The BEHAVIORAL APPROACH to SYSTEMS and CONTROL

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Problematique

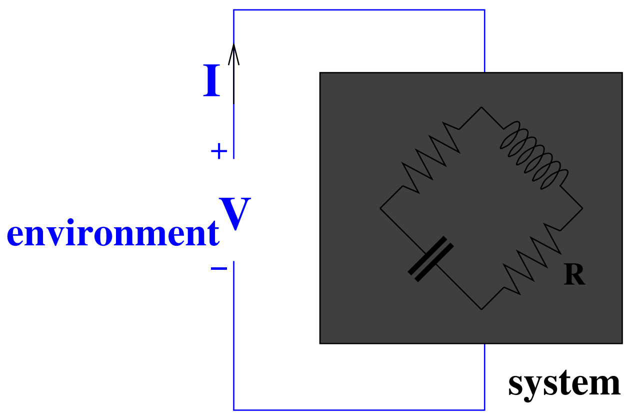
Develop a suitable *mathematical* framework for discussing dynamical systems

aimed at modeling, analysis, and synthesis.

$\leadsto$ control, signal processing, system identification, . . .
$\leadsto$ engineering systems, economics, physics, . . .

Motivational Examples

Electrical circuit



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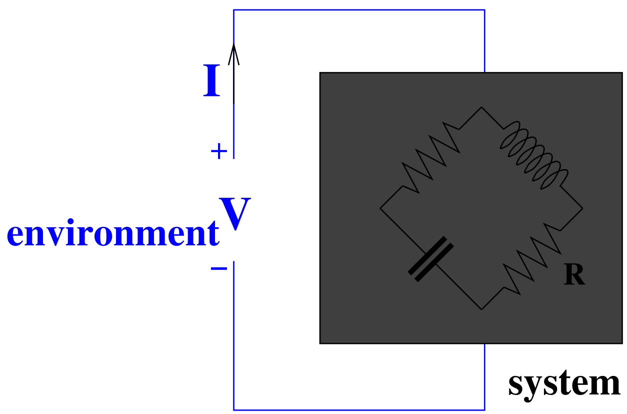

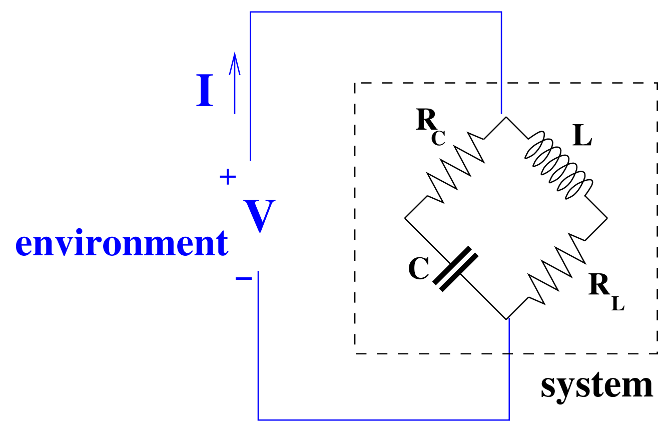

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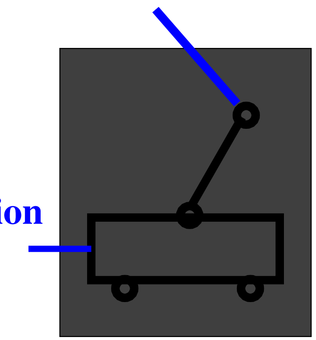
Electromechanical system



!! between the positions, forces, torque, angle, voltages, currents

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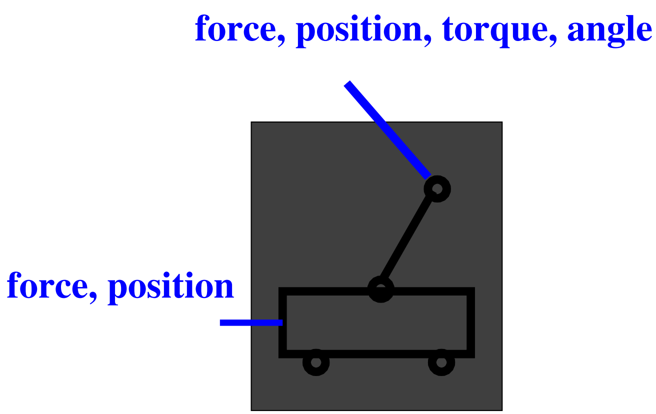

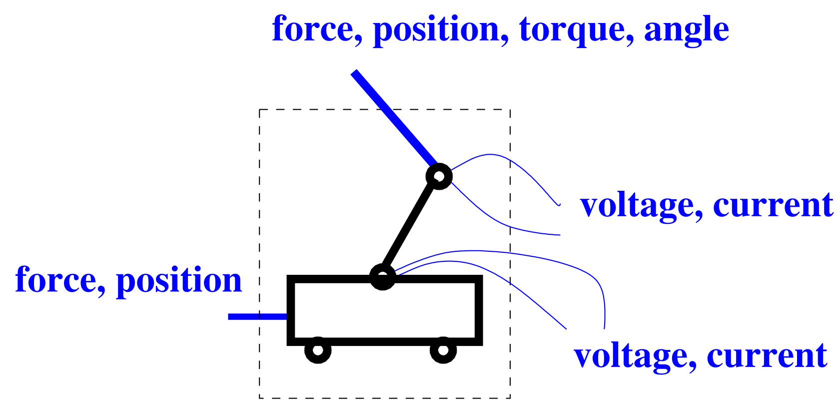

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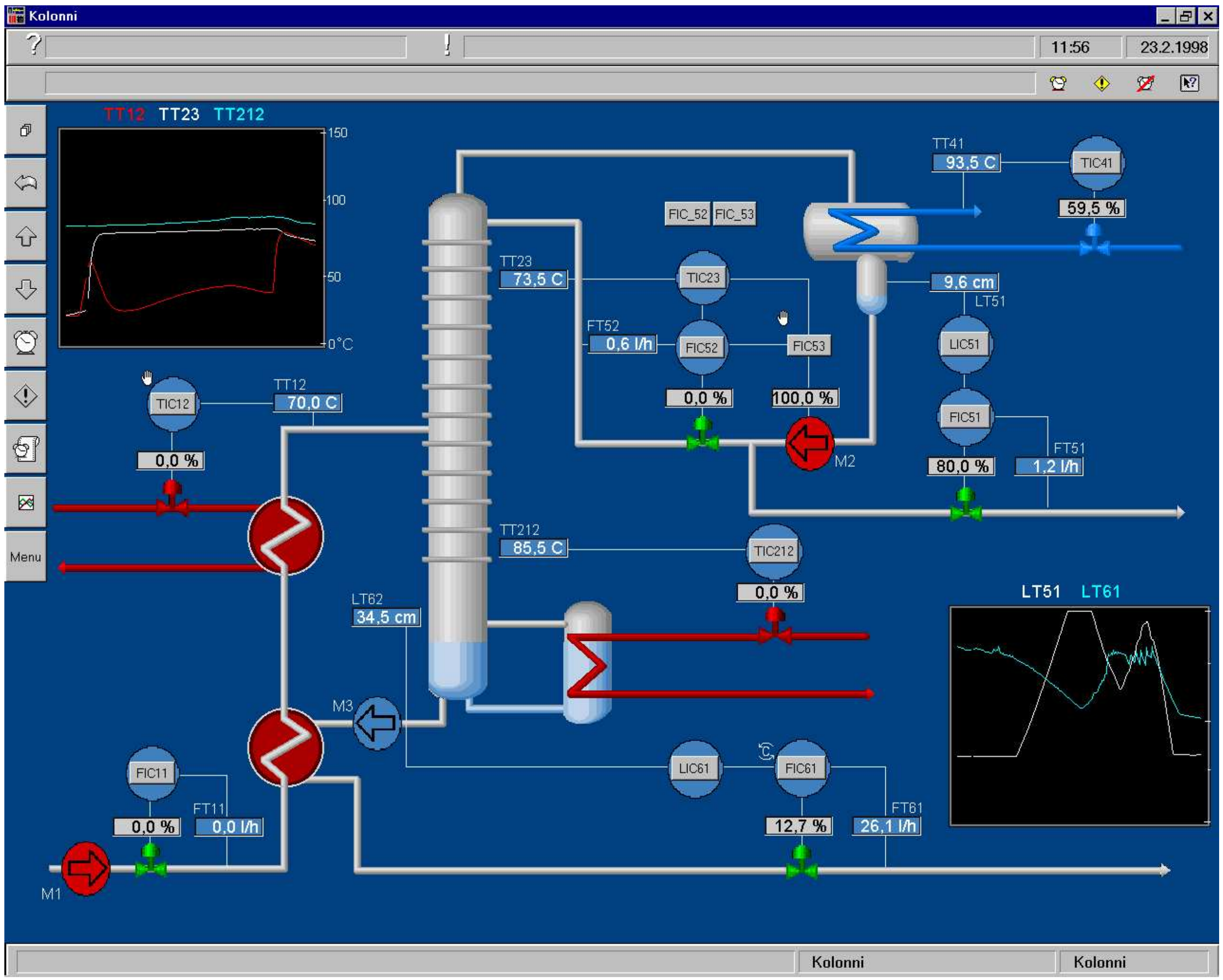
Distillation column



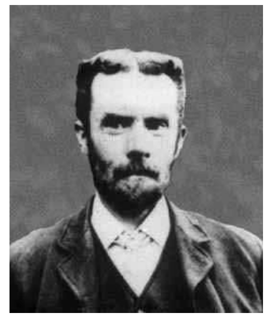
Features: Systems are typically

- **dynamical**
- **open,** they interact with their environment
- **interconnected,** with many subsystems
- **modular,** consisting of standard components

We are looking for a mathematical framework that is adapted to these features, and hence to **computer assisted modeling.**

Historical remarks

Early 20-th century: emergence of the notion of a transfer function (Rayleigh, Heaviside).



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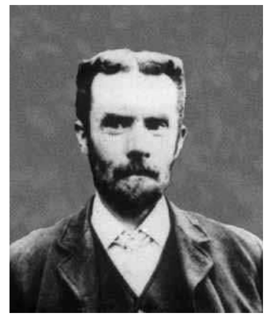

Since the 1920's: routinely used in circuit theory (Foster, Brune, Cederbaum, $\cdots$)

$\leadsto$ impedances, admittances, scattering matrices, etc.

1930's: control embraces transfer functions

(Nyquist, Bode, $\cdots$)

$\rightsquigarrow$ **plots and diagrams, classical control.**

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Around 1950: **Wiener sanctifies the notion of a blackbox,**
attempts nonlinear generalization (via Volterra series).

<u>1960's</u>: Kalman's state space ideas (incl. controllability, observability, recursive filtering, state models and representations) come in vogue

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$\rightsquigarrow$ **input/state/output systems, and the ubiquitous**

$$\boxed{\frac{d}{dt}x = Ax + Bu, \quad y = Cx + Du,}$$

<u>1960's</u>: Kalman's state space ideas (incl. controllability, observability, recursive filtering, state models and representations) come in vogue

or its nonlinear counterpart

$$\boxed{\frac{d}{dt}x = f(x, u), \quad y = h(x, u).}$$

These mathematical structures, transfer functions, + their discrete-time analogs, are nowadays the basic models used in control and signal processing (cfr. MATLAB©).

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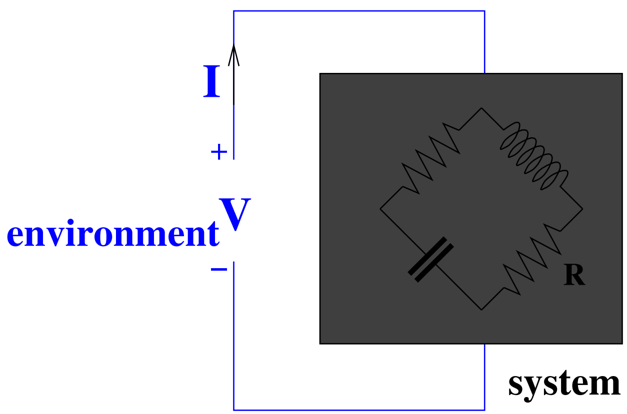
All these theories: input/output; **cause ⇒ effect.**

Beyond input/output

What's wrong with input/output thinking?

Let's look at examples: Our electrical circuit.



Is V the input? Or I ? Or both, or are they both outputs?

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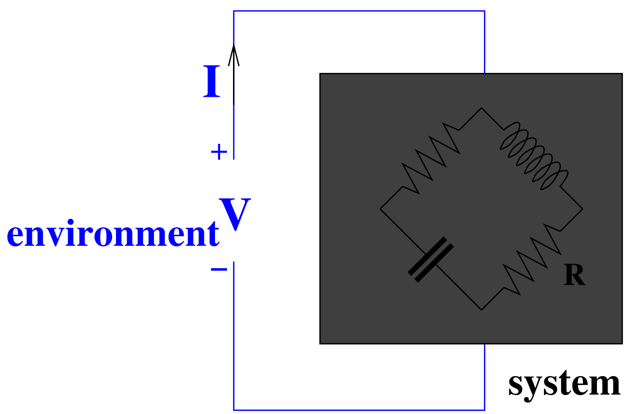

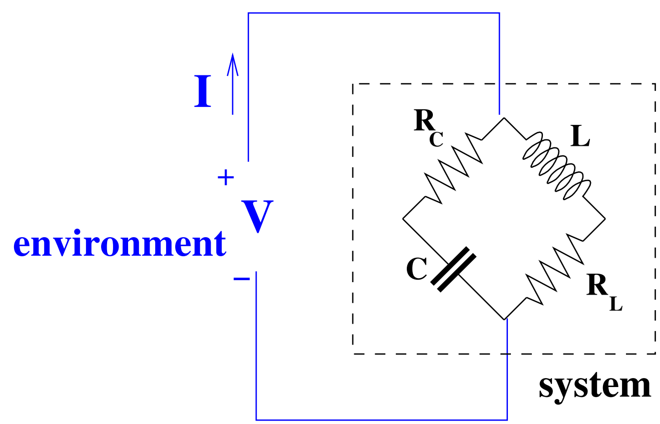

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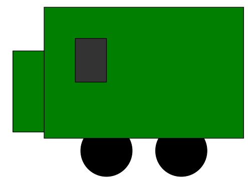
An automobile:



External terminals:

wind, tires, steering wheel, gas/brake pedal.

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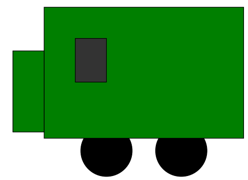


External terminals:

wind, tires, steering wheel, gas/brake pedal.

What are the inputs?

at the wind terminal: the force,

at the tires: forces, or, more likely, positions?

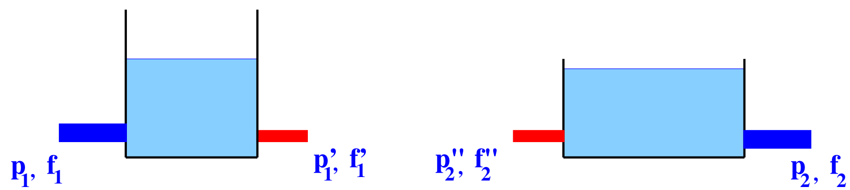
at the steering wheel: the torque or the angle?

at the gas-, or brake-pedal: the force or the position?

Difficulty: at each terminal there are many (typically paired) interconnection variables!

Input/output is awkward in modeling interconnections.

Consider a two-tank example.



Reasonable input choices: the pressures,
output choices: the flows.

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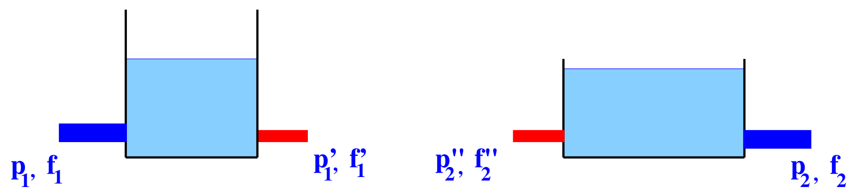

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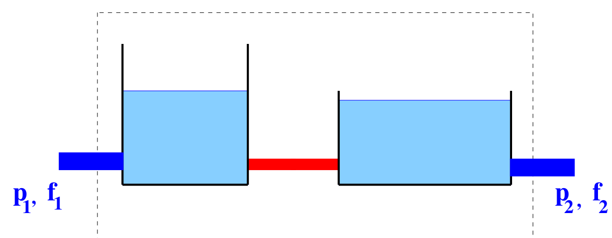

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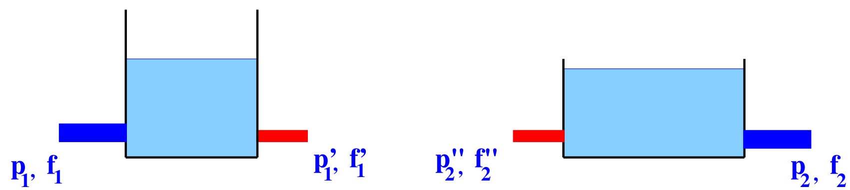

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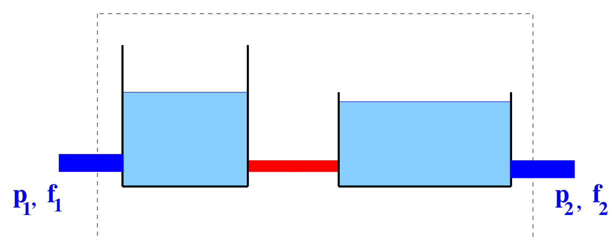

Interconnection: $p_1' = p_2'', \quad f_1' + f_2'' = 0$

input=input; output=output! $\not\Leftrightarrow$ **SIMULINK©**

Interconnections contradicting SIMULINK© are in fact

the rule, not the exception,

in mechanics, fluidics, heat transfer, etc.

Mathematical difficulties

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Is it a parameterized map[†] $(u(\cdot), \alpha) \mapsto y(\cdot)$ **?**
All sorts of new difficulties...

† **Cfr. the book of Kalman, Falb, and Arbib**

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Construct the state!

But from what?

From the system model!

What system?

Conclusions

$\ast$ **for physical systems** ($\Rightarrow\!\Leftarrow$ **signal processors**) $\ast$

- **External variables are basic, but <u>what 'drives' what</u>, not.**

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$*$ for physical systems ($\Rightarrow\!\Leftarrow$ signal processors) $*$

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- **External variables are basic, but <u>what 'drives' what</u>, not.**
- **It is impossible to make an a priori, fixed, input/output selection for off-the-shelf modeling.**
- **What can be the input, and the output should be deduced from a dynamical model. Therefore, we need a more general notion of 'system', of 'dynamical model'.**

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- **What can be the input, and the output should be deduced from a dynamical model. Therefore, we need a more general notion of 'system', of 'dynamical model'.**
- **Interconnection, variable sharing, rather that input selection, is the basic mechanism by which a system interacts with its environment.**

Conclusions

$*$ for physical systems ($\Rightarrow\!\!\Leftarrow$ signal processors) $*$

- External variables are basic, but what 'drives' what, not.
- It is impossible to make an a priori, fixed, input/output selection for off-the-shelf modeling.
- What can be the input, and the output should be deduced from a dynamical model. Therefore, we need a more general notion of 'system', of 'dynamical model'.
- Interconnection, variable sharing, rather that input selection, is the basic mechanism by which a system interacts with its environment.

$\Rightarrow$ We need a better framework for discussing 'open' systems!

$\rightsquigarrow$ Behavioral systems.

The basic concepts

Behavioral systems

A dynamical system = $\boldsymbol{\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})}$

$\mathbb{T} \subseteq \mathbb{R}$, **the *time-axis***

$\mathbb{W}$, **the *signal space***

$\mathfrak{B} \subseteq \mathbb{W}^{\mathbb{T}}$ **: *the behavior***

The basic concepts

Behavioral systems

A dynamical system = $\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$

$\mathbb{T} \subseteq \mathbb{R}$, **the *time-axis*** **(= the relevant time instances),**

$\mathbb{W}$, **the *signal space*** **(= where the variables take on their values),**

$\mathfrak{B} \subseteq \mathbb{W}^{\mathbb{T}}$: ***the behavior*** **(= the admissible trajectories).**

$$\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$$

For a trajectory $w : \mathbb{T} \to \mathbb{W}$, we thus have:

$w \in \mathfrak{B}$: the model allows the trajectory w,

$w \notin \mathfrak{B}$: the model forbids the trajectory w.

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Usually, $\mathbb{T} = \mathbb{R}$, or $[0, \infty)$ (in continuous-time systems),
or $\mathbb{Z}$, or $\mathbb{N}$ (in discrete-time systems).

Usually, $\mathbb{W} \subseteq \mathbb{R}^{\mathtt{w}}$ (in lumped systems),
a function space (in distributed systems),
or a finite set (in DES).

Emphasis later today: $\mathbb{T} = \mathbb{R}$, $\mathbb{W} = \mathbb{R}^{\mathtt{w}}$,
$\mathfrak{B}$ = sol'ns of system of linear constant coefficient ODE's.

Examples

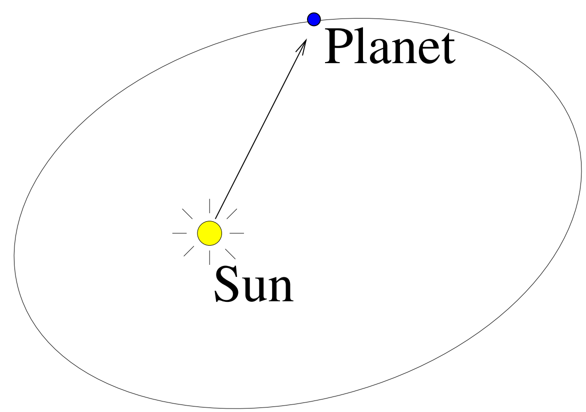
1. Planetary orbits

Examples

1. Planetary orbits

$\mathbb{T} = \mathbb{R}$ **(time),**
$\mathbb{W} = \mathbb{R}^3$ **(position),**
$\mathfrak{B} =$ **planetary orbits** $\cong$ **Kepler's laws:**

$$\text{ellipses, = areas in = time, } \frac{(\text{period})^2}{(\text{axis})^3} = \text{constant.}$$



Examples

1. Planetary orbits

2. Input / output systems

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1. Planetary orbits

2. Input / output systems

$$f_1\left(y(t), \frac{d}{dt}y(t), \frac{d^2}{dt^2}y(t), \ldots, t\right) = f_2\left(u(t), \frac{d}{dt}u(t), \frac{d^2}{dt^2}u(t), \ldots, t\right)$$

Examples

1. Planetary orbits

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3. Flows

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1. Planetary orbits

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$$\frac{d}{dt}x(t) = f(x(t)),$$

$\mathfrak{B} =$ **all state trajectories.**

... of very marginal value as a paradigm for dynamics ...

Modeling closed systems by tearing and zooming

$\rightsquigarrow$ **open systems.**

Examples

1. ***Planetary orbits***

2. ***Input / output systems***

3. ***Flows***

4. ***Observed flows***

Examples

1. Planetary orbits

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$$\frac{d}{dt}x(t) = f(x(t)); \quad y(t) = h(x(t)),$$

$\mathfrak{B} =$ **all possible output trajectories.**

Examples

1. ***Planetary orbits***

2. ***Input / output systems***

3. ***Flows***

4. ***Observed flows***

5. ***Convolutional codes***

Examples

1. *Planetary orbits*

2. *Input / output systems*

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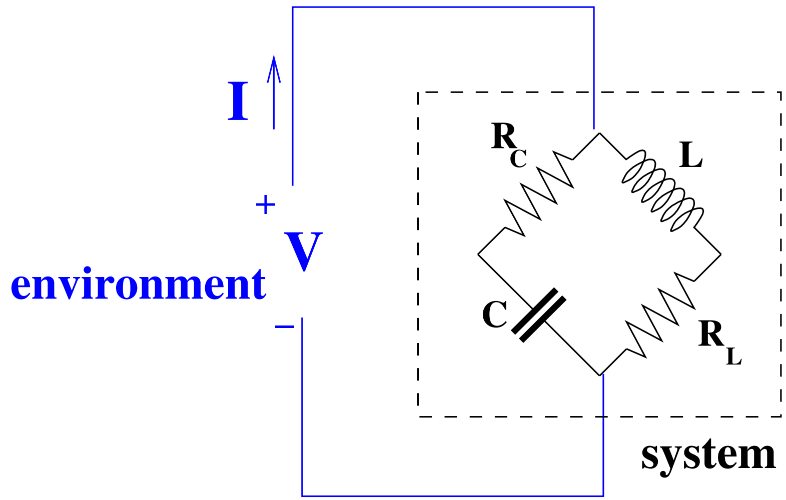
5. *Convolutional codes*

6. *Formal languages*

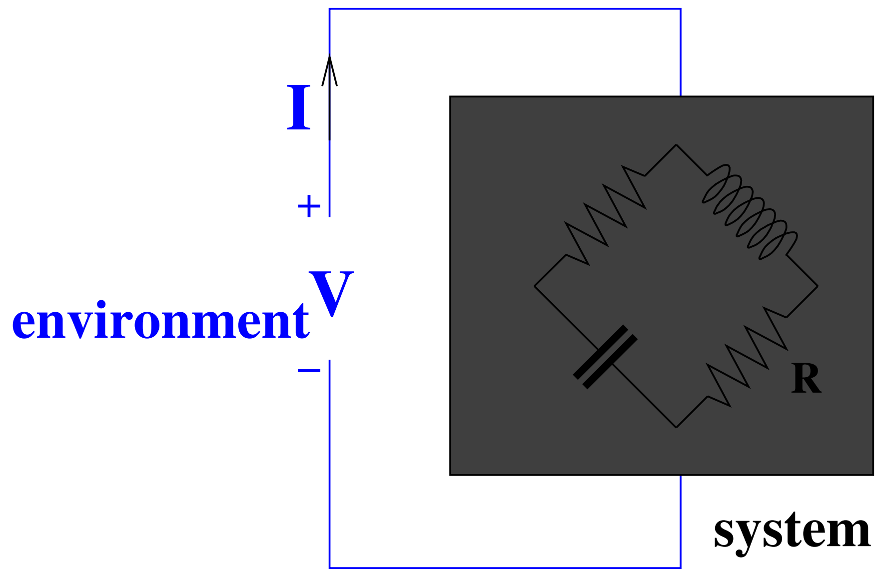
Latent variable systems

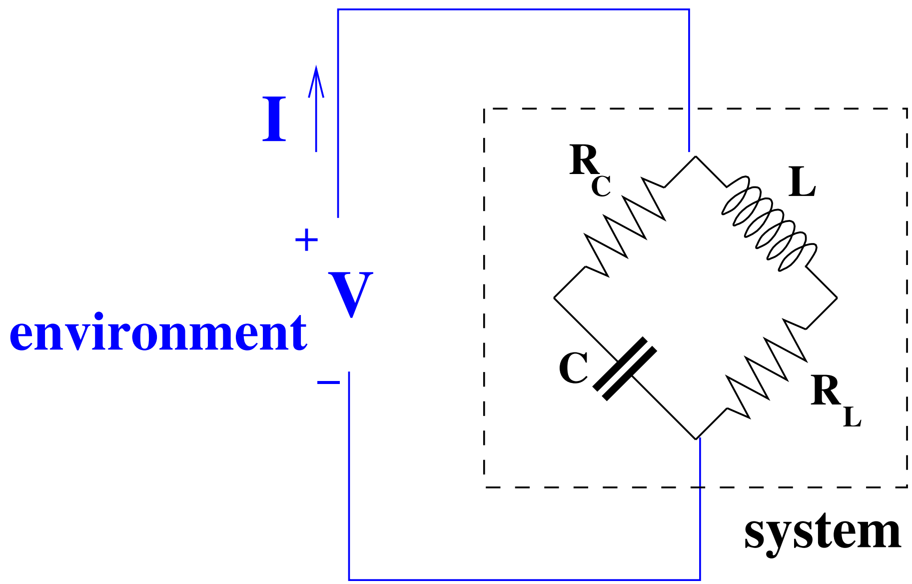
Consider once again our electrical RLC - circuit:



!! Model the relation between V and I !!

How does this modeling proceed?


I
+
V
environment
−
R
system


I
+
V
environment
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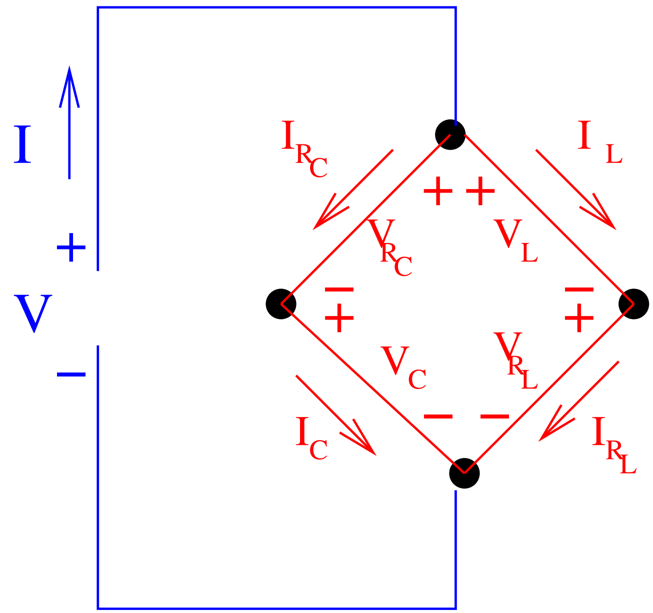
R_C
L
C
R_L
system



The circuit graph

Introduce the following additional variables:

the voltage across and the current in each branch:

$$V_{R_C}, I_{R_C}, V_C, I_C, V_{R_L}, I_{R_L}, V_L, I_L.$$

System equations

Constitutive equations (CE):

$$V_{R_C} = R_C I_{R_C}, \; V_{R_L} = R_L I_{R_L}, \; C\frac{d}{dt}V_C = I_C, \; L\frac{d}{dt}I_L = V_L$$

Kirchhoff's voltage laws (KVL):

$$V = V_{R_C} + V_C, \; V = V_L + V_{R_L}, \; V_{R_C} + V_C = V_L + V_{R_L}$$

Kirchhoff's current laws (KCL):

$$I = I_{R_C} + I_L, \; I_{R_C} = I_C, \; I_L = I_{R_L}, \; I_C + I_{R_L} = I$$

The preceding is a complete model, but it is not an explicit relation the between V and I. Here it is:

Case 1: $CR_C \neq \frac{L}{R_L}$.

$$(\frac{R_C}{R_L} + (1 + \frac{R_C}{R_L})CR_C\frac{d}{dt} + CR_C\frac{L}{R_L}\frac{d^2}{dt^2})V$$
$$= (1 + CR_C\frac{d}{dt})(1 + \frac{L}{R_L}\frac{d}{dt})R_C I.$$

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Case 2: $CR_C = \frac{L}{R_L}$.

$$\boxed{(\frac{R_C}{R_L} + CR_C\frac{d}{dt})V = (1 + CR_C\frac{d}{dt})R_C I}$$

These are the exact relations between V and I !

First principles models invariably contain **auxiliary variables,** in addition to the variables the model aims at.

$\rightsquigarrow$ **Manifest** and **latent** variables.

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Manifest = the variables the model aims at,

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$\rightsquigarrow$ Manifest and latent variables.

Manifest = the variables the model aims at,
Latent = auxiliary variables.

We want to capture this in mathematical definitions.

Latent variable systems

A dynamical system with latent variables **=**

$$\Sigma_L = (\mathbb{T}, \mathbb{W}, \mathbb{L}, \mathfrak{B}_{\text{full}})$$

$\mathbb{T} \subseteq \mathbb{R}$, **the** ***time-axis***
$\mathbb{W}$, **the** ***signal space***
$\mathbb{L}$, **the** ***latent variable space***

$$\mathfrak{B}_{\text{full}} \subseteq (\mathbb{W} \times \mathbb{L})^{\mathbb{T}}$$ **:** ***the full behavior***

Latent variable systems

A dynamical system with latent variables **=**

$$\Sigma_L = (\mathbb{T}, \mathbb{W}, \mathbb{L}, \mathfrak{B}_{\text{full}})$$

$\mathbb{T} \subseteq \mathbb{R}$, **the *time-axis* (= the set of relevant time instances).**
$\mathbb{W}$, **the *signal space* (= the variables that the model aims at).**
$\mathbb{L}$, **the *latent variable space* (= auxiliary modeling variables).**

$$\mathfrak{B}_{\text{full}} \subseteq (\mathbb{W} \times \mathbb{L})^{\mathbb{T}}$$ **: *the full behavior***

(= the pairs $(w, \ell) : \mathbb{T} \to \mathbb{W} \times \mathbb{L}$ that the model declares possible).

The manifest behavior

Call the elements of $\mathbb{W}$ ***'manifest' variables*** ,

those of $\mathbb{L}$ ***'latent' variables*** .

The latent variable system $\Sigma_L = (\mathbb{T}, \mathbb{W}, \mathbb{L}, \mathfrak{B}_{\text{full}})$ induces the *manifest system* $\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$, with *manifest behavior*

$$\mathfrak{B} = \{w : \mathbb{T} \to \mathbb{W} \mid \exists\ \ell : \mathbb{T} \to \mathbb{L} \text{ such that } (w, \ell) \in \mathfrak{B}_{\text{full}}\}$$

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In convenient equations for $\mathfrak{B}$, the latent variables are ***'eliminated'.***

Examples

1. The RLC - circuit **before elimination.**

Examples

1. The RLC - circuit

2. Models obtained by tearing and zooming

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3. Input / state / output systems

$$\frac{d}{dt}x(t) = f(x(t), u(t)); \quad y(t) = h(x(t), u(t)),$$

$\mathbb{T} = \mathbb{R}, \mathbb{W} = \mathbb{U} \times \mathbb{Y}, \mathbb{L} = \mathbb{X},$

$\mathfrak{B}_{\text{full}} =$ all $(u, y, x) : \mathbb{R} \to \mathbb{U} \times \mathbb{Y} \times \mathbb{X}$

that satisfy these equations,

$\mathfrak{B} =$ all (input / output)-pairs.

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4. Trellis diagrams

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Latent variables = the transition nodes;
the language generated = the manifest behavior

Examples

1. The RLC - circuit

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6. Grammars

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Central notions:

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- **(Full) behavioral equations**
  $\rightsquigarrow$ a specification of the (full) behavior.

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- Distinction between manifest and latent variables
  $\rightsquigarrow$ manifest behavior specifies what the model aims at.
- First principles models $\rightsquigarrow$ latent variables.
- (Full) behavioral equations
  $\rightsquigarrow$ a specification of the (full) behavior.
- Equivalent equations
  $:\Leftrightarrow$ the manifest behaviors are equal.

Linear differential systems

We now discuss the fundamentals of the theory of systems

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3. *differential,* meaning
   $\mathfrak{B}$ consists of the sol'ns of a system of differential eq'ns.

$$\rightsquigarrow \quad \boxed{R_0 w + R_1 \frac{d}{dt} w + \cdots + R_n \frac{d^n}{dt^n} w = 0,}$$

with $R_0, R_1, \cdots, R_n \in \mathbb{R}^{\bullet \times \mathtt{w}}$.

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with $R_0, R_1, \cdots, R_n \in \mathbb{R}^{\bullet \times w}$. With the polynomial matrix

$$R(\xi) = R_0 + R_1 \xi + \cdots + R_n \xi^n,$$

we obtain the short notation

$$\boxed{R(\frac{d}{dt}) w = 0.}$$

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But, the theory has also been developed for PDE's†.

† **by Oberst, Zerz, Shankar, Pillai, e.a.**

$\mathtt{n}$-D systems

$\mathbb{T} = \mathbb{R}^{\mathtt{n}}$, $\mathtt{n}$ **independent variables,**
$\mathbb{W} = \mathbb{R}^{\mathtt{w}}$, $\mathtt{w}$ **dependent variables,**
$\mathfrak{B}$ = **the sol'ns of a system of linear constant coeff. of PDE's.**

Let $R \in \mathbb{R}^{\bullet \times \mathtt{w}}[\xi_1, \cdots, \xi_{\mathtt{n}}]$, **and consider**

$$R(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathtt{n}}})w = 0 \quad (*)$$

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Define its behavior by

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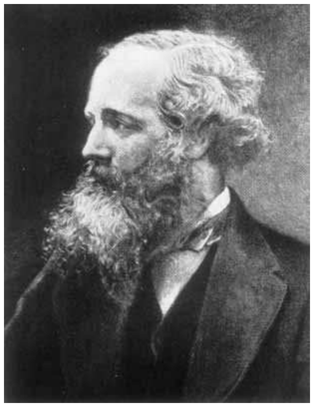
$\mathfrak{C}^{\infty}(\mathbb{R}^{\mathtt{n}}, \mathbb{R}^{\mathtt{w}})$ **mainly for convenience, but important for some results.**

Example: Maxwell's Equations



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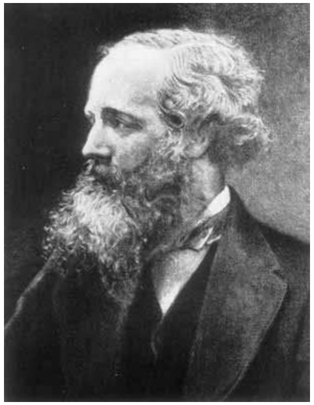

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$\mathbb{T} = \mathbb{R} \times \mathbb{R}^3$ **(time and space),**

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(electric field, magnetic field, current density, charge density),

$\mathbb{W} = \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}$,

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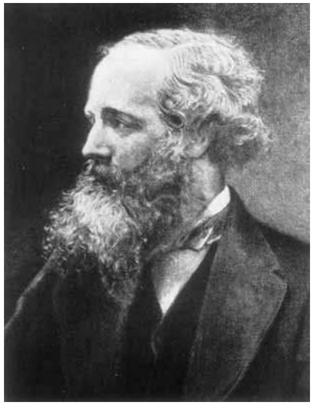

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Note: 10 variables, 8 equations! $\Rightarrow \exists$ free variables.

NOMENCLATURE

$\mathfrak{L}_{\mathtt{n}}^{\mathtt{w}}$: **the set of such systems with** $\mathtt{n}$ **independent, and** $\mathtt{w}$ **dependent variables**

$\mathfrak{L}^{\bullet}$: **with any - finite - number of (in)dependent variables**

Elements of $\mathfrak{L}^{\bullet}$ **:** ***'linear differential systems'***

$R(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathtt{n}}})w = 0$: **a *kernel representation* of the corresponding** $\Sigma \in \mathfrak{L}^{\bullet}$ **or** $\mathfrak{B} \in \mathfrak{L}^{\bullet}$

Algebraization of $\mathfrak{L}^\bullet$

Note that

$$R(\frac{d}{dt})w = 0$$

and

$$U(\frac{d}{dt})R(\frac{d}{dt})w = 0$$

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¿¿ $\exists$ 'intrinsic' characterization of $\mathfrak{B} \in \mathfrak{L}_{\mathtt{n}}^{\mathtt{w}}$??

Define the ***annihilators*** **of** $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\mathtt{n}}$ **by**

$$\mathfrak{N}_{\mathfrak{B}} := \{n \in \mathbb{R}^{\mathtt{w}}[\xi_1, \cdots, \xi_{\mathtt{n}}] \mid n^\top(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathtt{n}}})\mathfrak{B} = 0\}.$$

$\mathfrak{N}_{\mathfrak{B}}$ **is clearly an** $\mathbb{R}[\xi_1, \cdots, \xi_{\mathtt{n}}]$ **sub-module of** $\mathbb{R}^{\mathtt{w}}[\xi_1, \cdots, \xi_{\mathtt{n}}]$.

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Let $< R >:=$ the sub-module of $\mathbb{R}^{\mathtt{w}}[\xi_1, \cdots, \xi_{\mathtt{n}}]$ generated by the transposes of the rows of R. Obviously $< R > \subseteq \mathfrak{N}_{\mathfrak{B}}$. But, indeed:

$$\boxed{\mathfrak{N}_{\mathfrak{B}} = < R >!}$$

Define the *annihilators* of $\mathfrak{B} \in \mathfrak{L}_{\mathrm{n}}^{\mathtt{w}}$ by

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Note: Depends on $\mathfrak{C}^\infty$; false for compact support sol'ns: for any $p \neq 0$, $p(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathrm{n}}})w = 0$ has only $w = 0$ as compact support sol'n.

Conclusion

(i) $\boxed{\mathfrak{L}_{\mathtt{n}}^{\mathtt{w}} \xleftrightarrow{1:1} \text{sub-modules of } \mathbb{R}^{\mathtt{w}}[\xi_1, \cdots, \xi_{\mathtt{n}}]}$

Conclusion

(i) $\mathfrak{L}_{\mathtt{n}}^{\mathtt{w}} \overset{1:1}{\longleftrightarrow}$ sub-modules of $\mathbb{R}^{\mathtt{w}}[\xi_1, \cdots, \xi_{\mathtt{n}}]$

(ii) $R_1(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathtt{n}}})w = 0$ and $R_2(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathtt{n}}})w = 0$

define the same system iff

$$< R_1 > \; = \; < R_2 > .$$

i.e., iff $\exists \; F_1, F_2 \in \mathbb{R}^{\bullet \times \bullet}[\xi_1, \cdots, \xi_{\mathtt{n}}]$ such that

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(iii) For $\mathrm{n} = 1, \;\; \exists$ a 'minimal' R of full row rank, and $R \mapsto UR$, U unimodular, generates all minimal kernel representations.

Elimination

First principle models $\rightsquigarrow$ latent variables. In the case of systems described by linear constant coefficient PDE's: $\rightsquigarrow$

$$\boxed{R(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathrm{n}}})w = M(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathrm{n}}})\ell}$$

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This <u>is</u> the natural model class to start a study of finite dimensional linear time-invariant systems! Much more so than

$$\frac{d}{dt}x = Ax + Bu, \quad y = Cx + Du.$$

But is it(s manifest behavior) really a differential system ??

Consider $R(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathrm{n}}})w = M(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathrm{n}}})\ell$.

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Full behavior:

$$\mathfrak{B}_{\mathrm{full}} = \{(w, \ell) \in \mathfrak{C}^\infty(\mathbb{R}^{\mathtt{n}}, \mathbb{R}^{\mathtt{w}+\ell}) \mid \cdots\}.$$

belongs to $\mathfrak{L}_{\mathtt{n}}^{\mathtt{w}+\ell}$**, by definition.**

Its manifest behavior equals

$$\mathfrak{B} = \{w \in \mathfrak{C}^\infty(\mathbb{R}^{\mathtt{n}}, \mathbb{R}^{\mathtt{w}}) \mid \exists \ \ell \ \text{such that} \ \cdots\}.$$

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The fundamental principle states that

$$F(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathrm{n}}})x = y$$

$F \in \mathbb{R}^{\mathrm{n}_1 \times \mathrm{n}_2}[\xi_1, \cdots, \xi_{\mathrm{n}}]$, $y \in \mathfrak{C}^\infty(\mathbb{R}^{\mathrm{n}}, \mathbb{R}^{\mathrm{n}_1})$ **is solvable for** $x \in \mathfrak{C}^\infty(\mathbb{R}^{\mathrm{n}}, R^{\mathrm{n}_2})$ **iff**

$$n \in \mathbb{R}^{\mathrm{n}_1}[\xi_1, \cdots, \xi_{\mathrm{n}}] \wedge n^\top F = 0 \quad \Rightarrow \quad n^\top(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathrm{n}}})y = 0.$$

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Respect the uncontrollable!

$\rightsquigarrow$ **the elimination algorithms**

Calculations via transfer f'ns may give erroneous results.

Example: Consider the RLC circuit.

First principles modeling ($\cong$ CE's, KVL, & KCL)

$\leadsto$ 15 behavioral equations.

Include both the port and the branch voltages and currents.

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Why can the port behavior be described by a system of linear constant coefficient differential equations?

Because:

1. The CE's, KVL, & KCL are all linear constant coefficient differential equations.
2. The elimination theorem[†].

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Why is there *exactly one* equation? Passivity!

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Elimination theorem $\Rightarrow$

this exercise would be exact & successful.

Remarks:

- **Number of eq'ns (for $n = 1$: constant coeff. lin. ODE's)**
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- **Not generalizable to smooth nonlinear systems.**
  Why are differential equations models so prevalent?

It follows from all this that $\mathfrak{L}^\bullet$ has very nice properties. It is closed under:

- **Intersection:** $(\mathfrak{B}_1, \mathfrak{B}_2 \in \mathfrak{L}^{\mathtt{w}}_{\mathtt{n}}) \Rightarrow (\mathfrak{B}_1 \cap \mathfrak{B}_2 \in \mathfrak{L}^{\mathtt{w}}_{\mathtt{n}})$.
- **Addition:** $(\mathfrak{B}_1, \mathfrak{B}_2 \in \mathfrak{L}^{\mathtt{w}}_{\mathtt{n}}) \Rightarrow (\mathfrak{B}_1 + \mathfrak{B}_2 \in \mathfrak{L}^{\mathtt{w}}_{\mathtt{n}})$.
- **Projection:** $(\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}_1+\mathtt{w}_2}_{\mathtt{n}}) \Rightarrow (\Pi_{w_1}\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}_1}_{\mathtt{n}})$.
- **Action of a linear differential operator:**
$$(\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}_1}_{\mathtt{n}}, P \in \mathbb{R}^{\mathtt{w}_2 \times \mathtt{w}_1}[\xi_1, \cdots, \xi_{\mathtt{n}}]) \Rightarrow (P(\tfrac{\partial}{\partial x_1}, \cdots, \tfrac{\partial}{\partial x_{\mathtt{n}}})\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}_2}_{\mathtt{n}}).$$
- **Inverse image of a linear differential operator:**
$$(\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}_2}_{\mathtt{n}}, P \in \mathbb{R}^{\mathtt{w}_2 \times \mathtt{w}_1}[\xi_1, \cdots, \xi_{\mathtt{n}}]) \Rightarrow (P(\tfrac{\partial}{\partial x_1}, \cdots, \tfrac{\partial}{\partial x_{\mathtt{n}}}))^{-1}\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}_1}_{\mathtt{n}}).$$

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Controllability $:\Leftrightarrow$

system trajectories must be ‘patch-able’, ‘concatenable’.

Case $n = 1$:

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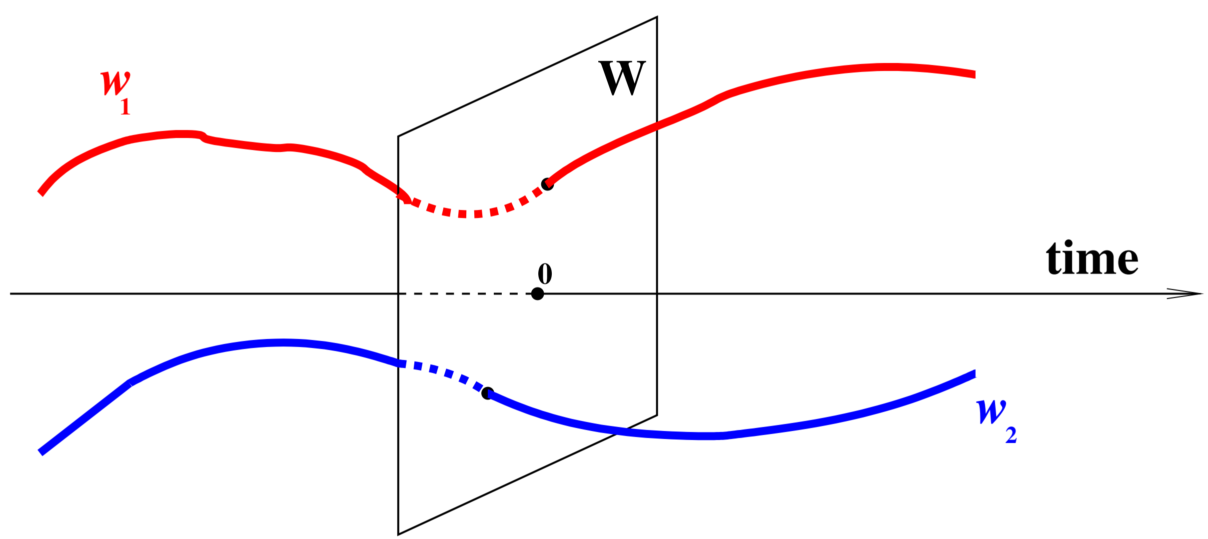
Case $\mathrm{n} = 1$:

Consider two arbitrary elements $w_1, w_2 \in \mathfrak{B}$



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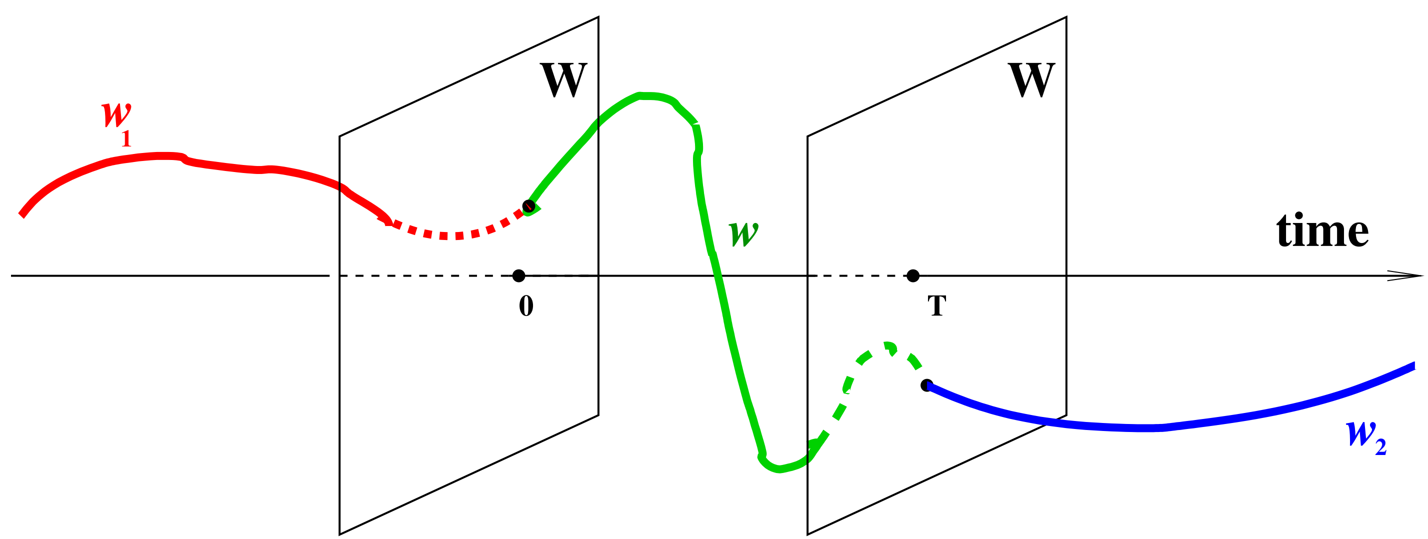
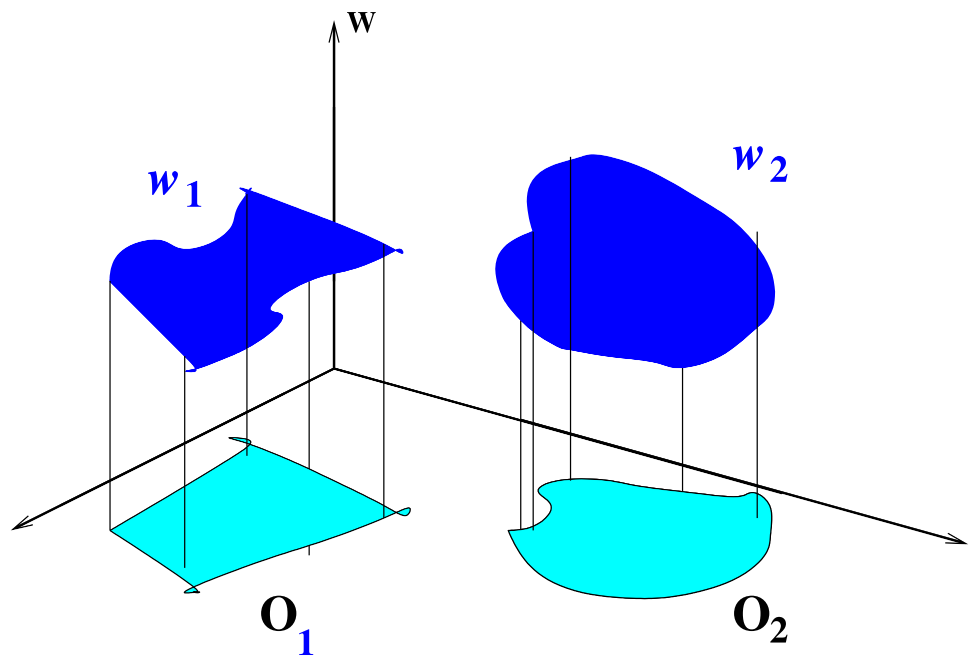
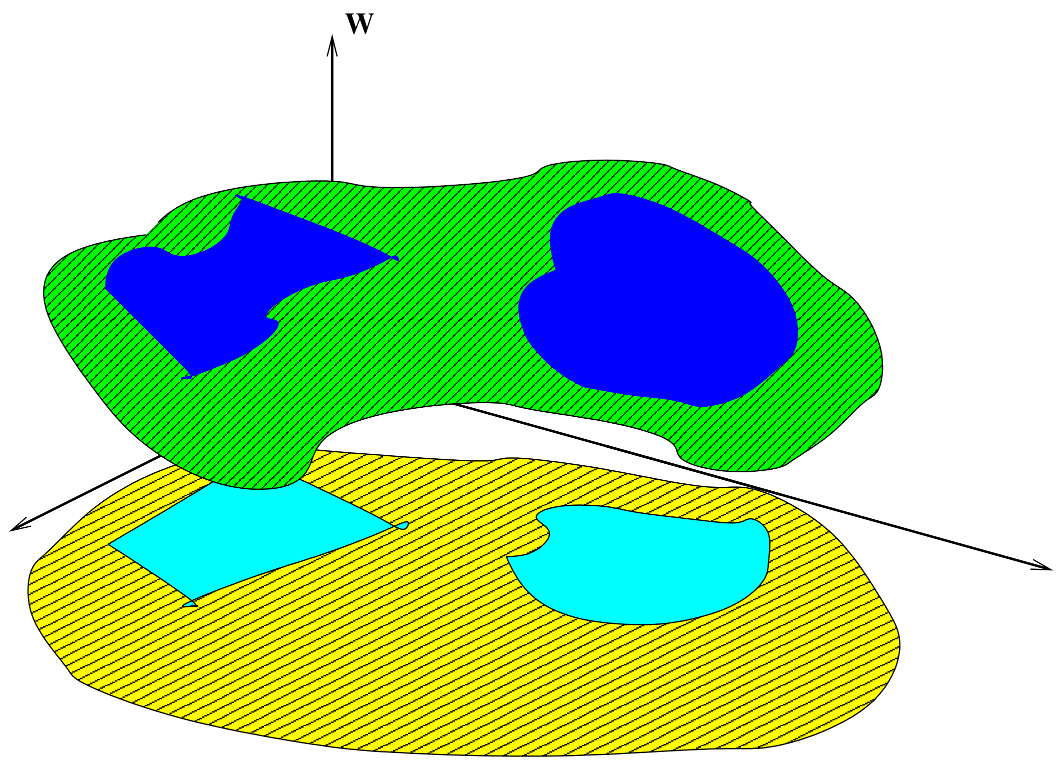


General n:

Consider arbitrary patches of two solutions:



Controllability := patchability



Is the system defined by

$$R_0 w + R_1 \frac{d}{dt} w + \cdots + R_{\mathtt{n}} \frac{d^{\mathtt{n}}}{dt^{\mathtt{n}}} w = 0,$$

with $w = (w_1, w_2, \cdots, w_{\mathtt{w}})$ **and** $R_0, R_1, \cdots, R_{\mathtt{n}} \in \mathbb{R}^{\bullet \times \mathtt{w}}$,
i.e., $R(\frac{d}{dt})w = 0$, **controllable?**

Is the system defined by

$$R_0 w + R_1 \frac{d}{dt} w + \cdots + R_{\mathtt{n}} \frac{d^{\mathtt{n}}}{dt^{\mathtt{n}}} w = 0,$$

with $w = (w_1, w_2, \cdots, w_{\mathtt{w}})$ **and** $R_0, R_1, \cdots, R_{\mathtt{n}} \in \mathbb{R}^{\bullet \times \mathtt{w}}$,
i.e., $R(\frac{d}{dt})w = 0$, **controllable?**

We are looking for conditions on the polynomial matrix R**, and algorithms in the coefficient matrices** $R_0, R_1, \cdots, R_{\mathtt{n}}$**.**

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Non-example: $R \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi], \quad \det(R) \neq$ constant.

Image representations

Representations of $\mathfrak{L}_{\mathtt{n}}^{\mathtt{w}}$:

$$\boxed{R(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathtt{n}}})w = 0}$$

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$$R(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathtt{n}}})w = M(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathtt{n}}})\ell$$

called a *'latent variable' representation* of the manifest behavior

$$\mathfrak{B} = (R(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathtt{n}}}))^{-1} M(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathtt{n}}})\mathfrak{C}^{\infty}(\mathbb{R}^{\mathtt{n}}, \mathbb{R}^{\ell}).$$

Missing link:

$$w = M(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathrm{n}}})\ell$$

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¿¿ Which kernels are also images ??

Theorem: The following are equivalent for $\mathfrak{B} \in \mathfrak{L}_{\mathtt{n}}^{\mathtt{w}}$:

1. $\mathfrak{B}$ is controllable,
2. $\mathfrak{B}$ admits an image representation,
3. for any $a \in \mathbb{R}^{\mathtt{w}}[\xi_1, \cdots, \xi_{\mathtt{n}}]$,
$$a^{\top}[\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathtt{n}}}]\mathfrak{B} \text{ equals } 0 \text{ or all of } \mathfrak{C}^{\infty}(\mathbb{R}^{\mathtt{n}}, \mathbb{R}),$$
4. $\mathbb{R}^{\mathtt{w}}[\xi_1, \cdots, \xi_{\mathtt{n}}]/\mathfrak{N}_{\mathfrak{B}}$ is torsion free,

etc.

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The following equations in the *scalar potential* $\phi : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$ and the *vector potential* $\vec{A} : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$, generate exactly the solutions to Maxwell's equations:

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\begin{aligned}
\vec{E} &= -\frac{\partial}{\partial t}\vec{A} - \nabla\phi, \\
\vec{B} &= \nabla \times \vec{A}, \\
\vec{j} &= \varepsilon_0 \frac{\partial^2}{\partial t^2}\vec{A} - \varepsilon_0 c^2 \nabla^2 \vec{A} + \varepsilon_0 c^2 \nabla(\nabla \cdot \vec{A}) + \varepsilon_0 \frac{\partial}{\partial t}\nabla\phi, \\
\rho &= -\varepsilon_0 \frac{\partial}{\partial t}\nabla \cdot \vec{A} - \varepsilon_0 \nabla^2 \phi.
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Proves controllability. Illustrates the interesting connection

controllability $\Leftrightarrow \exists$ potential!

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- Algorithm: R + syzygies + Gröbner basis

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- $\exists$ partial results for nonlinear systems
- Kalman controllability is a straightforward special case

Observability

Consider the system $\Sigma = (\mathbb{T}, \mathbb{W}_1 \times \mathbb{W}_2, \mathfrak{B})$.

Each element of the behavior $\mathfrak{B}$ hence consists of
a pair of trajectories (w_1, w_2).

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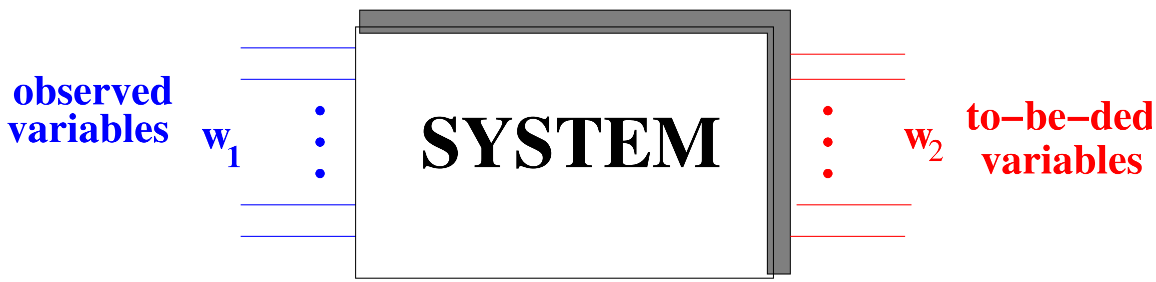
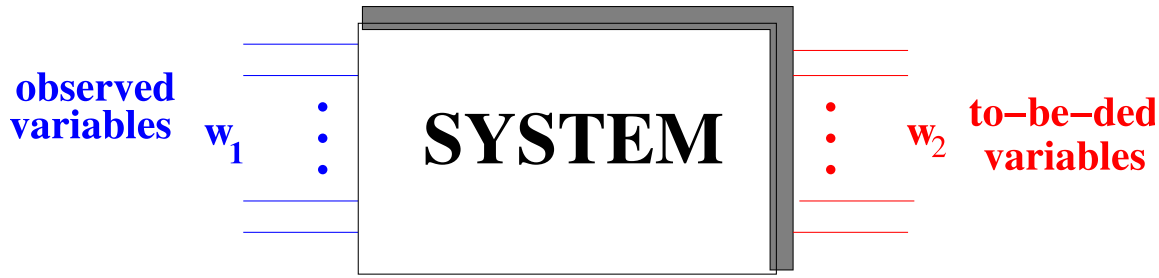


w_1 : observed; w_2 : to-be-deduced.



w_2 **is said to be** ***observable*** **from** w_1

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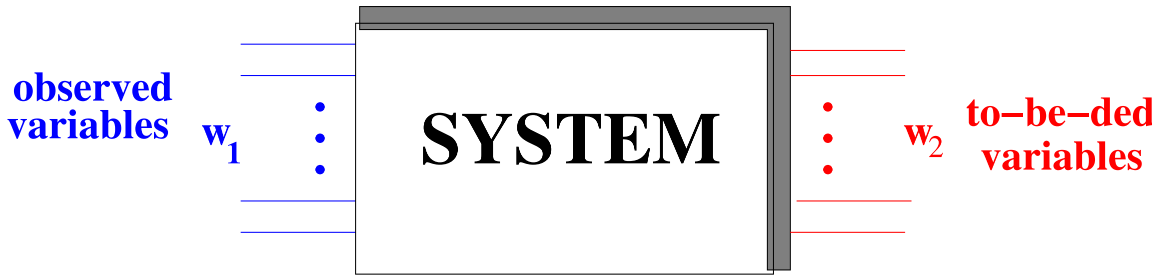

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i.e., if, on $\mathfrak{B}$, there exists a map $w_1 \mapsto w_2$.

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Equivalently, if and only if there exists 'consequences'
(i.e. elements of $\mathfrak{N}_{\mathfrak{B}}$**) of the form** $w_2 = F(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n})w_1$**.**

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Observability is analogous (but not 'dual') to controllability.

Controllability & Observability

Call a latent variable systems

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Observable image representations of - of course - controllable systems are sometimes called differentially 'flat'†.

† Cfr. Fliess c.s.

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- **Observers, Control**
- Quadratic differential forms, dissipative systems, $\mathcal{H}_\infty$-control

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- Observability := deducing one variable from another

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Examples:

- **Probability and the theory of stochastic processes as an axiomatization of uncertainty.**
- **The development of input/output ideas in system theory and control - often these axiomatics are implicit, but nevertheless much very present.**
- **QM.**

Thank you for your kind attention

Details & copies of the lecture frames are available from/at

`Jan.Willems@esat.kuleuven.ac.be`

`http://www.esat.kuleuven.ac.be/~jwillems`