words, if $\vec{E},\vec{j}$ is of compact support and satisfies

$$\begin{aligned} \varepsilon_0 \frac{\partial}{\partial t}\nabla\cdot\vec{E} + \nabla\cdot\vec{j} &= 0, \\ \varepsilon_0 \frac{\partial^2}{\partial t^2}\vec{E} + \varepsilon_0 c^2 \nabla\times\nabla\times\vec{E} + \frac{\partial}{\partial t}\vec{j} &= 0, \end{aligned}$$

then

$$\int_{\mathbb{R}} \left(\int_{\mathbb{R}^3} (-\vec{E}\cdot\vec{j}) \; dxdydz \right) dt = 0.$$

OUTLINE

- **Lyapunov theory and dissipative dynamical systems**
- **Linear differential systems**
- **Dissipative distributed systems**
- **Local dissipation law**
- **The factorization equation**

LOCAL DISSIPATION LAW

Dissipativity $:\Leftrightarrow$

$$\int_{\mathbb{R}} \left(\int_{\mathbb{R}^3} Q_{\Phi}(w)\, dxdydz \right) dt \geq 0 \qquad \text{for all } w \in \mathfrak{B} \cap \mathfrak{D}.$$

LOCAL DISSIPATION LAW

Dissipativity $:\Leftrightarrow$

$$\int_{\mathbb{R}} \left(\int_{\mathbb{R}^3} Q_\Phi(w)\, dxdydz \right) dt \geq 0 \qquad \text{for all } w \in \mathfrak{B} \cap \mathfrak{D}.$$

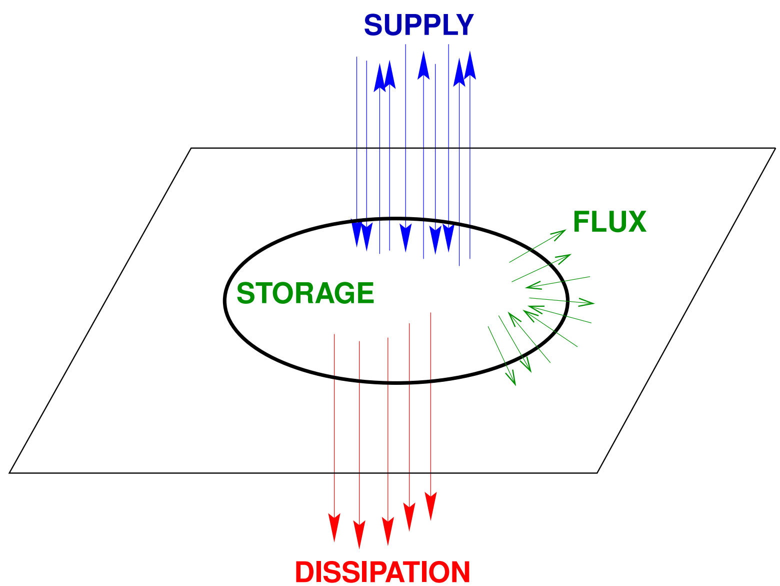
Can this be reinterpreted as: As the system evolves,
energy is locally stored, and dissipated or redistributed over time and space?

!! Invent storage and flux, locally defined in time and space, such that in every spatial domain there holds:

$$\boxed{\frac{d}{dt}\,\textbf{Storage + Spatial flux} \leq \textbf{Supply.}}$$

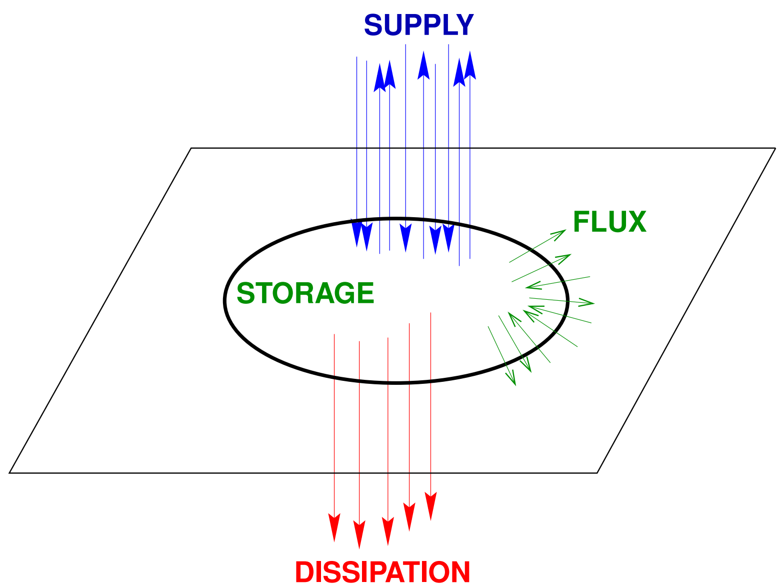
!! Invent storage and flux, locally defined in time and space, such that in every spatial domain there holds:

$$\boxed{\frac{d}{dt}\ \text{Storage} + \text{Spatial flux} \leq \text{Supply.}}$$



!! Invent storage and flux, locally defined in time and space, such that in every spatial domain there holds:

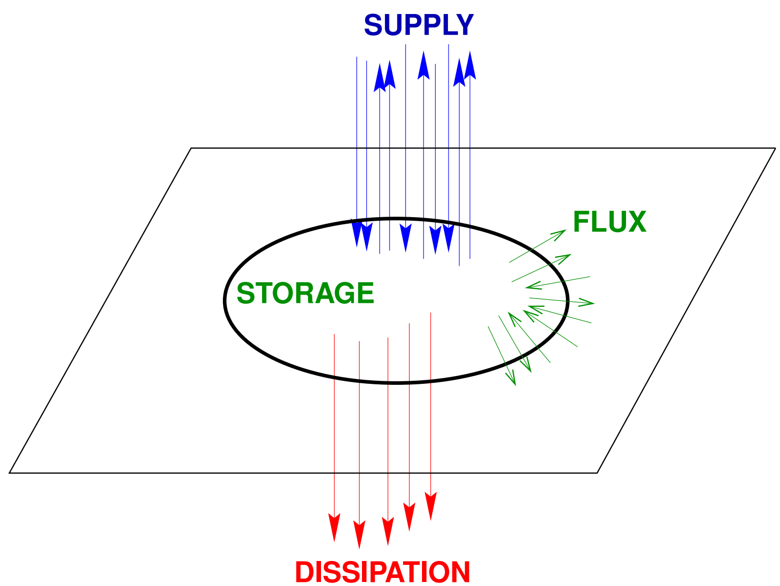
$$\frac{d}{dt}\text{ Storage + Spatial flux} \leq \text{Supply.}$$



Supply = partly stored + partly radiated + partly dissipated.

!! Invent storage and flux, locally defined in time and space, such that in every spatial domain there holds:

$$\frac{d}{dt}\ \text{Storage} + \text{Spatial flux} \leq \text{Supply.}$$



¡¡ Construct internal energy, internal entropy as a local function !!

MAIN RESULT (stated for $\mathtt{n} = 4$)

Theorem: Assume $\mathtt{n} = 4$: independent variables $x, y, z; t$: space and time. Let $\mathfrak{B} \in \mathfrak{L}_4^{\mathtt{w}}$ be controllable. Then

$$\int_{\mathbb{R}} \left(\int_{\mathbb{R}^3} Q_\Phi(w) \, dxdydz \right) dt \geq 0 \qquad \text{for all } w \in \mathfrak{B} \cap \mathfrak{D}.$$

MAIN RESULT (stated for $\mathtt{n} = 4$)

Theorem: Assume $\mathtt{n} = 4$: independent variables $x, y, z; t$: space and time. Let $\mathfrak{B} \in \mathfrak{L}_4^{\mathtt{w}}$ be controllable. Then

$$\int_{\mathbb{R}} \left(\int_{\mathbb{R}^3} Q_\Phi(w)\, dxdydz \right) dt \geq 0 \qquad \text{for all } w \in \mathfrak{B} \cap \mathfrak{D}.$$

if and only if

$\exists$ an image representation $w = M(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \frac{\partial}{\partial t})\ell$ of $\mathfrak{B}$, and QDF's S, the ***storage,*** and F_x, F_y, F_z, the ***flux,***

MAIN RESULT (stated for $\mathtt{n} = 4$)

Theorem: Assume $\mathtt{n} = 4$: independent variables $x, y, z; t$: space and time. Let $\mathfrak{B} \in \mathfrak{L}_4^{\mathtt{w}}$ be controllable. Then

$$\int_{\mathbb{R}} \left(\int_{\mathbb{R}^3} Q_\Phi(w)\, dxdydz \right) dt \geq 0 \qquad \text{for all } w \in \mathfrak{B} \cap \mathfrak{D}.$$

if and only if

$\exists$ an image representation $w = M(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \frac{\partial}{\partial t})\ell$ of $\mathfrak{B}$, and QDF's S, the *storage,* and F_x, F_y, F_z, the *flux,* such that the *local dissipation law*

$$\boxed{\frac{\partial}{\partial t} S(\ell) + \frac{\partial}{\partial x} F_x(\ell) + \frac{\partial}{\partial y} F_y(\ell) + \frac{\partial}{\partial z} F_z(\ell) \leq Q_\Phi(w)}$$

holds for all (w, ℓ) that satisfy $w = M(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \frac{\partial}{\partial t})\ell$.

Note:

the local law involves (possibly unobservable, - i.e., hidden!) latent variables (the ℓ's).

EXAMPLE: ENERGY STORED IN EM FIELDS

Maxwell's equations are dissipative (in fact, conservative) with respect to $-\vec{E} \cdot \vec{j},$ **the rate of energy supplied.**

EXAMPLE: ENERGY STORED IN EM FIELDS

Maxwell's equations are dissipative (in fact, conservative) with respect to $-\vec{E}\cdot\vec{j},$ **the rate of energy supplied.**

Introduce the *stored energy density,* S**, and**

the *energy flux density* (the *Poynting vector*), $\vec{F}$**,**

$$S(\vec{E},\vec{B}) := \frac{\varepsilon_0}{2}\vec{E}\cdot\vec{E} + \frac{\varepsilon_0 c^2}{2}\vec{B}\cdot\vec{B},$$

$$\vec{F}(\vec{E},\vec{B}) := \varepsilon_0 c^2 \vec{E}\times\vec{B}.$$

EXAMPLE: ENERGY STORED IN EM FIELDS

Maxwell's equations are dissipative (in fact, conservative) with respect to $-\vec{E}\cdot\vec{j}$, **the rate of energy supplied.**

Introduce the *stored energy density,* S, and

the *energy flux density* (the *Poynting vector*), $\vec{F}$,

$$S(\vec{E},\vec{B}) := \frac{\varepsilon_0}{2}\vec{E}\cdot\vec{E} + \frac{\varepsilon_0 c^2}{2}\vec{B}\cdot\vec{B},$$

$$\vec{F}(\vec{E},\vec{B}) := \varepsilon_0 c^2 \vec{E}\times\vec{B}.$$

The following is a local conservation law for Maxwell's equations:

$$\boxed{\frac{\partial}{\partial t}S(\vec{E},\vec{B}) + \nabla\cdot\vec{F}(\vec{E},\vec{B}) = -\vec{E}\cdot\vec{j}.}$$

EXAMPLE: ENERGY STORED IN EM FIELDS

Maxwell's equations are dissipative (in fact, conservative) with respect to $-\vec{E}\cdot\vec{j}$, **the rate of energy supplied.**

Introduce the *stored energy density,* S, **and**

the *energy flux density* (the *Poynting vector*), $\vec{F}$,

$$S(\vec{E},\vec{B}) := \frac{\varepsilon_0}{2}\vec{E}\cdot\vec{E} + \frac{\varepsilon_0 c^2}{2}\vec{B}\cdot\vec{B},$$

$$\vec{F}(\vec{E},\vec{B}) := \varepsilon_0 c^2 \vec{E}\times\vec{B}.$$

The following is a local conservation law for Maxwell's equations:

$$\boxed{\frac{\partial}{\partial t}S(\vec{E},\vec{B}) + \nabla\cdot\vec{F}(\vec{E},\vec{B}) = -\vec{E}\cdot\vec{j}.}$$

Involves $\vec{B}$, **unobservable from the energy variables** $\vec{E}$ **and** $\vec{j}$.

OUTLINE

- Lyapunov theory and dissipative dynamical systems
- Linear differential systems
- Dissipative distributed systems
- Local dissipation law
- The factorization equation

OUTLINE of the PROOF

Using controllability and image representations, we may assume, WLOG:

$$\mathfrak{B} = \mathfrak{C}^{\infty}(\mathbb{R}^{\mathtt{n}}, \mathbb{R}^{\mathtt{w}})$$

$$\int_{\mathbb{R}^{\mathrm{n}}} Q_{\Phi}(w) \geq 0 \text{ for all } w \in \mathfrak{D}$$

$$\Updownarrow \quad \textbf{(Parseval)}$$

$$\Phi(-i\omega, i\omega) \geq 0 \text{ for all } \omega \in \mathbb{R}^{\mathrm{n}}$$

$$\Phi(-i\omega, i\omega) \geq 0 \text{ for all } \omega \in \mathbb{R}^{\mathrm{n}}$$

$$\Updownarrow$$ **(Factorization equation)**

$$\exists\ D : \quad \Phi(-\xi, \xi) = D^{\top}(-\xi)D(\xi)$$

$$\exists\ D: \quad \Phi(-\xi,\xi) = D^\top(-\xi)D(\xi)$$

$$\Updownarrow \quad \textbf{(easy)}$$

$$\exists\ \Psi: \quad (\zeta+\eta)^\top \Psi(\zeta,\eta) = \Phi(\zeta,\eta) - D^\top(\zeta)D(\eta)$$

$$\exists\ \Psi : \quad (\zeta + \eta)^\top \Psi(\zeta, \eta) = \Phi(\zeta, \eta) - D^\top(\zeta) D(\eta)$$

$\Updownarrow$ **(clearly)**

$\exists\ \Psi : \quad \nabla \cdot Q_\Psi(w) \leq Q_\Phi(w)$ **for all** $w \in \mathfrak{C}^\infty$

Assuming factorizability:

Global dissipation $:\Leftrightarrow$

$$\int_{\mathbb{R}^n} Q_\Phi(w) \geq 0 \text{ for all } w \in \mathfrak{D}$$

$$\Updownarrow$$

$$\exists\ \Psi : \quad \nabla \cdot Q_\Psi(w) \leq Q_\Phi(w) \text{ for all } w \in \mathfrak{C}^\infty$$

$\Leftrightarrow$: **Local dissipation**

Assuming factorizability:

Global dissipation :⇔

$$\int_{\mathbb{R}^{\mathrm{n}}} Q_{\Phi}(w) \geq 0 \text{ for all } w \in \mathfrak{D}$$

$$\Updownarrow$$

$$\exists\ \Psi : \quad \nabla \cdot Q_{\Psi}(w) \leq Q_{\Phi}(w) \text{ for all } w \in \mathfrak{C}^{\infty}$$

⇔: **Local dissipation**

This argument is valid for $\mathrm{n} = 1$...

THE FACTORIZATION EQ'N

Consider

$$\boxed{X^\top(-\xi)X(\xi) = Y(\xi)}$$

with $Y \in \mathbb{R}^{\bullet\times\bullet}[\xi]$ given, and X the unknown. ***Solvable??***

THE FACTORIZATION EQ'N

Consider

$$X^\top(-\xi)X(\xi) = Y(\xi)$$

with $Y \in \mathbb{R}^{\bullet\times\bullet}[\xi]$ given, and X the unknown. ***Solvable??***

$\cong$

$$X^\top(\xi)X(\xi) = Y(\xi)$$

with $Y \in \mathbb{R}^{\bullet\times\bullet}[\xi]$ given, and X the unknown.

Under what conditions on Y does there exist a solution X?

THE FACTORIZATION EQ'N

Consider

$$X^\top(-\xi)X(\xi) = Y(\xi)$$

with $Y \in \mathbb{R}^{\bullet\times\bullet}[\xi]$ given, and X the unknown. ***Solvable??***

$\cong$

$$X^\top(\xi)X(\xi) = Y(\xi)$$

with $Y \in \mathbb{R}^{\bullet\times\bullet}[\xi]$ given, and X the unknown.

Under what conditions on Y does there exist a solution X?

Scalar case: !! write the real polynomial Y as a sum of squares

$$Y = x_1^2 + x_2^2 + \cdots + x_k^2.$$

$$\boxed{X^\top(\xi)X(\xi) = Y(\xi)}$$

Y **is a given polynomial matrix;** X **is the unknown.**

$$X^\top(\xi)X(\xi) = Y(\xi)$$

Y is a given polynomial matrix; X is the unknown.

For $\mathtt{n} = 1$ and $Y \in \mathbb{R}[\xi]$, solvable (for $X \in \mathbb{R}^2[\xi]$) iff

$$Y(\alpha) \geq 0 \qquad \text{for all } \alpha \in \mathbb{R}.$$

$$X^\top(\xi)X(\xi) = Y(\xi)$$

Y is a given polynomial matrix; X is the unknown.

For $\mathtt{n} = 1$, and $Y \in \mathbb{R}^{\bullet\times\bullet}[\xi]$, it is well-known (but non-trivial) that this factorization equation is solvable (with $X \in \mathbb{R}^{\bullet\times\bullet}[\xi]$!) iff

$$Y(\alpha) = Y^\top(\alpha) \geq 0 \qquad \text{for all } \alpha \in \mathbb{R}.$$

$$X^\top(\xi)X(\xi) = Y(\xi)$$

Y **is a given polynomial matrix;** X **is the unknown.**

For $\mathtt{n} > 1$**, and under this obvious symmetry and positivity requirement,**

$$Y(\alpha) = Y^\top(\alpha) \geq 0 \qquad \text{for all } \alpha \in \mathbb{R}^{\mathtt{n}},$$

this equation can nevertheless in general <u>not</u> be solved over the polynomial matrices, for $X \in \mathbb{R}^{\bullet\times\bullet}[\xi]$**,**

$$X^\top(\xi)X(\xi) = Y(\xi)$$

Y is a given polynomial matrix; X is the unknown.

For $\mathtt{n} > 1$, and under this obvious symmetry and positivity requirement,

$$Y(\alpha) = Y^\top(\alpha) \geq 0 \qquad \text{for all } \alpha \in \mathbb{R}^{\mathtt{n}},$$

this equation can nevertheless in general not be solved over the polynomial matrices, for $X \in \mathbb{R}^{\bullet\times\bullet}[\xi]$, but it can be solved over the matrices of rational functions, i.e., for $X \in \mathbb{R}^{\bullet\times\bullet}(\xi)$.

This factorizability is a simple consequence of Hilbert's 17-th pbm!

!! Solve $p = p_1^2 + p_2^2 + \cdots + p_k^2, \; p$ **given**

This factorizability is a simple consequence of Hilbert's 17-th pbm!

!! Solve $p = p_1^2 + p_2^2 + \cdots + p_k^2,\ p$ **given**

A polynomial $p \in \mathbb{R}[\xi_1, \cdots, \xi_n]$, **with** $p(\alpha_1, \ldots, \alpha_n) \geq 0$ **for all** $(\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$ **can in general not be expressed as a sum of squares of polynomials, with the** p_i**'s** $\in \mathbb{R}[\xi_1, \cdots, \xi_n]$.

This factorizability is a simple consequence of Hilbert's 17-th pbm!

!! Solve $p = p_1^2 + p_2^2 + \cdots + p_{\mathrm{k}}^2, \; p$ given

But a rational function (and hence a polynomial) $p \in \mathbb{R}(\xi_1, \cdots, \xi_{\mathrm{n}})$, with $p(\alpha_1, \ldots, \alpha_{\mathrm{n}}) \geq 0,$ for all $(\alpha_1, \ldots, \alpha_{\mathrm{n}}) \in \mathbb{R}^{\mathrm{n}},$ **can** be expressed as a sum of squares of ($\mathrm{k} = 2^{\mathrm{n}}$) rational functions, with the p_i's $\in \mathbb{R}(\xi_1, \cdots, \xi_{\mathrm{n}})$.

$\Rightarrow$ **solvability of the factorization eq'n**

$$\Phi(-i\omega, i\omega) \geq 0 \text{ for all } \omega \in \mathbb{R}^{\mathrm{n}}$$

$$\Updownarrow$$ **(Factorization equation)**

$$\exists\ D : \quad \Phi(-\xi, \xi) = D^{\top}(-\xi)D(\xi)$$

over the rational functions, i.e., with D a matrix with elements in $\mathbb{R}(\xi_1, \cdots, \xi_{\mathrm{n}})$.

$\Rightarrow$ **solvability of the factorization eq'n**

$$\Phi(-i\omega, i\omega) \geq 0 \text{ for all } \omega \in \mathbb{R}^{\mathrm{n}}$$

$$\Updownarrow \qquad \boxed{\text{(Factorization equation)}}$$

$$\exists \; D : \quad \Phi(-\xi, \xi) = D^{\top}(-\xi) D(\xi)$$

over the rational functions, i.e., with D a matrix with elements in $\mathbb{R}(\xi_1, \cdots, \xi_{\mathrm{n}})$.

The need to introduce rational functions in this factorization and an image representation of $\mathfrak{B}$ (to reduce the pbm to $\mathfrak{C}^{\infty}$) are the causes of the unavoidable presence of (possibly unobservable, i.e., 'hidden') latent variables in the local dissipation law.

UNIQUENESS

Non-uniqueness of the storage function stems from 3 sources

UNIQUENESS

Non-uniqueness of the storage function stems from 3 sources

1. **The non-uniqueness of the latent variable ℓ in various (non-observable) image representations.**

UNIQUENESS

Non-uniqueness of the storage function stems from 3 sources

1. **The non-uniqueness of the latent variable ℓ in various (non-observable) image representations.**
2. **The non-uniqueness of D in the factorization equation**
$$\Phi(-\xi,\xi) = D^{\top}(-\xi)D(\xi)$$

UNIQUENESS

Non-uniqueness of the storage function stems from 3 sources

1. **The non-uniqueness of the latent variable ℓ in various (non-observable) image representations.**
2. **The non-uniqueness of D in the factorization equation** $\Phi(-\xi,\xi) = D^\top(-\xi)D(\xi)$
3. **The non-uniqueness (in the case $\mathtt{n} > 1$) of the solution Ψ of**

$$(\zeta+\eta)^\top \Psi(\zeta,\eta) = \Phi(\zeta,\eta) - D^\top(\zeta)D(\eta)$$

UNIQUENESS

Non-uniqueness of the storage function stems from 3 sources

1. **The non-uniqueness of the latent variable ℓ in various (non-observable) image representations.**
2. **The non-uniqueness of D in the factorization equation**
$$\Phi(-\xi,\xi) = D^\top(-\xi)D(\xi)$$
3. **The non-uniqueness (in the case $\mathtt{n} > 1$) of the solution Ψ of**

$$(\zeta+\eta)^\top \Psi(\zeta,\eta) = \Phi(\zeta,\eta) - D^\top(\zeta)D(\eta)$$

For conservative systems, $\Phi(-\xi,\xi) = 0$, whence $D = 0$, but, when $\mathtt{n} > 1$, the third source of non-uniqueness remains.

The non-uniqueness is very real, even for EM fields.

The non-uniqueness is very real, even for EM fields. Cfr.

The ambiguity of the field energy

... There are, in fact, an infinite number of different possibilities for u **[the internal energy]** ***and*** S **[the flux]** ***... It is sometimes claimed that this problem can be resolved using the theory of gravitation ... as yet nobody has done such a delicate experiment ... So we will follow the rest of the world - besides, we believe that it*** **[our choice]** ***is probably perfectly right.***

The Feynman Lectures on Physics, Volume II, page 27-6.

SUMMARY

- **The theory of dissipative systems centers artound the construction of the storage function**

SUMMARY

- **The theory of dissipative systems centers artound the construction of the storage function**
- **global dissipation $\Leftrightarrow \exists$ local dissipation law**

SUMMARY

- The theory of dissipative systems centers artound the construction of the storage function
- **global dissipation $\Leftrightarrow \exists$ local dissipation law**
- Involves hidden latent variables (e.g. $\vec{B}$ in Maxwell's eq'ns)

SUMMARY

- **The theory of dissipative systems centers artound the construction of the storage function**
- **global dissipation $\Leftrightarrow$ $\exists$ local dissipation law**
- **Involves hidden latent variables (e.g. $\vec{B}$ in Maxwell's eq'ns)**
- **The proof $\cong$ Hilbert's 17-th problem**

SUMMARY

- The theory of dissipative systems centers artound the construction of the storage function
- global dissipation $\Leftrightarrow \exists$ local dissipation law
- Involves hidden latent variables (e.g. $\vec{B}$ in Maxwell's eq'ns)
- The proof $\cong$ Hilbert's 17-th problem
- Neither controllability nor observability are good generic system theoretic assumptions for physical models

<u>Reference</u>: H. Pillai and JCW, Dissipative distributed systems,
***SIAM Journal on Control and Optimization*,**
Volume 40, pages 1406-1430, 2002.

The manuscript & copies of the lecture frames will be available from/at

`Jan.Willems@esat.kuleuven.ac.be`
`http://www.esat.kuleuven.ac.be/∼jwillems`

Reference: **H. Pillai and JCW, Dissipative distributed systems,**
SIAM Journal on Control and Optimization**,**
Volume 40, pages 1406-1430, 2002.

The manuscript & copies of the lecture frames will be available from/at

`Jan.Willems@esat.kuleuven.ac.be`
`http://www.esat.kuleuven.ac.be/∼jwillems`

Thank you for your attention !