DISSIPATIVE DYNAMICAL SYSTEMS

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THEME

A dissipative system absorbs 'supply' (e.g., energy).

How do we formalize this?

Involves the storage function.

How is it constructed? Is it unique?

$\rightsquigarrow$ **KYP, LMI's, ARE's.**

Where is this notion applied in systems and control?

OUTLINE

1. Lyapunov theory
2. !! Dissipative systems !!
3. Physical examples
4. Construction of the storage function
5. LQ theory $\rightsquigarrow$ LMI's, etc.
6. Applications in systems and control
7. Dissipativity for behavioral systems
8. Polynomial matrix factorization
9. Recapitulation

LYAPUNOV THEORY

LYAPUNOV FUNCTIONS

Consider the classical 'dynamical system', the *flow*

$$\Sigma: \quad \frac{d}{dt}x = f(x)$$

with $x \in \mathbb{X} = \mathbb{R}^{\mathrm{n}}$, the *state space*, $f : \mathbb{X} \to \mathbb{X}$. Denote the set of solutions $x : \mathbb{R} \to \mathbb{X}$ by $\mathfrak{B}$, the 'behavior'. The function

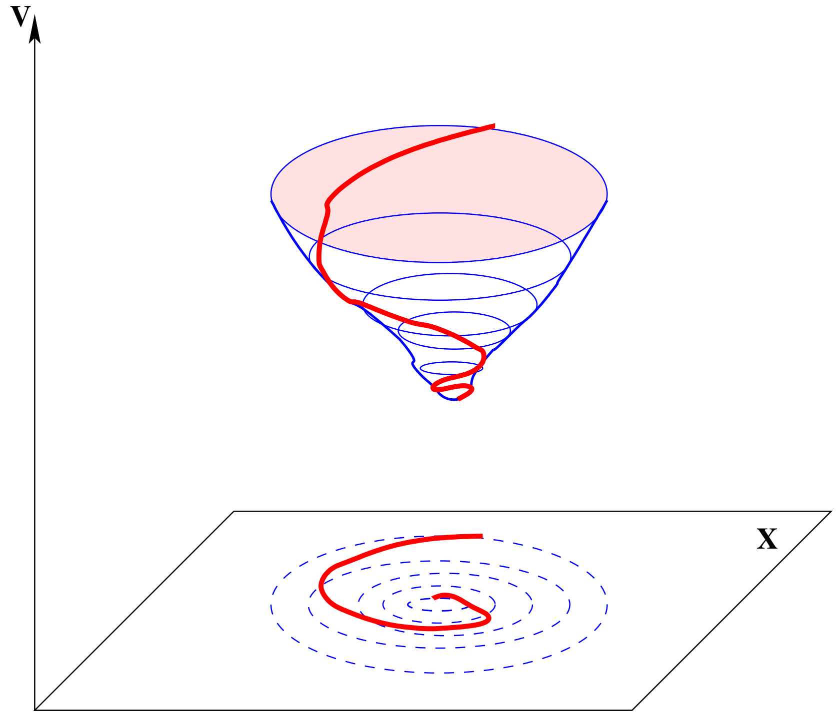
$$V : \mathbb{X} \to \mathbb{R}$$

is said to be a Lyapunov function for Σ if along $x \in \mathfrak{B}$

$$\frac{d}{dt} V(x(\cdot)) \leq 0$$

Equivalent to $\quad \overset{\bullet}{V}^{\Sigma} := \nabla V \cdot f \leq 0$

Typical Lyapunov 'theorem':



$$V(0) = 0, \quad \text{and} \quad V(x) > 0, \quad \overset{\bullet}{V}^{\Sigma}(x) < 0 \text{ for } 0 \neq x \in \mathbb{X}$$

$$\Rightarrow$$

$\forall \; x \in \mathfrak{B}$, there holds $x(t) \to 0$ for $t \to \infty$ 'global stability'

Refinements: **LaSalle's invariance principle.**

Converse: **Kurzweil's thm.**

LQ theory

$$\text{for } \frac{d}{dt}x = Ax, \quad V(x) = x^\top X x \ \rightsquigarrow \ \overset{\bullet}{V}^{\Sigma}(x) = x^\top Y x$$

$$\rightsquigarrow \quad \boxed{A^\top X + XA = Y}$$ **(Matrix) 'Lyapunov equation'**

A linear system is (asymptotically) stable iff it has a quadratic positive definite Lyapunov function

$$\Leftrightarrow \ \exists \ \text{sol'n } X = X^\top > 0, \ Y = Y^\top < 0 \,.$$

Basis for most stability results in control, physics, adaptation, even numerical analysis, system identification.

Lyapunov functions play a remarkably central role in the field.

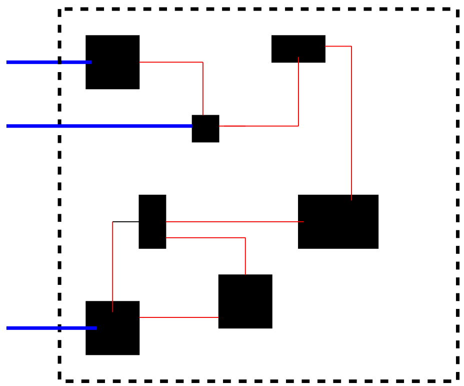
Aleksandr Mikhailovich Lyapunov (1857-1918)

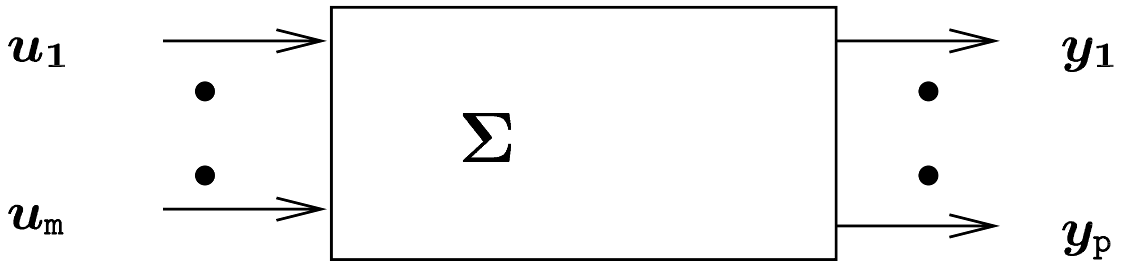
Studied mechanics, differential equations.

Introduced Lyapunov's 'second method' in his Ph.D. thesis (1899).

DISSIPATIVE SYSTEMS

A much more appropriate starting point for the study of dynamics are 'open' systems. $\rightsquigarrow$





INPUT/STATE/OUTPUT SYSTEMS

Consider the 'dynamical system'

$$\Sigma : \quad \frac{d}{dt}x = f(x,u), \quad y = h(x,u).$$

$u \in \mathbb{U} = \mathbb{R}^{\mathtt{m}}, y \in \mathbb{Y} = \mathbb{R}^{\mathtt{p}}, x \in \mathbb{X} = \mathbb{R}^{\mathtt{n}}$: **the input, output, state.**

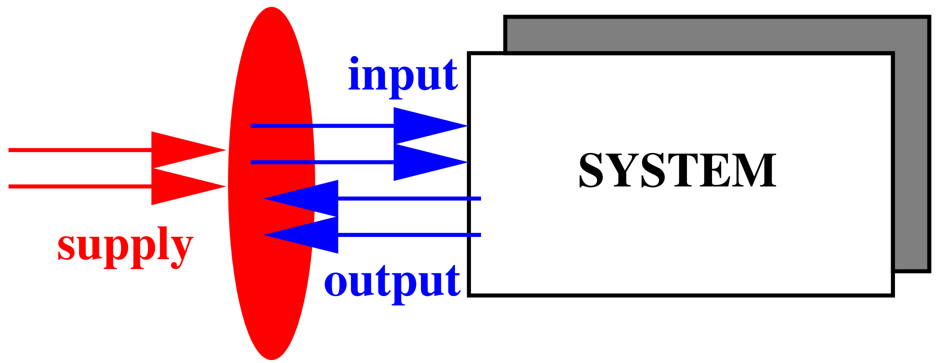
Behavior $\mathfrak{B} =$ **all sol'ns** $(u,y,x) : \mathbb{R} \to \mathbb{U} \times \mathbb{Y} \times \mathbb{X}$.

Let

$$s : \mathbb{U} \times \mathbb{Y} \to \mathbb{R}$$

be a function, called the *supply rate*.

$s(u, y)$ models something like the power delivered to the system when the input value is u and output value is y.



DISSIPATIVITY

Σ is said to be ***dissipative*** **w.r.t. the supply rate** s if $\exists$

$$V : \mathbb{X} \to \mathbb{R},$$

called the ***storage function,*** such that

$$\frac{d}{dt} V(x(\cdot)) \leq s(u(\cdot), y(\cdot))$$

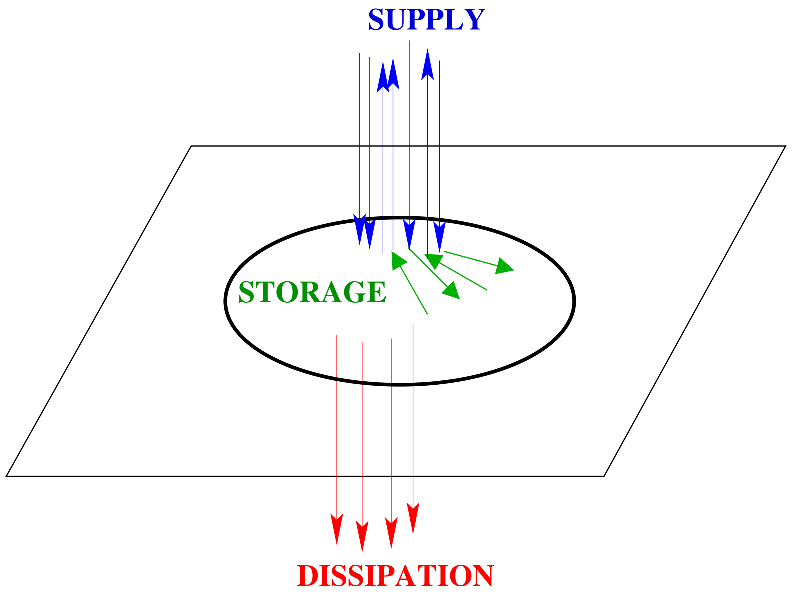
along input/output/state trajectories ($\forall\ (u(\cdot), y(\cdot), x(\cdot)) \in \mathfrak{B}$).

This inequality is called the ***dissipation inequality.***

Equivalent to $\overset{\bullet}{V}^{\Sigma}(x, u) := \nabla V(x) \cdot f(x, u) \leq s(u, h(x, u))$
for all $(u, x) \in \mathbb{U} \times \mathbb{X}$.

If equality holds: **'conservative' system.**

Dissipativity :$\Leftrightarrow$ Increase in storage $\leq$ Supply.



Special case: 'closed system': $\rightsquigarrow$ $\boxed{s = 0}$ **then**

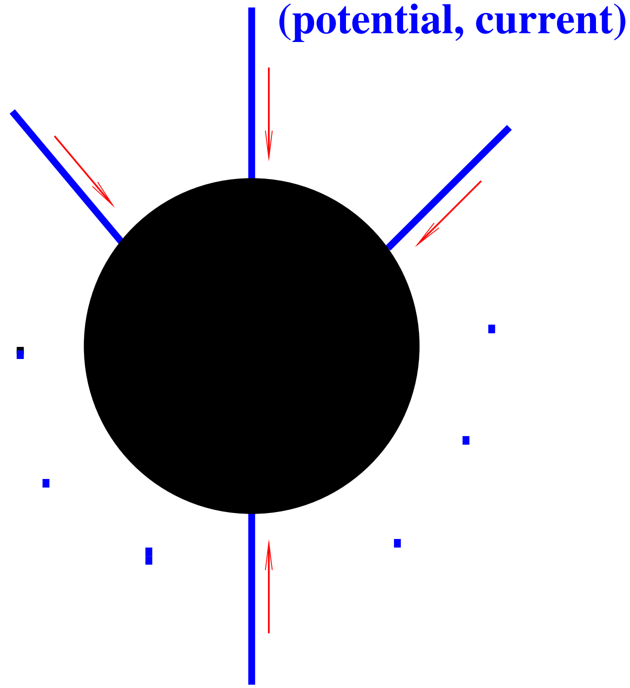
dissipativity $\leftrightarrow$ V **is a Lyapunov function.**

Dissipativity is a natural generalization of Lyapunov theory to open systems.

Stability for closed systems $\simeq$ **Dissipativity for open systems.**

PHYSICAL EXAMPLES

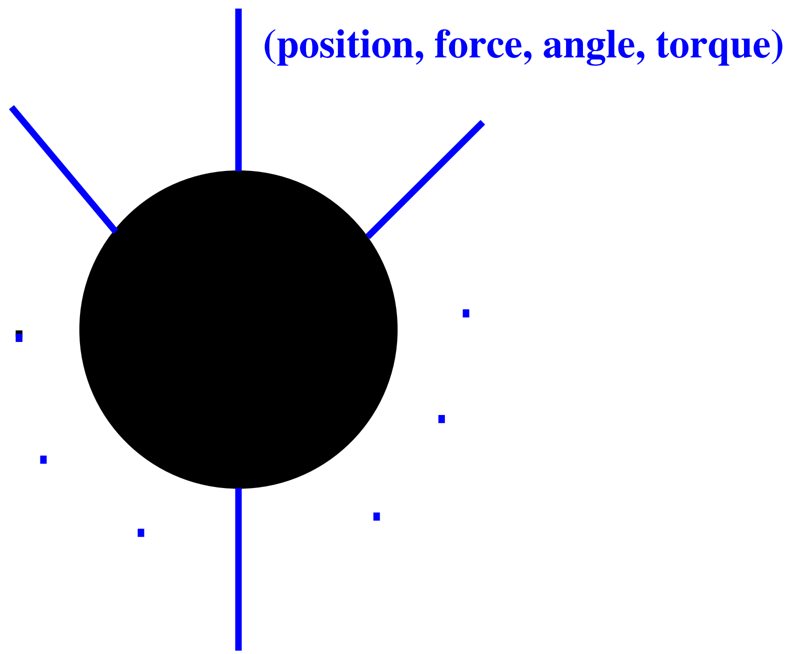
Electrical circuit:



Dissipative w.r.t. $\Sigma_{\ell=1}^{N} V_\ell I_\ell$ **(electrical power).**

System	Supply	Storage
Electrical circuit	$V^\top I$ V : voltage I : current	energy in capacitors and inductors
etc.	etc.	etc.

<u>Mechanical device:</u>



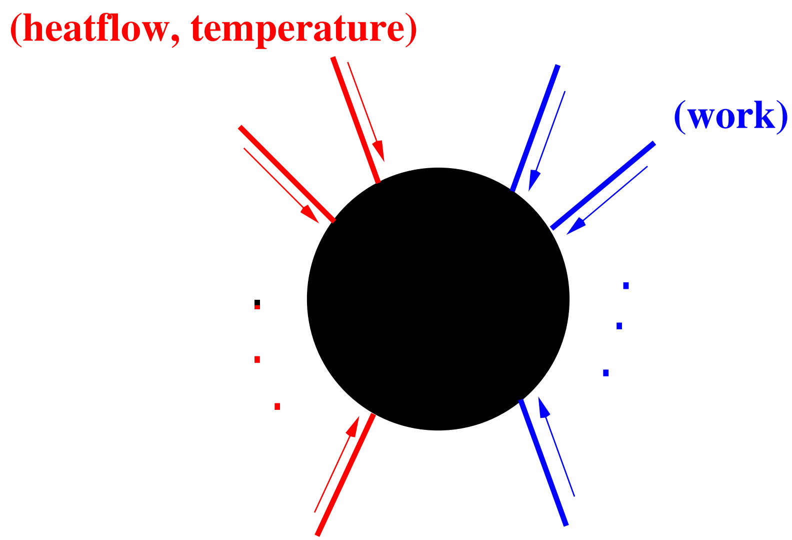
Dissipative w.r.t.

$$\Sigma_{\ell=1}^{\mathtt{N}}\left(\left(\frac{d}{dt}q_\ell\right)^\top F_\ell + \left(\frac{d}{dt}\theta_\ell\right)^\top T_\ell\right)$$

(mechanical power)

System	Supply	Storage
Electrical circuit	$V^\top I$ V : voltage I : current	energy in capacitors and inductors
Mechanical system	$F^\top v + (\frac{d}{dt}\theta)^\top T$ F : force, v : velocity θ: angle, T : torque	potential + kinetic energy
etc.	etc.	etc.

Thermodynamic system:



Conservative w.r.t. $\Sigma_{\ell=1}^{N} Q_\ell + \Sigma_{\ell=1}^{N'} W_\ell,$

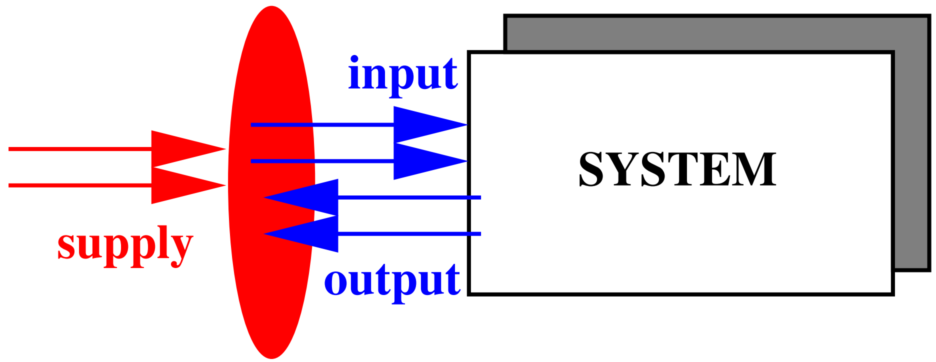
Dissipative w.r.t. $-\Sigma_{\ell=1}^{N} \frac{Q_\ell}{T\ell}.$

System	Supply	Storage
Electrical circuit	$V^\top I$ V : voltage I : current	energy in capacitors and inductors
Mechanical system	$F^\top v + (\frac{d}{dt}\theta)^\top T$ F : force, v : velocity θ: angle, T : torque	potential + kinetic energy
Thermodynamic system	$Q + W$ Q : heat, W : work	internal energy
Thermodynamic system	$-Q/T$ Q : heat, T : temp.	entropy
etc.	etc.	etc.

THE CONSTRUCTION OF STORAGE FUNCTIONS

Central question:

Given (a representation of) Σ, the dynamics, and given s, the supply rate, is the system dissipative w.r.t. s, i.e., does there exist a storage function V such that the dissipation inequality holds?



Assume known dynamics,

Given the system history, how much 'energy' is stored?

Assume henceforth that a number of (reasonable) conditions hold:

$f(0,0) = 0, h(0,0) = 0, s(0,0) = 0;$
Maps and functions (including V) smooth;
State space $\mathbb{X}$ of Σ **‘connected’:**
every state reachable from every other state;
Observability.

‘Thm’: Let Σ and s be given.

Then Σ is dissipative w.r.t. s iff

$$\boxed{\oint s(u(\cdot), y(\cdot)) \;\; dt \geq 0}$$

for all **periodic** $(u(\cdot), y(\cdot), x(\cdot)) \in \mathfrak{B}$.

The AVAILABLE STORAGE and the REQUIRED SUPPLY

Two universal storage functions:

1. The available storage

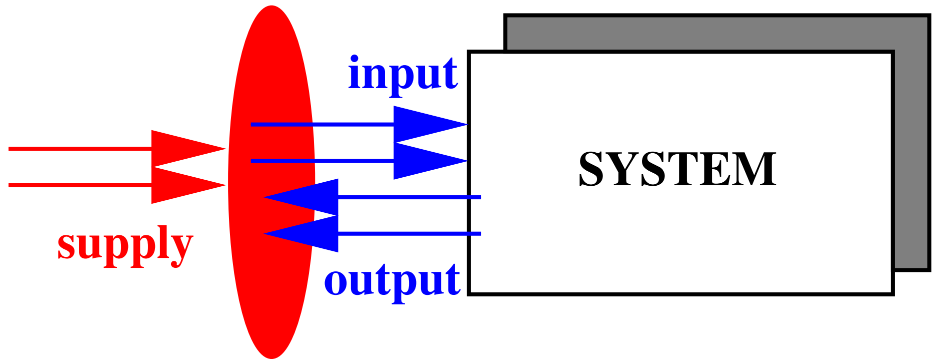
$$V_{\text{available}}(x_0) :=$$

$$\sup_{(u(\cdot),y(\cdot),x(\cdot))\in\mathfrak{B},x(0)=x_0,x(\infty)=0} \ \{-\int_0^{+\infty} s(u(\cdot),y(\cdot)) \ dt\}$$

2. The required supply

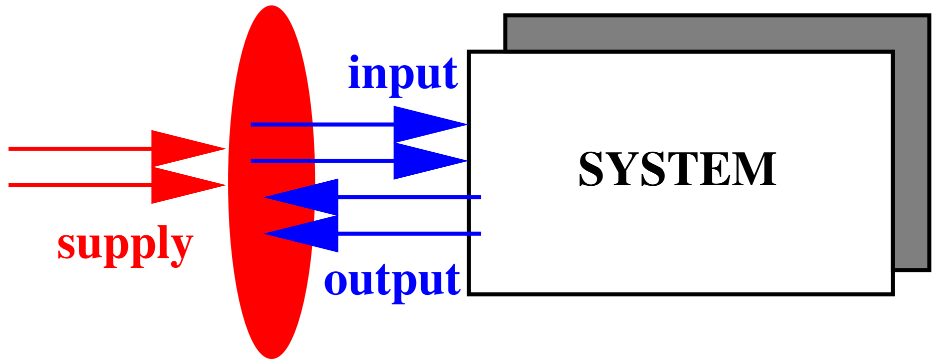
$$V_{\text{required}}(x_0) :=$$

$$\inf_{(u(\cdot),y(\cdot),x(\cdot))\in\mathfrak{B},x(-\infty)=0,x(0)=x_0} \ \{\int_{-\infty}^{0} s(u(\cdot),y(\cdot)) \ dt\}$$



!! Maximize the supply extracted, starting in fixed initial state

$\rightsquigarrow$ **available storage.**



!! Minimize the supply needed to set up a fixed initial state

$\rightsquigarrow$ **required supply.**

Storage f'ns form convex set, every storage function satisfies

$$\boxed{V_{\mathbf{available}} \le V \le V_{\mathbf{required}}.}$$

LINEAR SYSTEMS with QUADRATIC SUPPLY RATES

Assume Σ linear, time-invariant, finite-dimensional:

$$\frac{d}{dt}x = Ax + Bu, \quad y = Cx,$$

and s quadratic: e.g.,

$$s : (u, y) \mapsto ||u||^2 - ||y||^2.$$

E.g., for circuits $u = \frac{V+I}{2}$, $y = \frac{V-I}{2}$, etc.

Assume (A, B) controllable, (A, C) observable.
$G(s) := D + C(Is - A)^{-1}B$, the transfer function of Σ.

Theorem: The following are equivalent:

1. Σ is dissipative w.r.t. s (i.e., there exists a storage function V),
2. $\forall (u(\cdot), y(\cdot), x(\cdot)) \in \mathfrak{B} \cap \mathcal{L}_2$,
$$\boxed{||u(\cdot)||_{\mathcal{L}_2} \geq ||y(\cdot)||_{\mathcal{L}_2},}$$
3. $\boxed{||G(i\omega)|| \leq 1}$ for all $\omega \in \mathbb{R}$,
4. $\exists$ a quadratic storage f'n, $V(x) = x^\top K x, K = K^\top$,

5. there exists a solution $K = K^\top$ to the Linear Matrix Inequality (LMI)

$$\begin{bmatrix} A^\top K + KA + C^\top C & KB \\ B^\top K & -I \end{bmatrix} \leq 0,$$

6. there exists a solution $K = K^\top$ to the Algebraic Riccati Inequality (ARIneq)

$$A^\top K + KA + KBB^\top K + C^\top C \leq 0,$$

7. there exists a solution $K = K^\top$ to the Algebraic Riccati Equation (ARE)

$$A^\top K + KA + KBB^\top K + C^\top C = 0.$$

Solution set (of LMI, ARineq) is convex, compact, and attains its infimum and its supremum:

$$\boxed{K^- \leq K \leq K^+}$$

These extreme sol'ns K^- and K^+ corresponding to the available storage and the required supply, themselves satisfy the ARE.

Extensive theory, relation with other system representations, many applications, well-understood (also algorithmically).

Connection with optimal LQ control, semi-definite programming, $\mathcal{H}_\infty$ control, etc.

Important refinement: Existence of a $V \geq 0$ (i.e., bounded from below)

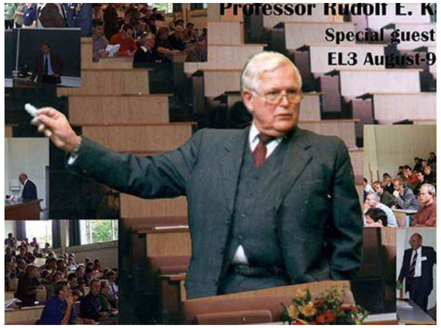
$$\rightsquigarrow \int_{-\infty}^{0} s(u(\cdot), y(\cdot)) \; dt \geq 0.$$

In LQ case $\Leftrightarrow$

- $\int_{-\infty}^{0} ||u(\cdot)||^2 \, dt \geq \int_{-\infty}^{0} ||y(\cdot)||^2 \, dt,$
- $\sup_{\{s \in \mathbb{C} | \mathrm{Re}(s) > 0\}} ||G(s)|| =: ||G||_{\mathcal{H}_\infty} \leq 1,$

  Note def. of $\mathcal{H}_\infty$-norm !
- $\exists$ sol'n $K = K^\top \geq 0$ to LMI, ARineq, ARE.

$\rightsquigarrow$ KYP-lemma.

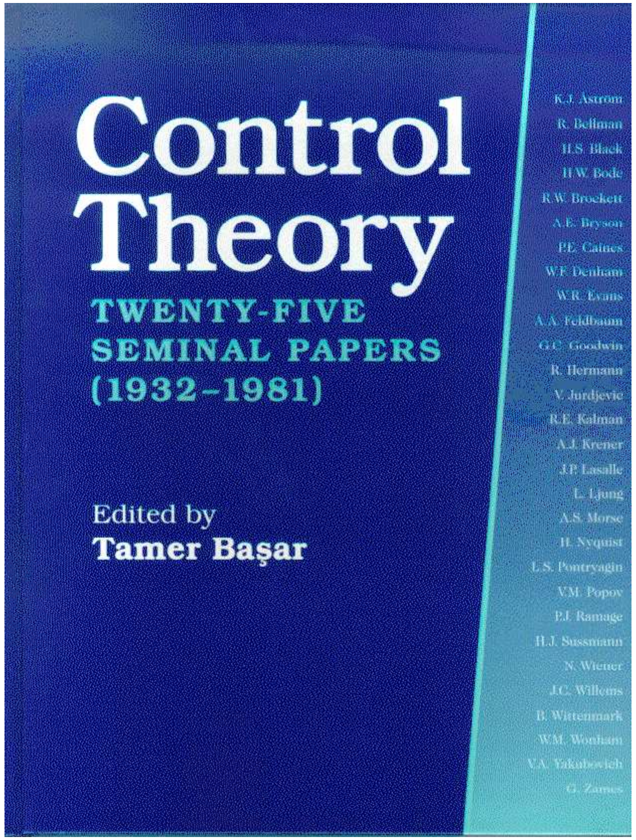


APPLICATIONS

- **Synthesis of RLC-circuits**
- **Robust stability**

  (‘the interconnection of dissipative systems is stable’)
- **Stabilization (by ‘passivation’)**
- **Robust stabilization (by making the loop dissipative), $\mathcal{H}_\infty$-control**
- **Norm estimation (e.g., bounding the balanced reduction error)**
- **Covariance generation**
- $\cdots$

Dissipative systems (and **LMI's** which emerged from this) play a remarkably central role in the field.

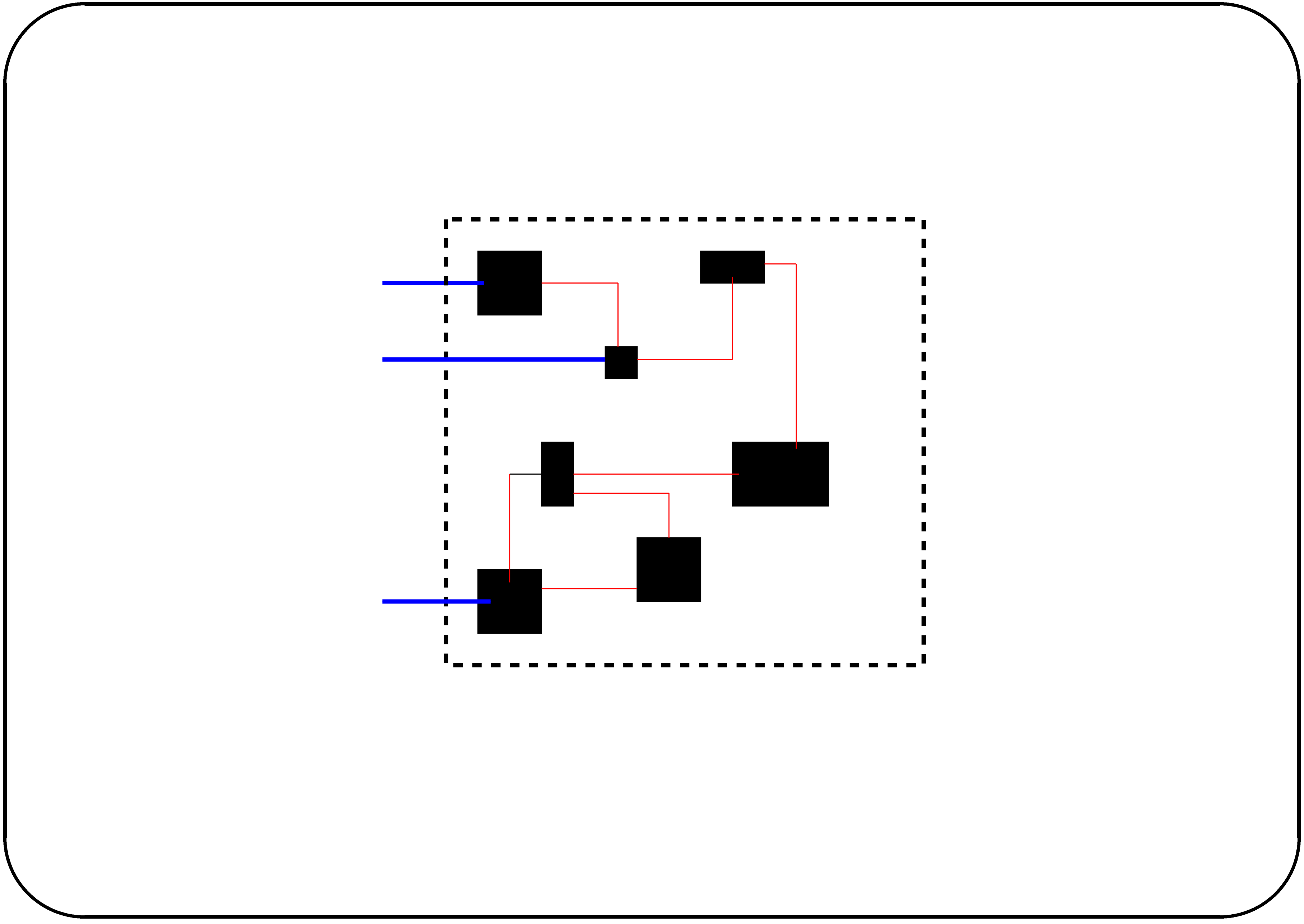


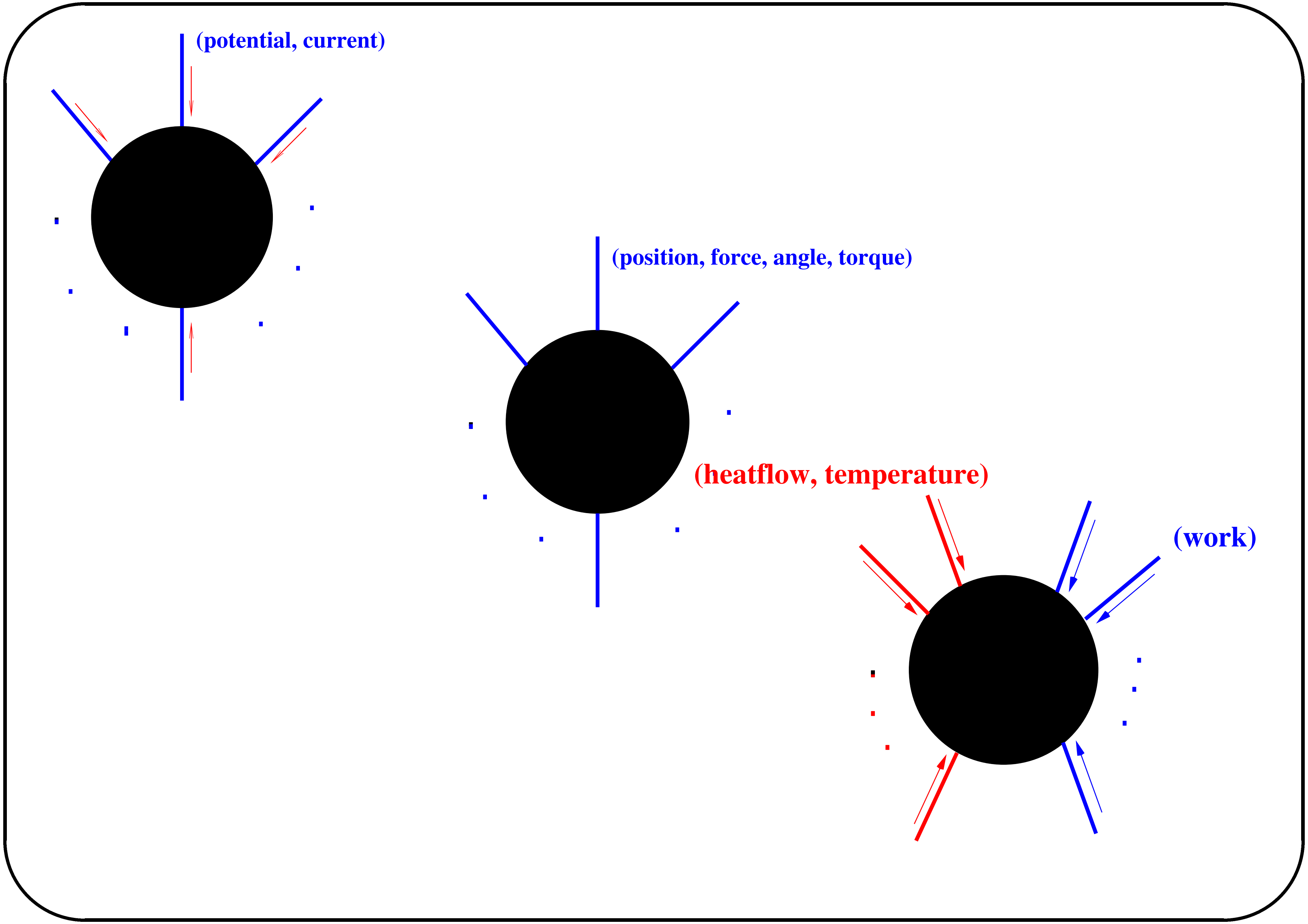
BEHAVIORAL SYSTEMS

The input/output, nor input/state/output approach are not logical starting points for studying

- **(open) physical systems**
- **interconnected systems**
- **dissipative systems**
- $\cdots$

$\leadsto$ **'behavioral systems'**




(potential, current)
(position, force, angle, torque)
(heatflow, temperature)
(work)

BEHAVIORAL SYSTEMS

A dynamical system **=** $\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$

$\mathbb{T} \subseteq \mathbb{R}$, **the *time-axis* (= the relevant time instances),**

$\mathbb{W}$, **the *signal space* (= where the variables take on their values),**

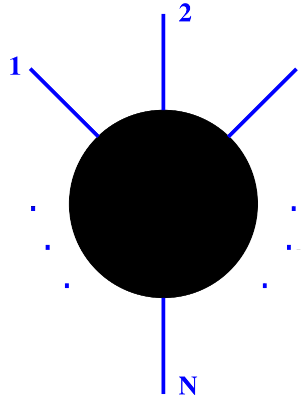
$\mathfrak{B} \subseteq \mathbb{W}^{\mathbb{T}}$ **: *the behavior*** **(= the admissible trajectories).**

$$\boxed{\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})}$$

For a trajectory $w : \mathbb{T} \to \mathbb{W}$, we thus have:

$w \in \mathfrak{B}$ **: the model allows the trajectory w,**

$w \notin \mathfrak{B}$ **: the model forbids the trajectory w.**



Today: $\mathbb{T} = \mathbb{R}$,

$\mathbb{W} = \mathbb{R}^{\mathtt{w}}$,

$\mathfrak{B}$ **= sol's of system of linear constant coefficient ODE's.**

DIFFERENTIAL SYSTEMS

Consider

$$R_0 w + R_1 \frac{d}{dt} w + \cdots + R_n \frac{d^n}{dt^n} w = 0,$$

with $R_0, R_1, \cdots, R_n \in \mathbb{R}^{\bullet \times \mathtt{w}}$.

Combined with the polynomial matrix

$$R(\xi) = R_0 + R_1 \xi + \cdots + R_n \xi^n,$$

we obtain the short notation

$$\boxed{R(\frac{d}{dt}) w = 0.}$$

$$R(\frac{d}{dt})w = 0.$$

defines the system with

$\mathbb{T} = \mathbb{R}$, **time,**

$\mathbb{W} = \mathbb{R}^{\mathtt{w}}$, $\mathtt{w}$ **dependent variables,**

$\mathfrak{B} =$ **sol'ns of a linear const. coeff. system of diff. eq'ns.**

A 'differential system'; **Notation:** $\mathfrak{L}^{\mathtt{w}}, \mathfrak{L}^{\bullet}$

For example,

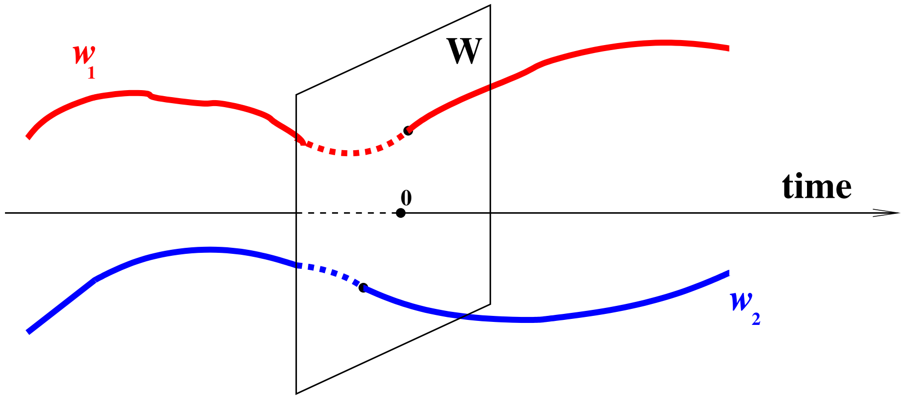
$$P(\frac{d}{dt})y = Q(\frac{d}{dt})u, \quad w = (u, y)$$

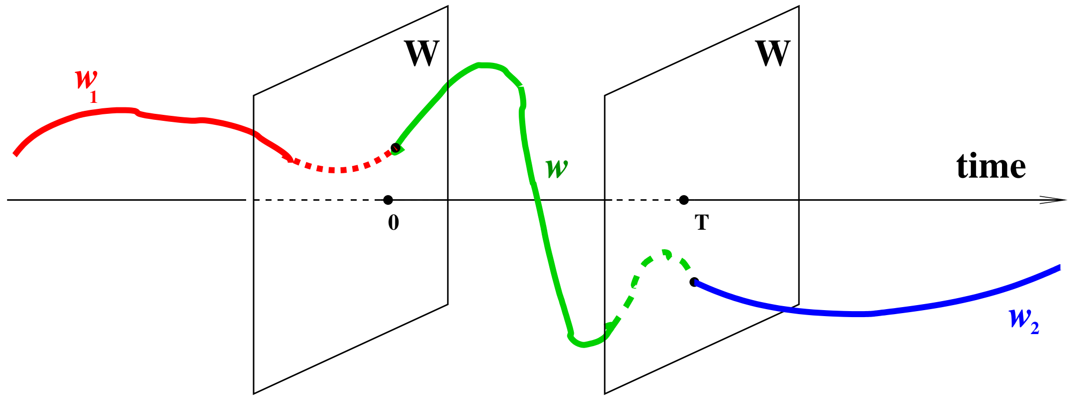
$$\frac{d}{dt}x = Ax + Bu, \; y = Cx + Du, \quad w = (u, x, y) \text{ or } (u, y)$$

CONTROLLABILITY

Controllability $\Leftrightarrow$

system trajectories must be ‘patch-able’, ‘concatenable’.





Is the system defined by

$$R_0 w + R_1 \frac{d}{dt} w + \cdots + R_{\mathrm{n}} \frac{d^{\mathrm{n}}}{dt^{\mathrm{n}}} w = 0,$$

with $w = (w_1, w_2, \cdots, w_{\mathtt{w}})$ **and** $R_0, R_1, \cdots, R_{\mathrm{n}} \in \mathbb{R}^{\bullet \times \mathtt{w}}$,

i.e.,

$$R(\frac{d}{dt}) w = 0,$$

controllable?

We are looking for conditions on the polynomial matrix R **and algorithms in the coefficient matrices** $R_0, R_1, \cdots, R_{\mathrm{n}}$.

Thm: The following are equivalent:

1. $R(\frac{d}{dt})w = 0$ defines a **controllable** system
2. $\boxed{\text{rank}(R(\lambda)) \text{ is independent of } \lambda \text{ for } \lambda \in \mathbb{C}.}$

Example: $$r_1(\frac{d}{dt})w_1 = r_2(\frac{d}{dt})w_2 \quad (w_1, w_2 \text{ scalar})$$

is controllable if and only if r_1 and r_2 have no common factor.

Representations of $\mathfrak{L}^\bullet$:

$$R(\tfrac{d}{dt})w = 0$$

called a *'kernel' representation* of $\mathfrak{B} = \ker(R(\frac{d}{dt}))$

Another representation:

$$w = M(\tfrac{d}{dt})\ell$$

called an *'image' representation* of $\mathfrak{B} = \mathrm{im}(M(\frac{d}{dt}))$.

Elimination theorem $\Rightarrow$ every image is also a kernel.

¿¿ Which kernels are also images ??

Theorem: The following are equivalent for $\mathfrak{B} \in \mathfrak{L}^{\bullet}$:

1. $\mathfrak{B}$ is controllable
2. $\mathfrak{B}$ admits an image representation
3. $\cdots$

QDF's

The quadratic map acting on $w : \mathbb{R} \to \mathbb{R}^{\mathtt{w}}$ and its derivatives, defined by

$$w \mapsto \sum_{k,\ell} (\frac{d^k}{dx^k} w)^\top \Phi_{k,\ell} (\frac{d^\ell}{dx^\ell} w)$$

is called *quadratic differential form* (QDF).
$\Phi_{k,\ell} \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$**; WLOG:** $\Phi_{k,\ell} = \Phi_{\ell,k}^\top$**.**

Introduce the 2-variable polynomial matrix Φ

$$\Phi(\zeta,\eta) = \sum_{k,\ell} \Phi_{k,\ell} \zeta^k \eta^\ell .$$

Denote the QDF as Q_Φ. QDF's are parametrized by $\mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$.

DISSIPATIVE BEHAVIORAL SYSTEMS

We consider only controllable linear differential systems and QDF's for supply rates.

E.g., $V^\top I$ for electrical circuits, $F^\top \frac{d}{dt} q$ for mechanical systems, ...

Definition: $\mathfrak{B} \in \mathfrak{L}^\bullet$, controllable, is said to be *dissipative* with respect to the supply rate Q_Φ (a QDF) if

$$\boxed{\int_{\mathbb{R}} Q_\Phi(w)\, dt \geq 0}$$

for all $w \in \mathfrak{B}$ of compact support.

In any trajectory from rest back to rest, supply is absorbed.

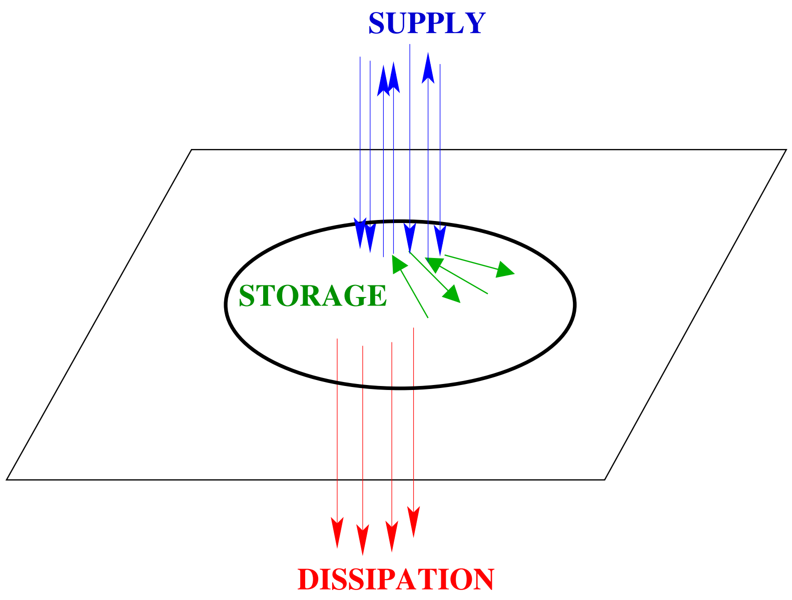
STORAGE FUNCTION

Dissipativity: $\Leftrightarrow$ $\int_{\mathbb{R}} Q_{\Phi}(w)\, dt \geq 0$ for $w \in \mathfrak{B}$ compact supp.

Can this be reinterpreted as: As the system evolves,

some supply is stored, some is dissipated?

!! Invent storage, such that: $\frac{d}{dt}$ Storage $\leq$ Supply.



MAIN RESULT

Theorem: Let $\mathfrak{B} \in \mathfrak{L}^\bullet$ be controllable, and Q_Φ be a QDF. Then

$$\int_{\mathbb{R}} Q_\Phi(w)\, dt \geq 0 \qquad \text{for all } w \in \mathfrak{B} \text{ of compact support}$$

if and only if

there exists a QDF, Q_Ψ, the **storage function** such that

$$\boxed{\frac{d}{dt} Q_\Psi(w)(t) \leq Q_\Phi(w)(t)}$$

for all $w \in \mathfrak{B}$ and $t \in \mathbb{R}$.

Note: The computation of Ψ is an LMI involving R (or M) and Φ!

OUTLINE of the PROOF

Using controllability and the existence of an image representation, reduce to case that w is 'free'.

Now consider, for a given (smooth) $w : \mathbb{R} \to \mathbb{R}^{\mathtt{w}}$,

$$\text{infimum} \int_{-\infty}^{0} Q_{\Phi}(\hat{w})\, dt,$$

with infimum taken over all $\hat{w} \in \mathfrak{B}$ such that $\hat{w}(t) = w(t)$ for $t \geq 0$.

$\rightsquigarrow$ **the 'available storage'.**

Prove that this infimum is a QDF, $Q_{\Psi}(w)(0)$, and that it qualifies as a storage function.

This proof provides (but does not rely on!) a simple proof of the following (known) factorization result for polynomial matrices. Consider

$$\boxed{X^\top(\xi)X(\xi) = Y(\xi)}$$

Y **is a given real polynomial matrix;** X **is the unknown.**

For $Y \in \mathbb{R}[\xi]$**, a scalar, this eq'n is solvable (for** $X \in \mathbb{R}^2[\xi]$ **) iff**

$$Y(\alpha) \geq 0 \qquad \text{for all } \alpha \in \mathbb{R}.$$

For $Y \in \mathbb{R}^{\bullet\times\bullet}[\xi]$**, it is solvable (with** $X \in \mathbb{R}^{\bullet\times\bullet}[\xi]$ **!) iff**

$$Y(\alpha) = Y^\top(\alpha) \geq 0 \qquad \text{for all } \alpha \in \mathbb{R}.$$

Btw: For multivariable polynomials, and under the obvious symmetry and positivity requirement,

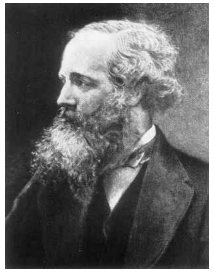
$$Y(\alpha) = Y^\top(\alpha) \geq 0 \qquad \text{for all } \alpha \in \mathbb{R}^{\mathtt{n}},$$

this equation can nevertheless in general not be solved over the polynomial matrices, for $X \in \mathbb{R}^{\bullet \times \bullet}[\xi]$, but it can be solved over the matrices of rational functions, i.e., for $X \in \mathbb{R}^{\bullet \times \bullet}(\xi)$.

This is Hilbert's 17-th pbm!

Remarks

- **Very important refinement:**
  $\int_{-\infty}^{0} Q_\Phi(w)\,dt \geq 0 \Leftrightarrow \exists\ \Psi$ **such that** $Q_\Psi(w) \geq 0.$
- **The storage function is always a state function.**
  Not so for discrete-time systems (Kaneko).
- **Generalized to systems describes by PDE's. Uses factorizability for multivariable polynomials. Constructs stored energy and flux (the 'Poynting vector') for Maxwell's eq'ns.**



- **Applies to $\mathcal{H}_\infty$ problem in behavioral setting, with the famous 'coupling condition' of two storage functions.**

RECAP

The notion of a dissipative system:

- **Generalization of ‘Lyapunov function’ to open systems**
- **Central concept in control theory: many applications to feedback stability, stabilization, robust ($\mathcal{H}_\infty$-) control, adaptive control, system identification, passivation control**
- **Stimulated emergence of LMI’s, semi-definite programming**
- **Other applications: system norm estimates, passive electrical circuit synthesis procedures, covariance generation**
- **Combined with behavioral systems, dissipativity forms a natural systems concept for the analysis of open physical systems**
- **Notable special case: second law of thermodynamics**
- **Forms a tread through modern system theory**

More info, copy sheets? **Surf to**

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Thank you !