CONSERVED and ZERO-MEAN QUANTITIES in OSCILLATORY SYSTEMS

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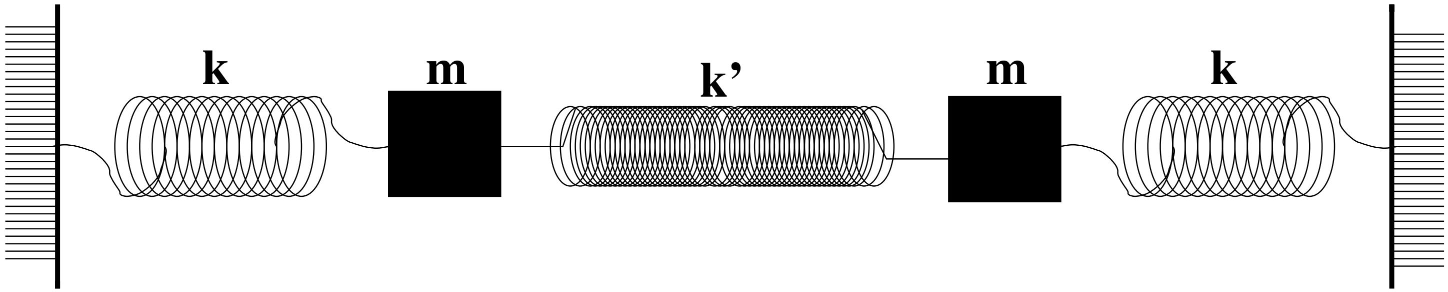
System Theory Day, University of Groningen **June 27, 2003**

Joint research with Jan

Let us start by looking at the responses of some simple mechanical systems.

Simulations



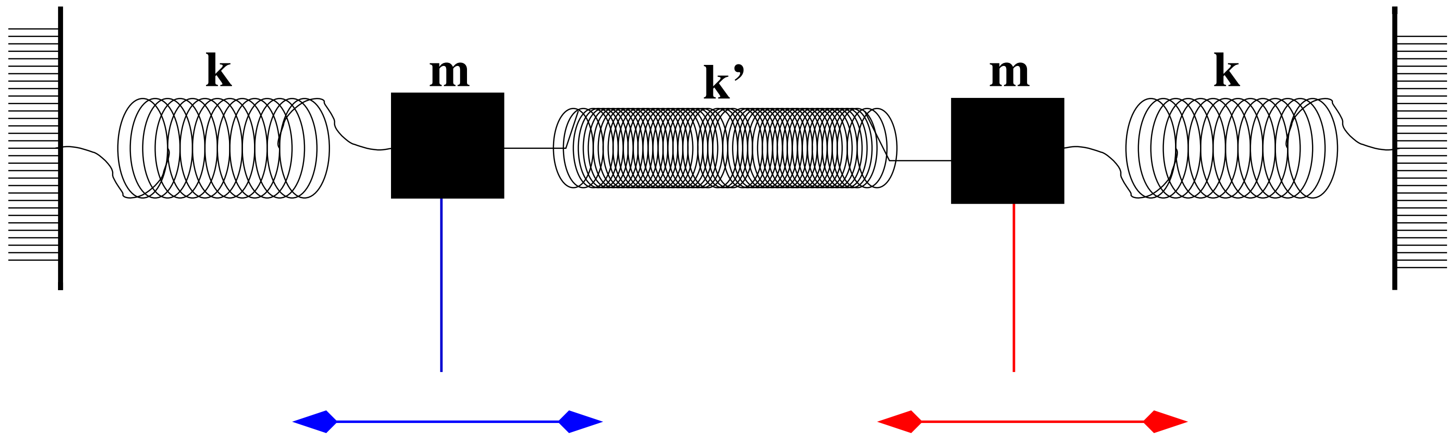
Take $m = 13,\ k = 7,\ k' = 0,001.$

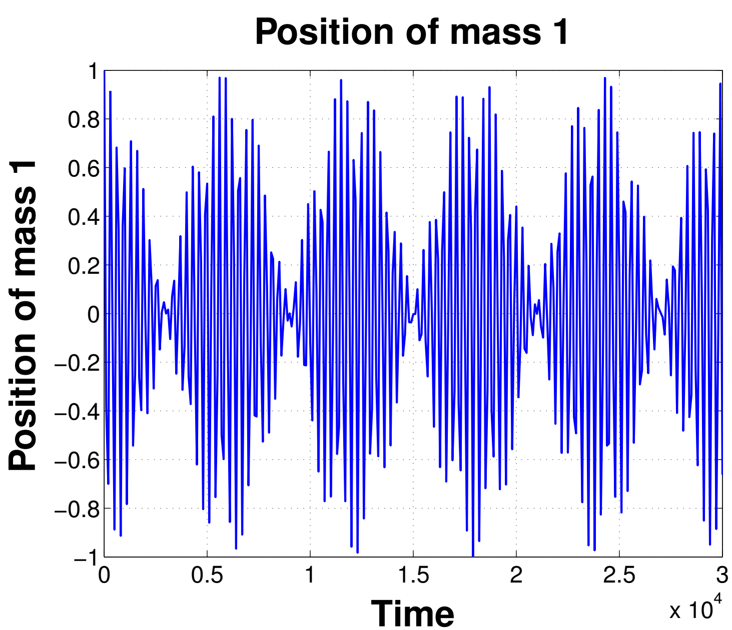
at $t = 0$: **displacement left mass = 1,**

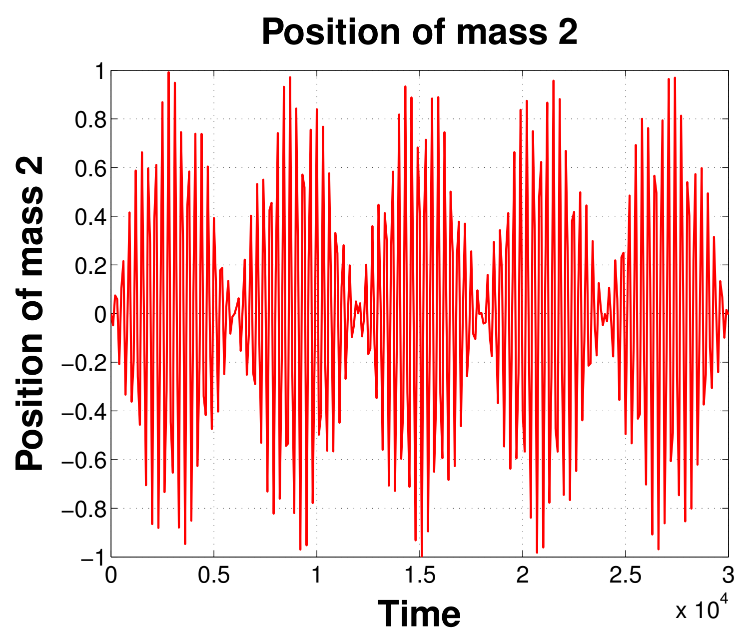
velocities, displacements of other mass = 0.

Compute the displacements of both masses.

Simulations

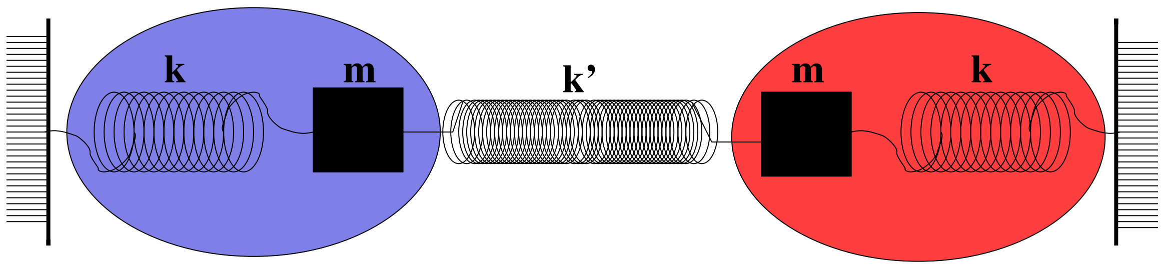






Simulations

Compute actual energy, and average energy, in the two oscillators.



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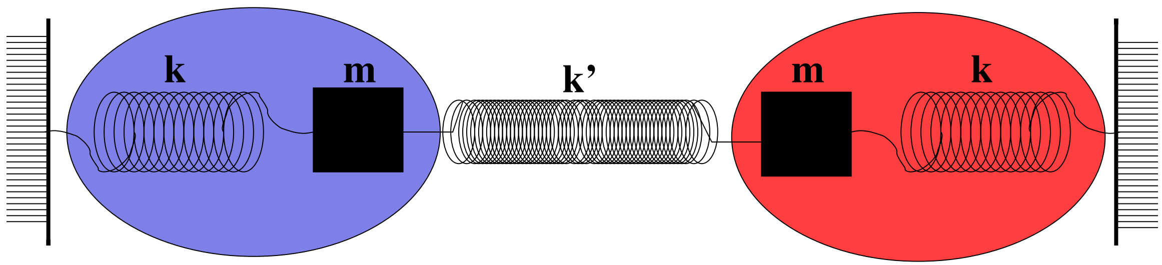
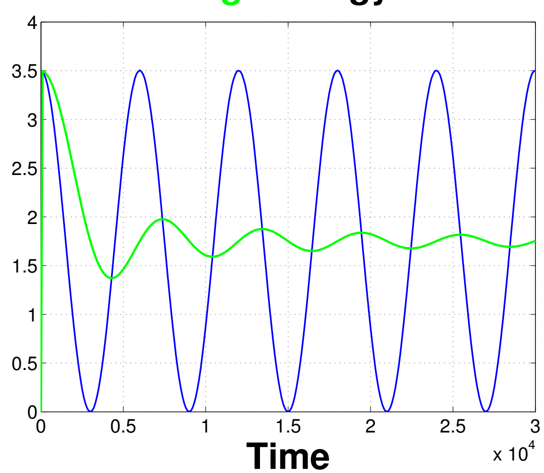
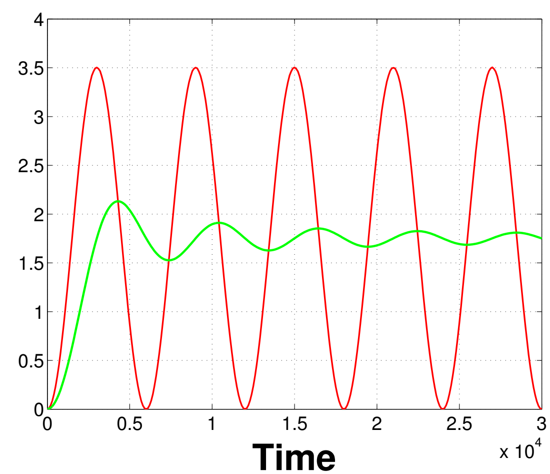


Actual and average energy in oscillator 1



Actual and average energy in oscillator 2



Note that the averages are asymptotically equal - 'equipartition'!

Simulations

Compute the total energy in the system.

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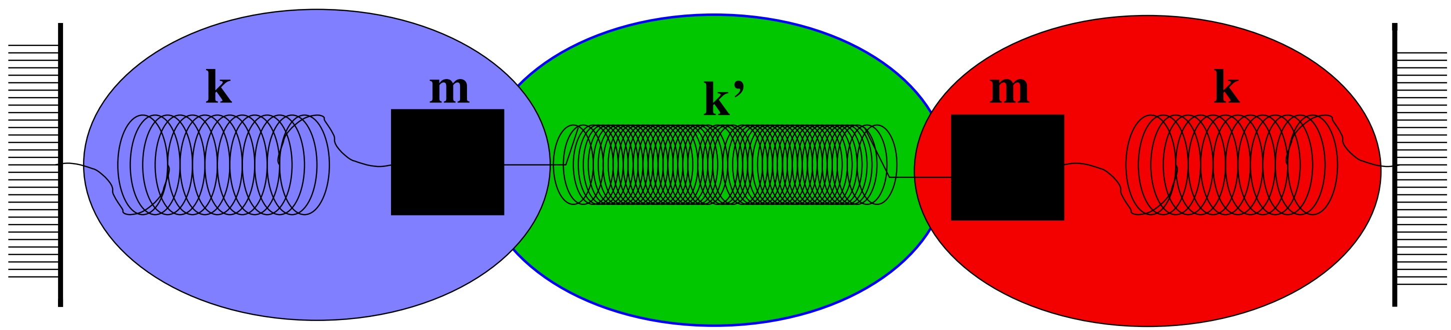

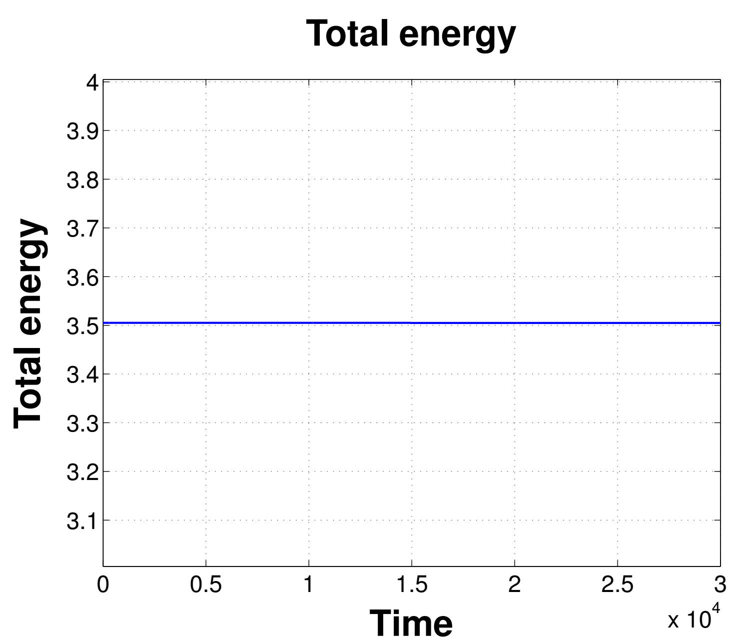

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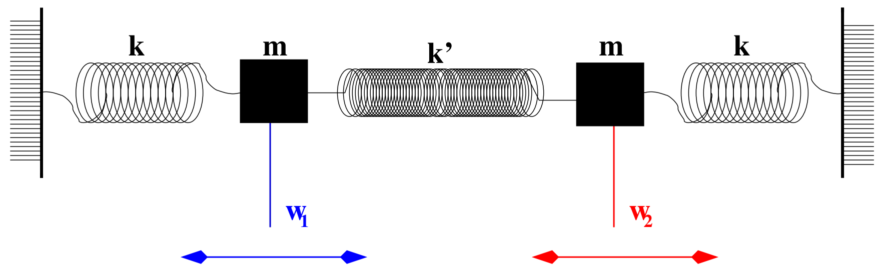

Now compute the evolution of the following quadratic expression:

$$(2k + k')(w_1^2 + w_2^2) + 2(k - k')w_1 w_2 + 2m\left(\left(\frac{dw_1}{dt}\right)^2 + \left(\frac{dw_2}{dt}\right)^2 + \frac{dw_1}{dt}\frac{dw_2}{dt}\right)$$

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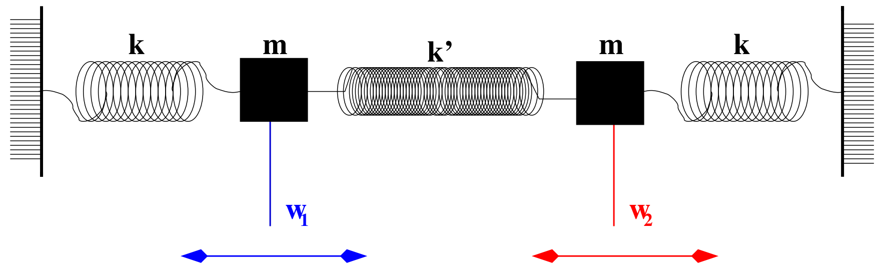

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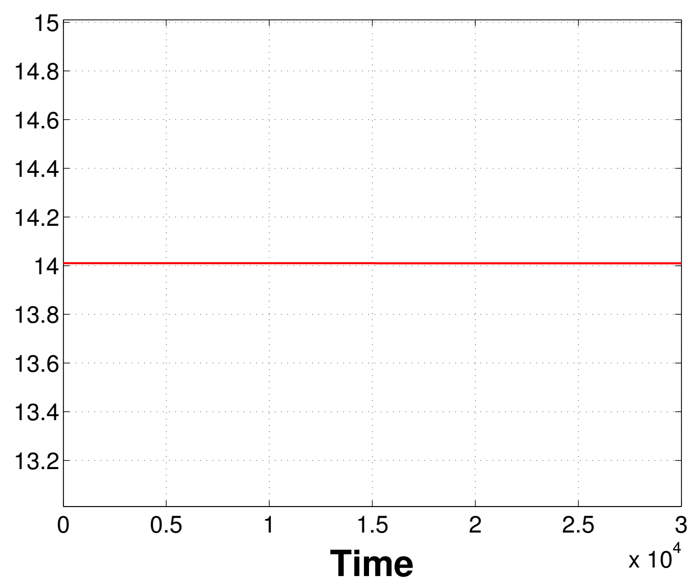

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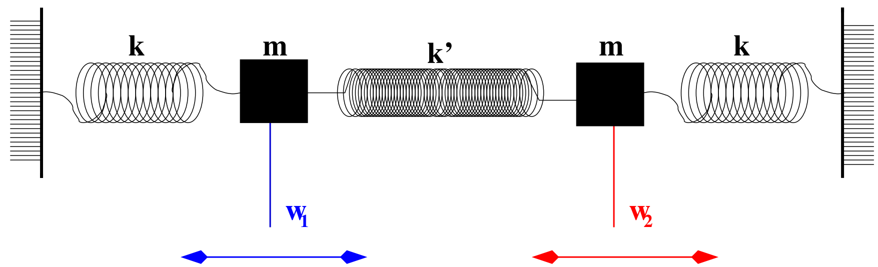

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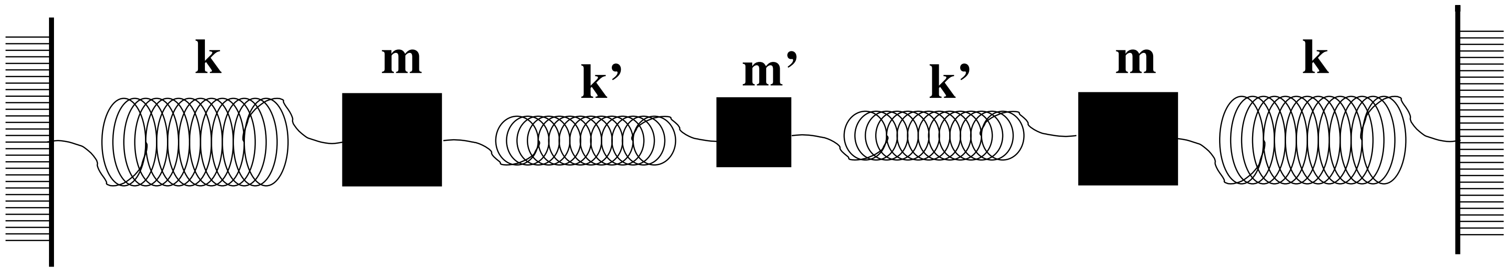
There are other conserved quantities than the total energy!

Is this always the case?

Simulations



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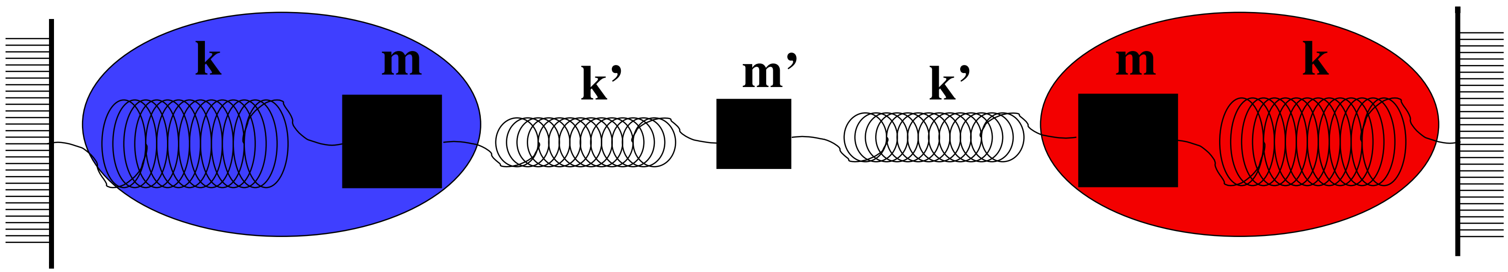

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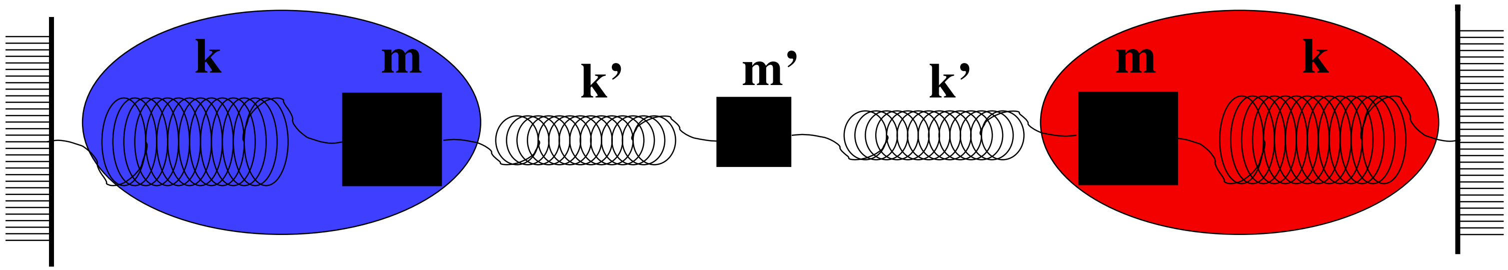

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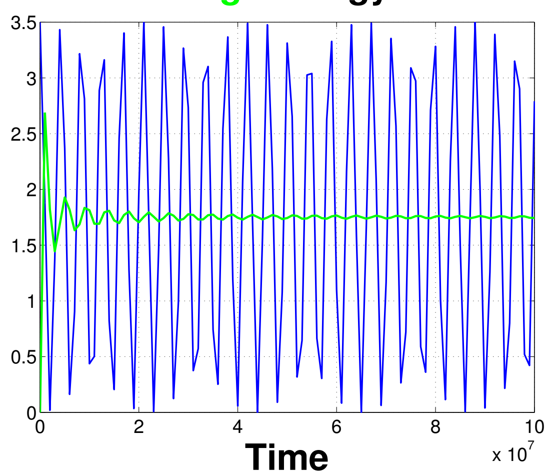

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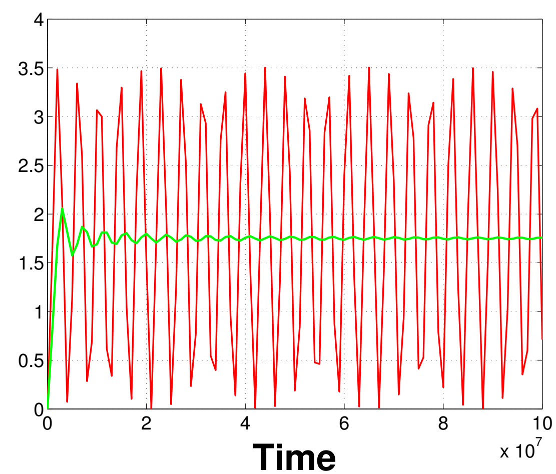

The averages are again asymptotically equal - 'equipartition'!

CONJECTURE

The average energy in symmetrically coupled identical oscillators is the same for each oscillator.

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The average energy in
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We call this the

Deterministic 'Equipartition of Energy' principle,

following a very nice idea of Bernstein and Bhat (CDC 2002)

usual statistical average $\mapsto$ time-average.

Setting the stage

A linear differential system (or *'behavior'*) is *'oscillatory'* if all trajectories in $\mathfrak{B}$ are quasi-periodic.

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Formal definition:

The behavior $\mathfrak{B}$ defines a linear ***'oscillatory system'*** **:$\Leftrightarrow$**

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$$R \in \mathbb{R}^{\bullet \times \mathtt{w}}[\xi]$$

- **every solution $w : \mathbb{R} \to \mathbb{R}^{\mathtt{w}}$ is bounded (on $(-\infty, \infty)$!!).**

Setting the stage

Examples of oscillatory behaviors:

- **any vector of displacements and velocities in a spring-mass mechanical system.**
- **any vector of voltages and currents in any capacitor or inductor in a LC (and LCTG) electrical circuit.**
- **the behavior of any output of $\frac{d}{dt}x = Ax, y = Cx$ with A oscillatory (some positive definite quadratic form $x^\top Qx$ is invariant).**
  Cfr. (linearized) Lagrangian or Hamiltonian mechanics.

Setting the stage

For simplicity of exposition (and WLOG!), today, mostly $\mathtt{w} = 1$.

<u>Proposition:</u>

$$R(\frac{d}{dt})w = 0 \qquad 0 \neq R \in \mathbb{R}[\xi]$$

defines an oscillatory system

if and only if

all the roots of R are distinct and on the imaginary axis.

For simplicity of exposition, today:

R has no roots in origin $\rightarrow$ R is an even polynomial.

Setting the stage

Hence we consider, with slight abuse of notation,

$$\boxed{R(\frac{d^2}{dt^2})w = 0.}$$

with $0 \neq R \in \mathbb{R}[\xi]$, roots real, negative, and distinct.
Set $\mathtt{n} := \text{degree}(R)$, (whence $\dim(\mathfrak{B}) = 2\mathtt{n}$).

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So, we effectively assume that each element of $\mathfrak{B}$ looks like

$$w(t) = \Sigma_{\mathrm{k}=1,\ldots,\mathrm{n}} A_{\mathrm{k}} \sin(\omega_{\mathrm{k}} t + \phi_{\mathrm{k}}),$$

with the $-\omega_{\mathrm{k}}^2$'s the roots of R, and the A_{k}'s, ϕ_{k}'s arbitrary.
'quasi-periodic' functions

QDF's

The real two-variable polynomial

$$\Phi(\zeta,\eta) = \Sigma_{k,k'} \Phi_{k,k'} \zeta^{k} \eta^{k'}$$

induces the map

$$w \in \mathfrak{C}^{\infty}(\mathbb{R},\mathbb{R}) \;\mapsto\; \Sigma_{k,k'} \left(\frac{d^{k}}{dt^{k}} w\right) \Phi_{k,k'} \left(\frac{d^{k'}}{dt^{k'}} w\right) \in \mathfrak{C}^{\infty}(\mathbb{R},\mathbb{R}),$$

called a *a 'quadratic differential form'* (QDF), denoted as Q_Φ.

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Let $\mathfrak{B}$ be a behavior. Call $\Phi_1, \Phi_2 \in \mathbb{R}[\zeta,\eta]$ *'$\mathfrak{B}$-equivalent'* :⇔

$$w \in \mathfrak{B} \;\Rightarrow\; Q_{\Phi_1}(w) = Q_{\Phi_2}(w)$$

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==================

Let $\mathfrak{B}(\dim=2n)$ be oscillatory. Each mod-$\mathfrak{B}$ eq. class contains exactly one Q_{Φ} with the highest k,k' in the above sum $<2n$.

Hence the QDF's modulo $\mathfrak{B}\cong$ the symmetric $2n\times 2n$ matrices,

$\cong$ the quadratic forms in $w,\frac{d}{dt}w,\ldots,\frac{d^{2n-1}}{dt^{2n-1}}w$.

Conserved and Zero-mean QDF's

Let $\mathfrak{B}$ be an oscillatory system.

Call Q_Φ *'conserved'* :$\Leftrightarrow$

$$\boxed{w \in \mathfrak{B} \quad \Rightarrow \quad \frac{d}{dt}Q_\Phi(w) = 0}$$

Conserved and Zero-mean QDF's

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Call Q_Φ **'trivially zero-mean'** :⇔

$$w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}), \text{ quasi-periodic} \quad \Rightarrow \quad \text{limit}_{T \to \infty} \frac{1}{T} \int_0^T Q_\Phi(w)(t)\, dt = 0$$

zero-mean for all oscillatory systems, not just $\mathfrak{B}$.

Conserved and Zero-mean QDF's

Note that for any $\Phi \in \mathbb{R}[\zeta, \eta]$, and any $w \in \mathfrak{B}$, the limit

$$\text{limit}_{\,T \to \infty} \quad \frac{1}{T} \int_0^T Q_\Phi(w)(t)\,dt$$

exists, and is obviously a conserved QDF.

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$$\text{limit}_{T\to\infty} \frac{1}{T}\int_0^T Q_\Phi(w)(t)\,dt =: Q_{\Phi_{\text{average}}}(w)(0)$$

exists, and is obviously a conserved QDF.

This yields the decomposition

$$Q_\Phi = Q_{\Phi_{\text{average}}} + (Q_\Phi - Q_{\Phi_{\text{average}}})$$

into the sum of a conserved QDF $\bigoplus$ a zero-mean QDF.

Conserved and Zero-mean QDF's

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- **What are the dimensions of the linear subspaces of conserved, zero-mean, trivially zero mean, and intrinsically zero-mean QDF's modulo $\mathfrak{B}$?**

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Call the QDF’s in a suitable complement *‘intrinsically zero mean’*.

- **What are the dimensions of the linear subspaces of conserved, zero-mean, trivially zero mean, and intrinsically zero-mean QDF’s modulo $\mathfrak{B}$?**
- **Given a representation of $\mathfrak{B}$, oscillatory, and a $\Phi \in \mathbb{R}[\zeta, \eta]$, how can we decide whether it is conserved, zero-mean, trivially zero mean, or intrinsically zero-mean? Parametrizations?**

Main Results

Let $\mathfrak{B}$ be an oscillatory system, $\dim(\mathfrak{B}) = 2\mathrm{n}$**.**

Recall that the QDF's modulo $\mathfrak{B}$
$\cong$ the real symmetric matrices of dimension $2\mathrm{n} \times 2\mathrm{n}$.

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Each QDF mod $\mathfrak{B}$ admits a unique decomposition as the sum of conserved $\bigoplus$ trivially zero-mean $\bigoplus$ intrinsically zero mean QDF.

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Moreover,

$$
\begin{aligned}
\dim(\text{conserved}) &= \mathrm{n} \\
\dim(\text{zero-mean}) &= 2\mathrm{n}^2 \\
\dim(\text{trivially zero-mean}) &= \mathrm{n}(2\mathrm{n}-1) \\
\dim(\text{intrinsically zero mean}) &= \mathrm{n} \\
\dim(\text{QDF's modulo } \mathfrak{B}) &= \mathrm{n}(2\mathrm{n}+1)
\end{aligned}
$$

Main Results

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Recall that the QDF's modulo $\mathfrak{B}$

$\cong$ the real symmetric matrices of dimension $2\mathtt{n} \times 2\mathtt{n}$.

$\exists$ an $\mathtt{n}$-dimensional subspace of conserved QDF's !!?

$\exists$ an $\mathtt{n}$-dimensional subspace of intrinsically zero-mean QDF's !!?

Main Results

We have explicit rules for verifying whether a QDF is conserved, zero-mean, trivially zero mean, or intrinsically zero-mean, and explicit parametrizations

==============================

Main Results

Q_Φ **is conserved** $\Leftrightarrow \exists\, X \in \mathbb{R}[\zeta, \eta]$ **such that**

$$(\zeta + \eta)\Phi(\zeta, \eta) = X(\zeta, \eta)R(\eta^2) + R(\zeta^2)X(\eta, \zeta)$$

Main Results

Q_Φ **is zero-mean** $\Leftrightarrow \Phi(-\xi, \xi)$ **has** $R(\xi^2)$ **as a factor,**

Q_Φ **is trivially zero-mean** $\Leftrightarrow \Phi(-\xi, \xi) = 0.$

Main Results

$$\frac{R(\zeta^2)\eta F(\eta^2) + \zeta F(\zeta^2) R(\eta^2)}{\zeta + \eta} =: \mathcal{C}(\zeta, \eta)$$

with $F \in \mathbb{R}[\xi]$ generates exactly all the conserved QDF's.

$\mathcal{C} >_{\mathfrak{B}} 0$ iff the roots of F interlace those of R: non-empty interior.

Each such $\mathcal{C}$ is of the form

$$\mathcal{C}(\zeta, \eta) = \mathcal{C}_0(\zeta^2, \eta^2) + \zeta\eta\, \mathcal{C}_1(\zeta^2, \eta^2)$$

'Potential' + 'Kinetic'.

Main Results

$$(\zeta + \eta)\mathbb{R}[\zeta, \eta]$$

equals exactly the trivially zero-mean QDF's.

Main Results

$$\frac{R(\zeta^2)\eta F(\eta^2) - \zeta F(\zeta^2) R(\eta^2)}{\zeta - \eta} =: \mathcal{N}(\zeta, \eta)$$

with $F \in \mathbb{R}[\xi]$ generates a choice for the

intrinsically zero-mean QDF's.

Each such $\mathcal{N}$ is of the form

$$\mathcal{N}(\zeta, \eta) = \mathcal{C}_0(\zeta^2, \eta^2) - \zeta\eta\, \mathcal{C}_1(\zeta^2, \eta^2)$$

'Potential' - 'Kinetic'.

Main Results

$$\boldsymbol{R}(-\zeta\eta)\boldsymbol{F}(-\zeta\eta) =: \mathcal{N}(\zeta, \eta)$$

with $F \in \mathbb{R}[\xi]$ generates yet another choice for the intrinsically zero-mean QDF's.

Polynomials in $\zeta\eta$.

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We end with the a general THEOREM (stated for 2 variables only):

Consider a system in $\mathfrak{L}^2$: 2 real variables w_1, w_2. Assume

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1. **$\mathfrak{B}$ is 'oscillatory'**
2. **$\mathfrak{B}$ is 'permutation symmetric' i.e.,**

$$(w_1, w_2) \in \mathfrak{B} \Leftrightarrow (w_2, w_1) \in \mathfrak{B}$$

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3. w_2 is 'observable' from w_1 i.e.,

$$(w_1, w_2'), (w_1, w_2'') \in \mathfrak{B} \Leftrightarrow w_2' = w_2''.$$

No 'decoupling'.

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Let Q_Φ be any QDF. Then $\boxed{Q_\Phi(w_1) - Q_\Phi(w_2)}$ **is zero-mean.**

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Any 'sign-symmetric' QDF has zero mean!

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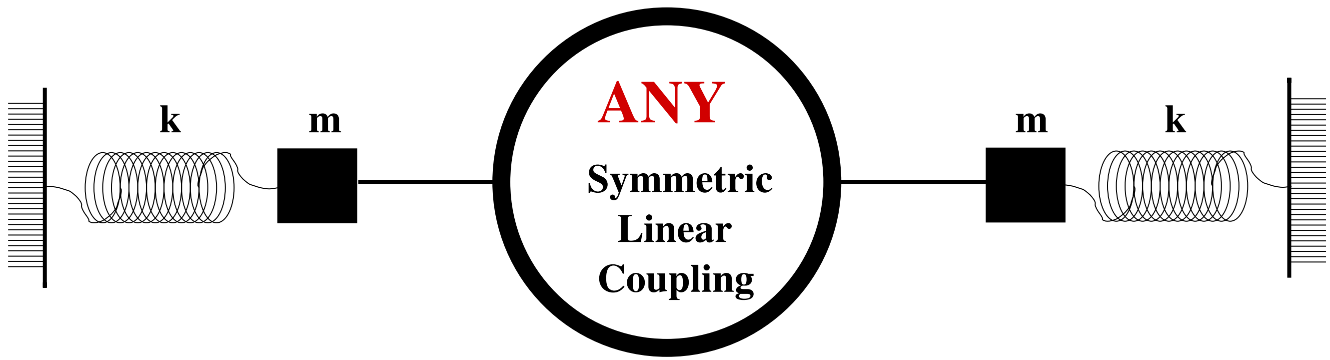
Extends to systems with $\mathtt{w} > 2$ variables.

Equipartition of Energy

Immediate corollary:



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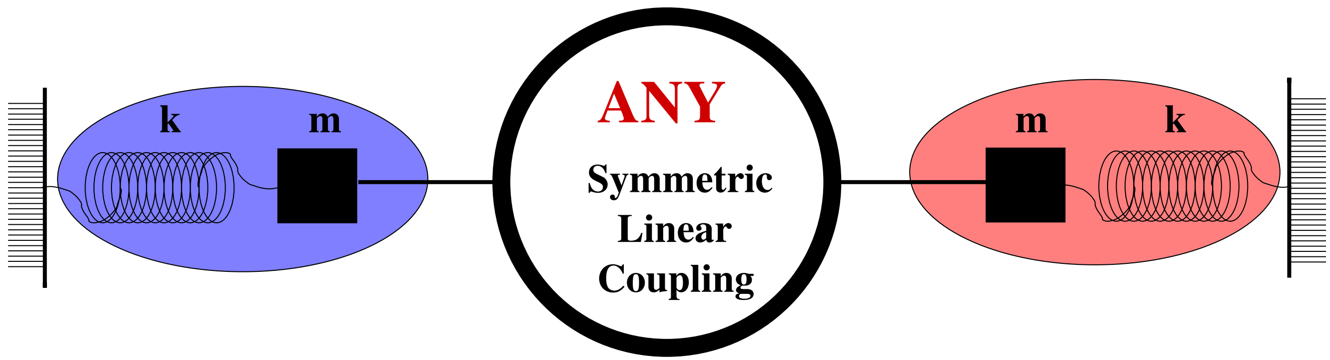
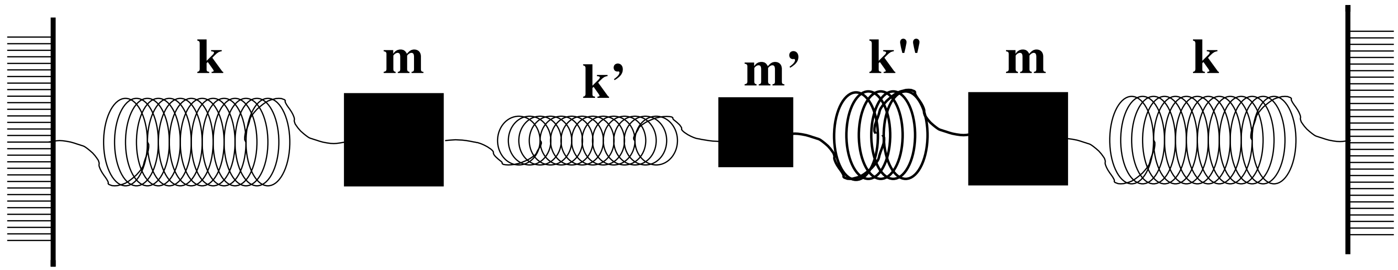


The average energy in symmetrically coupled identical oscillators is the same for each oscillator.

$\leadsto$ **a very general deterministic 'Equipartition of Energy' principle.**

Simulations



Take $m = 13,\ m' = 10,\ k = 7,\ k' = 2,\ k" = 1.$

at $t = 0$: displacement left mass = 1,

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Note that the symmetry is 'broken'.

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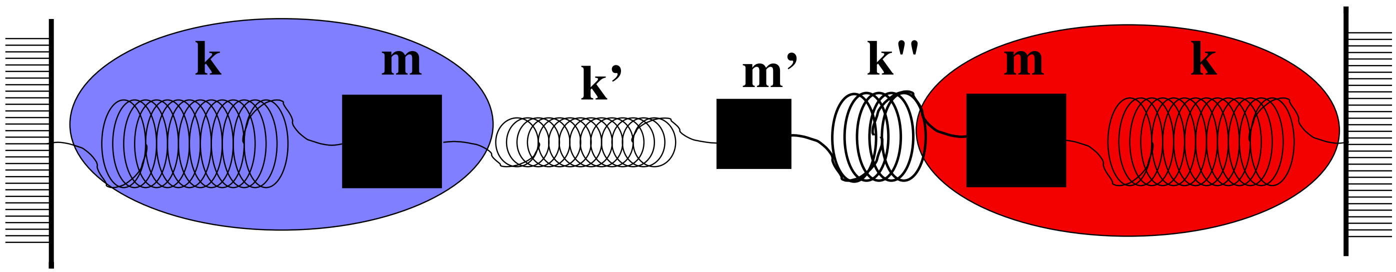

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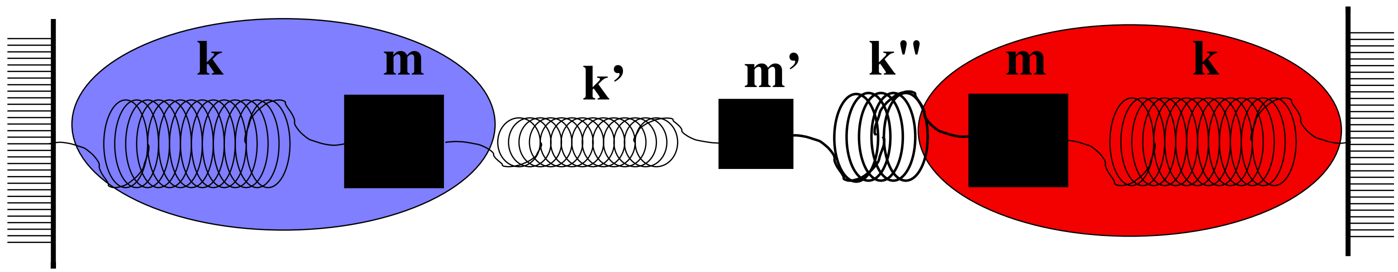
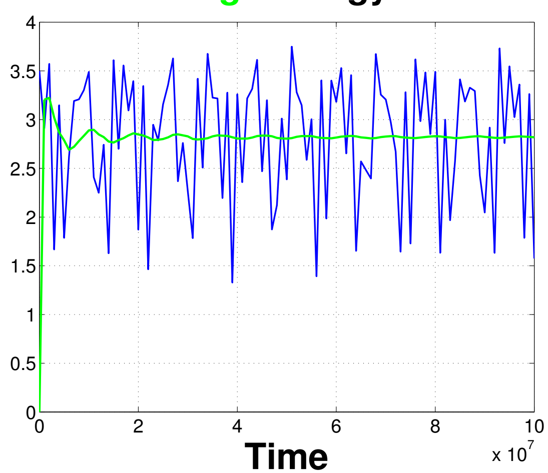
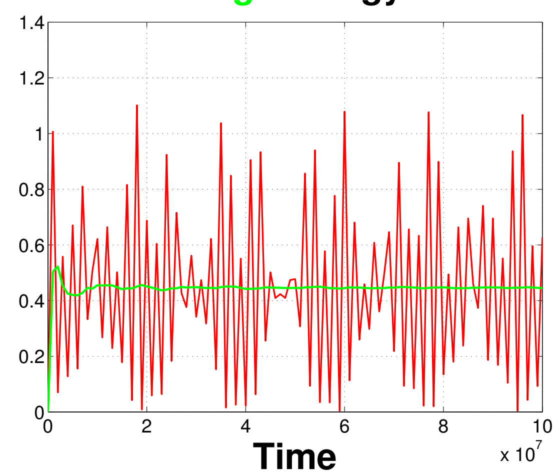


Actual and average energy in oscillator 1



Actual and average energy in oscillator 2



No equipartition!

Concluding Remarks

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- **Note the very effective and transparent use of QDF’s.**
  It is the proper mathematical tool for this class of problems.

<u>Reference</u>: An article is in the process of being manufactured.

Copies of the lecture frames will soon be available from/at

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`http://www.esat.kuleuven.ac.be/~jwillems`

Thank you for your attention !