# The CANONICAL CONTROLLER and its REGULARITY

Jan C. Willems, K.U. Leuven, Belgium
Agung Julius, Un. of Twente, NL
Madhu N. Belur, IIT Bombay, India

CDC2003, Maui

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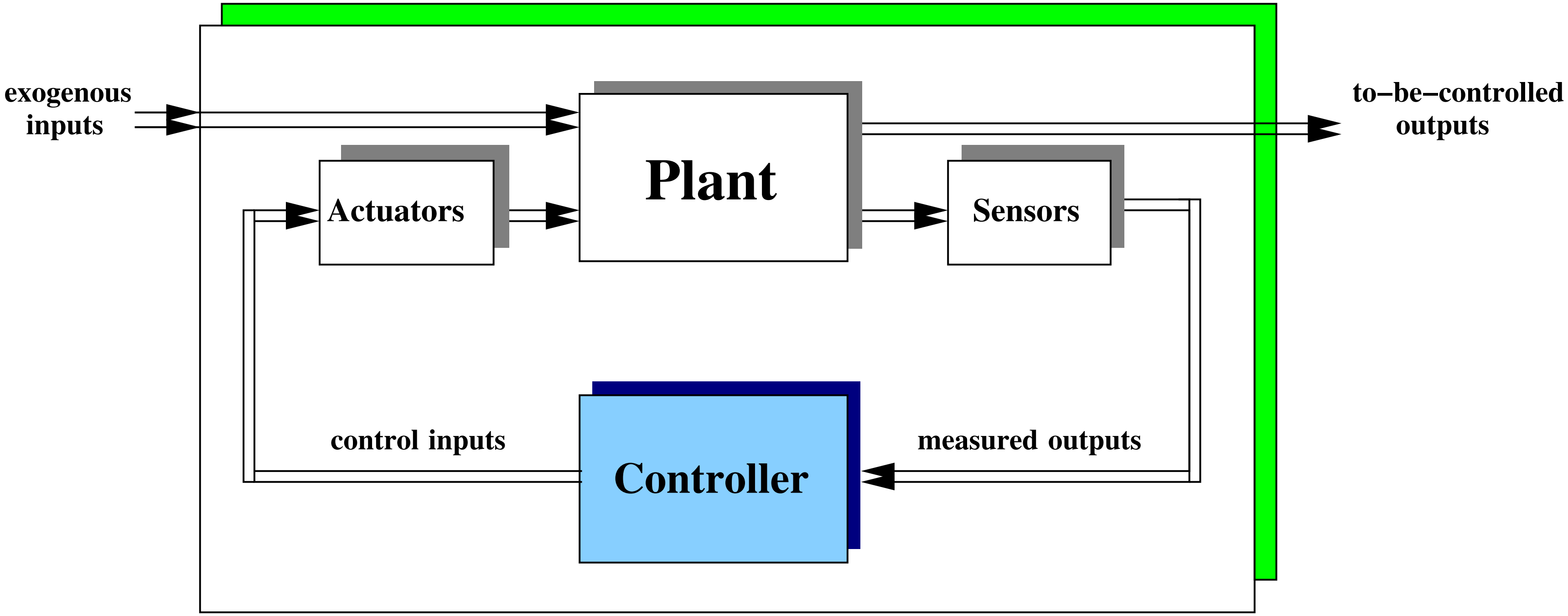
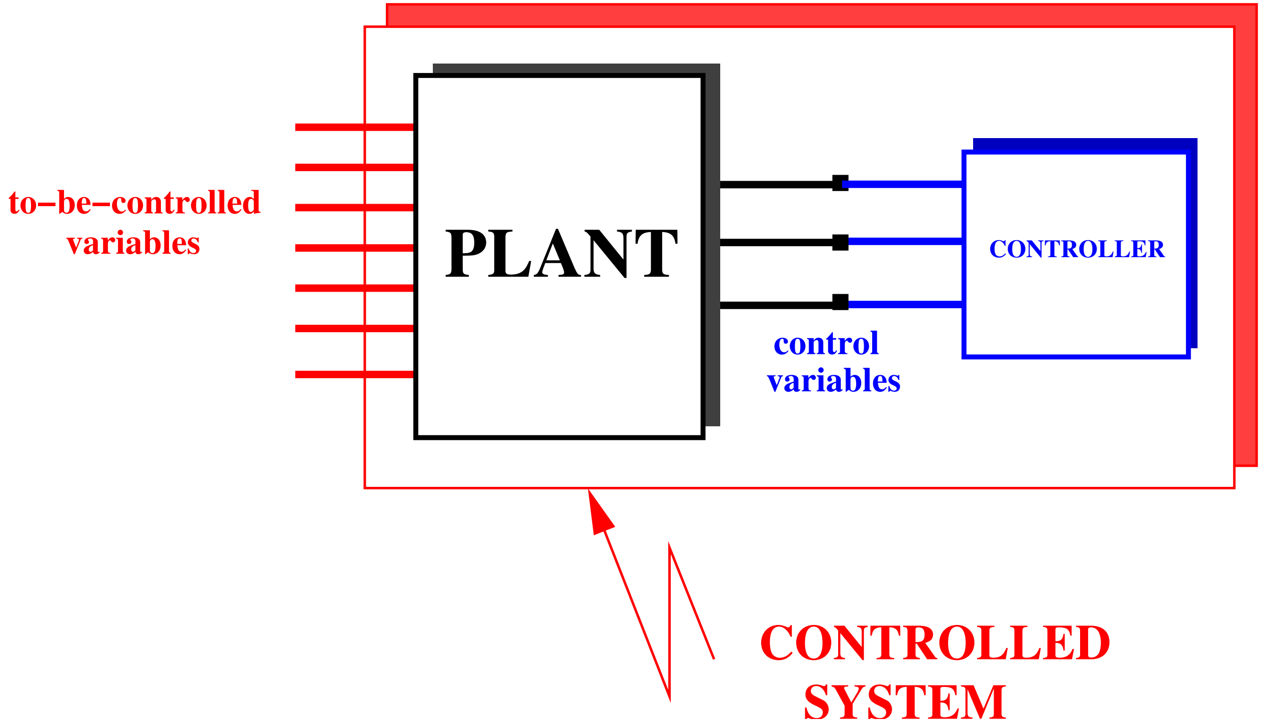
# Feedback control

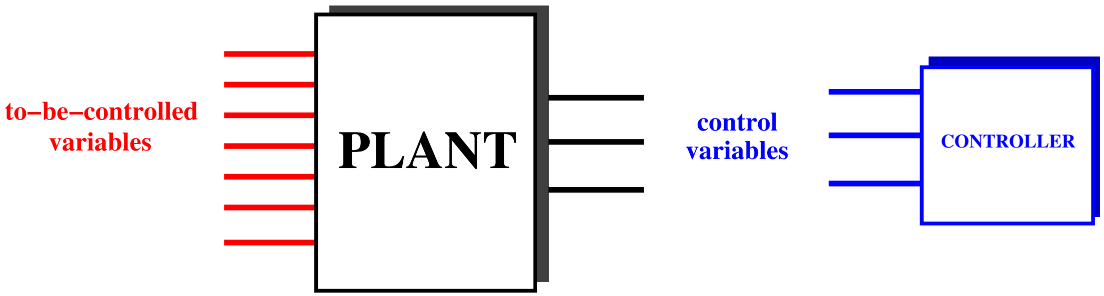


# Behavioral control



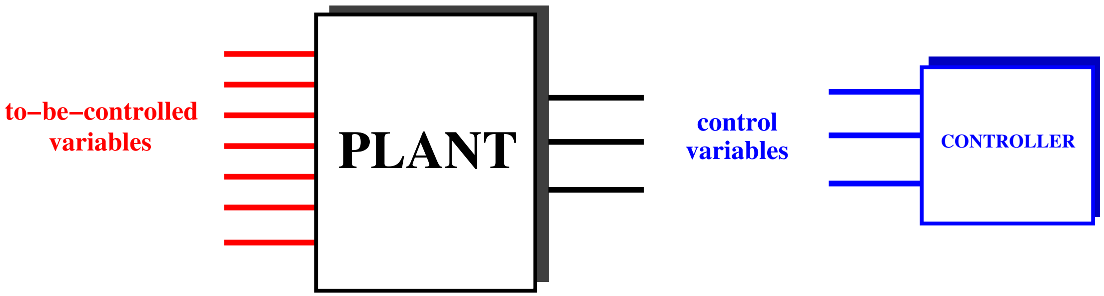

**Control as interconnection**

# Behavioral control

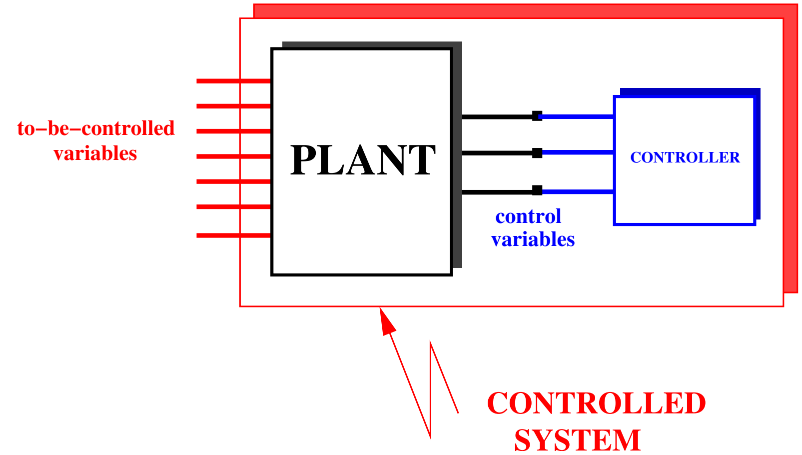



## Before interconnection

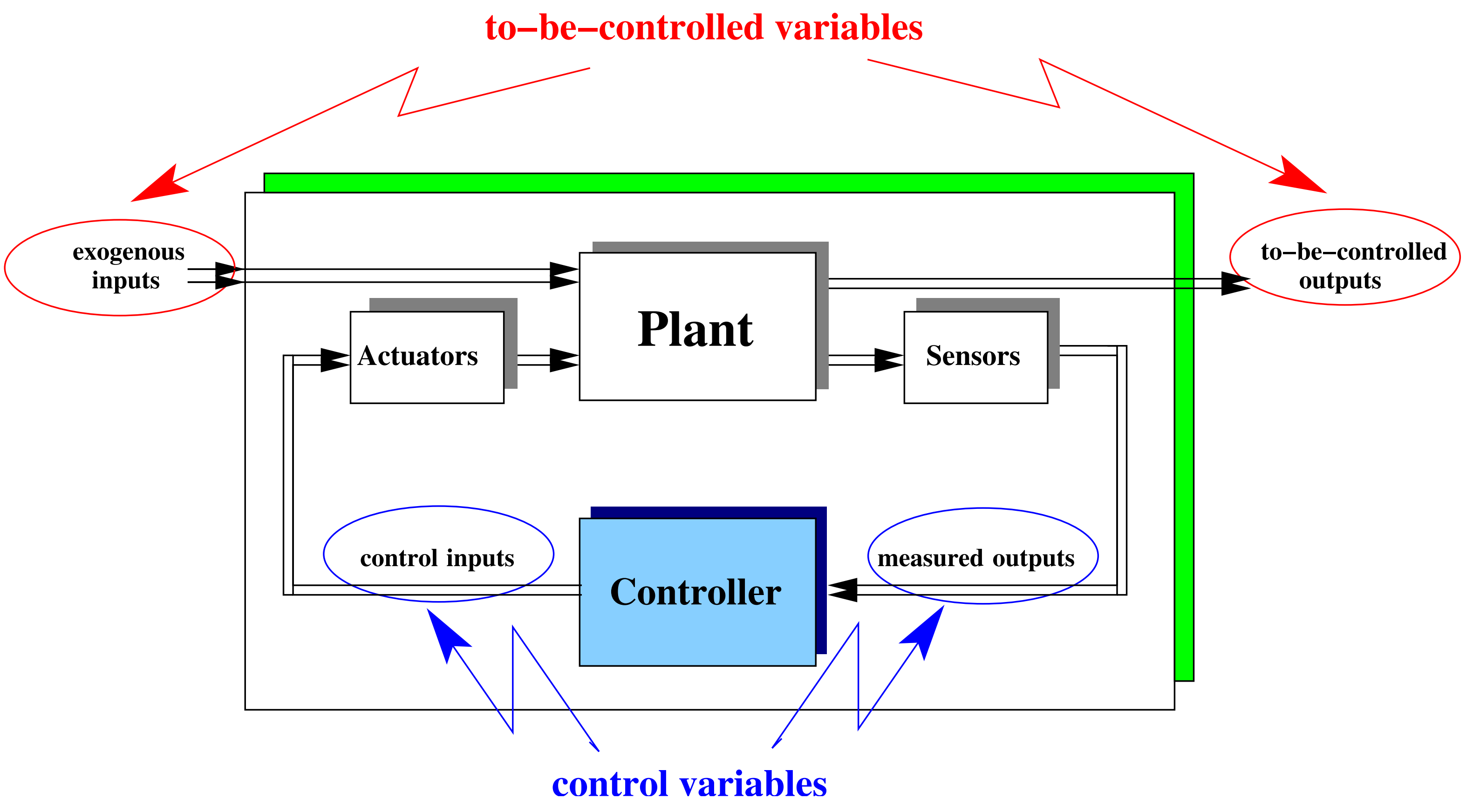
# Behavioral control




**Before interconnection**

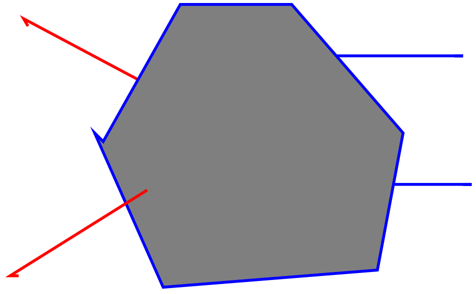



**After interconnection**

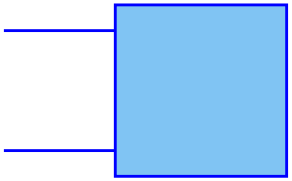


to−be−controlled variables
exogenous inputs
Plant
Actuators
Sensors
to−be−controlled outputs
control inputs
Controller
measured outputs
control variables

# ‘Example’



Mechanical object



Mechanical object

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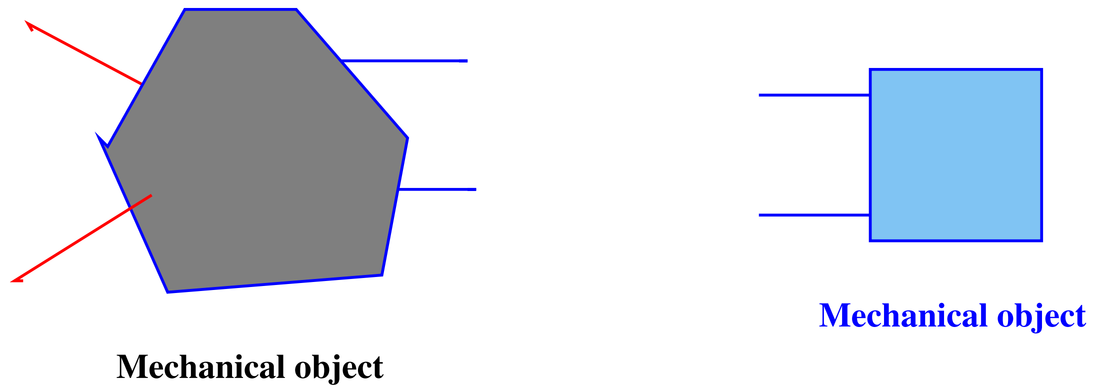


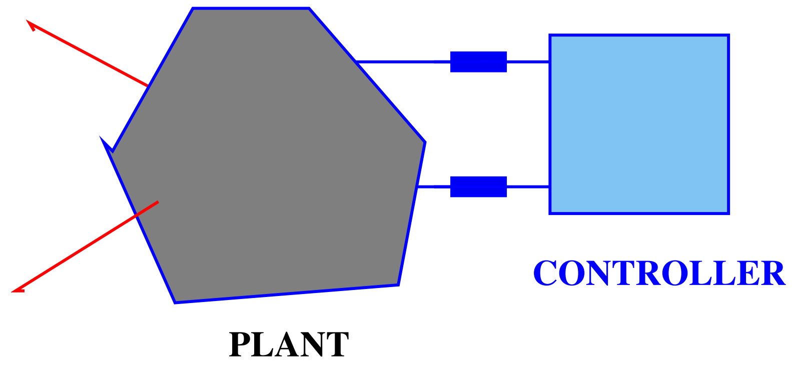

# Mathematization

Domain of the to-be-controlled variables: $\mathbb{W}$

Domain of the control variables: $\mathbb{C}$

Typically: families of time-signals

## Mathematization

**Full plant behavior:**

$$\mathcal{P}_{\text{full}} = \{(w, c) \in \mathbb{W} \times \mathbb{C} \mid \textbf{allowed by plant laws}\}$$

**Controller:**

$$\mathcal{C} = \{c \in \mathbb{C} \mid \textbf{allowed by controller laws}\}$$

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**Controlled behavior:**

$$\mathcal{K} := \{w \in \mathbb{W} \mid \exists\, c \in \mathbb{C} \text{ such that } (w, c) \in \mathcal{P}_{\text{full}} \text{ and } c \in \mathcal{C}\}.$$

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**such that** $(\boldsymbol{w}, \boldsymbol{c}) \in \mathcal{P}_{\text{full}}$ **and** $\boldsymbol{c} \in \mathcal{C}\}$.

**We say that** ***$\mathcal{C}$ implements $\mathcal{K}$*****,**

**and that $\mathcal{K}$ is implementable**

# Mathematatization

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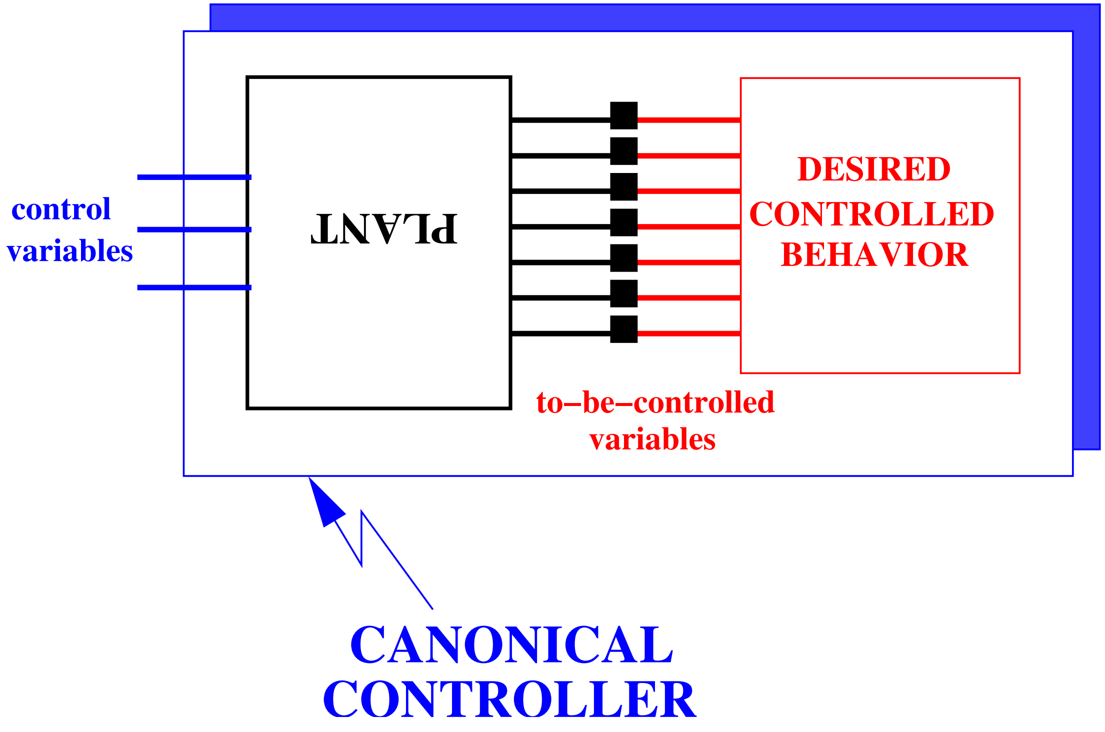
We say that $\mathcal{C}$ *implements* $\mathcal{K}$,

and that $\mathcal{K}$ is implementable

**Questions:**

- Which $\mathcal{C}$ implements the *desired controlled behavior* $\mathcal{D}$?
- Given $\mathcal{P}_{\mathrm{full}}$, which $\mathcal{K} \subseteq \mathbb{W}$ are implementable?

# van der Schaft's canonical controller



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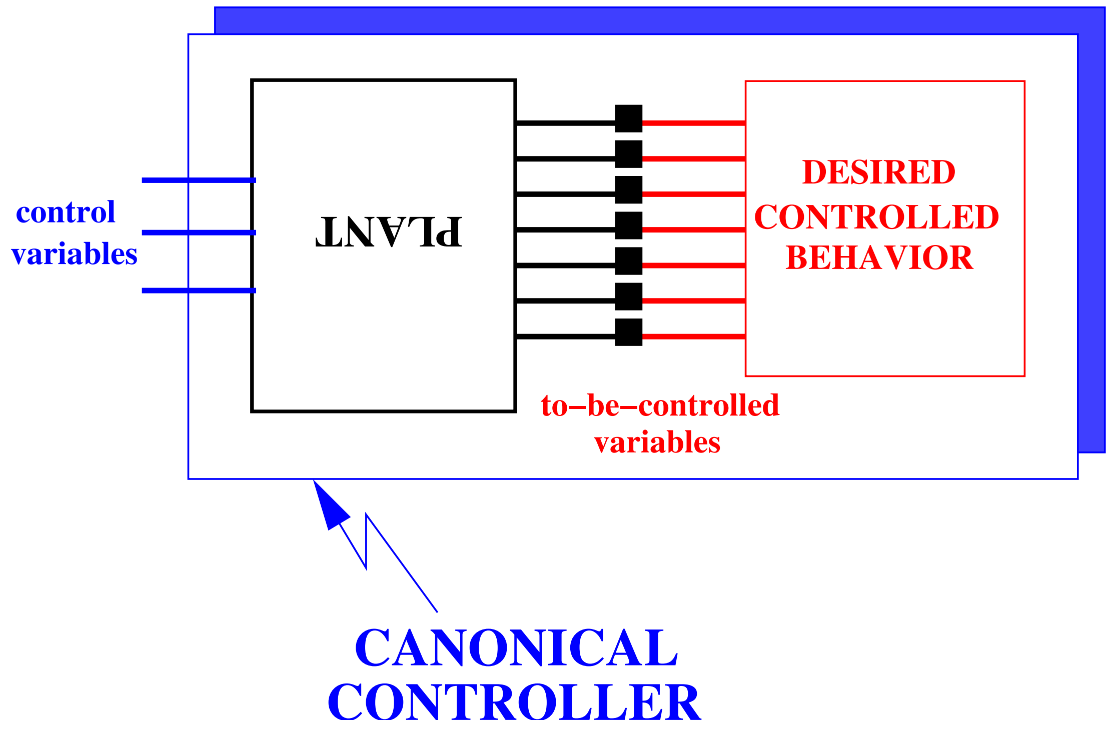
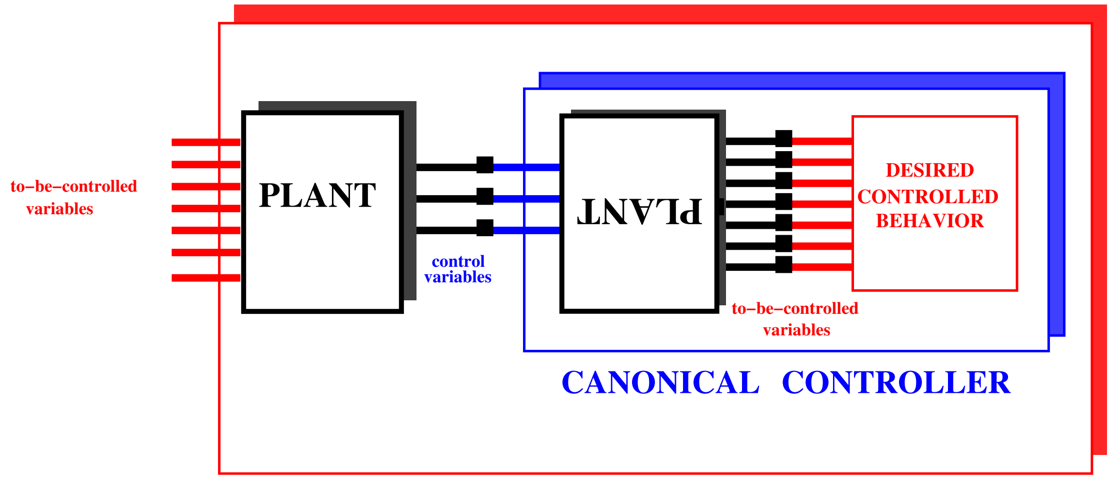



$$\mathcal{C}_{\text{canonical}} := \{c \in \mathbb{C} \mid \exists\, v \in \mathcal{D} \text{ such that } (v, c) \in \mathcal{P}_{\text{full}} \text{ and } v \in \mathcal{D}\}.$$

# van der Schaft’s canonical controller

## The canonically controlled system:



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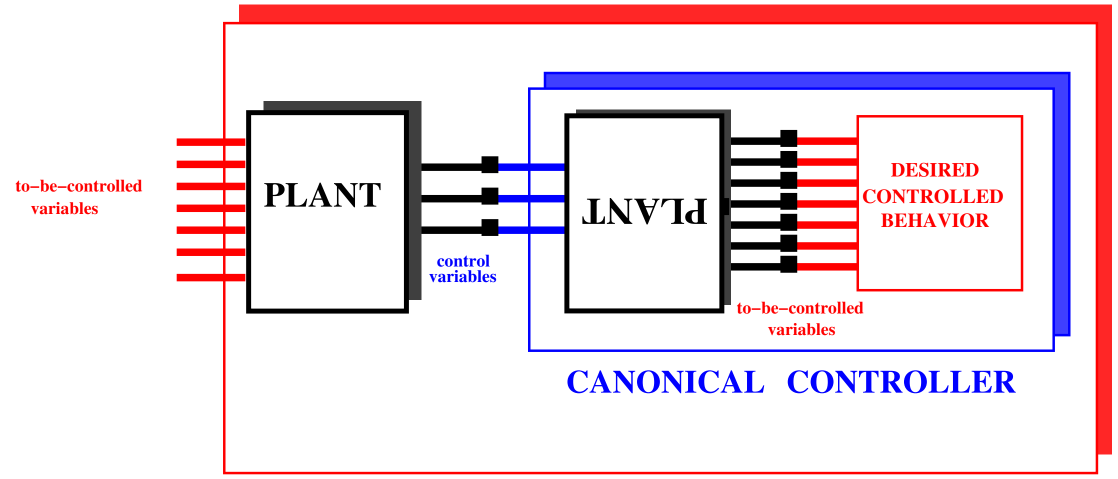


## This is the internal model principle at work!

# van der Schaft's canonical controller

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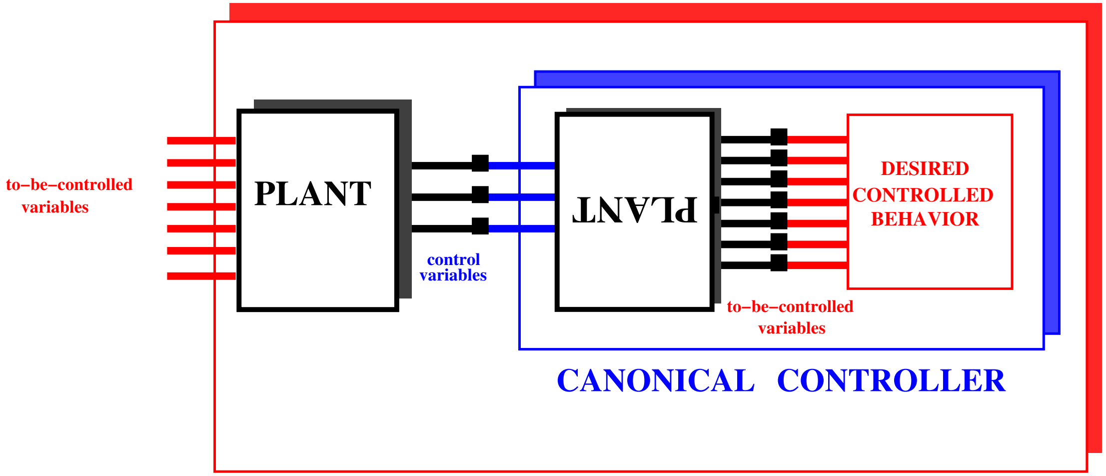


## Theorem:

**$\mathcal{D}$ is implementable $\Leftrightarrow$ $\mathcal{C}_{\text{canonical}}$ implements it.**

# van der Schaft's canonical controller

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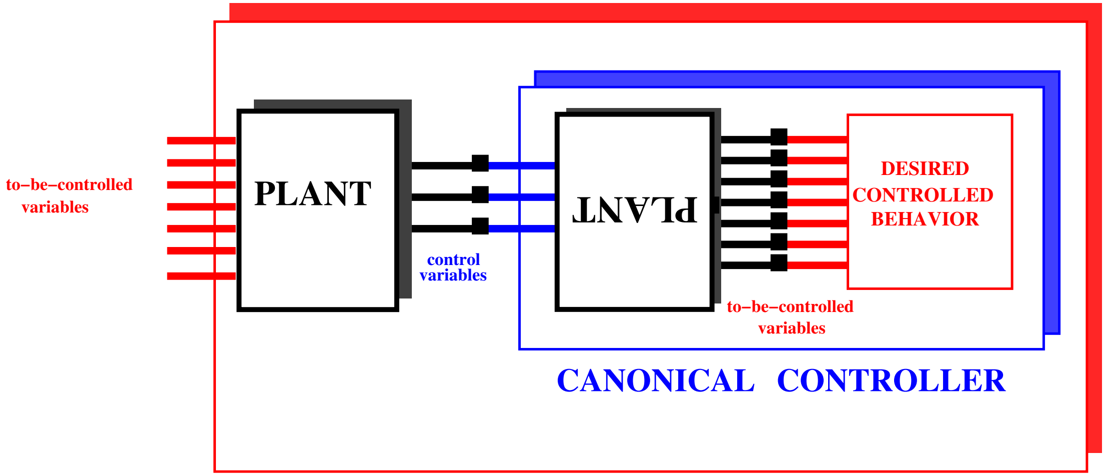


## Theorem:

**$\mathcal{D}$ is implementable $\Leftrightarrow$ $\mathcal{C}_{\text{canonical}}$ implements it.**

**Does $\mathcal{C}_{\text{canonical}}$ have good properties?**

## LTIS

**We henceforth restrict attention to**

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**linear time-invariant differential systems.**

**The *behavior* $\mathfrak{B}$ belongs to $\mathfrak{L}^{\mathtt{w}}$**

**:$\Leftrightarrow$**

$\exists$ **a polynomial matrix $R \in \mathbb{R}^{\bullet \times \mathtt{w}}[\xi]$ such that**

$$\mathfrak{B} = \{w \in \mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \mid R(\tfrac{d}{dt})w = 0\}.$$

## Control of LTIS

**Plant:**

$$\mathcal{P}_{\text{full}} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}}.$$

**Controller:**

$$\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}.$$

**Controlled system:**

$$\mathcal{K} = \{w \in \mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \mid \exists\, c \in \mathcal{C} : (w, c) \in \mathcal{P}_{\text{full}}\}.$$

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**By the ‘elimination theorem’**

$$\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}$$

## Implementability

***Which behaviors $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}$ can be implemented by attaching a controller $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$ to a given plant $\mathcal{P}_{\text{full}} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}}$?***

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**This question has a very concrete and intuitive answer.**

**Theorem: Let $\mathcal{P}_{\mathrm{full}} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}}$ be given.**

**The behavior $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}$ is implementable if and only if**

$$\mathcal{N} \subseteq \mathcal{K} \subseteq \mathcal{P}$$

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**where $\mathcal{N} \in \mathfrak{L}^{\mathtt{w}}$ is the *hidden behavior* defined by**

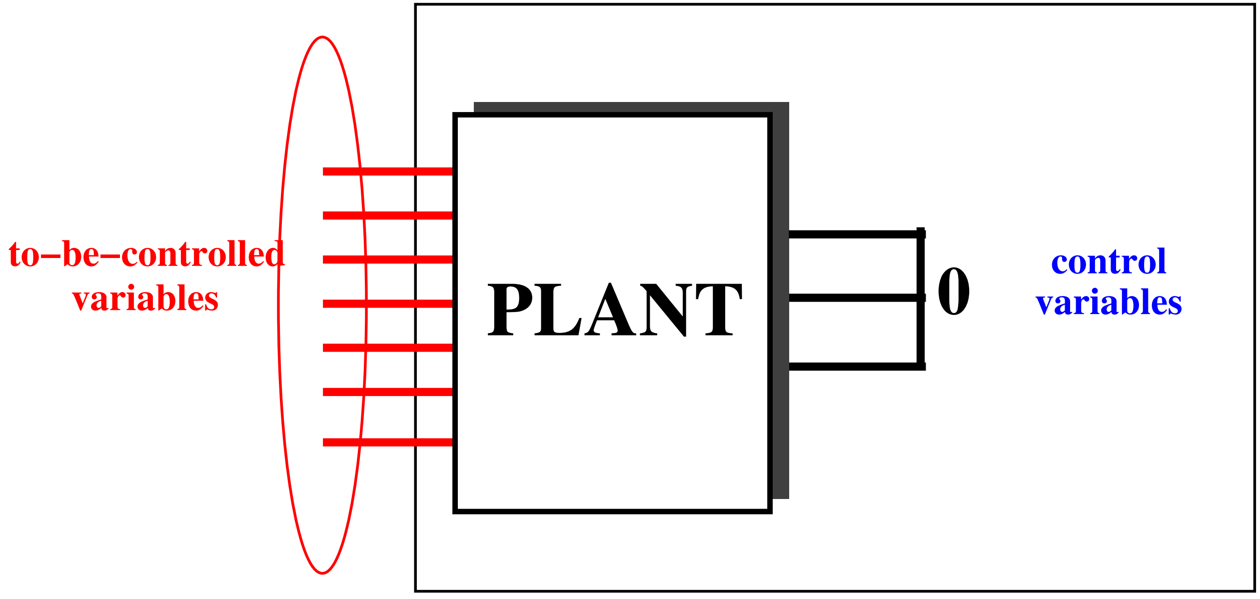
$$\mathcal{N} := \{w \in \mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \mid (w, 0) \in \mathcal{P}_{\text{full}}\},$$

**and $\mathcal{P}$ is the *manifest plant behavior* defined by**

$$\mathcal{P} := \{w \in \mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \mid \exists\, c : (w, c) \in \mathcal{P}_{\text{full}}\}.$$

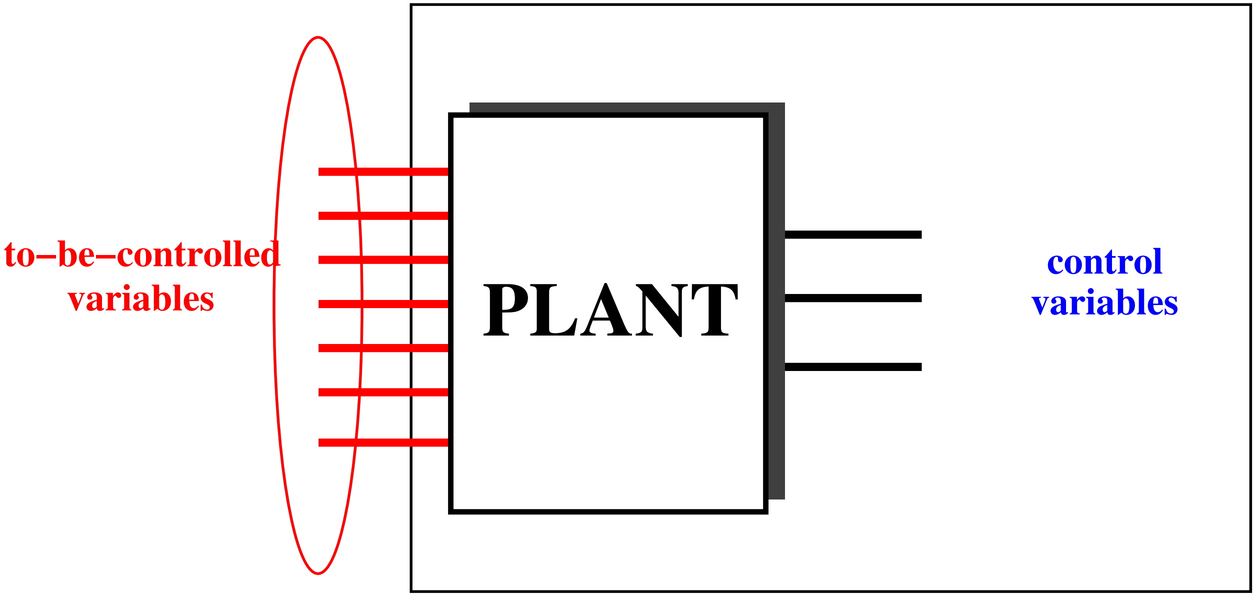
# Implementability

$\mathcal{N} \in \mathfrak{L}^{\mathtt{w}}$, the ***hidden behavior***



# Implementability

$\mathcal{P} \in \mathfrak{L}^{\mathtt{w}}$, the ***manifest plant behavior***



# The effect of the canonical controller

**Theorem:** Consider $\mathcal{P}_{\rm full} \in \mathfrak{L}^{\tt w+c}$ and $\mathcal{D} \in \mathfrak{L}^{\tt w}$. The controlled behavior implemented by the associated canonical controller $\mathcal{C}_{\rm canonical} \in \mathfrak{L}^{\tt c}$ is

$$\mathcal{K} = \mathcal{N} + \mathcal{D} \cap \mathcal{P}$$

with $\mathcal{N}$ the hidden and $\mathcal{P}$ the manifest plant behavior.

# The effect of the canonical controller

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The controlled behavior implemented by the associated canonical controller $\mathcal{C}_{\text{canonical}} \in \mathfrak{L}^{\mathtt{c}}$ is

$$\mathcal{K} = \mathcal{N} + \mathcal{D} \cap \mathcal{P}$$

with $\mathcal{N}$ the hidden and $\mathcal{P}$ the manifest plant behavior.

**Corollary:** The canonical controller implements $\mathcal{D} \in \mathfrak{L}^{\mathtt{w}}$ if and only if $\mathcal{N} \subseteq \mathcal{D} \subseteq \mathcal{P}$
i.e. if and only if $\mathcal{D}$ is implementable.

# Regularity

The ***full controlled behavior*** $\mathcal{K}_{\mathrm{full}} \subseteq \mathcal{P}_{\mathrm{full}}$ **is defined by**

$$\mathcal{K}_{\mathrm{full}} := \{(w, c) \in \mathcal{P}_{\mathrm{full}} \mid c \in \mathcal{C}\}.$$

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Consider the maps $\mathtt{m}, \mathtt{p} : \mathfrak{L}^{\mathtt{w}} \to \{0, 1, \ldots, \mathtt{w}\}$
with $\mathtt{m}(\mathfrak{B})$ the number of input variables,
and $\mathtt{p}(\mathfrak{B})$ the number of output variables in $\mathfrak{B}$.

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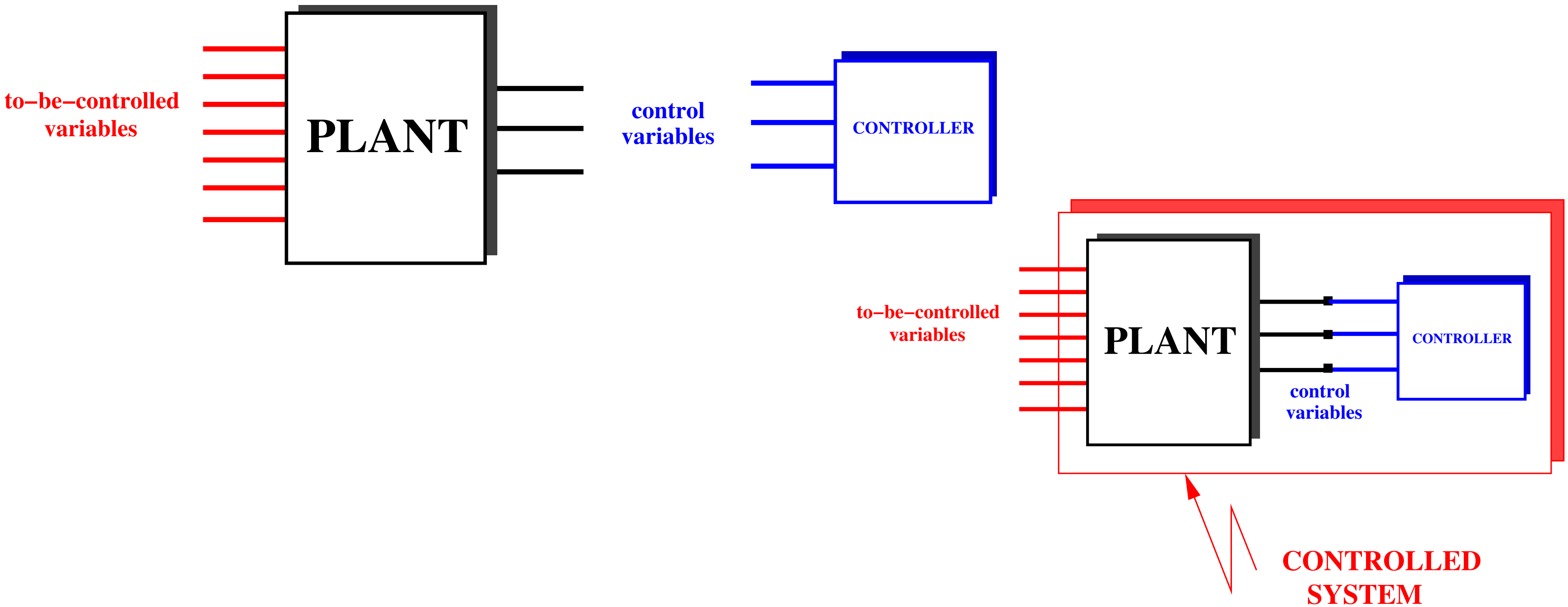
The controller $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$ is said to be regular if

$$\mathtt{p}(\mathcal{K}_{\text{full}}) = \mathtt{p}(\mathcal{P}_{\text{full}}) + \mathtt{p}(\mathcal{C}).$$

# Regularity

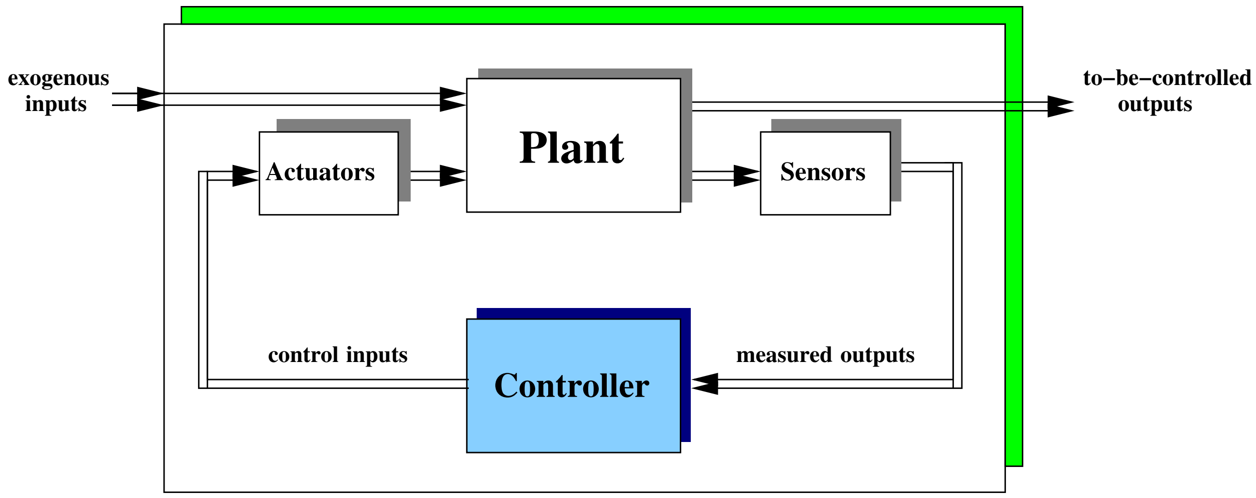



**Regularity :=**

**if the controller has $p$ bound (i.e. output) variables, then the plant looses $p$ free variables after interconnection.**

# Regularity

A controller is regular if and only if it can be realized as a feedback controller with a (possibly non-proper) transfer function from an output to an input in $\mathcal{P}_{\text{full}}$ for an input/output partition of $c$.



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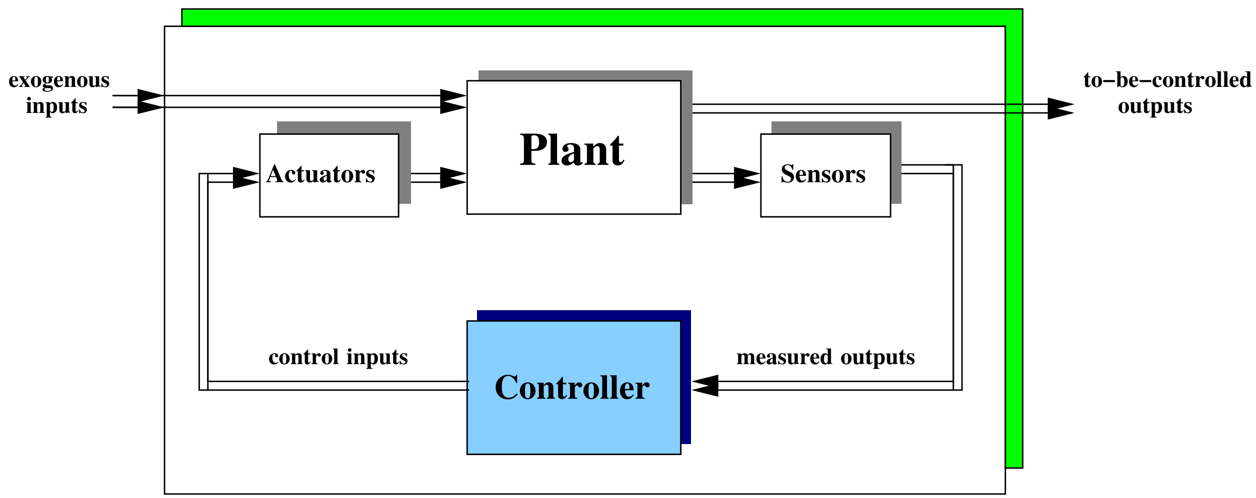


$\Rightarrow$ **A controller is regular if and only if it can be viewed as an ‘intelligent controller’ that processes sensor inputs outputs into actuator inputs.**

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$\Rightarrow$ **A controller is regular if and only if it can be viewed as an 'intelligent controller' that processes sensor inputs outputs into actuator inputs.**

**If $\mathcal{P}$ is controllable, then every implementable $\mathcal{K}$ is regularly implementable.**

# Is the canonical controller regular?

**Theorem:** Equivalent for a $\mathcal{P}_{\text{full}} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}}$:

- $\mathcal{P}_c = \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{c}})$: the control variables are **free**;

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- Every controller is regular;
- The canonical controller is regular.

$\Rightarrow$ The canonical controller is regular

if and only if every controller is regular.

Hence the canonical controller is maximally irregular.

Thank you

Thank you

Thank you

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