FROM TIME SERIES
to
BALANCED REPRESENTATION
Part I: Theory

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The problem

Given an observed vector time-series

$$\begin{bmatrix} \tilde{u}(1) \\ \tilde{y}(1) \end{bmatrix}, \begin{bmatrix} \tilde{u}(2) \\ \tilde{y}(2) \end{bmatrix}, \ldots, \begin{bmatrix} \tilde{u}(T) \\ \tilde{y}(T) \end{bmatrix}$$

find a model for the system that produced these data.

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find a model for the system that produced these data.

In particular, a deterministic linear time-invariant model

in balanced form.

Motivation

∃ **algorithms from**

$$\begin{bmatrix}\tilde{u}(1)\\ \tilde{y}(1)\end{bmatrix},\ldots,\begin{bmatrix}\tilde{u}(t_1)\\ \tilde{y}(t_1)\end{bmatrix},\ldots,\begin{bmatrix}\tilde{u}(t_2)\\ \tilde{y}(t_2)\end{bmatrix},\ldots,\begin{bmatrix}\tilde{u}(T)\\ \tilde{y}(T)\end{bmatrix}$$

$\downarrow\downarrow$ **to** $\downarrow\downarrow$

$$\tilde{x}(t_1)\,,\ \ldots\ ,\tilde{x}(t_2)$$

Motivation

$\exists$ **algorithms from**

$$\begin{bmatrix} \tilde{u}(1) \\ \tilde{y}(1) \end{bmatrix}, \ldots, \begin{bmatrix} \tilde{u}(t_1) \\ \tilde{y}(t_1) \end{bmatrix}, \ldots, \begin{bmatrix} \tilde{u}(t_2) \\ \tilde{y}(t_2) \end{bmatrix}, \ldots, \begin{bmatrix} \tilde{u}(T) \\ \tilde{y}(T) \end{bmatrix}$$

$\downarrow\downarrow$ **to** $\downarrow\downarrow$

$$\tilde{x}(t_1), \ \ldots \ , \tilde{x}(t_2)$$

Solve (LS)

$$\begin{bmatrix} \tilde{x}(t_1+1) & \cdots & \tilde{x}(t_2) \\ y(t_1) & \cdots & y(t_2-1) \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \tilde{x}(t_1) & \cdots & \tilde{x}(t_2-1) \\ u(t_1) & \cdots & u(t_2-1) \end{bmatrix}$$

This yields a state representation.

Motivation

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This yields a state representation. $\leadsto$ Reduce the state dimension, by reducing the row dimension of

$$\begin{bmatrix} \tilde{x}(t_1) & \ldots & \tilde{x}(t_2) \end{bmatrix}.$$

Motivation

Solve (LS)

$$\begin{bmatrix} \tilde{x}(t_1+1) & \cdots & \tilde{x}(t_2) \\ y(t_1) & \cdots & y(t_2-1) \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \tilde{x}(t_1) & \cdots & \tilde{x}(t_2-1) \\ u(t_1) & \cdots & u(t_2-1) \end{bmatrix}$$

This yields a state representation. $\rightsquigarrow$ Reduce the state dimension, by reducing the row dimension of

$$\begin{bmatrix} \tilde{x}(t_1) & \ldots & \tilde{x}(t_2) \end{bmatrix}.$$

This leads to the problem:

Construct $\tilde{x}(t_1), \ldots, \tilde{x}(t_2)$ in a balanced basis.

'Subspace methods' do this.

A ‘sequential’ zero input response series

A ‘sequential’ zero input response series

$$x(1) \xrightarrow{u(1)} x(2) \xrightarrow{u(2)} \dots \xrightarrow{u(t-1)} x(t) \xrightarrow{u(t)} x(t+1) \xrightarrow{u(t+1)} \dots$$

A ‘sequential’ zero input response series

$$\begin{array}{cccccccccc}
x(1) & \xrightarrow{u(1)} & x(2) & \xrightarrow{u(2)} & \cdots & \xrightarrow{u(t-1)} & x(t) & \xrightarrow{u(t)} & x(t+1) & \xrightarrow{u(t+1)} \cdots \\
\downarrow \mathbf{0} & & \downarrow \mathbf{0} & & \cdots & & \downarrow \mathbf{0} & & \downarrow \mathbf{0} & \cdots \\
y_1(1) & & y_2(1) & & \cdots & & y_t(1) & & y_{t+1}(1) & \cdots
\end{array}$$

A 'sequential' zero input response series

$$
\begin{array}{ccccccccc}
x(1) & \xrightarrow{u(1)} & x(2) & \xrightarrow{u(2)} & \cdots & \xrightarrow{u(t-1)} & x(t) & \xrightarrow{u(t)} & x(t+1) & \xrightarrow{u(t+1)} & \cdots \\
\downarrow 0 & & \downarrow 0 & & \cdots & & \downarrow 0 & & \downarrow 0 & & \cdots \\
y_1(1) & & y_2(1) & & \cdots & & y_t(1) & & y_{t+1}(1) & & \cdots \\
\downarrow 0 & & \downarrow 0 & & \cdots & & \downarrow 0 & & \downarrow 0 & & \cdots \\
y_1(2) & & y_2(2) & & \cdots & & y_t(2) & & y_{t+1}(2) & & \cdots
\end{array}
$$

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x(1) & \xrightarrow{u(1)} & x(2) & \xrightarrow{u(2)} & \cdots & \xrightarrow{u(t-1)} & x(t) & \xrightarrow{u(t)} & x(t+1) & \xrightarrow{u(t+1)} & \cdots \\
\downarrow 0 & & \downarrow 0 & & \cdots & & \downarrow 0 & & \downarrow 0 & & \cdots \\
y_1(1) & & y_2(1) & & \cdots & & y_t(1) & & y_{t+1}(1) & & \cdots \\
\downarrow 0 & & \downarrow 0 & & \cdots & & \downarrow 0 & & \downarrow 0 & & \cdots \\
y_1(2) & & y_2(2) & & \cdots & & y_t(2) & & y_{t+1}(2) & & \cdots \\
\downarrow 0 & & \downarrow 0 & & \cdots & & \downarrow 0 & & \downarrow 0 & & \cdots \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
\downarrow 0 & & \downarrow 0 & & \cdots & & \downarrow 0 & & \downarrow 0 & & \cdots \\
y_1(t') & & y_2(t') & & \cdots & & y_t(t') & & y_{t+1}(t') & & \cdots \\
\downarrow 0 & & \downarrow 0 & & \cdots & & \downarrow 0 & & \downarrow 0 & & \cdots \\
y_1(t'+1) & & y_2(t'+1) & & \cdots & & y_t(t'+1) & & y_{t+1}(t'+1) & & \cdots \\
\downarrow 0 & & \downarrow 0 & & \cdots & & \downarrow 0 & & \downarrow 0 & & \cdots \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots
\end{array}
$$

A 'sequential' zero input response series

Organized into the matrix

$$Y_0 := \begin{bmatrix} y_1(1) & y_2(1) & \cdots & y_t(1) & y_{t+1}(1) & \cdots \\ y_1(2) & y_2(2) & \cdots & y_t(2) & y_{t+1}(2) & \cdots \\ . & . & \cdots & . & . & \cdots \\ y_1(t') & y_2(t') & \cdots & y_t(t') & y_{t+1}(t') & \cdots \\ . & . & \cdots & . & . & \cdots \end{bmatrix}$$

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Note

$$x(t+1) = Ax(t) + Bu(t); \qquad \text{for some } u(\cdot)$$

$$Y_0 = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{t'-1} \\ \vdots \end{bmatrix} \begin{bmatrix} x(1) & x(2) & \cdots & x(t) & \cdots \end{bmatrix}$$

How does deterministic subspace identification work ?

Deterministic subspace identification

There are basically five steps. Use the data

$$\begin{bmatrix} \tilde{u}(1) \\ \tilde{y}(1) \end{bmatrix}, \begin{bmatrix} \tilde{u}(2) \\ \tilde{y}(2) \end{bmatrix}, \ldots, \begin{bmatrix} \tilde{u}(T) \\ \tilde{y}(T) \end{bmatrix}$$

to compute (an estimate of)

Deterministic subspace identification

1. a **sequential zero input response series** matrix of the system that produced the data $\rightsquigarrow$ Y_0

Deterministic subspace identification

1. a **sequential zero input response series** $\rightsquigarrow$ Y_0

2. the **impulse response matrix** $H : \mathbb{Z}_+ \to \mathbb{R}^{p \times m}$
   of the system that produced the data
   $\rightsquigarrow$ **the Hankel matrix** $\mathfrak{H}$

Deterministic subspace identification

1. a **sequential zero input response series** $\rightsquigarrow$ Y_0
2. the **impulse response matrix** $\rightsquigarrow$ $\mathfrak{H}$
3. an **SVD** of this Hankel matrix $\mathfrak{H} = U \Sigma V^\top$

Deterministic subspace identification

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3. an **SVD** of this Hankel matrix $\mathfrak{H} = U\Sigma V^\top$

4. the **balanced state trajectory**

$$\begin{bmatrix} x(1) & x(2) & \cdots & x(t) & \cdots \end{bmatrix} = \sqrt{\Sigma^{-1}}\,U^\top Y_0$$

Deterministic subspace identification

1. a **sequential zero input response series** $\rightsquigarrow$ Y_0
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3. an **SVD** of this Hankel matrix $\mathfrak{H} = U\Sigma V^\top$
4. the **balanced state trajectory**

$$\begin{bmatrix} x(1) & x(2) & \cdots & x(t) & \cdots \end{bmatrix} = \sqrt{\Sigma^{-1}}\, U^\top Y_0$$

5. (LS) solve, with u, y, x a (data ind.) system traj.

$$\begin{bmatrix} x(2) & x(3) & \cdots & x(t+1) & \cdots \\ y(1) & y(2) & \cdots & y(t) & \cdots \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x(1) & x(2) & \cdots & x(t) & \cdots \\ u(1) & u(2) & \cdots & u(t) & \cdots \end{bmatrix}$$

This yields a desired balanced state representation.

The question is

How do we compute all these responses, starting from the data ?

The model class

Time axis = $\mathbb{N}$ **(discrete-time systems)**

σ = **'backward shift'** $\rightsquigarrow$ $(\sigma f)(t) := f(t+1)$

Model class:

$$\begin{aligned} \sigma x &= Ax + Bu \\ y &= Cx + Du \\ w &= \begin{bmatrix} u \\ y \end{bmatrix} \end{aligned}$$

Notation: $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$; **impulse response matrix**

$$H : \mathbb{Z}_+ \to \mathbb{R}^{\mathtt{p}\times\mathtt{m}}; \quad H(0) = D, H(t) = CA^{t-1}B.$$

The model class

But, for good reasons, the (equivalent) representation as a system of linear difference equations

$$R_0 w(t) + R_1 w(t+1) + \cdots + R_L w(t+\ell) = 0 \qquad w = \begin{bmatrix} u \\ y \end{bmatrix}$$

is often to be preferred.

The model class

But, for good reasons, the (equivalent) representation as a system of linear difference equations

$$R_0 w(t) + R_1 w(t+1) + \cdots + R_L w(t+\ell) = 0 \qquad w = \begin{bmatrix} u \\ y \end{bmatrix}$$

is often to be preferred. With the polynomial matrix

$$R(\xi) = R_0 + R_1 \xi + \cdots + R_\ell \xi^\ell$$

these equations can be written as

$$\boxed{R(\sigma) w = 0}$$

The behavior of $R(\sigma)w = 0$

Call

$$\begin{aligned}\mathfrak{B} &= \{w : \mathbb{N} \to \mathbb{R}^{\mathtt{w}} \mid R(\sigma)w = 0\} \\ &= \ker(R(\sigma))\end{aligned}$$

the ***'behavior'.***

The behavior of $R(\sigma)w = 0$

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$$\begin{aligned}\mathfrak{B} &= \{w : \mathbb{N} \to \mathbb{R}^{\mathtt{w}} \mid R(\sigma)w = 0\} \\ &= \ker(R(\sigma))\end{aligned}$$

the ***'behavior'.***

Any subset of $(\mathbb{R}^{\mathtt{w}})^{\mathbb{N}}$ which is

linear, shift-invariant, and closed

allows such a representation.

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the ***'behavior'.*** **Consider also its** ***'annihilators'***

$$\mathfrak{N}_{\mathfrak{B}} = \{n \in \mathbb{R}^{\mathtt{w}}(\xi) \mid n^{\top}(\sigma)\mathfrak{B} = 0\}$$

$$n_0 + n_1\xi + \cdots + n_\ell \xi^\ell \in \mathfrak{N}_{\mathfrak{B}} :\Leftrightarrow$$

$$n_0^\top w(t) + n_1^\top w(t+1) + \cdots + n_\ell^\top w(t+\ell) = 0$$

for all $w \in \mathfrak{B}$ **and** $t \in \mathbb{N}$

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the ***'behavior'.*** Consider also its ***'annihilators'***

$$\mathfrak{N}_{\mathfrak{B}} = \{n \in \mathbb{R}^{\mathtt{w}}(\xi) \mid n^{\top}(\sigma)\mathfrak{B} = 0\}$$

Note: (the transpose of) each row of R belongs to $\mathfrak{N}_{\mathfrak{B}}$.

$\mathfrak{N}_{\mathfrak{B}}$ = the module generated by the transposes of the rows of R.

The behavior generated by $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$

Given $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$, define its behavior as

$$\mathfrak{B} = \{ w = \begin{bmatrix} u \\ y \end{bmatrix} \mid \exists\, x \text{ such that } \sigma x = Ax + Bu,\ y = Cx + Du. \}$$

The behavior generated by $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$

Given $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$, define its behavior as

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Any $\mathfrak{B} = \ker(R(\sigma))$ allows an **observable** repr. $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$. Assumed henceforth.

In behavioral theory

observability $\Leftrightarrow$ **minimality** of the state repr.

Properties and invariants of $\mathfrak{B}$

Each notion has a version for each representation, $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$, $\mathfrak{B}$, **and** $\mathfrak{N}_{\mathfrak{B}}$.

We give the most convenient one.

Properties and invariants of $\mathfrak{B}$

Controllability

$\mathtt{m}(\mathfrak{B}), \mathtt{p}(\mathfrak{B}), \mathtt{n}(\mathfrak{B})$ **:= input, output, state dimension**

Properties and invariants of $\mathfrak{B}$

$\ell(\mathfrak{B})$:= the *lag* in $\mathfrak{B}$

= the degree of R in a *'shortest lag'* repr.

$$R(\sigma)w = 0$$

= the observability index

= the narrowest window through which 'legality' of $w \in \mathfrak{B}$ can be determined.

There holds:

$$\ell(\mathfrak{B}) \leq \mathtt{n}(\mathfrak{B})$$

with = in the single output case.

Properties and invariants of $\mathfrak{B}$

$\ell(\mathfrak{B})$:= the *lag* in $\mathfrak{B}$

$\mathfrak{B}|_{[1,\Delta]}$:= the behavior restr. to the interval $[1,\Delta]$
= the ‘legal’ prefixes of length Δ

Properties and invariants of $\mathfrak{B}$

$\ell(\mathfrak{B})$:= the *lag* in $\mathfrak{B}$

$\mathfrak{B}|_{[1,\Delta]}$:= the behavior restr. to the interval $[1, \Delta]$

= the 'legal' prefixes of length Δ

$\mathfrak{N}_{\mathfrak{B}}^{\mathtt{d}}$:= the annihilators of degree $\leq \mathtt{d}$.

$$n_0^\top w(t) + n_1^\top w(t+1) + \cdots + n_{\mathtt{d}}^\top w(t+\mathtt{d}) = 0$$

for all $w \in \mathfrak{B}$ and $t \in \mathbb{N}$.

Properties and invariants of $\mathfrak{B}$

It follows that

$$w \in \mathfrak{B} \Leftrightarrow \begin{bmatrix} n_0^\top & \cdots & n_{\ell(\mathfrak{B})}^\top \end{bmatrix} \begin{bmatrix} w(t) \\ \vdots \\ w(t+\ell(\mathfrak{B})) \end{bmatrix} = 0$$

for all $n \in \mathfrak{N}_{\mathfrak{B}}^{\ell(\mathfrak{B})}, t \in \mathbb{N}$

$$\Leftrightarrow \begin{bmatrix} w(t) \\ \vdots \\ w(t+\ell(\mathfrak{B})) \end{bmatrix} \in \mathfrak{B}|_{[1,\ell(\mathfrak{B})+1]}$$

for all $t \in \mathbb{N}$.

Properties and invariants of $\mathfrak{B}$

Hence, if $\Delta > \ell(\mathfrak{B})$,

$$w \in \mathfrak{B} \Leftrightarrow \begin{bmatrix} n_0^\top & \cdots & n_{\Delta-1}^\top \end{bmatrix} \begin{bmatrix} w(t) \\ \vdots \\ w(t+\Delta-1) \end{bmatrix} = 0$$

for all $n \in \mathfrak{N}_{\mathfrak{B}}^{\Delta-1}, t \in \mathbb{N}$

$$\Leftrightarrow \begin{bmatrix} w(t) \\ \vdots \\ w(t+\Delta-1) \end{bmatrix} \in \mathfrak{B}|_{[1,\Delta]}$$ for all $t \in \mathbb{N}$.

Properties and invariants of $\mathfrak{B}$

Hence, if $\Delta > \ell(\mathfrak{B})$, $\mathfrak{B}$ is uniquely determined by its '**short**' sequences and '**short**' annihilators

$$\mathfrak{B}|_{[1,\Delta]} \quad \text{and} \quad \mathfrak{N}_{\mathfrak{B}}^{\Delta-1}.$$

Properties and invariants of $\mathfrak{B}$

Another consequence. Consider

$$\left(\begin{bmatrix}\tilde{u}'(1)\\ \tilde{y}'(1)\end{bmatrix},\ldots,\begin{bmatrix}\tilde{u}'(t-\Delta)\\ \tilde{y}'(t-\Delta)\end{bmatrix},\begin{bmatrix}\tilde{u}'(t-\Delta+1)\\ \tilde{y}'(t-\Delta+1)\end{bmatrix},\ldots,\begin{bmatrix}\tilde{u}'(t)\\ \tilde{y}'(t)\end{bmatrix}\right)\in\mathfrak{B}|_{[1,t]}$$

$$\left(\begin{bmatrix}\tilde{u}''(1)\\ \tilde{y}''(1)\end{bmatrix},\ldots,\begin{bmatrix}\tilde{u}''(\Delta)\\ \tilde{y}''(\Delta)\end{bmatrix},\begin{bmatrix}\tilde{u}''(\Delta+1)\\ \tilde{y}''(\Delta+1)\end{bmatrix},\quad\cdots\quad,\quad\begin{bmatrix}\tilde{u}''(t)\\ \tilde{y}''(t)\end{bmatrix}\right)\in\mathfrak{B}|_{[1,t]}$$

Properties and invariants of $\mathfrak{B}$

Another consequence. Assume suffix' = prefix''.

$$\left(\begin{bmatrix}\tilde{u}'(1)\\ \tilde{y}'(1)\end{bmatrix},\ldots,\begin{bmatrix}\tilde{u}'(t-\Delta)\\ \tilde{y}'(t-\Delta)\end{bmatrix},\begin{bmatrix}\tilde{u}'(t-\Delta+1)\\ \tilde{y}'(t-\Delta+1)\end{bmatrix},\ldots,\begin{bmatrix}\tilde{u}'(t)\\ \tilde{y}'(t)\end{bmatrix}\right)$$

$$/\!/$$

$$\left(\begin{bmatrix}\tilde{u}''(1)\\ \tilde{y}''(1)\end{bmatrix},\ldots,\begin{bmatrix}\tilde{u}''(\Delta)\\ \tilde{y}''(\Delta)\end{bmatrix},\begin{bmatrix}\tilde{u}''(\Delta+1)\\ \tilde{y}''(\Delta+1)\end{bmatrix},\ldots,\begin{bmatrix}\tilde{u}''(t)\\ \tilde{y}''(t)\end{bmatrix}\right)$$

Properties and invariants of $\mathfrak{B}$

Another consequence. Assume suffix' = prefix".

Then their linking

$$\left(\begin{bmatrix}\tilde{u}'(1)\\ \tilde{y}'(1)\end{bmatrix},\ldots,\begin{bmatrix}\tilde{u}'(t-\Delta)\\ \tilde{y}'(t-\Delta)\end{bmatrix},\begin{bmatrix}\tilde{u}'(t-\Delta+1)\\ \tilde{y}'(t-\Delta+1)\end{bmatrix},\ldots,\begin{bmatrix}\tilde{u}'(t)\\ \tilde{y}'(t)\end{bmatrix},\begin{bmatrix}\tilde{u}''(\Delta+1)\\ \tilde{y}''(\Delta+1)\end{bmatrix},\ldots,\begin{bmatrix}\tilde{u}''(t)\\ \tilde{y}''(t)\end{bmatrix}\right)$$

belongs to $\mathfrak{B}|_{[1,2t-\Delta]}$,

if $\Delta \geq \ell(\mathfrak{B})$**, hence if** $\Delta \geq \mathtt{n}(\mathfrak{B})$**.**

Fundamental lemma

Key question

Assume that the vector time-series

$$\begin{bmatrix} \tilde{u}(1) \\ \tilde{y}(1) \end{bmatrix}, \begin{bmatrix} \tilde{u}(2) \\ \tilde{y}(2) \end{bmatrix}, \ldots, \begin{bmatrix} \tilde{u}(T) \\ \tilde{y}(T) \end{bmatrix}$$

has been produced by $\mathfrak{B}$.

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has been produced by $\mathfrak{B}$.

Then, of course, the vectors

$$\begin{bmatrix} \tilde{u}(1) \\ \tilde{y}(1) \\ \vdots \\ \tilde{u}(\Delta) \\ \tilde{y}(\Delta) \end{bmatrix}, \begin{bmatrix} \tilde{u}(2) \\ \tilde{y}(2) \\ \vdots \\ \tilde{u}(\Delta+1) \\ \tilde{y}(\Delta+1) \end{bmatrix}, \ldots, \begin{bmatrix} \tilde{u}(T-\Delta+1) \\ \tilde{y}(T-\Delta+1) \\ \vdots \\ \tilde{u}(T) \\ \tilde{y}(T) \end{bmatrix}$$

belong to $\mathfrak{B}|_{[1,\Delta]}$.

Key question

$$\begin{bmatrix} \tilde{u}(1) \\ \tilde{y}(1) \\ \vdots \\ \tilde{u}(\Delta) \\ \tilde{y}(\Delta) \end{bmatrix}, \begin{bmatrix} \tilde{u}(2) \\ \tilde{y}(2) \\ \vdots \\ \tilde{u}(\Delta+1) \\ \tilde{y}(\Delta+1) \end{bmatrix}, \cdots, \begin{bmatrix} \tilde{u}(T-\Delta+1) \\ \tilde{y}(T-\Delta+1) \\ \vdots \\ \tilde{u}(T) \\ \tilde{y}(T) \end{bmatrix}$$

Under what conditions on

$$\begin{bmatrix} \tilde{u}(1) \\ \tilde{y}(1) \end{bmatrix}, \begin{bmatrix} \tilde{u}(2) \\ \tilde{y}(2) \end{bmatrix}, \cdots, \begin{bmatrix} \tilde{u}(T) \\ \tilde{y}(T) \end{bmatrix} \textit{ and } \mathfrak{B}$$

do they span $\mathfrak{B}|_{[1,\Delta]}$ ***and hence, if*** $\Delta > \ell(\mathfrak{B})$, ***determine the generating behavior*** $\mathfrak{B}$ ***?***

Persistency of excitation

The vector time-series

$$\tilde{u}(1), \tilde{u}(2), \ldots \tilde{u}(T)$$

is said to be ***persistently exciting of order L*** if the Hankel matrix

$$\begin{bmatrix} \tilde{u}(1) & \tilde{u}(2) & \tilde{u}(3) & \cdots & \tilde{u}(T-L+1) \\ \tilde{u}(2) & \tilde{u}(3) & \tilde{u}(4) & \cdots & \tilde{u}(T-L+2) \\ \tilde{u}(3) & \tilde{u}(4) & \tilde{u}(5) & \cdots & \tilde{u}(T-L+3) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \tilde{u}(L) & \tilde{u}(L+1) & \tilde{u}(L+2) & \cdots & \tilde{u}(T) \end{bmatrix}$$

is of full row rank. Pers. of exc. $\Leftrightarrow$ no linear relations.

Fundamental lemma

Assume that the observed vector time-series

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has been generated by a **controllable** finite dimensional linear time-invariant system $\rightsquigarrow$ behavior $\mathfrak{B}$.

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has been generated by a controllable finite dimensional linear time-invariant system $\rightsquigarrow$ behavior $\mathfrak{B}$. Then the vectors

$$\begin{bmatrix}\tilde{u}(1)\\ \tilde{y}(1)\\ \vdots \\ \tilde{u}(\Delta)\\ \tilde{y}(\Delta)\end{bmatrix}, \begin{bmatrix}\tilde{u}(2)\\ \tilde{y}(2)\\ \vdots \\ \tilde{u}(\Delta+1)\\ \tilde{y}(\Delta+1)\end{bmatrix}, \ldots, \begin{bmatrix}\tilde{u}(T-\Delta+1)\\ \tilde{y}(T-\Delta+1)\\ \vdots \\ \tilde{u}(T)\\ \tilde{y}(T)\end{bmatrix}$$

span $\mathfrak{B}|_{[1,\Delta]}$ if $\tilde{u}(1), \ldots, \tilde{u}(T)$ is persistently exc. of order

???

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$$\begin{bmatrix} \tilde{u}(1) \\ \tilde{y}(1) \end{bmatrix}, \begin{bmatrix} \tilde{u}(2) \\ \tilde{y}(2) \end{bmatrix}, \ldots, \begin{bmatrix} \tilde{u}(T) \\ \tilde{y}(T) \end{bmatrix}$$

has been generated by a controllable finite dimensional linear time-invariant system $\rightsquigarrow$ behavior $\mathfrak{B}$. Then the vectors

$$\begin{bmatrix} \tilde{u}(1) \\ \tilde{y}(1) \\ \vdots \\ \tilde{u}(\Delta) \\ \tilde{y}(\Delta) \end{bmatrix}, \begin{bmatrix} \tilde{u}(2) \\ \tilde{y}(2) \\ \vdots \\ \tilde{u}(\Delta+1) \\ \tilde{y}(\Delta+1) \end{bmatrix}, \ldots, \begin{bmatrix} \tilde{u}(T-\Delta+1) \\ \tilde{y}(T-\Delta+1) \\ \vdots \\ \tilde{u}(T) \\ \tilde{y}(T) \end{bmatrix}$$

span $\mathfrak{B}|_{[1,\Delta]}$ if $\tilde{u}(1), \ldots, \tilde{u}(T)$ is persistently exc. of order

$$\boxed{\Delta + \mathrm{n}(\mathfrak{B})}$$

Fundamental lemma

Hence, under the assumptions of

1. controllability and 2. persistency of excitation,

the span (& hence left annihilators) of the data vectors

$$\begin{bmatrix} \tilde{u}(1) \\ \tilde{y}(1) \\ \tilde{u}(2) \\ \tilde{y}(2) \\ \vdots \\ \tilde{u}(\Delta) \\ \tilde{y}(\Delta) \end{bmatrix}, \begin{bmatrix} \tilde{u}(2) \\ \tilde{y}(2) \\ \tilde{u}(3) \\ \tilde{y}(3) \\ \vdots \\ \tilde{u}(\Delta+1) \\ \tilde{y}(\Delta+1) \end{bmatrix}, \ldots, \begin{bmatrix} \tilde{u}(T-\Delta+1) \\ \tilde{y}(T-\Delta+1) \\ \tilde{u}(T-\Delta+2) \\ \tilde{y}(T-\Delta+2) \\ \vdots \\ \tilde{u}(T) \\ \tilde{y}(T) \end{bmatrix}$$

determines $\mathfrak{B}$, provided 3. $\boxed{\Delta > \ell(\mathfrak{B})}$

Conclusion

Under reasonable conditions

(contr., Δ suff. large, persistency of excitation),

the data matrix

$$\begin{bmatrix} \tilde{U} \\ \tilde{Y} \end{bmatrix} = \begin{bmatrix} \tilde{u}(1) & \tilde{u}(2) & \cdots & \tilde{u}(T-\Delta+1) \\ \vdots & \vdots & \vdots & \vdots \\ \tilde{u}(\Delta) & \tilde{u}(\Delta+1) & \cdots & \tilde{u}(T) \\ \tilde{y}(1) & \tilde{y}(2) & \cdots & \tilde{y}(T-\Delta+1) \\ \vdots & \vdots & \vdots & \vdots \\ \tilde{y}(\Delta) & \tilde{y}(\Delta+1) & \cdots & \tilde{y}(T) \end{bmatrix}$$

has the ‘correct’ span and the ‘correct’ left kernel.

Conclusion

Under reasonable conditions the data matrix has the **'correct'** span and the **'correct'** left kernel.

$\Rightarrow$ <u>any</u> response, in particular, seq. zero input resp., impulse resp., etc., can be obtained by solving

$$\begin{bmatrix} u(1) \\ \vdots \\ u(\Delta) \\ y(1) \\ \vdots \\ y(\Delta) \end{bmatrix} = \begin{bmatrix} \tilde{U} \\ \tilde{Y} \end{bmatrix} g$$

Conclusion

and linking and solving, with $\mathrm{n}_{\max} \geq \ell(\mathfrak{B})$ or $\mathrm{n}(\mathfrak{B})$,

$$\begin{bmatrix} u(1) \\ \vdots \\ u(\Delta - \mathrm{n}_{\max}) \\ u(\Delta - \mathrm{n}_{\max} + 1) \\ \vdots \\ u(\Delta) \\ y(1) \\ \vdots \\ y(\Delta - \mathrm{n}_{\max}) \\ y(\Delta - \mathrm{n}_{\max} + 1) \\ \vdots \\ y(\Delta) \end{bmatrix} \quad \nearrow \quad \nearrow \quad \begin{bmatrix} u(\Delta - \mathrm{n}_{\max} + 1) \\ \vdots \\ u(\Delta) \\ u(\Delta + 1) \\ \vdots \\ u(2\Delta - \mathrm{n}_{\max}) \\ y(\Delta - \mathrm{n}_{\max} + 1) \\ \vdots \\ y(\Delta) \\ y(\Delta + 1) \\ \vdots \\ y(2\Delta - \mathrm{n}_{\max}) \end{bmatrix} = \begin{bmatrix} \tilde{U} \\ \tilde{Y} \end{bmatrix} g$$

Conclusion

and proceeding recursively

$$
\begin{bmatrix}
u(\Delta' + 1) \\ \vdots \\ u(\Delta' + n_{max}) \\ u(\Delta' + n_{max} + 1) \\ \vdots \\ u(\Delta' + \Delta) \\ y(\Delta' + n_{max}) \\ \vdots \\ y(\Delta) \\ y(\Delta' + n_{max} + 1) \\ \vdots \\ y(\Delta' + \Delta)
\end{bmatrix}
\begin{matrix} \nearrow \\ \nearrow \end{matrix}
\begin{bmatrix}
u(\Delta' + n_{max} + 1) \\ \vdots \\ u(\Delta + \Delta') \\ u(\Delta' + \Delta + 1) \\ \vdots \\ u(\Delta + \Delta' + n_{max}) \\ y(\Delta' + n_{max} + 1) \\ \vdots \\ y(\Delta + \Delta') \\ y(\Delta' + \Delta + 1) \\ \vdots \\ y(\Delta + \Delta' + n_{max})
\end{bmatrix}
= \begin{bmatrix} \tilde{U} \\ \tilde{Y} \end{bmatrix} g
$$

Conclusion

This way, an arbitrary long sequence

$$\left(\begin{bmatrix} u(1) \\ y(1) \end{bmatrix}, \begin{bmatrix} u(2) \\ y(2) \end{bmatrix}, \ldots, \begin{bmatrix} u(t) \\ y(t) \end{bmatrix} \right) \in \mathfrak{B}|_{[1,t]}$$

can be obtained.

Note: These algorithms allow nicely for (LS) approximate computations.

An idea of the proof

Assume

$$\begin{bmatrix} \tilde{u}(1) \\ \tilde{y}(1) \end{bmatrix}, \begin{bmatrix} \tilde{u}(2) \\ \tilde{y}(2) \end{bmatrix}, \ldots, \begin{bmatrix} \tilde{u}(T) \\ \tilde{y}(T) \end{bmatrix} \in \mathfrak{B}_{[1,T]}.$$

An idea of the proof

Assume

$$\begin{bmatrix} \tilde{u}(1) \\ \tilde{y}(1) \end{bmatrix}, \begin{bmatrix} \tilde{u}(2) \\ \tilde{y}(2) \end{bmatrix}, \ldots, \begin{bmatrix} \tilde{u}(T) \\ \tilde{y}(T) \end{bmatrix} \in \mathfrak{B}_{[1,T]}.$$

SISO case $\rightsquigarrow$ $\mathfrak{B}$ **determined by**

$$p_{\mathrm{n}} y(t+\mathrm{n}) + p_{\mathrm{n}-1} y(t+\mathrm{n}-1) + \cdots + p_0 y(t)$$
$$= q_{\mathrm{n}} u(t+\mathrm{n}) + q_{\mathrm{n}-1} u(t+\mathrm{n}-1) + \cdots + q_0 u(t)$$

$$\boxed{p(\sigma) y = q(\sigma) u}$$

$$\begin{aligned} p(\xi) &= p_0 + p_1 \xi + \cdots + p_{\mathrm{n}} \xi^{\mathrm{n}}, \qquad \text{with } p_{\mathrm{n}} \neq 0, \\ q(\xi) &= q_0 + q_1 \xi + \cdots + q_{\mathrm{n}} \xi^{\mathrm{n}}. \end{aligned}$$

An idea of the proof

Assume

$$\begin{bmatrix} \tilde{u}(1) \\ \tilde{y}(1) \end{bmatrix}, \begin{bmatrix} \tilde{u}(2) \\ \tilde{y}(2) \end{bmatrix}, \ldots, \begin{bmatrix} \tilde{u}(T) \\ \tilde{y}(T) \end{bmatrix} \in \mathfrak{B}_{[1,T]}.$$

$$\boxed{p(\sigma)y = q(\sigma)u}$$

$$\begin{aligned} p(\xi) &= p_0 + p_1\xi + \cdots + p_{\mathrm{n}}\xi^{\mathrm{n}}, \qquad \text{with } p_{\mathrm{n}} \neq 0, \\ q(\xi) &= q_0 + q_1\xi + \cdots + q_{\mathrm{n}}\xi^{\mathrm{n}}. \end{aligned}$$

$$\mathfrak{N}_{\mathfrak{B}} = \text{span}\left\{ \begin{bmatrix} -q(\xi) \\ p(\xi) \end{bmatrix}, \begin{bmatrix} -\xi q(\xi) \\ \xi p(\xi) \end{bmatrix}, \ldots, \begin{bmatrix} -\xi^{\mathrm{k}} q(\xi) \\ \xi^{\mathrm{k}} p(\xi) \end{bmatrix}, \ldots \right\}$$

An idea of the proof

Data matrix:

$$\begin{bmatrix} \tilde{u}(1) & \tilde{u}(2) & \cdots & \tilde{u}(T-\Delta+1) \\ \tilde{u}(2) & \tilde{u}(3) & \cdots & \tilde{u}(T-\Delta+2) \\ \vdots & \vdots & \vdots & \vdots \\ \tilde{u}(\Delta) & \tilde{u}(\Delta+1) & \cdots & \tilde{u}(T) \\ \tilde{y}(1) & \tilde{y}(2) & \cdots & \tilde{y}(T-\Delta+1) \\ \tilde{y}(2) & \tilde{y}(3) & \cdots & \tilde{y}(T-\Delta+2) \\ \vdots & \vdots & \vdots & \vdots \\ \tilde{y}(\Delta) & \tilde{y}(\Delta+1) & \cdots & \tilde{y}(T) \end{bmatrix}$$

An idea of the proof

For $\Delta = \mathtt{n} + 1$, the left kernel contains

$$\begin{bmatrix} -q_0 & -q_1 & \cdots & -q_{\mathtt{n}} & p_0 & p_1 & \cdots & p_{\mathtt{n}} \end{bmatrix}.$$

An idea of the proof

For $\Delta > \mathrm{n}+1,$ the left kernel contains the rows of

$$\begin{bmatrix} -q_0 & \cdots & -q_{\mathrm{n}} & 0 & \cdots & 0 & p_0 & \cdots & p_{\mathrm{n}} & 0 & \cdots & 0 \\ 0 & -q_0 & \cdots & -q_{\mathrm{n}} & \cdots & 0 & 0 & p_0 & \cdots & p_{\mathrm{n}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & -q_0 & \cdots & -q_{\mathrm{n}} & 0 & \cdots & 0 & p_0 & \cdots & p_{\mathrm{n}} \end{bmatrix} \begin{matrix} \leftarrow \text{row } 1 \\ \leftarrow \text{row } 2 \\ \\ \leftarrow \text{row } \Delta - \mathrm{n} \end{matrix}$$

An idea of the proof

For $\Delta > n + 1$, the left kernel contains the rows of

$$\begin{bmatrix} -q_0 & \cdots & -q_n & 0 & \cdots & 0 & p_0 & \cdots & p_n & 0 & \cdots & 0 \\ 0 & -q_0 & \cdots & -q_n & \cdots & 0 & 0 & p_0 & \cdots & p_n & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & -q_0 & \cdots & -q_n & 0 & \cdots & 0 & p_0 & \cdots & p_n \end{bmatrix}$$

← row 1
← row 2
← row $\Delta - n$

Assume that the kernel contains another vector, not in their span

$$\begin{bmatrix} r_0 & \cdots & \cdot & \cdot & \cdots & r_{\Delta-1} & s_0 & \cdots & \cdot & \cdot & \cdots & s_{\Delta-1} \end{bmatrix}$$

An idea of the proof

Extend the data matrix to a larger window:

$$\begin{bmatrix} \tilde{u}(1) & \tilde{u}(2) & \cdots & \tilde{u}(T-\Delta'+1) \\ \tilde{u}(2) & \tilde{u}(3) & \cdots & \tilde{u}(T-\Delta'+2) \\ \vdots & \vdots & \vdots & \vdots \\ \tilde{u}(\Delta') & \tilde{u}(\Delta'+1) & \cdots & \tilde{u}(T) \\ \tilde{y}(1) & \tilde{y}(2) & \cdots & \tilde{y}(T-\Delta'+1) \\ \tilde{y}(2) & \tilde{y}(3) & \cdots & \tilde{y}(T-\Delta'+2) \\ \vdots & \vdots & \vdots & \vdots \\ \tilde{y}(\Delta') & \tilde{y}(\Delta'+1) & \cdots & \tilde{y}(T) \end{bmatrix}$$

An idea of the proof

Then the left kernel contains the rows of

$$\begin{bmatrix} -q_0 & \cdots & -q_n & 0 & \cdots & 0 & p_0 & \cdots & p_n & 0 & \cdots & 0 \\ 0 & -q_0 & \cdots & -q_n & \cdots & 0 & 0 & p_0 & \cdots & p_n & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & -q_0 & \cdots & -q_n & 0 & \cdots & 0 & p_0 & \cdots & p_n \end{bmatrix} \begin{matrix} \leftarrow \text{row } 1 \\ \leftarrow \text{row } 2 \\ \\ \leftarrow \text{row } \Delta' - n \end{matrix}$$

An idea of the proof

Then the left kernel contains the rows of

$$\begin{bmatrix} -q_0 & \cdots & -q_n & 0 & \cdots & 0 & p_0 & \cdots & p_n & 0 & \cdots & 0 \\ 0 & -q_0 & \cdots & -q_n & \cdots & 0 & 0 & p_0 & \cdots & p_n & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & -q_0 & \cdots & -q_n & 0 & \cdots & 0 & p_0 & \cdots & p_n \end{bmatrix} \begin{matrix} \leftarrow \text{row } 1 \\ \leftarrow \text{row } 2 \\ \\ \leftarrow \text{row } \Delta' - n \end{matrix}$$

$$\begin{bmatrix} r_0 & \cdot & r_{\Delta-1} & 0 & \cdot & 0 & s_0 & \cdot & s_{\Delta-1} & 0 & \cdot & 0 \\ 0 & r_0 & \cdot & r_{\Delta-1} & \cdot & 0 & 0 & s_0 & \cdot & s_{\Delta-1} & \cdot & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdot & 0 & r_0 & \cdot & r_{\Delta-1} & 0 & \cdot & 0 & s_0 & \cdot & s_{\Delta-1} \end{bmatrix} \begin{matrix} \leftarrow \text{row } 1 \\ \leftarrow \text{row } 2 \\ \\ \leftarrow \text{row } \Delta' - \Delta + 1 \end{matrix}$$

An idea of the proof

Then the left kernel contains the rows of

$$\begin{bmatrix} -q_0 & \cdots & -q_n & 0 & \cdots & 0 & p_0 & \cdots & p_n & 0 & \cdots & 0 \\ 0 & -q_0 & \cdots & -q_n & \cdots & 0 & 0 & p_0 & \cdots & p_n & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & -q_0 & \cdots & -q_n & 0 & \cdots & 0 & p_0 & \cdots & p_n \end{bmatrix} \begin{matrix} \leftarrow \text{row } 1 \\ \leftarrow \text{row } 2 \\ \\ \leftarrow \text{row } \Delta' - n \end{matrix}$$

$$\begin{bmatrix} r_0 & \cdot & r_{\Delta-1} & 0 & \cdot & 0 & s_0 & \cdot & s_{\Delta-1} & 0 & \cdot & 0 \\ 0 & r_0 & \cdot & r_{\Delta-1} & \cdot & 0 & 0 & s_0 & \cdot & s_{\Delta-1} & \cdot & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdot & 0 & r_0 & \cdot & r_{\Delta-1} & 0 & \cdot & 0 & s_0 & \cdot & s_{\Delta-1} \end{bmatrix} \begin{matrix} \leftarrow \text{row } 1 \\ \leftarrow \text{row } 2 \\ \\ \leftarrow \text{row } \Delta' - \Delta + 1 \end{matrix}$$

If all rows were linearly independent, then at each extension step, the rank of the data matrix remains constant. But, persistency of excitation $\Rightarrow$ the rank increases by 1. $\rightsquigarrow$ conflict, when $\Delta' = \Delta + n$.

An idea of the proof

Then the left kernel contains the rows of

$$\begin{bmatrix} -q_0 & \cdots & -q_n & 0 & \cdots & 0 & p_0 & \cdots & p_n & 0 & \cdots & 0 \\ 0 & -q_0 & \cdots & -q_n & \cdots & 0 & 0 & p_0 & \cdots & p_n & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & -q_0 & \cdots & -q_n & 0 & \cdots & 0 & p_0 & \cdots & p_n \end{bmatrix} \begin{matrix} \leftarrow \text{row } 1 \\ \leftarrow \text{row } 2 \\ \\ \leftarrow \text{row } \Delta' - \mathrm{n} \end{matrix}$$

$$\begin{bmatrix} r_0 & \cdot & r_{\Delta-1} & 0 & \cdot & 0 & s_0 & \cdot & s_{\Delta-1} & 0 & \cdot & 0 \\ 0 & r_0 & \cdot & r_{\Delta-1} & \cdot & 0 & 0 & s_0 & \cdot & s_{\Delta-1} & \cdot & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdot & 0 & r_0 & \cdot & r_{\Delta-1} & 0 & \cdot & 0 & s_0 & \cdot & s_{\Delta-1} \end{bmatrix} \begin{matrix} \leftarrow \text{row } 1 \\ \leftarrow \text{row } 2 \\ \\ \leftarrow \text{row } \Delta'-\Delta+1 \end{matrix}$$

Therefore one of the rows of the second matrix must be linearly dependent on the rows preceding it and the rows of the first matrix.

Written in polynomial notation, this yields

An idea of the proof

$$f(\xi)\left[r(\xi) \quad | \quad s(\xi)\right] = h(\xi)\left[-q(\xi) \quad | \quad p(\xi)\right]$$

with, without loss of generality, f and h co-prime.

An idea of the proof

$$f(\xi)\left[r(\xi) \quad | \quad s(\xi)\right] = h(\xi)\left[-q(\xi) \quad | \quad p(\xi)\right]$$

with, without loss of generality, f and h co-prime.
This means that f must be a factor of both p and q.

An idea of the proof

$$f(\xi)\left[r(\xi) \quad | \quad s(\xi)\right] = h(\xi)\left[-q(\xi) \quad | \quad p(\xi)\right]$$

with, without loss of generality, f and h co-prime.
This means that f must be a factor of both p and q.

If degree$(f) > 0$,

this contradicts the fact that $\mathfrak{B}$ is controllable.

An idea of the proof

$$f(\xi)\left[r(\xi) \quad | \quad s(\xi)\right] = h(\xi)\left[-q(\xi) \quad | \quad p(\xi)\right]$$

with, without loss of generality, f and h co-prime.
This means that f must be a factor of both p and q.

If degree$(f) > 0$,

this contradicts the fact that $\mathfrak{B}$ is controllable.

Whence, $f = 1$, but then

$$\left[r(\xi) \quad | \quad s(\xi)\right] = h(\xi)\left[-q(\xi) \quad | \quad p(\xi)\right]$$

An idea of the proof

and hence

$$\begin{bmatrix} r_0 & \cdots & \cdot & \cdot & \cdots & r_{\Delta-1} & s_0 & \cdots & \cdot & \cdot & \cdots & s_{\Delta-1} \end{bmatrix}$$

is in the span of the rows of

$$\begin{bmatrix} -q_0 & \cdots & -q_n & 0 & \cdots & 0 & p_0 & \cdots & p_n & 0 & \cdots & 0 \\ 0 & -q_0 & \cdots & -q_n & \cdots & 0 & 0 & p_0 & \cdots & p_n & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & -q_0 & \cdots & -q_n & 0 & \cdots & 0 & p_0 & \cdots & p_n \end{bmatrix} \begin{matrix} \leftarrow \text{row } 1 \\ \leftarrow \text{row } 2 \\ \\ \leftarrow \text{row } \Delta - n \end{matrix}$$

An idea of the proof

and hence

$$\begin{bmatrix} r_0 & \cdots & \cdot & \cdot & \cdots & r_{\Delta-1} & s_0 & \cdots & \cdot & \cdot & \cdots & s_{\Delta-1} \end{bmatrix}$$

is in the span of the rows of

$$\begin{bmatrix} -q_0 & \cdots & -q_{\mathrm{n}} & 0 & \cdots & 0 & p_0 & \cdots & p_{\mathrm{n}} & 0 & \cdots & 0 \\ 0 & -q_0 & \cdots & -q_{\mathrm{n}} & \cdots & 0 & 0 & p_0 & \cdots & p_{\mathrm{n}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & -q_0 & \cdots & -q_{\mathrm{n}} & 0 & \cdots & 0 & p_0 & \cdots & p_{\mathrm{n}} \end{bmatrix} \begin{matrix} \leftarrow \text{row } 1 \\ \leftarrow \text{row } 2 \\ \\ \leftarrow \text{row } \Delta - \mathrm{n} \end{matrix}$$

Therefore, the data matrix had the ’correct’ kernel to begin with. **QED**

Van Overschee - De Moor from this perspective

Van Overschee - De Moor subspace identification

1. How can we compute a sequential zero input response series?

Van Overschee - De Moor subspace identification

Define the ‘past’ and ‘future’ input and output data matrices by

$$\begin{bmatrix} \tilde{U}_{\text{p}} \\ \tilde{Y}_{\text{p}} \\ \tilde{U}_{\text{f}} \\ \tilde{Y}_{\text{f}} \end{bmatrix} = \begin{bmatrix} \tilde{u}(1) & \tilde{u}(2) & \cdots & \tilde{u}(T-2\Delta+1) \\ \vdots & \vdots & \vdots & \vdots \\ \tilde{u}(\Delta) & \tilde{u}(\Delta+1) & \cdots & \tilde{u}(T-\Delta) \\ \tilde{y}(1) & \tilde{y}(2) & \cdots & \tilde{y}(T-2\Delta+1) \\ \vdots & \vdots & \vdots & \vdots \\ \tilde{y}(\Delta) & \tilde{y}(\Delta+1) & \cdots & \tilde{y}(T-\Delta) \\ \tilde{u}(\Delta+1) & \tilde{u}(\Delta+2) & \cdots & \tilde{u}(T-\Delta+1) \\ \vdots & \vdots & \vdots & \vdots \\ \tilde{u}(2\Delta) & \tilde{u}(2\Delta+1) & \cdots & \tilde{u}(T) \\ \tilde{y}(\Delta+1) & \tilde{y}(\Delta+2) & \cdots & \tilde{y}(T-\Delta+1) \\ \vdots & \vdots & \vdots & \vdots \\ \tilde{y}(2\Delta) & \tilde{y}(2\Delta+1) & \cdots & \tilde{y}(T) \end{bmatrix}$$

Assume $\mathtt{n}(\mathfrak{B}) \ll \Delta \ll T$ & pers. of excitation, as needed.

Van Overschee - De Moor subspace identification

1. Solve for $??$ (through G) in

$$\begin{bmatrix} \tilde{U}_{\mathrm{p}} \\ \tilde{Y}_{\mathrm{p}} \\ O \\ ?? \end{bmatrix} = \begin{bmatrix} \tilde{U}_{\mathrm{p}} \\ \tilde{Y}_{\mathrm{p}} \\ \tilde{U}_{\mathrm{f}} \\ \tilde{Y}_{\mathrm{f}} \end{bmatrix} G$$

Van Overschee - De Moor subspace identification

1. Solve for $??$ (through G) in

$$\begin{bmatrix} \tilde{U}_{\mathrm{p}} \\ \tilde{Y}_{\mathrm{p}} \\ O \\ ?? \end{bmatrix} = \begin{bmatrix} \tilde{U}_{\mathrm{p}} \\ \tilde{Y}_{\mathrm{p}} \\ \tilde{U}_{\mathrm{f}} \\ \tilde{Y}_{\mathrm{f}} \end{bmatrix} G$$

$??$ is obviously a seq. zero input resp. $\rightsquigarrow$ Y_0.

Van Overschee - De Moor subspace identification

1. Solve for $??$ (through G) in

$$\begin{bmatrix} \tilde{U}_{\mathrm{p}} \\ \tilde{Y}_{\mathrm{p}} \\ O \\ ?? \end{bmatrix} = \begin{bmatrix} \tilde{U}_{\mathrm{p}} \\ \tilde{Y}_{\mathrm{p}} \\ \tilde{U}_{\mathrm{f}} \\ \tilde{Y}_{\mathrm{f}} \end{bmatrix} G$$

$??$ is obviously a seq. zero input resp. $\rightsquigarrow$ Y_0.

Y_0 = the oblique projection of the row span of $\tilde{Y}_{\mathrm{f}}$,

along the row span of $\tilde{U}_{\mathrm{f}}$, onto the row span of $\begin{bmatrix} \tilde{U}_{\mathrm{p}} \\ \tilde{Y}_{\mathrm{p}} \end{bmatrix}$!

Van Overschee - De Moor subspace identification

2. How can we compute (an estimate of) the Hankel matrix?

2. How can we compute (an estimate of) the Hankel matrix?

By solving for G in:

Van Overschee - De Moor subspace identification

$$\begin{bmatrix} O_{m\times m} & O_{m\times m} & \cdots & O_{m\times m} & I_{m\times m} \\ O_{m\times m} & O_{m\times m} & \cdots & I_{m\times m} & O_{m\times m} \\ \vdots & \vdots & \nearrow & \vdots & \vdots \\ O_{m\times m} & I_{m\times m} & \cdots & O_{m\times m} & O_{m\times m} \\ I_{m\times m} & O_{m\times m} & \cdots & O_{m\times m} & O_{m\times m} \\ \star & \star & \cdots & \star & \star \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \star & \star & \cdots & \star & \star \\ O_{m\times m} & O_{m\times m} & \cdots & O_{m\times m} & O_{m\times m} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ O_{m\times m} & O_{m\times m} & \cdots & O_{m\times m} & O_{m\times m} \\ \hat{H}(1) & \hat{H}(2) & \cdots & \hat{H}(\Delta-1) & \hat{H}(\Delta) \\ \hat{H}(2) & \hat{H}(3) & \cdots & \hat{H}(\Delta) & \hat{H}(\Delta+1) \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \hat{H}(\Delta-1) & \hat{H}(\Delta) & \cdots & \hat{H}(2\Delta-3) & \hat{H}(2\Delta-2) \\ \hat{H}(\Delta) & \hat{H}(\Delta+1) & \cdots & \hat{H}(2\Delta-2) & \hat{H}(2\Delta-1) \end{bmatrix} = \begin{bmatrix} \tilde{U}_{\mathsf{p}} \\ \tilde{Y}_{\mathsf{p}} \\ \tilde{U}_{\mathsf{f}} \\ \tilde{Y}_{\mathsf{f}} \end{bmatrix} G$$

Van Overschee - De Moor subspace identification

Or, since the columns of $\begin{bmatrix} \tilde{U}_{\mathsf{p}} \\ \tilde{Y}_{\mathsf{p}} \\ O \\ Y_0 \end{bmatrix}$ are spanned by the

columns of $\begin{bmatrix} \tilde{U}_{\mathsf{p}} \\ \tilde{Y}_{\mathsf{p}} \\ \tilde{U}_{\mathsf{f}} \\ \tilde{Y}_{\mathsf{f}} \end{bmatrix}$, by solving for G' in

Van Overschee - De Moor subspace identification

$$
\begin{bmatrix}
O_{m\times m} & O_{m\times m} & \cdots & O_{m\times m} & I_{m\times m} \\
O_{m\times m} & O_{m\times m} & \cdots & I_{m\times m} & O_{m\times m} \\
\vdots & \vdots & \iddots & \vdots & \vdots \\
O_{m\times m} & I_{m\times m} & \cdots & O_{m\times m} & O_{m\times m} \\
I_{m\times m} & O_{m\times m} & \cdots & O_{m\times m} & O_{m\times m} \\
\star & \star & \cdots & \star & \star \\
\vdots & \vdots & \cdots & \vdots & \vdots \\
\star & \star & \cdots & \star & \star \\
O_{m\times m} & O_{m\times m} & \cdots & O_{m\times m} & O_{m\times m} \\
\vdots & \vdots & \cdots & \vdots & \vdots \\
O_{m\times m} & O_{m\times m} & \cdots & O_{m\times m} & O_{m\times m} \\
\hat{H}(1) & \hat{H}(2) & \cdots & \hat{H}(\Delta-1) & \hat{H}(\Delta) \\
\hat{H}(2) & \hat{H}(3) & \cdots & \hat{H}(\Delta) & \hat{H}(\Delta+1) \\
\vdots & \vdots & \cdots & \vdots & \vdots \\
\hat{H}(\Delta-1) & \hat{H}(\Delta) & \cdots & \hat{H}(2\Delta-3) & \hat{H}(2\Delta-2) \\
\hat{H}(\Delta) & \hat{H}(\Delta+1) & \cdots & \hat{H}(2\Delta-2) & \hat{H}(2\Delta-1)
\end{bmatrix}
=
\begin{bmatrix}
\tilde{U}_p \\ \tilde{Y}_p \\ O \\ Y_0
\end{bmatrix}
G'
$$

Van Overschee - De Moor subspace identification

Solution $\rightsquigarrow$

$$G' = \tilde{U}_{\mathrm{p}}^{\top}(\tilde{U}_{\mathrm{p}}\tilde{U}_{\mathrm{p}}^{\top})^{-1}J$$

where

$$J = \begin{bmatrix} O_{\mathtt{m}\times\mathtt{m}} & O_{\mathtt{m}\times\mathtt{m}} & \cdots & O_{\mathtt{m}\times\mathtt{m}} & I_{\mathtt{m}\times\mathtt{m}} \\ O_{\mathtt{m}\times\mathtt{m}} & O_{\mathtt{m}\times\mathtt{m}} & \cdots & I_{\mathtt{m}\times\mathtt{m}} & O_{\mathtt{m}\times\mathtt{m}} \\ \cdot & \cdot & \nearrow & \cdot & \cdot \\ O_{\mathtt{m}\times\mathtt{m}} & I_{\mathtt{m}\times\mathtt{m}} & \cdots & O_{\mathtt{m}\times\mathtt{m}} & O_{\mathtt{m}\times\mathtt{m}} \\ I_{\mathtt{m}\times\mathtt{m}} & O_{\mathtt{m}\times\mathtt{m}} & \cdots & O_{\mathtt{m}\times\mathtt{m}} & O_{\mathtt{m}\times\mathtt{m}} \end{bmatrix}$$

Van Overschee - De Moor subspace identification

Solution $\rightsquigarrow$

$$G' = \tilde{U}_{\mathrm{p}}^{\top}(\tilde{U}_{\mathrm{p}}\tilde{U}_{\mathrm{p}}^{\top})^{-1}J$$

$\rightsquigarrow$ **the following estimate of the Hankel matrix $\mathfrak{H}$:**

$$\hat{\mathfrak{H}} := \begin{bmatrix} \hat{H}(1) & \hat{H}(2) & \cdots & \hat{H}(\Delta-1) & \hat{H}(\Delta) \\ \hat{H}(2) & \hat{H}(3) & \cdots & \hat{H}(\Delta) & \hat{H}(\Delta+1) \\ . & . & \cdots & . & . \\ \hat{H}(\Delta-1) & \hat{H}(\Delta) & \cdots & \hat{H}(2\Delta-3) & \hat{H}(2\Delta-2) \\ \hat{H}(\Delta) & \hat{H}(\Delta+1) & \cdots & \hat{H}(2\Delta-2) & \hat{H}(2\Delta-1) \end{bmatrix}$$

$$\hat{\mathfrak{H}} = Y_0\,\tilde{U}_{\mathrm{p}}^{\top}(\tilde{U}_{\mathrm{p}}\tilde{U}_{\mathrm{p}}^{\top})^{-1}J$$

Van Overschee - De Moor subspace identification

Solution $\rightsquigarrow$

$$G' = \tilde{U}_{\mathsf{p}}^{\top}(\tilde{U}_{\mathsf{p}}\tilde{U}_{\mathsf{p}}^{\top})^{-1}J$$

$\rightsquigarrow$ **the following estimate of the Hankel matrix** $\mathfrak{H}$**:**

$$\hat{\mathfrak{H}} = Y_0\,\tilde{U}_{\mathsf{p}}^{\top}(\tilde{U}_{\mathsf{p}}\tilde{U}_{\mathsf{p}}^{\top})^{-1}J$$

Note: no new eq'ns to be solved, once we have Y_0**.**

Van Overschee - De Moor subspace identification

3. SVD this Hankel matrix $\rightsquigarrow$ $\hat{\mathfrak{H}} = U\Sigma V^\top$

Van Overschee - De Moor subspace identification

3. **SVD** this Hankel matrix $\rightsquigarrow$ $\hat{\mathfrak{H}} = U \Sigma V^\top$

4. Obtain the **balanced state trajectory**

$$\begin{bmatrix} \hat{x}(\Delta + 1) & \hat{x}(\Delta + 2) & \cdots & \hat{x}(T - \Delta + 1) \end{bmatrix} = \sqrt{\Sigma^{-1}} U^\top Y_0$$

Van Overschee - De Moor subspace identification

3. SVD this Hankel matrix $\rightsquigarrow$ $\hat{\mathfrak{H}} = U\Sigma V^\top$

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$$\begin{bmatrix} \hat{x}(\Delta+1) & \hat{x}(\Delta+2) & \cdots & \hat{x}(T-\Delta+1) \end{bmatrix} = \sqrt{\Sigma^{-1}}\, U^\top Y_0$$

$\hat{x}(\Delta+1), \hat{x}(\Delta+2), \cdots, \hat{x}(T-\Delta+1)$ are estimates of a balanced state traj. separating the 'past' and 'future'.

Van Overschee - De Moor subspace identification

$$\begin{bmatrix} \tilde{U}_{\mathrm{p}} \\ \tilde{Y}_{\mathrm{p}} \\ \hat{X} \\ \tilde{U}_{\mathrm{f}} \\ \tilde{Y}_{\mathrm{f}} \end{bmatrix} = \begin{bmatrix} \tilde{u}(1) & \tilde{u}(2) & \cdots & \tilde{u}(T-2\Delta+1) \\ \vdots & \vdots & \cdots & \vdots \\ \tilde{u}(\Delta) & \tilde{u}(\Delta+1) & \cdots & \tilde{u}(T-\Delta) \\ \tilde{y}(1) & \tilde{y}(2) & \cdots & \tilde{y}(T-2\Delta+1) \\ \vdots & \vdots & \cdots & \vdots \\ \tilde{y}(\Delta) & \tilde{y}(\Delta+1) & \cdots & \tilde{y}(T-\Delta) \\ \hat{x}(\Delta+1) & \hat{x}(\Delta+2) & \cdots & \hat{x}(T-\Delta+1) \\ \tilde{u}(\Delta+1) & \tilde{u}(\Delta+2) & \cdots & \tilde{u}(T-\Delta+1) \\ \vdots & \vdots & \cdots & \vdots \\ \tilde{u}(2\Delta) & \tilde{u}(2\Delta+1) & \cdots & \tilde{u}(T) \\ \tilde{y}(\Delta+1) & \tilde{y}(\Delta+2) & \cdots & \tilde{y}(T-\Delta+1) \\ \vdots & \vdots & \cdots & \vdots \\ \tilde{y}(2\Delta) & \tilde{y}(2\Delta+1) & \cdots & \tilde{y}(T) \end{bmatrix}$$

Van Overschee - De Moor subspace identification

3. **SVD** this Hankel matrix $\rightsquigarrow$ $\hat{\mathfrak{H}} = U\Sigma V^\top$

4. Obtain the **balanced state trajectory**

$$\begin{bmatrix} \hat{x}(\Delta+1) & \hat{x}(\Delta+2) & \cdots & \hat{x}(T-\Delta+1) \end{bmatrix} = \sqrt{\Sigma^{-1}}\, U^\top Y_0$$

5. Compute the (LS) sol'n of the **linear equations**

$$\begin{bmatrix} \hat{x}(\Delta+2) & \cdots & \hat{x}(T-\Delta+1) \\ \tilde{y}(\Delta+1) & \cdots & \tilde{y}(T-\Delta) \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \hat{x}(\Delta+1) & \cdots & \hat{x}(T-\Delta) \\ \tilde{u}(\Delta+1) & \cdots & \tilde{u}(T-\Delta) \end{bmatrix}$$

This solution yields the desired **balanced** system.

More on this, and other algorithms, in Ivan's talk

Summary

- **From data to balanced state representation:** sequential zero input response series
  $\longrightarrow$ **Hankel matrix**
  $\longrightarrow$ **SVD**
  $\longrightarrow$ **balanced state trajectory**
  $\longrightarrow$ **est. of syst. parameters.**

Summary

- From data to balanced state representation: sequential zero input response series
  $\longrightarrow$ Hankel matrix
  $\longrightarrow$ SVD
  $\longrightarrow$ balanced state trajectory
  $\longrightarrow$ est. of syst. parameters.

- Algorithms that pass from $\tilde{u}, \tilde{y}$ *directly* to a state resp. $\tilde{x}$ and, from there, to (an est. of) $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$: known for some time. Difficulty:

  arrive *directly* at a *balanced* model.

Summary

- From data to balanced state representation:
  sequential zero input response series
  $\longrightarrow$ Hankel matrix
  $\longrightarrow$ SVD
  $\longrightarrow$ balanced state trajectory
  $\longrightarrow$ est. of syst. parameters.
- The algorithms may be viewed as part of the research question:
  Develop algorithms that pass from a given system representation directly to a balanced state representation, or reduction.

Summary

- From data to balanced state representation: sequential zero input response series

  $\longrightarrow$ Hankel matrix

  $\longrightarrow$ SVD

  $\longrightarrow$ balanced state trajectory

  $\longrightarrow$ est. of syst. parameters.

- Under reasonable conditions, every system response can be obtained by solving a linear equation involving the Hankel matrix of the data.

Summary

- From data to balanced state representation:
  sequential zero input response series
  $\longrightarrow$ Hankel matrix
  $\longrightarrow$ SVD
  $\longrightarrow$ balanced state trajectory
  $\longrightarrow$ est. of syst. parameters.
- Under reasonable conditions, every system response can be obtained by solving a linear equation involving the Hankel matrix of the data.
- These insights will be used for setting up effective algorithms for subspace-like identification.

Summary

- From data to balanced state representation: sequential zero input response series

  $\longrightarrow$ Hankel matrix

  $\longrightarrow$ SVD

  $\longrightarrow$ balanced state trajectory

  $\longrightarrow$ est. of syst. parameters.

- Under reasonable conditions, every system response can be obtained by solving a linear equation involving the Hankel matrix of the data.
- The combined stochastic/deterministic case from this vantage point is our next target.

Thank you

Thank you

Thank you

Thank you

Thank you

Thank you

Thank you

Thank you