CONTROL as INTERCONNECTION

Madhu N. Belur (University of Groningen)

A. Agung Julius (University of Twente)

Jan C. Willems (SISTA)

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Program

- Part 1: Problem formulation
  - Jan Willems (SCD-SISTA, KU Leuven)
- Part 2: Controller implementability
  - Agung Julius (Twente University, NL)
- Part 3: Synthesis of dissipative systems
  - Madhu Belur (University of Groningen, NL)

Dissipativity

Let $\boldsymbol{\Sigma} = \boldsymbol{\Sigma}^T \in \mathbb{R}^{\mathtt{w} \times \mathtt{w}}$ be nonsingular.
A controllable behavior $\mathfrak{B}$ is called dissipative with respect to $\boldsymbol{\Sigma}$ if

$$\int_{-\infty}^{+\infty} \boldsymbol{w}^T \boldsymbol{\Sigma} \boldsymbol{w} \, \boldsymbol{dt} \geqslant \mathbf{0} \text{ for all } \boldsymbol{w} \in \mathfrak{B} \cap \mathfrak{D}$$

We say $\mathfrak{B}$ is $\boldsymbol{\Sigma}$-dissipative (on $\mathbb{R}$).

Half line dissipativity

$\mathfrak{B}$ is called Σ-dissipative on $\mathbb{R}_-$ if

$$\int_{-\infty}^{0} w^T \Sigma w \, dt \geqslant 0 \text{ for all } w \in \mathfrak{B} \cap \mathfrak{D}$$

Example: $\mathcal{H}_\infty$ norm

Consider $\boldsymbol{\Sigma} = \begin{bmatrix} \boldsymbol{I_d} & \boldsymbol{0} \\ \boldsymbol{0} & -\boldsymbol{I_f} \end{bmatrix}$

and $\mathfrak{B}$ described by a transfer function $\boldsymbol{G}$ acting on input $\boldsymbol{d}$ and output $\boldsymbol{f}$.

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Then $\boldsymbol{\Sigma}$-dissipativity of $\mathfrak{B}$ on $\mathbb{R}_-$ is, by definition,

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$\mathfrak{B}$ is Σ-dissipative on $\mathbb{R}_-$ if and only if $||G||_{\mathcal{H}_\infty} \leqslant 1$.

$$||G||_{\mathcal{H}_\infty} := \sup_{\omega \in \mathbb{R}} \sigma_{\max}(G(i\omega)).$$

Example: Passive systems

Consider $\Sigma = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$

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$$\int_{-\infty}^{0} u^T y \, dt \geqslant 0 \text{ for all } (u, y) \in \mathfrak{B} \cap \mathfrak{D}.$$

$\mathfrak{B}$ is Σ-dissipative on $\mathbb{R}_-$ if and only if G is stable and positive real, i.e.

$$G(\lambda) + G(\bar{\lambda}) \geqslant 0 \text{ for all } \lambda \text{ with positive real part.}$$

Stability and half-line dissipativity

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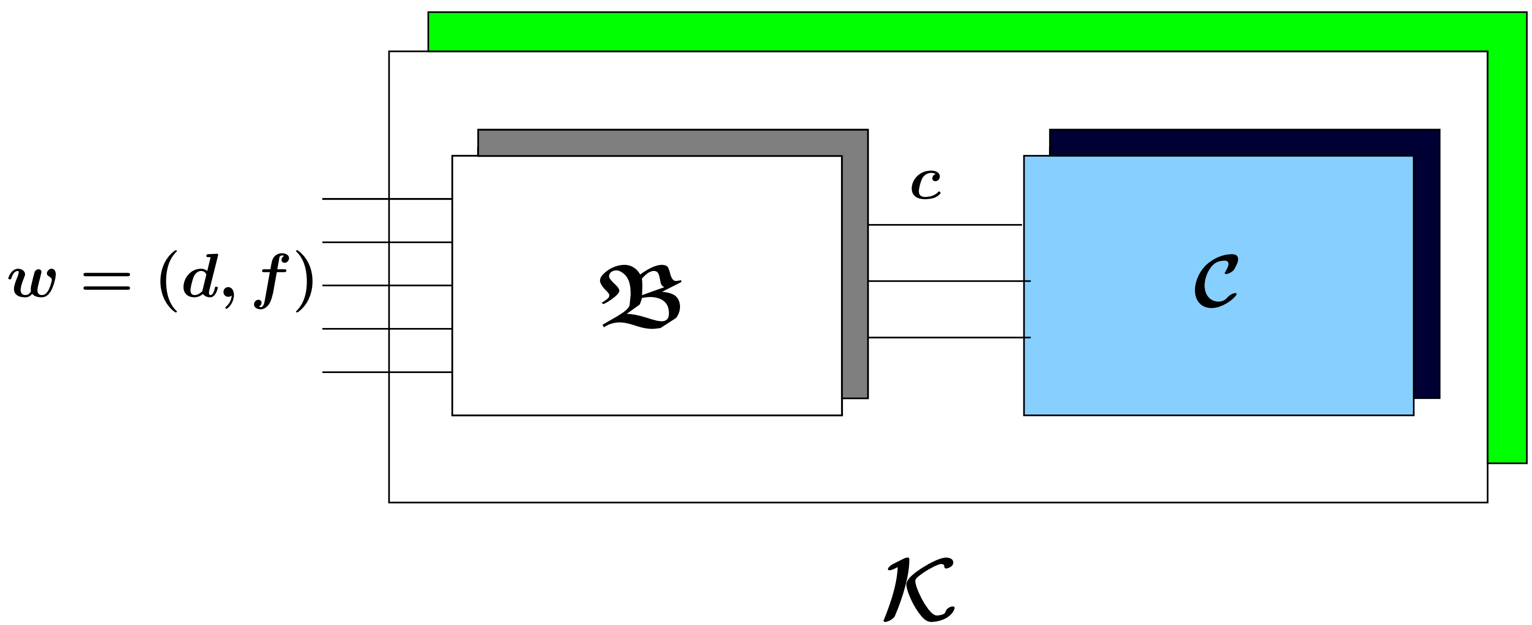
Energy upto any time instant has been absorbed.

Dissipativity synthesis problem formulation

Assume (for the sake of exposition)

$$\Sigma = \begin{bmatrix} I_d & 0 \\ 0 & -I_f \end{bmatrix}$$



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Some necessary conditions

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Moreover,
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$\mathcal{K} \subset \mathcal{P}$ implies that $\mathcal{P}^\perp \subset \mathcal{K}^\perp$ and hence
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Sufficient ?

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Storage functions!!

Quadratic differential form

Let $w \in \mathfrak{B}$. A quadratic differential form Q_Ψ in w is a quadratic function of w and a finite number of its derivatives:

$$Q_\Psi(w) := \begin{bmatrix} w \\ \frac{d}{dt}w \\ \vdots \\ (\frac{d}{dt})^k w \\ \vdots \end{bmatrix}^T \begin{bmatrix} \Psi_{00} & \Psi_{01} & \cdots & \Psi_{0l} & \cdots \\ \Psi_{10} & \Psi_{11} & \cdots & \Psi_{1l} & \cdots \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \Psi_{k0} & \Psi_{k1} & \cdots & \Psi_{kl} & \cdots \\ \vdots & & & \vdots & \end{bmatrix} \begin{bmatrix} w \\ \frac{d}{dt}w \\ \vdots \\ (\frac{d}{dt})^l w \\ \vdots \end{bmatrix}$$

Dissipativity and storage function

Recall that $\mathfrak{B}$ is called Σ-dissipative if

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Theorem: $\mathfrak{B}$ is dissipative if and only if there exists a quadratic differential form Q_{Ψ} such that $\frac{d}{dt} Q_{\Psi}(w) \leqslant w^T \Sigma w$ for all $w \in \mathfrak{B}$.

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The global property of dissipativiness holds if and only if

a particular local property holds.

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Q_Ψ is called a storage function for $\mathfrak{B}$ with respect to Σ.

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Moreover, $\mathfrak{B}$ is dissipative on $\mathbb{R}_-$ $\Leftrightarrow$ there exists a storage function Q_Ψ such that

$$Q_\Psi(w) \geqslant 0 \text{ for all } w \in \mathfrak{B}.$$

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Construct the quadratic differential form:

$$Q_{\mathrm{cpl}}(w_1, w_2) := Q_{\Psi_{\mathcal{N}}}(w_1) - Q_{\Psi_{\mathcal{P}^{\perp}}}(w_2) + L_{\Psi}(w_1, w_2)$$

L_{Ψ} is a cross-term which comes from orthogonality of $\mathcal{N}$ and $\mathcal{P}^{\perp}$.

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1. $\mathcal{N}$ is Σ-dissipative,
2. $\mathcal{P}^{\perp}$ is $(-\Sigma)$-dissipative,
3. the coupling QDF $Q_{\mathbf{cpl}}$ on $\mathcal{N} \times \mathcal{P}^{\perp}$

$$Q_{\mathbf{cpl}}(w_1, w_2) = Q_{\Psi_{\mathcal{N}}}(w_1) - Q_{\Psi_{\mathcal{P}^{\perp}}}(w_2) + L_{\Psi_{(\mathcal{N},\mathcal{P}^{\perp})}}(w_1, w$$

is non-negative for all $w_1 \in \mathcal{N}$ and $w_2 \in \mathcal{P}^{\perp}$.

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All conditions are actually LMI's!

State space $\mathcal{H}_\infty$-control problem

Let $\mathfrak{B}$ be described by

$$\begin{aligned}
\frac{d}{dt}x &= Ax + Bu + Gd_1 \\
y &= Cx + d_2 \\
z &= \begin{bmatrix} Hx \\ u \end{bmatrix}
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A state-space representation for $\mathcal{N}$ is:

$$
\begin{array}{rcll}
\frac{d}{dt}x & = & Ax & + \; Gd_1 \\
d_2 & = & -Cx & \\
z & = & \begin{bmatrix} Hx \\ 0 \end{bmatrix} &
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State space $\mathcal{H}_\infty$-control problem

Under what conditions, does

$$\int_{-\infty}^{0} |d_1|^2 + |Cx|^2 - |Hx|^2 \geqslant 0$$

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$$\int_{-\infty}^{0} |d_1|^2 + |Cx|^2 - |Hx|^2 \geqslant 0$$

This is equivalent to the existence of a positive definite solution P to the following Riccati equation:

$$A^T P + PA - PGG^T P - C^T C + H^T H = 0$$

State space $\mathcal{H}_\infty$-control problem

Similarly, $-\Sigma$-dissipativity of $\mathcal{P}^\perp$ is equivalent to existence of a negative definite solution Q to the dual Riccati equation.

$$AQ + QA^T - QH^THQ - GG^T + BB^T = 0$$

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The coupling condition is

$$\rho(PQ) \geqslant 1$$

for some solutions P and Q of the above Riccati equations, respectively.

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Thank you for your attention