



THE BEHAVIORAL APPROACH

to

SYSTEMS and CONTROL

Jan C. Willems
ESAT-SCD (SISTA)
University of Leuven, Belgium

Problematique:

Develop a suitable *mathematical* framework for
discussing dynamical systems

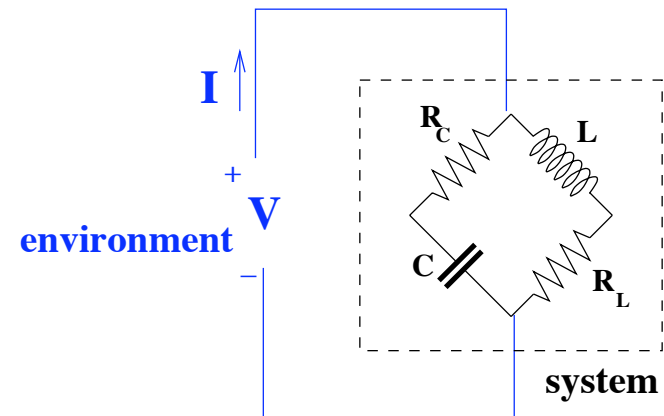
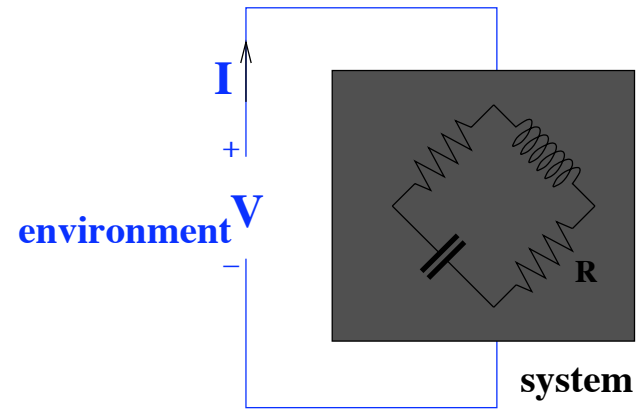
aimed at **modeling**, analysis, and synthesis.

~> control, signal processing, system identification, . . .

~> engineering systems, economics, physics, . . .

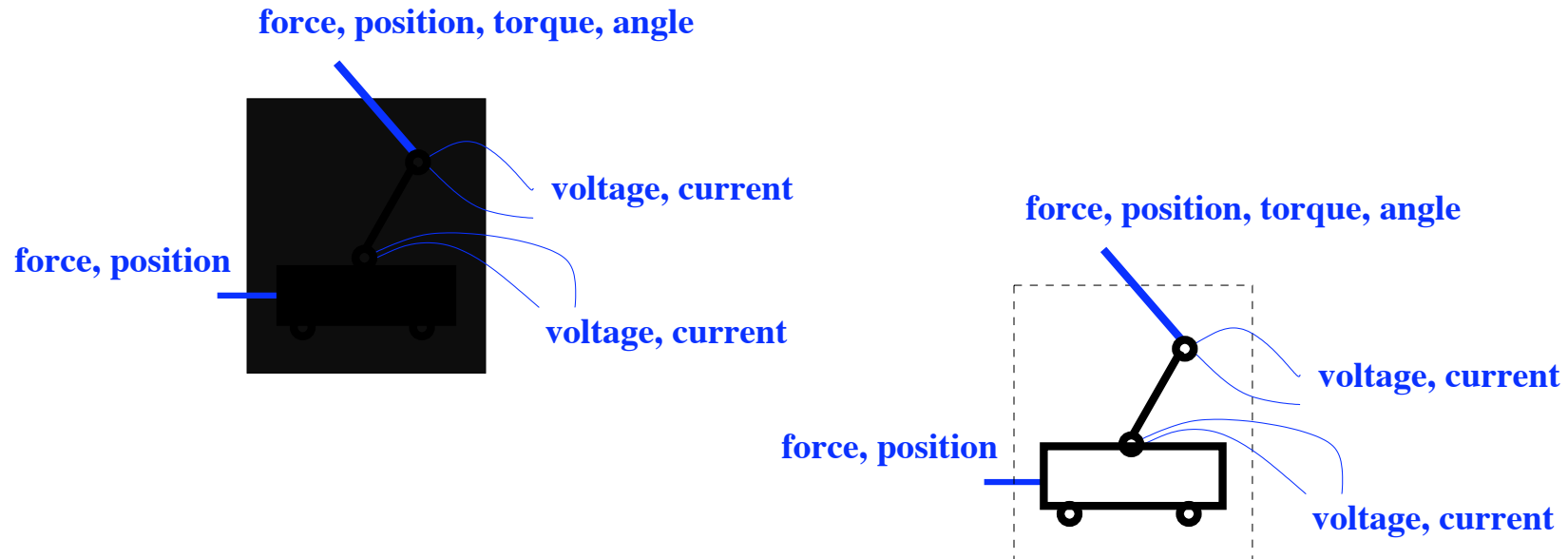
Motivational examples

Electrical circuit



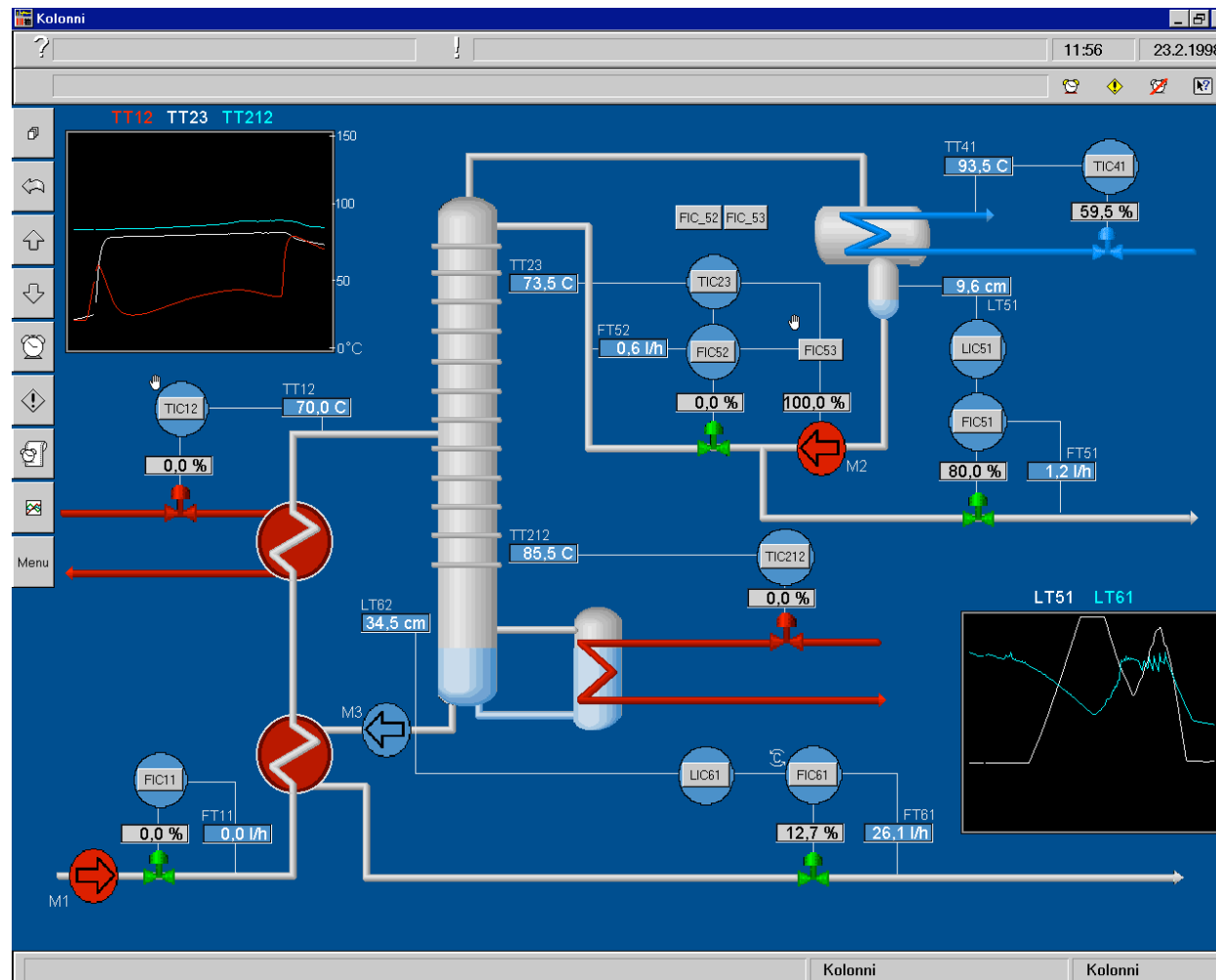
!! Model the relation between the voltage V and the current I

Electromechanical system



!! between the positions, forces, torque, angle, voltages, currents

Distillation column



Features: Systems are typically

dynamical

open, they interact with their environment

interconnected, with many subsystems

modular, consisting of standard components

We are looking for a mathematical framework that is adapted to these features, and hence to **computer assisted modeling**.

Historical remarks

Early 20-th century: emergence of the notion of a **transfer function**
(Rayleigh, Heaviside).



Since the 1920's: routinely used in **circuit theory**

~> impedances, admittances, scattering matrices, etc.

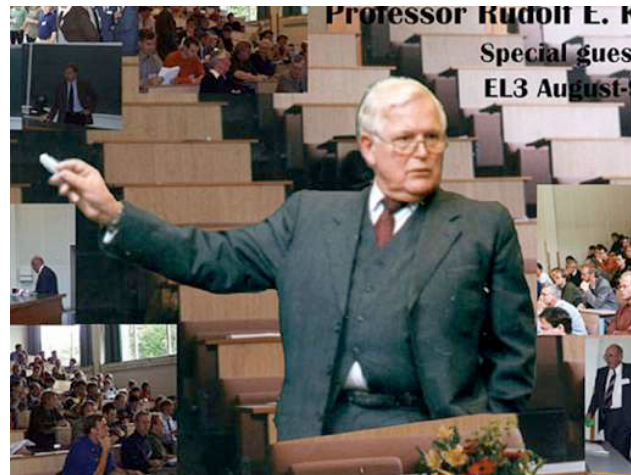
1930's: **control** embraces transfer functions

(Nyquist, Bode, . . .) ~> plots and diagrams, classical control.

Around 1950: Wiener sanctifies the notion of a **blackbox**, attempts nonlinear generalization (via **Volterra series**).



1960's: Kalman's **state space** ideas (incl. controllability, observability, recursive filtering, state models and representations) come in vogue



~> **input/state/output systems, and the ubiquitous**

$$\frac{d}{dt}x = Ax + Bu, \quad y = Cx + Du,$$

or its nonlinear counterpart

$$\frac{d}{dt}x = f(x, u), \quad y = h(x, u).$$

These mathematical structures, transfer functions, + their discrete-time analogs, are nowadays the basic models used in **control and signal processing (cfr. MATLAB[©]).**

All these theories: input/output; **cause \Rightarrow **effect**.**

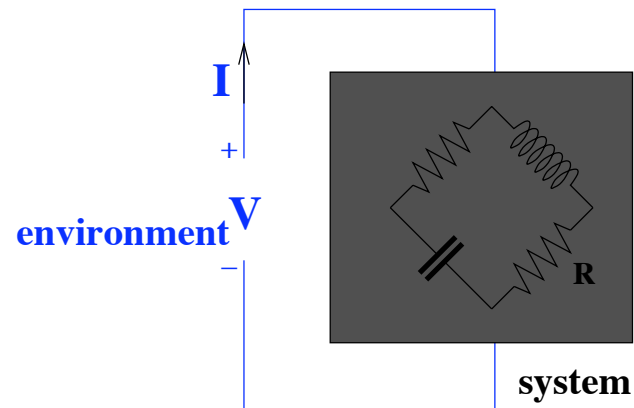
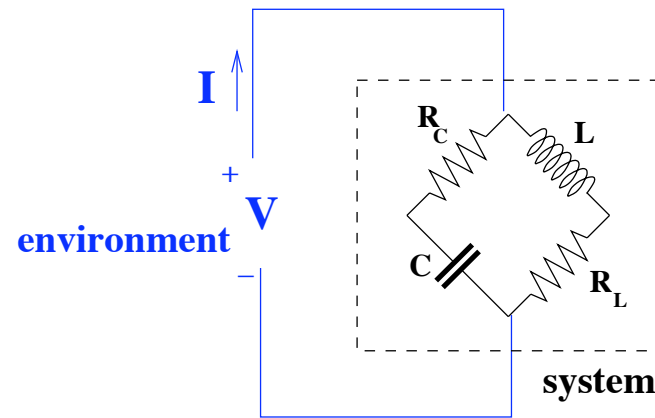


Beyond input/output

What's wrong with input/output thinking?

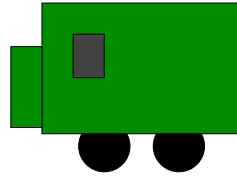
Let's look at examples:

Our electrical circuit.



Is V the input? Or I ? Or both, or are they both outputs?

An automobile:



External terminals:

wind, tires, steering wheel, gas/brake pedal.

What are the inputs?

at the wind terminal: **the force**,

at the tire terminals: **the forces**, or, more likely, **the positions?**

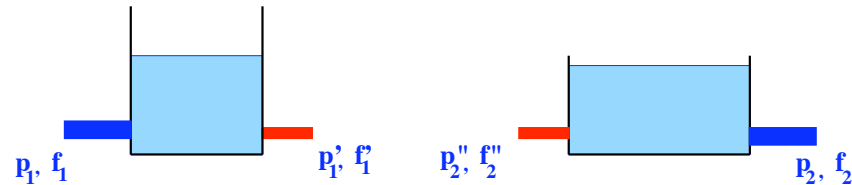
at the steering wheel: **the torque or the angle?**

at the gas-pedal, or the brake-pedal: **the force or the position?**

Difficulty: at each terminal there are **many** (typically paired)
interconnection variables

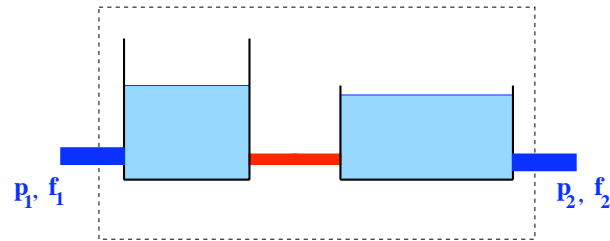
Input/output is awkward in modeling interconnections.

Consider a two-tank example.



Reasonable input choices: **the pressures**, output choices: **the flows**.

Assume that we model the interconnection of two tanks.



$$\text{Interconnection: } p_1' = p_2'', \quad f_1' + f_2'' = 0$$

input=input; output=output!

$\Rightarrow \Leftarrow$ SIMULINK[©]

Interconnections contradicting SIMULINK[©] are in fact **normal, not exceptions**, in mechanics, fluidics, heat transfer, etc.

Mathematical difficulties:

Is a system a **map** $u(\cdot) \mapsto y(\cdot)$?

How to incorporate **'initial conditions'**?

Is it a parametrized map $(u(\cdot), \alpha) \mapsto y(\cdot)$?

All sorts of new difficulties...

Construct the state!

But from what?

From the system model!

What system?

Conclusions * for physical systems ($\Rightarrow \Leftarrow$ signal processors) *

- External variables are basic, but what 'drives' what, is not.
- It is impossible to make an **a priori, fixed**, input/output selection for off-the-shelf modeling.
- What can be the input, and what can be the output should be **deduced** from a dynamical model. Therefore, **we need a more general notion of 'system', of 'dynamical model'**.
- **Interconnection**, rather than **input selection**, is the basic mechanism by which a system interacts with its environment.

\Rightarrow We need a better framework for discussing **'open'** systems!

\rightsquigarrow **Behavioral systems.**

The basic concepts

Behavioral systems

A dynamical system = $\Sigma = (\mathbb{T}, \mathbb{W}, \mathcal{B})$

$\mathbb{T} \subseteq \mathbb{R}$, the time-axis (= the relevant time instances),

\mathbb{W} , the signal space (= where the variables take on their values),

$\mathcal{B} \subseteq \mathbb{W}^{\mathbb{T}}$: the behavior (= the admissible trajectories).

$$\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$$

For a trajectory $w : \mathbb{T} \rightarrow \mathbb{W}$, we thus have:

$w \in \mathfrak{B}$: the model **allows** the trajectory w ,
 $w \notin \mathfrak{B}$: the model **forbids** the trajectory w .

Usually, $\mathbb{T} = \mathbb{R}$, or $[0, \infty)$ (in continuous-time systems),
or \mathbb{Z} , or \mathbb{N} (in discrete-time systems).

Usually, $\mathbb{W} \subseteq \mathbb{R}^w$ (in lumped systems),
a function space
(in distributed systems, with time a distinguished variable),
or a finite set (in DES).

Emphasis later today: $\mathbb{T} = \mathbb{R}$, $\mathbb{W} = \mathbb{R}^w$,
 $\mathfrak{B} =$ solutions of system of linear constant coefficient ODE's.

Examples

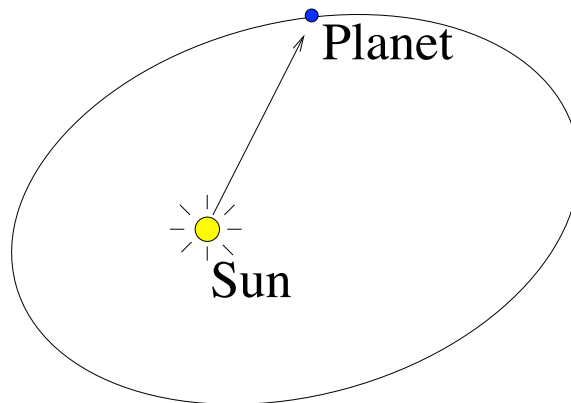
1. Planetary orbits

$T = \mathbb{R}$ (time),

$W = \mathbb{R}^3$ (position),

$\mathfrak{B} =$ planetary orbits \cong Kepler's laws:

ellipses, = areas in = time, $\frac{(\text{period})^2}{(\text{axis})^3} = \text{constant}$.



2. Input / output systems

$$\begin{aligned} f_1(y(t), \frac{d}{dt}y(t), \frac{d^2}{dt^2}y(t), \dots, t) \\ = f_2(u(t), \frac{d}{dt}u(t), \frac{d^2}{dt^2}u(t), \dots, t) \end{aligned}$$

$\mathbb{T} = \mathbb{R}$ (time),

$\mathbb{W} = \mathbb{U} \times \mathbb{Y}$ (input \times output signal spaces),

$\mathfrak{B} =$ **all input / output pairs.**

3. Flows

$$\frac{d}{dt}x(t) = f(x(t)),$$

\mathfrak{B} = all state trajectories.

... Of very marginal value as a paradigm for dynamics ...

Modeling **closed** systems by tearing and zooming

\rightsquigarrow **open** systems.

4. Observed flows

$$\frac{d}{dt}x(t) = f(x(t)); \quad y(t) = h(x(t)),$$

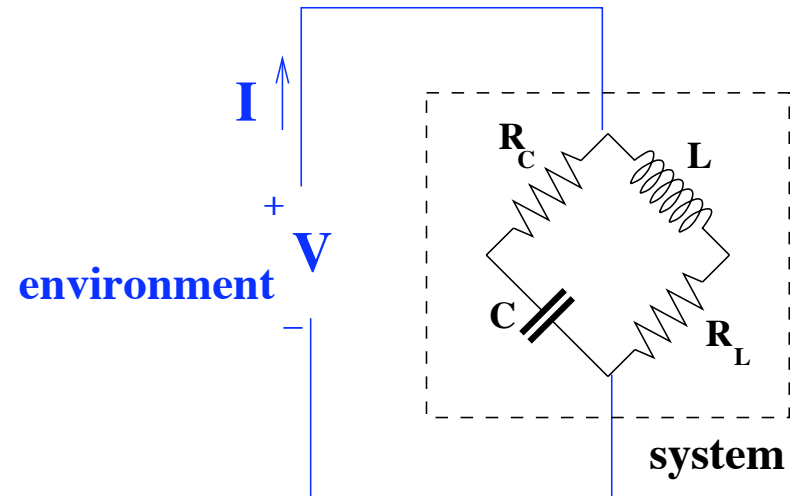
\mathfrak{B} = all possible output trajectories.

5. Convolutional codes

6. Formal languages

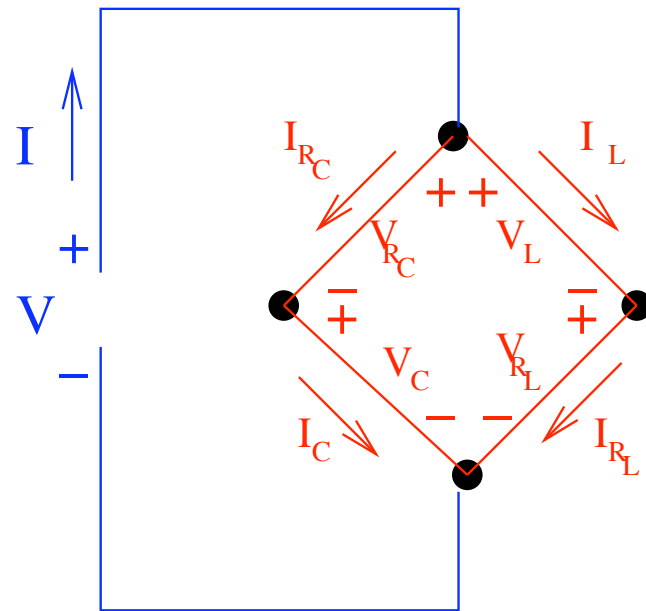
Latent variable systems

Consider our electrical RLC - circuit:



!! Model the relation between V and I !!

How does this modeling proceed?



The circuit graph

System equations

Introduce the following additional variables:

the **voltage across** and the **current in** each branch:

$$V_{RC}, I_{RC}, V_C, I_C, V_{RL}, I_{RL}, V_L, I_L.$$

Constitutive equations (CE):

$$V_{RC} = R_C I_{RC}, \quad V_{RL} = R_L I_{RL}, \quad C \frac{d}{dt} V_C = I_C, \quad L \frac{d}{dt} I_L = V_L$$

Kirchhoff's voltage laws (KVL):

$$V = V_{RC} + V_C, \quad V = V_L + V_{RL}, \quad V_{RC} + V_C = V_L + V_{RL}$$

Kirchhoff's current laws (KCL):

$$I = I_{RC} + I_L, \quad I_{RC} = I_C, \quad I_L = I_{RL}, \quad I_C + I_{RL} = I$$

The preceding is a complete model, but here is the

Relation between V and I .

Case 1: $CR_C \neq \frac{L}{R_L}$.

$$\left(\frac{R_C}{R_L} + \left(1 + \frac{R_C}{R_L}\right)CR_C \frac{d}{dt} + CR_C \frac{L}{R_L} \frac{d^2}{dt^2}\right)V$$
$$= \left(1 + CR_C \frac{d}{dt}\right)\left(1 + \frac{L}{R_L} \frac{d}{dt}\right)R_C I.$$

Case 2: $CR_C = \frac{L}{R_L}$.

$$\left(\frac{R_C}{R_L} + CR_C \frac{d}{dt}\right)V = \left(1 + CR_C \frac{d}{dt}\right)R_C I$$

Exact relations between V and I !

First principles models invariably contain auxiliary variables, in addition to the variables the model aims at.

↪ **Manifest** and **latent** variables.

Manifest = the variables the model aims at,

Latent = auxiliary variables.

We want to capture this in mathematical definitions.

A dynamical system with latent variables = $\Sigma_L = (\mathbb{T}, \mathbb{W}, \mathbb{L}, \mathcal{B}_{\text{full}})$

$\mathbb{T} \subseteq \mathbb{R}$, the *time-axis* (= the set of relevant time instances).

\mathbb{W} , the *signal space* (= the variables that the model aims at).

\mathbb{L} , the *latent variable space* (= the **auxiliary** modeling variables).

$\mathcal{B}_{\text{full}} \subseteq (\mathbb{W} \times \mathbb{L})^{\mathbb{T}}$: the full behavior

(= the pairs $(w, \ell) : \mathbb{T} \rightarrow \mathbb{W} \times \mathbb{L}$ that the model declares possible).

The manifest behavior

Call the elements of \mathbb{W} *'manifest' variables*,
those of \mathbb{L} *'latent' variables*.

The latent variable system $\Sigma_L = (\mathbb{T}, \mathbb{W}, \mathbb{L}, \mathcal{B}_{\text{full}})$ induces
the *manifest system* $\Sigma = (\mathbb{T}, \mathbb{W}, \mathcal{B})$, with *manifest behavior*

$$\mathcal{B} = \{w : \mathbb{T} \rightarrow \mathbb{W} \mid \exists \ell : \mathbb{T} \rightarrow \mathbb{L} \text{ such that } (w, \ell) \in \mathcal{B}_{\text{full}}\}$$

In convenient equations for \mathcal{B} , the latent variables are *'eliminated'*.

Examples

1. The RLC - circuit before elimination.

2. Models obtained by tearing and zooming

3. Input / state / output systems

$$\frac{d}{dt}\mathbf{x}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t)); \quad \mathbf{y}(t) = \mathbf{h}(\mathbf{x}(t), \mathbf{u}(t)),$$

$\mathbb{T} = \mathbb{R}, \mathbb{W} = \mathbb{U} \times \mathbb{Y}, \mathbb{L} = \mathbb{X},$

$\mathfrak{B}_{\text{full}} = \text{all } (\mathbf{u}, \mathbf{y}, \mathbf{x}) : \mathbb{R} \rightarrow \mathbb{U} \times \mathbb{Y} \times \mathbb{X} \text{ that satisfy these equations,}$

$\mathfrak{B} = \text{all (input / output)-pairs.}$

4. Trellis diagrams

5. Automata

Latent variables = nodes

6. Grammars

Recapitulation

Central notions:

The **behavior** \rightsquigarrow a model.

Manifest and latent variables \rightsquigarrow specifies what the model aims at.

First principles models \rightsquigarrow latent variables.

(Full) behavioral equations \rightsquigarrow a specification of the (full) behavior.

Equivalent equations $:\Leftrightarrow$ the manifest behavior is the same.

Linear differential systems

We now discuss the fundamentals of the theory of systems

$$\Sigma = (\mathbb{R}, \mathbb{R}^w, \mathfrak{B})$$

that are

1. **linear**, meaning
 $((w_1, w_2 \in \mathfrak{B}) \wedge (\alpha, \beta \in \mathbb{R})) \Rightarrow (\alpha w_1 + \beta w_2 \in \mathfrak{B});$
2. **time-invariant**, meaning
 $((w \in \mathfrak{B}) \wedge (t \in \mathbb{R})) \Rightarrow (\sigma^t w \in \mathfrak{B}),$
where σ^t denotes the backwards t -shift;
3. **differential**, meaning
 \mathfrak{B} consists of the solutions of a system of differential equations.

Yields

$$R_0 w + R_1 \frac{d}{dt} w + \cdots + R_n \frac{d^n}{dt^n} w = 0,$$

with $R_0, R_1, \dots, R_n \in \mathbb{R}^{\bullet \times w}$.

Combined with the polynomial matrix

$$R(\xi) = R_0 + R_1 \xi + \cdots + R_n \xi^n,$$

we obtain the short notation

$$R\left(\frac{d}{dt}\right)w = 0.$$

But, the theory has also been developed for PDE's.

n-D systems

$T = \mathbb{R}^n$, n independent variables,

$W = \mathbb{R}^w$, w dependent variables,

$\mathfrak{B} =$ **the solutions of a linear constant coefficient system of PDE's.**

Let $R \in \mathbb{R}^{\bullet \times w}[\xi_1, \dots, \xi_n]$, and consider

$$R\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right)w = 0 \quad (*)$$

Define its behavior

$$\mathfrak{B} = \left\{ w \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}^w) \mid (*) \text{ holds} \right\} = \ker\left(R\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right)\right)$$

$\mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}^w)$ **mainly** for convenience, but important for some results.

Example: *Maxwell's equations*



$$\begin{aligned}\nabla \cdot \vec{E} &= \frac{1}{\epsilon_0} \rho, \\ \nabla \times \vec{E} &= -\frac{\partial}{\partial t} \vec{B}, \\ \nabla \cdot \vec{B} &= 0, \\ c^2 \nabla \times \vec{B} &= \frac{1}{\epsilon_0} \vec{j} + \frac{\partial}{\partial t} \vec{E}.\end{aligned}$$

$\mathbb{T} = \mathbb{R} \times \mathbb{R}^3$ (time and space),

$w = (\vec{E}, \vec{B}, \vec{j}, \rho)$

(electric field, magnetic field, current density, charge density),

$\mathbb{W} = \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}$,

$\mathfrak{B} =$ set of solutions to these PDE's.

Note: 10 variables, 8 equations! $\Rightarrow \exists$ free variables.

NOMENCLATURE

\mathcal{L}_n^w : the set of such systems with n in-, w dependent variables

\mathcal{L}^\bullet : with any - finite - number of (in)dependent variables

Elements of \mathcal{L}^\bullet : *linear differential systems*

$R\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right)w = 0$: a *kernel representation* of the
corresponding $\Sigma \in \mathcal{L}^\bullet$ or $\mathfrak{B} \in \mathcal{L}^\bullet$

Algebraization of \mathcal{L}^\bullet

Note that

$$R\left(\frac{d}{dt}\right)w = 0$$

and

$$U\left(\frac{d}{dt}\right)R\left(\frac{d}{dt}\right)w = 0$$

have the same behavior if the polynomial matrix U is **uni-modular (i.e., when $\det(U)$ is a non-zero constant).**

$\Rightarrow R$ defines $\mathfrak{B} = \ker\left(R\left(\frac{d}{dt}\right)\right)$, but not vice-versa!

∴ ∃ ‘intrinsic’ characterization of $\mathfrak{B} \in \mathcal{L}_n^w$??

Define the **annihilators** of $\mathfrak{B} \in \mathcal{L}_n^w$ by

$$\mathfrak{N}_{\mathfrak{B}} := \{n \in \mathbb{R}^w[\xi_1, \dots, \xi_n] \mid n^\top \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \right) \mathfrak{B} = 0\}.$$

$\mathfrak{N}_{\mathfrak{B}}$ is clearly an $\mathbb{R}[\xi_1, \dots, \xi_n]$ sub-module of $\mathbb{R}^w[\xi_1, \dots, \xi_n]$.

Let $\langle R \rangle$ denote the sub-module of $\mathbb{R}^w[\xi_1, \dots, \xi_n]$ spanned by the transposes of the rows of R . Obviously $\langle R \rangle \subseteq \mathfrak{N}_{\mathfrak{B}}$. But, indeed:

$$\mathfrak{N}_{\mathfrak{B}} = \langle R \rangle!$$

Note: Depends on \mathcal{C}^∞ ; (\Leftarrow) false for compact support soln’s:

for any $p \neq 0$, $p\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right)w = 0$ has only $w = 0$

as compact support sol’n.

Conclusions:

(i) $\mathcal{L}_n^w \xleftrightarrow{1:1} \text{sub-modules of } \mathbb{R}^w[\xi_1, \dots, \xi_n]$

(ii)

$$R_1\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right)w = 0 \text{ and } R_2\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right)w = 0$$

define the same system iff

$$\langle R_1 \rangle = \langle R_2 \rangle .$$

Elimination

First principle models \rightsquigarrow **latent variables.** In the case of systems described by linear constant coefficient PDE's: \rightsquigarrow

$$R\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right)w = M\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right)l$$

with $R, M \in \mathbb{R}^{\bullet \times \bullet}[\xi]$.

This is the natural model class to start a study of finite dimensional linear time-invariant systems! Much more so than

$$\frac{d}{dt}x = Ax + Bu, \quad y = Cx + Du.$$

But is it(s manifest behavior) really a differential system ??

The full behavior of $R(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n})w = M(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n})\ell$,

$$\mathfrak{B}_{\text{full}} = \{ (w, \ell) \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}^{w+\ell}) \mid \\ R(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n})w = M(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n})\ell \}$$

belongs to $\mathfrak{L}_n^{w+\ell}$, by definition. Its manifest behavior equals

$$\mathfrak{B} = \{ w \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}^w) \mid \\ \exists \ell \text{ such that } R(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n})w = M(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n})\ell. \}$$

Does \mathfrak{B} belong to \mathfrak{L}_n^w ?

Theorem: It does!

Proof: The ‘fundamental principle’.

Example: Consider the RLC circuit.

First principles modeling (\cong CE's, KVL, & KCL)

\rightsquigarrow 15 behavioral equations.

These include both the **port and the **branch** voltages and currents.**

Why can the port behavior be described by a system of linear constant coefficient differential equations?

Because:

- 1. The CE's, KVL, & KCL are all linear constant coefficient differential equations.**
- 2. The elimination theorem.**

Why is there *exactly one* equation? Passivity!

Which PDE's describe (\vec{E}, \vec{j}) in Maxwell's equations ?

Eliminate \vec{B}, ρ from Maxwell's equations. Straightforward computation of the relevant left syzygy yields

$$\begin{aligned}\epsilon_0 \frac{\partial}{\partial t} \nabla \cdot \vec{E} + \nabla \cdot \vec{j} &= 0, \\ \epsilon_0 \frac{\partial^2}{\partial t^2} \vec{E} + \epsilon_0 c^2 \nabla \times \nabla \times \vec{E} + \frac{\partial}{\partial t} \vec{j} &= 0.\end{aligned}$$

Elimination theorem \Rightarrow this exercise would be exact & successful.

Remarks:

- **Number of equations for $n = 1$ (constant coeff. lin. ODE's)**
 \leq number of variables.

Elimination \Rightarrow fewer, higher order equations.

- **There exist effective computer algebra/Gröbner bases algorithms for elimination**

$$(R, M) \mapsto R'$$

- **Not generalizable to smooth nonlinear systems.**

Why are differential equations models so prevalent?

It follows from all this that \mathcal{L}^\bullet has very nice properties. It is **closed** under:

- **Intersection**: $(\mathcal{B}_1, \mathcal{B}_2 \in \mathcal{L}_n^w) \Rightarrow (\mathcal{B}_1 \cap \mathcal{B}_2 \in \mathcal{L}_n^w)$.
- **Addition**: $(\mathcal{B}_1, \mathcal{B}_2 \in \mathcal{L}_n^w) \Rightarrow (\mathcal{B}_1 + \mathcal{B}_2 \in \mathcal{L}_n^w)$.
- **Projection**: $(\mathcal{B} \in \mathcal{L}_n^{w_1+w_2}) \Rightarrow (\Pi_{w_1} \mathcal{B} \in \mathcal{L}_n^{w_1})$.

- **Action of a linear differential operator**:

$$\begin{aligned} (\mathcal{B} \in \mathcal{L}_n^{w_1}, P \in \mathbb{R}^{w_2 \times w_1}[\xi_1, \dots, \xi_n]) \\ \Rightarrow (P(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}) \mathcal{B} \in \mathcal{L}_n^{w_2}). \end{aligned}$$

- **Inverse image of a linear differential operator**:

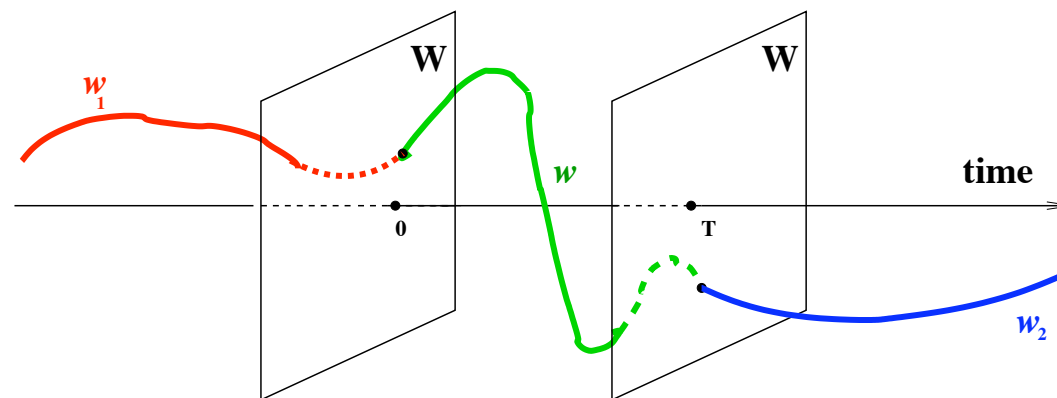
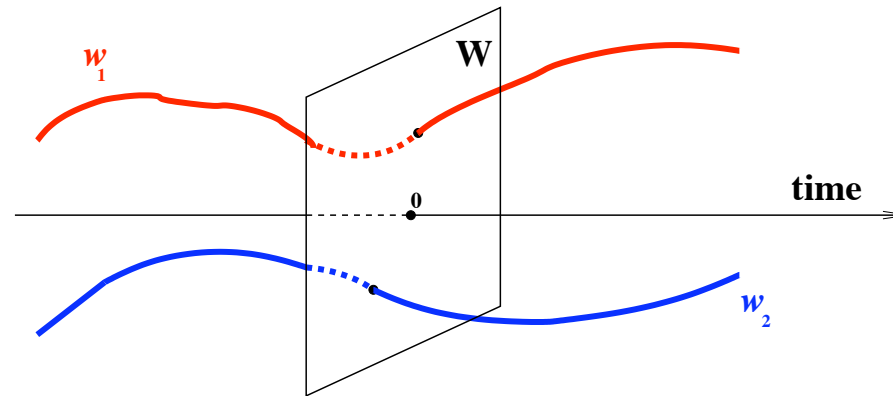
$$\begin{aligned} (\mathcal{B} \in \mathcal{L}_n^{w_2}, P \in \mathbb{R}^{w_2 \times w_1}[\xi_1, \dots, \xi_n]) \\ \Rightarrow (P(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}))^{-1} \mathcal{B} \in \mathcal{L}_n^{w_1}). \end{aligned}$$

Controllability

Controllability \Leftrightarrow

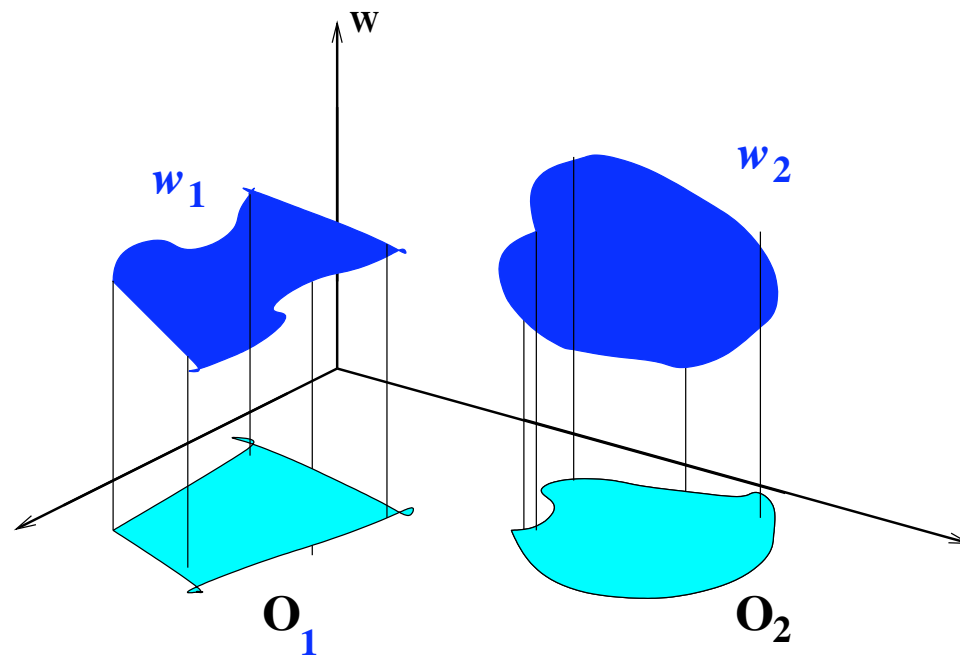
system trajectories must be 'patch-able', 'concatenable'.

Case $n = 1$

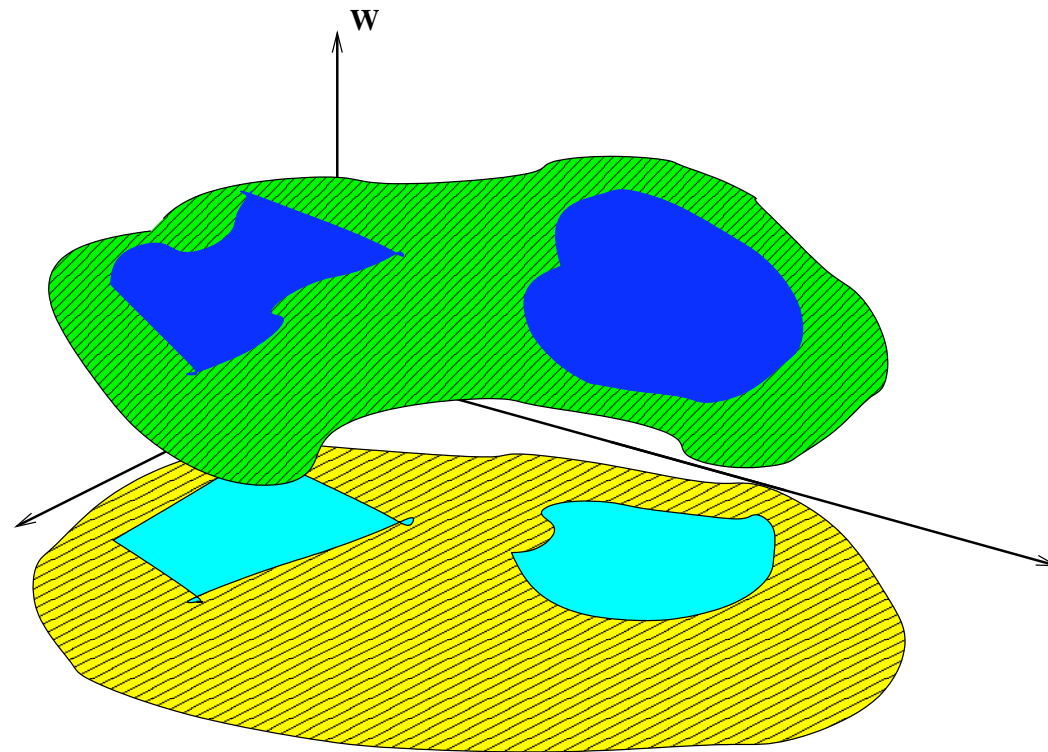


General n.

Consider two solutions:



Controllability = patchability:



Is the system defined by

$$R_0 w + R_1 \frac{d}{dt} w + \cdots + R_n \frac{d^n}{dt^n} w = 0,$$

with $w = (w_1, w_2, \dots, w_w)$ and $R_0, R_1, \dots, R_n \in \mathbb{R}^{g \times w}$,

i.e., $R(\frac{d}{dt})w = 0$, **controllable?**

We are looking for conditions on the polynomial matrix R
and algorithms in the coefficient matrices R_0, R_1, \dots, R_n .

Thm: $R\left(\frac{d}{dt}\right)w = 0$ defines a **controllable** system if and only if

$\text{rank}(R(\lambda))$ is independent of λ for $\lambda \in \mathbb{C}$.

Example: $r_1\left(\frac{d}{dt}\right)w_1 = r_2\left(\frac{d}{dt}\right)w_2$ (w_1, w_2 scalar)

is controllable if and only if **r_1 and r_2 have no common factor.**

Example: The electrical circuit is controllable unless

$$CR_C = \frac{L}{R_L} \text{ and } R_C = R_L.$$

Non-example: $R \in \mathbb{R}^{w \times w}[\xi]$, $\det(R) \neq \text{constant}$.

Image representations

Representations of \mathfrak{L}_n^w :

$$R\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right)w = 0$$

called a *'kernel' representation* of $\mathfrak{B} = \ker(R(\frac{d}{dt}))$;

$$R\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right)w = M\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right)\ell$$

called a *'latent variable' representation* of the manifest behavior

$$\mathfrak{B} = \left(R\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right)\right)^{-1} M\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right) \mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R}^\ell).$$

Missing link:

$$w = M\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right)\ell$$

called an *'image' representation* of $\mathfrak{B} = \text{im}\left(M\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right)\right)$.

Elimination theorem \Rightarrow **every image is also a kernel.**

?? Which kernels are also images ??

Theorem: The following are equivalent for $\mathfrak{B} \in \mathfrak{L}_n^w$:

1. \mathfrak{B} is **controllable**,

2. \mathfrak{B} admits an **image representation**,

3. for any $a \in \mathbb{R}^w[\xi_1, \dots, \xi_n]$,

$a^\top \left[\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \right] \mathfrak{B}$ equals 0 or all of $\mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$,

4. $\mathbb{R}^w[\xi_1, \dots, \xi_n] / \mathfrak{N}_{\mathfrak{B}}$ is **torsion free**,

etc.

Are Maxwell's equations controllable ?

The following equations in the *scalar potential* $\phi : \mathbb{R} \times \mathbb{R}^3 \rightarrow \mathbb{R}$ and the *vector potential* $\vec{A} : \mathbb{R} \times \mathbb{R}^3 \rightarrow \mathbb{R}^3$, generate exactly the solutions to Maxwell's equations:

$$\vec{E} = -\frac{\partial}{\partial t}\vec{A} - \nabla\phi,$$

$$\vec{B} = \nabla \times \vec{A},$$

$$\vec{j} = \varepsilon_0 \frac{\partial^2}{\partial t^2} \vec{A} - \varepsilon_0 c^2 \nabla^2 \vec{A} + \varepsilon_0 c^2 \nabla(\nabla \cdot \vec{A}) + \varepsilon_0 \frac{\partial}{\partial t} \nabla \phi,$$

$$\rho = -\varepsilon_0 \frac{\partial}{\partial t} \nabla \cdot \vec{A} - \varepsilon_0 \nabla^2 \phi.$$

Proves controllability. Illustrates the interesting connection

controllability $\Leftrightarrow \exists$ potential!

Remarks:

- Algorithm: R + syzygies + Gröbner basis
 \Rightarrow numerical test for on coefficients of R .
- \exists complete generalization to PDE's
- \exists partial results for nonlinear systems
- Kalman controllability is a straightforward special case

Observability

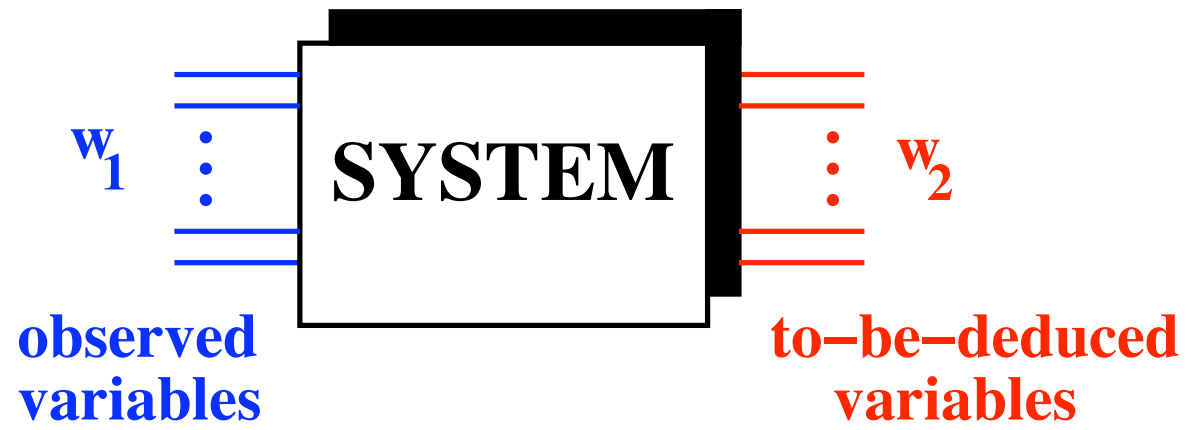
Consider the system $\Sigma = (\mathbb{T}, \mathbb{W}_1 \times \mathbb{W}_2, \mathfrak{B})$.

Each element of the behavior \mathfrak{B} hence consists of
a pair of trajectories (w_1, w_2) .

w_1 : observed; w_2 : to-be-deduced.

Recall: w_2 is said to be *observable* from w_1

if $((w_1, w'_2) \in \mathfrak{B}, \text{ and } (w_1, w''_2) \in \mathfrak{B}) \Rightarrow (w'_2 = w''_2)$,
i.e., if on \mathfrak{B} , there exists a map $w_1 \mapsto w_2$.



When is in

$$R_1\left(\frac{d}{dt}\right)w_1 = R_2\left(\frac{d}{dt}\right)w_2$$

w_2 observable from w_1 ?

If and only if $\text{rank}(R_2(\lambda)) = \text{coldim}(R_2)$ for all $\lambda \in \mathbb{C}$.

In general, if and only if there exists ‘consequences’ (i.e. elements of $\mathfrak{N}_{\mathfrak{g}}$) of the form $w_2 = F\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right)w_1$.

The RLC circuit is **observable** (**branch variables** observable from **external port variables**) iff $CR_C \neq \frac{L}{R_L}$.

\exists a complete theory (for constant coefficient ODE’s and PDE’s), including algorithms, observer design, etc.

Observability is **analogous** (but not ‘**dual**’) to controllability.

Further results

Many additional problem areas have been studied from the behavioral point of view.

- **System representations:** input/output representations, state representations and construction, model reduction, symmetries
- **System identification** \Rightarrow the most powerful unfalsified model **(MPUM)**, approximate system ID
- **Observers**
- **Control**
- **Quadratic differential forms, dissipative systems, \mathcal{H}_∞ -control**
- **Distributed parameter systems**

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Is it worth worrying about these 'axiomatizations'?

They have a deep and lasting influence! Especially in teaching.

Examples:

- **Probability** and the theory of stochastic processes as an axiomatization of **uncertainty**.
- The development of **input/output ideas** in system theory and control - often these axiomatizations are implicit, but nevertheless much very present.
- **QM.**

Thank you for your patience & attention

Details & copies of the lecture frames are available from/at

`Jan.Willems@esat.kuleuven.ac.be`

`http://www.esat.kuleuven.ac.be/~jwillems`